On complements of complete Kähler domains

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Chapter 1

Introduction

1.1 Background

In 1906, F. Hartogs observed that in $\mathbb{C}^2$ there exists such kind of domains: across their boundaries, every holomorphic functions can extend. In other words, not every domain in $\mathbb{C}^n$ with $n \geq 2$ is a domain of holomorphy, in contrary to the one dimensional case. This is the so-called Hartogs' extension theorem, which can be seen as one of the symbols of the independent development of several complex variables (henceforth, SCV).

Later H. Cartan and P. Thullen introduced the notion of holomorphic convexity and showed its equivalence to being a domain of holomorphy. In the theory of complex manifolds, holomorphic convexity leads naturally to the notion of Stein manifolds (introduced by K. Stein [40] in 1951). Immediately the characterization of holomorphic convexity / Steinness became an important topic in SCV and around it various theories have been developed.

For example, E. E. Levi tried to find equivalent conditions of being domains of holomorphy via the description of their boundaries, i.e., Levi pseudoconvexity. This is the well-known Levi problem, which was solved in the affirmative first by K. Oka in 1940s. Following from that, many powerful and influential results were obtained in this direction.

Another way to describe Stein manifolds is to use complete Kähler metrics, which was first considered by H. Grauert [20].

A complete Kähler manifold $(M, \mathrm{d}s^2)$ is a complex manifold M together with a complete Kähler metric $\mathrm{d}s^2$. It is known that every Stein manifold carries a complete Kähler metric, i.e., it is a complete Kähler domain. A natural question is whether or not the converse holds, i.e., if M is non-compact and complete Kähler, is M necessarily Stein?

It was observed by Grauert that the answer is negative. Instead, for any closed analytic subvariety A of a Stein manifold M, he constructed a complete Kähler metric on $M \setminus A$ (Satz A in [20]). In other words, in order to guarantee holomorphic convexity, besides existence of complete Kähler metrics, some additional assumptions are necessary , which generally divide into two kinds of approaches:

- Curvature assumptions on the complete Kähler metrics;
- Boundary regularity assumptions on the domains under consideration.

In the same paper, Grauert showed that if $\Omega \subset M$ carries a complete Kähler metric and has a C^ω boundary, then Ω is Stein (Satz C in [20]). Later the regularity assumption was reduced from C^ω to C^1 by T. Ohsawa [30].

In this article, we mainly follow the second line and study complete Kähler manifolds from the viewpoint of function theory. To be more precise, we study complete Kähler metrics by means of their potentials which are plurisubharmonic functions.

In SCV, plurisubharmonic functions form the natural counterpart of subharmonic functions in $\mathbb{C}$. They are invariant under biholomorphic mappings and possess many other good properties. Especially, in comparison with holomorphic functions, which are in some sense rigid, plurisubharmonic functions admit flexibility for modifications so that it is convenient to construct new functions as we desire.

At the same time, many notions in the classical potential theory can be generalized to several complex variables with subharmonic functions replaced by plurisubharmonic functions, e.g., pluripolar set, negligible sets, thin sets, etc. They are the important objects of study in the so-called pluripotential theory and play great roles in removable singularities and extension problems of analytic objects, etc.

We will focus on these small sets. First it is not difficult to see the following propositions:

(A) Except the trivial case (the whole domain), analytic varieties are complete pluripolar.

(B) Outside closed complete pluripolar sets one can construct complete Kähler metrics. In other words, closed complete pluripolar sets are contained in the complements of complete Kähler domains.

Their relations can be shown in the following graph:

$$\begin{array}{ccc}
\{\text{closed analytic varieties}\} & & \\
\cap & & \\
\{\text{closed complete pluripolar sets}\} & \subset & \{\text{pluripolar sets}\} \\
\cap & & \shortparallel \\
\{\text{complements of complete Kähler domains}\} & & \{\text{negligible sets}\}
\end{array}$$

The converse of the inclusions in the left column is of great interest.

On complex-analyticity of real submanifolds as complements of complete Kähler domains, Ohsawa [29] showed that for the two real codimensional case, merely C^1-regularity is sufficient. As a corollary, he also gave a partial answer to Nishino's problem [28], which can be seen as a partial converse to (A). It conjectured that if the graph of a continuous function is pluripolar, then the function is holomorphic. This problem was finally solved by N. Shcherbina [38].

However, K. Diederich and J. E. Fornæss later considered the higher codimensional case and showed that C^ω-regularity is necessary. As counterexamples, they constructed a closed C^∞ submanifold A of any real codimension $k \geq 3$ in a ball B, such that A is not complex-analytic and $B \setminus A$ admits a complete Kähler metric [15].

1.2 Main Results

As our first new result, we generalize Diederich–Fornæss' examples on open manifolds to the compact case. More precisely, for any $k \in \mathbb{N}, k \geq 3$, we construct a compact C^∞ submanifold A of real codimension k in $\mathbb{P}^n$, such that A is not complex-analytic and $\mathbb{P}^n \setminus A$ admits a complete Kähler metric (Theorem 4.1.5).

In light of the results in the classical potential theory, similar problems were posed for pluripotential theory, e.g., the equivalence between locally and globally pluripolar or complete pluripolar sets. However, different techniques were developed.

In 1978, B. Josefson first showed the equivalence between local and global pluripolarity in Stein manifolds [23]. Later E. Bedford and B. A. Taylor defined a new capacity with the help of complex Monge–Ampère operators and gave an alternative proof [6, 7]. At the same time, they got the equivalence between pluripolarity and negligibility (the right column in the graph above). The similar problem for closed complete pluripolar sets was solved by M. Colţoiu in 1990, i.e., in Stein manifolds, a closed locally complete pluripolar set is also globally complete pluripolar [11, 12].

Inspired by the fact (B) and Colţoiu's result, we consider the following problem:

Main Question. *Is it possible to patch up the potentials of complete Kähler metrics to obtain a global one?*

In other words, if a set is locally the complement of complete Kähler domains, is it globally the complement of some complete Kähler domain?

For a precise setting, we start with the potentials instead of complete Kähler metrics themselves. Otherwise, we need to extend the definition of the fundamental forms induced by these metrics so that we can solve the $\partial\overline{\partial}$-equations to obtain the potentials. The extension usually requires strong assumptions on the sets across which it is done. The known result is that the sets should be complete pluripolar [39, 19]. However, as the fact (B) mentioned above has shown, the existence of complete Kähler metrics outside closed complete pluripolar sets implies that the question has been solved in this case. So we choose a more general assumption and prove the following:

Theorem 1.2.1. *Assume M is a Stein manifold and $A \subset M$ is a closed subset. If $M \setminus A$ locally admits complete Kähler metrics in the following sense:*

- $\{U_i\}_{i\in\mathbb{N}}$ *is a locally finite open covering of M with $U_i \Subset M, i \in \mathbb{N}$;*
- *on each U_i, there exists $\varphi_i \in \mathrm{PSH}(U_i) \cap C^\infty(U_i \setminus A)$ such that $\partial\overline{\partial}\varphi_i$ gives a complete Kähler metric on $U_i \setminus A$ along $A \cap U_i$,*

then there exists a complete Kähler metric on $M \setminus A$ induced by a globally defined plurisubharmonic function on M. Moreover, this potential can be chosen to be bounded from below and smooth outside A. In particular, if every local potential is continuous, the global potential is also continuous.

The idea of the proof is as follows: we divide the problem into two cases depending on whether the potentials are bounded. For the first case, if all potentials are bounded, we use cut-off functions to extend their domains of definition to the whole of M. However, some negativities may be brought in this process. In order to remove them, we need to compose the strictly plurisubharmonic exhaustion function of M with a suitably chosen increasing convex function. By adding this term to the sum of extended potentials, we can find a global potential. For the second case, if on some open subset there exists a potential unbounded from below, we modify it to be bounded and reduce this case to the first one.

1.3 Notes

The article is organized as follows: In Chapter 2, we introduce some notions and recall basic properties. In Chapter 3, we discuss various kinds of small sets, such as pluripolar and complete pluripolar sets, negligible sets, the complements of complete Kähler domains, etc. In Chapter 4, an example of compact smooth, but not complex-analytic complements of complete Kähler domains is given. This generalizes an example given by Diederich–Fornæss. In Chapter 5, we prove the Main Theorem and pose some further questions.

The result in Chapter 4 has been accepted for publication and will appear in *Complex Analysis and Geometry, Springer Proceedings in Mathematics and Statistics.*

Chapter 2

Preliminaries

2.1 Plurisubharmonic functions

The main objects of this article are plurisubharmonic functions. First we recall the definition and some basic properties which will be used in the later part.

Definition 2.1.1 (plurisubharmonic function). *Assume M is a complex manifold of dimension n and $u : M \to [-\infty, +\infty)$. u is said to be plurisubharmonic if*

- *u is upper semicontinuous.*
- *its restriction $u|_C$ to any complex curve $C \hookrightarrow M$ is subharmonic.*

Suppose $u : M \to \mathbb{R}$ is twice continuously differentiable (henceforth, C^2). u is said to be strictly plurisubharmonic if its complex Hessian, which is an Hermitian form on TM defined by

$$\mathrm{H}u(z) = \sum \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}(z) \mathrm{d}z_i \otimes \mathrm{d}\overline{z}_j,$$

is positive definite everywhere.

The set of all plurisubharmonic functions on an open subset $U \subset M$ is usually denoted by $\mathrm{PSH}(U)$.

Like subharmonicity, plurisubharmonicity is also a local property.

Obviously, it is convenient to use complex Hessians to characterize smooth plurisubharmonic functions. For general plurisubharmonic functions, the following smoothing technique by convolution provides a useful approximation.

Let $\chi \in C^\infty(\mathbb{C}^n, [0, +\infty))$ with support in the closed unit ball in $\mathbb{C}^n$ such that $\int_{\mathbb{C}^n} \chi(z) \mathrm{d}\lambda(z) = 1$, where λ denotes the Lebesgue measure in $\mathbb{C}^n$. For $\epsilon > 0$, define $\chi_\epsilon(z) = \frac{1}{\epsilon^{2n}} \chi(\frac{z}{\epsilon})$.

Theorem 2.1.2. *Assume u is plurisubharmonic on an open subset $U \subset \mathbb{C}^n$. If $\epsilon > 0$ is such that $U_\epsilon := \{z \in U \mid \operatorname{dist}(z, \partial U) > \epsilon\} \neq \emptyset$, then*

$$u * \chi_\epsilon(z) = \int_{\mathbb{C}^n} \chi_\epsilon(z - w) u(w) \mathrm{d}\lambda(w)$$

is smooth and plurisubharmonic on U_ϵ. Moreover, $u * \chi_\epsilon$ monotonically decreases as $\epsilon \to 0$, and $\lim_{\epsilon \to 0} u * \chi_\epsilon(z) = u(z)$ for any $z \in U$.

Proposition 2.1.3. *The set of plurisubharmonic functions forms a convex cone in the vector space of semicontinuous functions, i.e.,*

- *if f is plurisubharmonic and $c > 0$, then $c \cdot f$ is plurisubharmonic;*
- *if f_1, f_2 are plurisubharmonic, then the sum $f_1 + f_2$ is plurisubharmonic.*

It is known that plurisubharmonicity is preserved by holomorphic substitutions.

Proposition 2.1.4. *Assume U and U' are open sets in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. If u is a plurisubharmonic function on U' and $f : U \to U'$ is a holomorphic mapping, then $u \circ f$ is plurisubharmonic on U.*

Proposition 2.1.5. *Assume $\{u_i\}_{i \in \mathbb{N}}$ is a family of plurisubharmonic functions on an open set $U \subset \mathbb{C}^n$, which are locally uniformly bounded from above. Define the upper envelope $u(z) = \sup_i u_i(z)$. Then its upper semicontinuous regularization*

$$u^*(z) = \limsup_{\zeta \to z} u(\zeta)$$

is plurisubharmonic on U.

Remark. For an uncountable family of plurisubharmonic functions $\{u_\alpha\}_{\alpha \in \Lambda}$, Choquet's lemma guarantees that there exists a countable subfamily $\{v_i = u_{\alpha(i)}\}$ such that its upper envelope v satisfies $v \leq u \leq u^* = v^*$.

A related notion is the negligible set, which exactly locates in the part modified in the process of upper semicontinuous regularization. The precise definition is as follows:

Definition 2.1.6 (negligible set)**.** *$E \subset \mathbb{C}^n$ is said to be negligible if there exists a family of plurisubharmonic functions $\{u_i\}_{i \in \mathbb{N}}$ which is locally uniformly bounded from above, such that*

$$E \subset \{z \in \mathbb{C}^n \mid u(z) < u^*(z)\},$$

where $u(z) = \sup_i u_i(z)$ and u^* is its upper semicontinuous regularization.

The following technique is often used to modify plurisubharmonic functions.

Theorem 2.1.7. *Assume $f : \mathbb{R} \to \mathbb{R}$ is an increasing convex function and u is plurisubharmonic on an open subset $U \subset \mathbb{C}^n$. Then $f \circ u$ is also plurisubharmonic on U.*

An important notion is the polar set of a plurisubharmonic function.

Definition 2.1.8 (pluripolar set). *$A \subset \mathbb{C}^n$ is said to be pluripolar if for any $z \in \mathbb{C}^n$, there exists a neighbourhood U of z and a plurisubharmonic function u on U, such that $u \not\equiv -\infty$ and $A \cap U \subset \{z \in U \mid u(z) = -\infty\}$.*
A is said to be complete pluripolar if the last " $\subset$ "is replaced by " $=$ ".

Example 2.1.9. Analytic varieties are complete pluripolar.

For a system of local defining functions $\{f_i\}$ of an analytic subset A, consider $\varphi := \log \sum_i |f_i|^2$.

Pluripolar sets are small sets, in the sense that they have zero Lebesgue measure, which follows easily from that they are local integrable, and they are of Hausdorff dimension at most $2n-2$ [1]. Their properties and relation with complete Kähler metrics will be studied in more details in Chapter 3.

2.2 Global regularization

In order to obtain global properties from local properties, the following patching and regularization procedure for continuous plurisubharmonic functions is efficient and important in different proofs.

Theorem 2.2.1 (Richberg's regularization [34], cf. [14]). *Assume M is a complex manifold and u is a continuous plurisubharmonic function on M. If u is strictly plurisubharmonic on an open subset $U \subset M$ with $\mathrm{H}u \geq \gamma$ for some continuous positive Hermitian form γ on U, then for any continuous function μ on U, $\mu > 0$, there exists a continuous plurisubharmonic function $\widetilde{u}$ on M such that*

- *$\widetilde{u}$ is C^∞ strictly plurisubharmonic on U with $u \leq \widetilde{u} \leq u + \mu$;*
- *$\mathrm{H}\widetilde{u} \geq (1-\mu)\gamma$ on U;*
- *$\widetilde{u} = u$ on $M \setminus U$.*

In particular, if u is strictly plurisubharmonic on M, then $\widetilde{u}$ can be chosen to be strictly plurisubharmonic on M as well.

For the proof, the following regularized max is the central ingredient in order to obtain a smooth patching function from merely continuous ones, which is also of independent interest.

Let $\theta \in C^\infty(\mathbb{R}, [0, +\infty))$ with support in $[-1, 1]$ such that $\int_{\mathbb{R}} \theta(h)\mathrm{d}h = 1$, $\int_{\mathbb{R}} h\theta(h)\mathrm{d}h = 0$. Set $\theta_\eta(h) := \frac{1}{\eta}\theta(\frac{h}{\eta})$ for $\eta > 0$.

Lemma 2.2.2. *For $\eta = (\eta_1, \ldots, \eta_p) \in (0, +\infty)^p$, let*

$$M_\eta(t_1, \ldots, t_p) := \int_{\mathbb{R}^p} \max_i \{t_i + h_i\} \prod_{j=1}^{p} \theta_{\eta_j}(h_j)\mathrm{d}h_1 \ldots \mathrm{d}h_p.$$

Then it satisfies the following properties:

- $M_\eta(t)$ *is nondecreasing in all variables, smooth and convex;*
- $\max\{t_1, \dots, t_p\} \le M_\eta(t_1, \dots, t_p) \le \max\{t_1 + \eta_1, \dots, t_p + \eta_p\}$*;*
- $M_\eta(t_1, \dots, t_p) = M_{(\eta_1,\dots,\hat{\eta}_j,\dots,\eta_p)}(t_1, \dots, \hat{t}_j, \dots, t_p)$ *if* $t_j + \eta_j \le \max_{k \ne j}\{t_k - \eta_k\}$*;*
- $M_\eta(t_1 + a, \dots, t_p + a) = M_\eta(t_1, \dots, t_p) + a, \forall a \in \mathbb{R}$*;*
- *if* $u_1, \dots, u_p$ *are plurisubharmonic functions and satisfy* $\mathrm{H}u_j \ge \gamma$ *for a continuous Hermitian form* γ*, then* $u := M_\eta(u_1, \dots, u_p)$ *is plurisubharmonic and satisfies* $\mathrm{H}u \ge \gamma$*.*

2.3 Convexity conditions

We will solve the main problem in the setting of Stein manifolds. They can be seen as the counterpart of domains of holomorphy in the category of complex manifolds and satisfy one geometric convexity condition - pseudoconvexity, which plays an important role in the proof.

Definition 2.3.1 (Stein manifold)**.** *Assume M is a complex manifold. M is said to be Stein if*

- *M is holomorphically convex, i.e., for any $K \Subset M$,*

$$\widehat{K}_{\mathcal{O}(M)} := \{z \in M \mid |f(z)| \le \sup_K |f|, \forall f \in \mathcal{O}(M)\} \Subset M.$$

- *M is holomorphically separable, i.e., for any $x, y \in M, x \ne y$, there exists $f \in \mathcal{O}(M)$ such that $f(x) \ne f(y)$.*
- *For every $p \in M$ there exist functions $f_1, \dots, f_n \in \mathcal{O}(M)$, whose differentials $\mathrm{d}f_j$ are $\mathbb{C}$-linearly independent at p.*

$\widehat{K}_{\mathcal{O}(M)}$ is called the holomorphically convex hull of K (with respect to M).

The third condition means that global holomorphic functions provide local charts at each point.

Remark. The holomorphically convex hull $\widehat{K}_{\mathcal{O}(M)}$ of K can be seen as an analogue of the geometrically convex hull $\widehat{K}$ of K, where in the place of the class of all holomorphic functions in the definition above, it is restricted to the class of all linear functions in $\mathbb{C}^n$. Similarly, the polynomial convex hull $\widehat{K}_p$ and plurisubharmonically convex hull $\widehat{K}_{\mathrm{PSH}(M)}$ can be defined in $\mathbb{C}^n$, respectively. They have the following inclusion relation:

$$\widehat{K} \supset \widehat{K}_p \supset \widehat{K}_{\mathcal{O}(M)} \supset \widehat{K}_{\mathrm{PSH}(M)}.$$

Example 2.3.2. An open set in $\mathbb{C}^n$ is Stein if and only if it is a domain of holomorphy.

Definition 2.3.3 (exhaustion function)**.** *Assume X is a topological space. $\psi : X \to [-\infty, +\infty)$ is said to be an exhaustion function if all of its sublevel sets are relatively compact in X, i.e.,*

$$\{z \in X \mid \psi(z) < c\} \Subset X, \forall c \in \mathbb{R}.$$

Definition 2.3.4 (pseudoconvexity)**.** *Assume M is a complex manifold. M is said to be*

- *pseudoconvex if there exists a smooth plurisubharmonic exhaustion function ψ on M.*
- *strongly pseudoconvex if there exists a smooth strictly plurisubharmonic exhaustion function ψ on M.*

Theorem 2.3.5. *Every holomorphically convex manifold is pseudoconvex. Every Stein manifold is strongly pseudoconvex.*

In fact, it is easy to see that for a holomorphically convex manifold M, there exists a sequence of holomorphically convex compact subsets $\{K_i\}_{i\in\mathbb{N}}$ exhausting M, i.e., $\cup_i K_i = M, \widehat{K_i}_{\mathcal{O}(M)} = K_i, K_i \subsetneq K_{i+1}$.

For any $a \in L_i := K_{i+2} \setminus \mathring{K}_{i+1}$, there exists $g_{i,a} \in \mathcal{O}(M)$ such that $\sup_{K_i} |g_{i,a}(z)| < 1, |g_{i,a}(a)| > 1$. Hence, $|g_{i,a}(z)| > 1$ in a neighbourhood of a. We can select finitely many $\{g_{i,a}\}_{a\in\Lambda_i}$ such that

$$h_i(z) := \max_{a\in\Lambda_i}\{|g_{i,a}(z)|\} > 1 \text{ on } L_i, h_i(z) < 1 \text{ on } K_i.$$

By choosing a large enough exponent p_i, we have

$$f_i(z) := \sum_{a\in\Lambda_i} |g_{i,a}(z)|^{2p_i} \geq i \text{ on } L_i, f_i(z) \leq 2^{-i} \text{ on } K_i.$$

It follows that

$$\psi(z) := \sum_{i\in\mathbb{N}} f_i(z)$$

converges uniformly to a C^ω plurisubharmonic function and exhausts M.

If M is Stein, then one can use the condition of holomorphically separation in Definition 2.3.1 to construct a smooth nonnegative strictly plurisubharmonic function u on M. Hence, $\psi' = \psi + u$ serves as a smooth strictly plurisubharmonic exhaustion function for M.

2.4 Complete Kähler manifolds

Assume (M, g) is a complex manifold together with an Hermitian metric given by

$$g = \mathrm{d}s^2 = \sum g_{i\bar{j}} \mathrm{d}z_i \otimes \mathrm{d}\bar{z}_j.$$

The length of a differential curve $\gamma : [a, b] \to M$ is defined by

$$\ell(\gamma) = \int_a^b |\gamma'(t)|_g = \int_a^b \sqrt{\sum g_{i\bar{j}}(\gamma(t))\gamma_i'(t)\overline{\gamma_j'(t)}}\mathrm{d}t.$$

The distance of two points p, q in M is defined by

$$d(p, q) = \inf \ell(\gamma_{pq}),$$

where γ_{pq} runs over all piecewise differential curves joining p and q, if they are in the same connected component of M, otherwise $d(p, q) = +\infty$.

This induced distance turns M to a metric space.

Definition 2.4.1 (complete metric)**.** *(M, g) is said to be complete if it is complete as a metric space with respect to the induced distance d, i.e., every Cauchy sequence in M converges to a point in M.*

To verify completeness, the following theorem is useful as a criterion.

Theorem 2.4.2 (Hopf–Rinow)**.** *Assume (M, g) is a connected Riemannian manifold. The following are equivalent:*

(A) Any closed and bounded subset of M is compact.

(B) M is complete as a metric space.

(C) M is geodesically complete, i.e., for any p in M, the exponential map $\exp_p$ is defined on the entire tangent space T_pM.

Furthermore, any one of the above implies that for any $p, q \in M$, there exists a length minimizing geodesic connecting p and q.

Remark. Condition (A) above is topologically equivalent to the following two conditions, which are sometimes easier for applications.

(D) If $\{K_i\}_{i\in\mathbb{N}}$ is a sequence of compact subsets of M which exhausts M, and $\{q_i\}_{i\in\mathbb{N}}$ is a sequence of points in M such that $q_i \notin K_i$ for all $i \in \mathbb{N}$, then $d(p, q_i) \to +\infty$ for any p in M.

(E) If γ is a non relatively compact differential curve, i.e., γ cannot be contained in any compact subset of M, then γ has $+\infty$ length with respect to g.

For every Hermitian metric as above, there exists an associate $(1, 1)$-form

$$\omega = \frac{\sqrt{-1}}{2} \sum g_{i\bar{j}}\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_j.$$

Definition 2.4.3 (Kähler metric)**.** *An Hermitian metric $\mathrm{d}s^2$ is said to be Kähler if its associate form ω is* d*-closed, i.e.,* $\mathrm{d}\omega = 0$.
A complex manifold admitting a Kähler metric is said to be a Kähler manifold.

It is easy to see that if φ is a strictly plurisubharmonic function on M, then $\partial\overline{\partial}\varphi$ is a Kähler metric on M. φ is called the Kähler potential of this metric.

Remark. The coefficient $\sqrt{-1}/2$ before $\partial\overline{\partial}g$ is to make the form ω real. But in this article, it is usually omitted for simplicity, since there is no difficulty to tell Hermitian forms or matrices from $(1,1)$-forms.

Another approach is to define

$$\mathrm{d}^c = \sqrt{-1}(\overline{\partial} - \partial),$$

which is also a real operator like d. Therefore, $\mathrm{dd}^c = 2\sqrt{-1}\partial\overline{\partial}$. This is the conventional notation for defining complex Monge–Ampère operators, which will be discussed later.

Example 2.4.4. The Euclidean metric in $\mathbb{C}^n$ is complete Kähler.

Example 2.4.5. The Fubini–Study metric in $\mathbb{P}^n$ is complete Kähler.

Example 2.4.6. The Poincaré metric on the punctured unit disc $\mathbb{D}^*$ is complete Kähler. Its potential is given by

$$-\log(-\log|z|).$$

Example 2.4.7. One important kind of Kähler metrics is the Bergman metric, which can be defined for any bounded domain.

Assume Ω is a bounded domain in $\mathbb{C}^n$ and $K_\Omega(z,w)$ is the Bergman kernel for Ω. Then the potential of the Bergman metric is

$$\log K_\Omega(z,z).$$

S. Kobayashi [26] posed a question: Which bounded pseudoconvex domains in $\mathbb{C}^n$ are complete with respect to the Bergman metric? So far, it is known that various additional assumptions can make the Bergman metric complete. For example, Ohsawa proved that the Bergman metric of any pseudoconvex domain with C^1-boundary is complete [31]. Any bounded hyperconvex domain (i.e., it admits a bounded continuous plurisubharmonic exhaustion function) is Bergman complete [5, 9].

Chapter 3

Small sets

3.1 Pluripolar sets

The structure of pluripolar sets may be very complicated, even in $\mathbb{C}$, which can be seen from the following example contained in [33].

Example 3.1.1. Assume $K \Subset \mathbb{C}$ has no isolated points and $\{z_n\}$ is a countable dense subset of K. Take $\{a_n\}$ a sequence of positive numbers such that $\sum_n a_n < +\infty$. Then

$$u(z) := \sum_n a_n \log|z - z_n|, z \in \mathbb{C}$$

is subharmonic on $\mathbb{C}$ and $u \not\equiv -\infty$.

In fact, if we take μ as a finite measure on $\mathbb{N}$ with $\mu(\{n\}) = a_n$ and define

$$\begin{array}{rcl} v : \mathbb{C} \times \mathbb{N} & \to & [-\infty, +\infty) \\ (z, n) & \mapsto & \log|z - z_n|, \end{array}$$

then

$$\int_{\mathbb{N}} v(z, n) \mathrm{d}\mu(n) = \sum_n a_n \log|z - z_n| = u(z), z \in \mathbb{C}.$$

It is easy to check that v is measurable on $\mathbb{C} \times \mathbb{N}$ and $z \mapsto \sup_n v(z, n)$ is locally bounded from above. It suffices to prove u is subharmonic on any $D \Subset \mathbb{C}$. By subtracting a constant, we can assume $v \leq 0$ on D. Consider a sequence $z_i' \to z$ in D. According to reverse Fatou's lemma,

$$\begin{array}{rcl} \limsup_{i\to\infty} u(z_i') & \leq & \int_{\mathbb{N}} \limsup_{i\to\infty} v(z_i', n) \mathrm{d}\mu(n) \\ & \leq & \int_{\mathbb{N}} v(z, n) \mathrm{d}\mu(n) = u(z). \end{array}$$

The second inequality follows because $v(\cdot, n)$ is subharmonic and upper semicontinuous for fixed n. This shows the upper semicontinuity of u on D.

Next, consider $\overline{D}(z,r) \subset D$ for $r > 0$ small enough. It follows that

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\mathrm{d}\theta &= \int_{\mathbb{N}} \frac{1}{2\pi}\int_0^{2\pi} v(z+re^{i\theta},n)\mathrm{d}\theta\mathrm{d}\mu(n) \\ &\geq \int_{\mathbb{N}} v(z,n)\mathrm{d}\mu(n) = u(z). \end{aligned}$$

This means u satisfies the submean inequality. $u(z) > -\infty$ for $z \in \mathbb{C}\setminus K$ implies $u \not\equiv -\infty$.

Furthermore, set $E := \{u(z) = -\infty\}$. Since $z_n \in E, \forall n \in \mathbb{N}$, E is a dense subset of K. It follows that

$$K \setminus E = \bigcup_{n\geq 1} \{u \in K \mid u(z) \geq -n\},$$

the right hand side of which is a countable union of closed nowhere dense sets, i.e., a meagre set (of Baire 1st category). If E is countable, then K must be meagre as well, which is a contradiction. Therefore, E is uncountable.

In this article, and also in many other papers, for simplicity, the pluripolar sets under consideration are always assumed to be closed.

3.2 Bounded potentials

It is known that outside a closed complete pluripolar set of $\mathbb{C}^n$, one can construct a complete Kähler metric in the following way.

Example 3.2.1. If $A \subset \mathbb{C}^n$ is a closed complete pluripolar set of the form

$$A = \{\varphi = -\infty\},$$

where φ is a plurisubharmonic function and smooth outside A, then there exists a complete Kähler metric on $\mathbb{C}^n \setminus A$.

In fact, its potential can be written as

$$\psi = h(\varphi),$$

where $h(t) := -\log(-t)\chi(t+3) + K\alpha(t)$, $\chi(t) \in C^\infty(\mathbb{R}, [0,1])$ with $\chi = 1$ on $(-\infty, 0]$ and $\chi = 0$ on $[1, +\infty)$, $\alpha(t) \in C^\infty(\mathbb{R}, [0, +\infty))$ with $\alpha = 0$ on $(-\infty, -4]$ and $\alpha''(t) > 0$ on $(-4, +\infty)$, $K > 0$ is chosen large enough such that $h(t)$ is increasing and convex.

In other words, this means every closed complete pluripolar set is the complement of some complete Kähler domain.

Remark. The construction above is suggested by Example 2.4.6. What we need for the completeness near the polar set A is the behavior of the function $-\log(-t)$ as $t \to -\infty$, so a cut-off function is used to extend its domain of definition to $\mathbb{R}$.

Note that the potential ψ is not bounded from below near A.

Sometimes bounded potentials (plurisubharmonic functions are always locally bounded from above, here boundedness means they are also bounded from below) are more convenient for applications. A detailed and skillful constructive proof of a bounded Kähler potential for the punctured complex plane $\mathbb{C}^*$ is contained in [15], the idea of which may trace back to Grauert [20].

Lemma 3.2.2. *There exists a continuous subharmonic function ϕ on $\mathbb{C}$ such that*

- *ϕ is radially symmetric, i.e., $\phi(z) = \phi(|z|)$;*
- *ϕ is smooth on $\mathbb{C}^*$;*
- *$\partial\overline{\partial}\phi$ gives a complete Kähler metric on $\mathbb{C}^*$.*

In fact, an alternative construction can be easily given by using $\frac{1}{\log(-t)}$ instead of $-\log(-t)$ in the representation of h in Example 3.2.1 and extending

$$\frac{1}{\log(-\log|z|)}$$

in a similar way . The same idea will be used in the proof of Main Theorem in Chapter 5.

3.3 Extension theorems and removable singularities

We have shown how to construct complete Kähler metrics outside closed complete pluripolar sets. Another close relation between these two notions is that pluripolar sets play important roles in the study of extensions and removable singularities of analytic objects, e.g., holomorphic functions, plurisubharmonic functions and closed positive forms or more general, currents. For example,

Theorem 3.3.1. *Assume M is a complex manifold and $A \subset M$ is a closed pluripolar set. If u is a plurisubharmonic function on $M \setminus A$ which is locally bounded from above near A, then it can extend uniquely into a plurisubharmonic function $\widetilde{u}$ on M.*

It is important to study the special kind of functions satisfying some desired growth rate.

For an open subset $D \subset \mathbb{C}^n$, let

$$A^2(D) := \mathcal{O}(D) \cap L^2(D).$$

Here $L^2(D)$ is the space of all square integrable functions in D, i.e., $\int_D |f|^2 \mathrm{d}\lambda < +\infty$, where λ is the Lebesgue measure in $\mathbb{C}^n$.

A classical result in the potential theory in $\mathbb{C}$ is the following

Theorem 3.3.2 ([8, 37], cf. [13])**.** *If $E \subset \mathbb{C}$ is closed, then the following are equivalent:*

(A) E is polar;

(B) $A^2(\mathbb{C} \setminus E) = \{0\}$;

(C) for any open set $D \supset E$, $A^2(D \setminus E) = A^2(D)$.

Later, J. Siciak proved the higher dimensional version:

Theorem 3.3.3 (Siciak [37])**.** *Assume M is a complex manifold and $A \subset M$ is a closed pluripolar set. Then every L^2 holomorphic function can extend across A.*

Remark. In the higher dimensional case, according to Hartogs' extension theorem, the converse does not hold any more.

For (A) $\Rightarrow$ (C) of Theorem 3.3.2, the original proof used an equivalent condition of polarity: a subset E of $\mathbb{C}$ is polar if and only if $c(E) = 0$, where $c(E)$ is the logarithmic capacity of E (see e.g., [8, 13, 33]). Recently, Chen–Wu–Wang [10] gave an alternative proof based on the gradient estimate for subharmonic functions, which also leads to an Ohsawa–Takegoshi type extension theorem for a single point in a bounded complete Kähler domain. Note that the original Ohsawa–Takegoshi extension occurs in pseudoconvex domains. And similar techniques are used to show the following

Theorem 3.3.4 (Chen–Wu–Wang [10])**.** *Assume $\Omega \subset \mathbb{C}^n$ is a domain and $E \subset \Omega$ is a closed pluripolar subset. Then every $\varphi \in W^{1,2}_{loc}(\Omega) \cap \mathrm{PSH}(\Omega \setminus E)$ can extend to a plurisubharmonic function on Ω.*

For our problem, if we want to obtain the potential of a complete Kähler metric by solving $\partial\overline{\partial}$-equations, the first step is to extend the domain of definition of the Kähler form. The following theorem on currents (roughly speaking, forms with distribution coefficients) will be needed:

Theorem 3.3.5 (Skoda–El Mir [39, 19, 36])**.** *Assume M is a complex manifold and $A \subset M$ is a closed complete pluripolar set. If a closed positive current Θ on $M \setminus A$ is locally integrable around A, then the trivial extension of Θ to M is closed.*

Since the assumption of complete pluripolarity of A is very strong, we choose to start with Kähler potentials instead.

3.4 Equivalence between local and global properties

In the classical potential theory in $\mathbb{R}^n$, the equivalences between local and global polar / complete polar sets of subharmonic functions are well known.

The similar problem on pluripolar sets was posed for the pluripotential theory, namely, Problem I of Lelong (due to A. Sadullaev [35] and E. Bedford [4]). It remained open for a long time until it was first solved by Josefson in 1978 [23]:

Theorem 3.4.1 (Josefson [23], cf. [22])**.** *Assume M is a Stein manifold. If $A \subset M$ is locally pluripolar, then A is globally pluripolar.*

Later Bedford and Taylor defined a new capacity by means of complex Monge–Ampère operators and gave an alternative and concise proof [6, 7].

Their method is very powerful, but one problem is that it is difficult to distinguish complete pluripolar sets from the others. The similar problem on complete pluripolar sets was solved by Colţoiu in 1990:

Theorem 3.4.2 (Colţoiu [11, 12])**.** *Assume M is a Stein manifold. If $A \subset M$ is closed and locally complete pluripolar, then A is globally complete pluripolar.*

The idea of his proof is: by composition with a well chosen increasing convex function, the plurisubharmonic functions for complete pluripolar sets can be modified to be of bounded differences between each other on common parts, then a corresponding series of continuous functions can be taken to add to these plurisubharmonic functions to make their maximum continuous. The strictly plurisubharmonic exhaustion function is modified to remove the negativity brought by these continuous functions, so that Richberg's regularization is applicable.

If we want to follow Bedford–Taylor's approach for the complete pluripolar case, a criterion for complete pluripolarity is necessary. The following notion which was first introduced by A. Zeriahi [42], is found to be useful.

Definition 3.4.3 (pluripolar hull)**.** *Assume $\Omega \subset \mathbb{C}^n$ is a domain and $E \subset \Omega$ is pluripolar. Then*

$$E^*_\Omega := \{z \in \Omega \mid u(z) = -\infty, \forall u \in \mathrm{PSH}(\Omega), u|_E = -\infty\}$$

is called the pluripolar hull of E with respect to Ω.

It is easy to prove the following proposition for complete pluripolar sets.

Proposition 3.4.4. *Assume $E \subset \Omega$ is complete pluripolar. Then $E = E^*_\Omega$ and E is of type G_δ.*

It was conjectured that the converse holds as well, but it still remains open. So far, the best result is given by Zeriahi [42] with an additional assumption on E.

Theorem 3.4.5 (Zeriahi [42])**.** *Assume Ω is a pseudoconvex domain and E is a pluripolar subset of Ω. If there exists F of type F_σ and G of type G_δ such that $F \subset E \subset E^*_\Omega \subset G$, then there exists $\widetilde{E} \subset \Omega$ such that $\widetilde{E}$ is complete pluripolar and $F \subset \widetilde{E} \subset G$.*
In particular, if E is of type F_σ and G_δ and $E = E^*_\Omega$, then E is complete pluripolar.

Chapter 4

Examples of complements of complete Kähler domains

4.1 History

Grauert first considered to connect complete Kähler metrics with Stein manifolds. One direction is easy.

Theorem 4.1.1. *Every Stein manifold admits complete Kähler metrics.*

Assume φ is the strictly plurisubharmonic exhaustion function for a Stein manifold M. Without loss of generality, we can assume $\varphi \geq 0$. (Otherwise consider $\exp \varphi$ instead of φ.) One can check that $\partial\overline{\partial}\varphi^2$ gives a complete Kähler metric on M.

For the converse, Grauert found that not every complex manifold M, $\dim_{\mathbb{C}} M \geq 2$, carrying a complete Kähler metric is Stein. Instead, for any closed analytic subvariety A of M, there exists a complete Kähler metric on $M \setminus A$ (Satz A in [20]).

One question arises from the above observation: what kind of condition can force the complement of a complete Kähler manifold to be complex-analytic?

The two real codimensional case was considered by Ohsawa:

Theorem 4.1.2 (Ohsawa [29])**.** *Assume M is a complex manifold, and $A \subset M$ is a closed C^1 submanifold of real codimension 2. If $M \setminus A$ admits a complete Kähler metric, then A is complex-analytic.*

Later Diederich and Fornæss considered the higher codimensional case and showed:

Theorem 4.1.3 (Diederich–Fornæss [15])**.** *Assume M is a complex manifold, and $A \subset M$ is a closed real-analytic submanifold of real codimension ≥ 3. If $M \setminus A$ admits a complete Kähler metric, then A is complex-analytic.*

Notice that in Ohsawa's result, the C^1-regularity condition is sufficient. In the contrary, in higher codimensional case, even smoothness is not able to guarantee the analyticity. In other words, real-analyticity is necessary, due to the following:

Theorem 4.1.4 (Diederich–Fornæss [15]). *For any $k \in \mathbb{N}, k \geq 3$, there exists a closed C^∞ submanifold A of real codimension k in a ball B, such that A is not complex-analytic and $B \setminus A$ admits a complete Kähler metric.*

The above examples were constructed on open manifolds. After some modifications, we generalize their example to the compact case and obtain the following result:

Theorem 4.1.5. *For any $k \in \mathbb{N}, k \geq 3$, there exists a compact C^∞ submanifold A of real codimension k in $\mathbb{P}^n$, such that A is not complex-analytic and $\mathbb{P}^n \setminus A$ admits a complete Kähler metric.*

4.2 Constructive proof of compact examples

Here we are going to directly construct compact examples of non-complex submanifolds of arbitrary real codimension ≥ 3 in the complements of complete Kähler domains.

Firstly, we construct a two real dimensional submanifold $A \subset \mathbb{C}^3$ as a graph over $S^1 \times S^1$ (instead of a graph over $\mathbb{R}^2$ cf. [15]) together with a complete Kähler metric near A.

The proof is based on the following key lemma.

Lemma 4.2.1. *Assume $f = F|_{S^1 \times S^1}$ where $F(z_1, z_2)$ is a polynomial on $\mathbb{C}^2$. Γ_f is the graph of f. Fix a point $p \notin \Gamma_f$ with $\operatorname{dist}(p, \Gamma_f) \geq 1$. For given $n \in \mathbb{N}, \epsilon > 0$, there exists a C^∞ strictly plurisubharmonic function ϕ on $\mathbb{C}^3$ and $h = H(z_1, z_2)|_{S^1 \times S^1}$ where H is a polynomial on $\mathbb{C}^2$ such that*

(A) $|\mathrm{D}^\alpha(h - f)| < \epsilon$ *on* $S^1 \times S^1, |\alpha| \leq n$;

(B) $|\phi| < \epsilon$ *on* $B_{2n} := \{z \mid |z| \leq 2n\}$;

(C) $|\mathrm{D}^\alpha \phi| < \epsilon$ *on* $B_{2n} \cap \{z \mid \operatorname{dist}(z, \Gamma_f) \geq \epsilon\}, |\alpha| \leq n$;

(D) the distance from p to $\Gamma_h \cap B_{2n}$ measured in B_{2n} with respect to the metric induced by $\partial\overline{\partial}\phi$ is no less than n.

Proof. Assume ϕ is a continuous subharmonic function on $\mathbb{C}$ as constructed in Lemma 3.2.2.

Let Z_F be the graph of F and $\phi_F := \phi(z_3 - F(z_1, z_2))$. Then ϕ_F is continuous plurisubharmonic on $\mathbb{C}^3$. One term $|z|^2$ can be added to ϕ_F to make it strictly plurisubharmonic. Then we can use Richberg's regularization to $\phi_F + |z|^2$ to get a continuous strictly plurisubharmonic function $\widetilde{\phi_F}$ on $\mathbb{C}^3$ such that

- $\widetilde{\phi_F}$ is smooth in an open neighbourhood of $B_{2n} \cap \{z \mid \operatorname{dist}(z, \Gamma_f) \geq \frac{\epsilon}{2}\}$;
- $\widetilde{\phi_F} = \phi_F + |z|^2$ on $B_{2n} \cap \{z \mid \operatorname{dist}(z, \Gamma_f) \leq \frac{\epsilon}{4}\}$.

We can scale $\widetilde{\phi_F}$ to satisfy the following properties:

- $|\widetilde{\phi_F}| < \frac{\epsilon}{16}$ on B_{2n};
- $|D^\alpha \widetilde{\phi_F}| < \frac{\epsilon}{16}$ on $B_{2n} \cap \{z \mid \operatorname{dist}(z, \Gamma_f) \geq \frac{\epsilon}{2}\}$ for all $|\alpha| \leq n$.

Then we use Richberg's regularization once more to $\widetilde{\phi_F}$ to get a C^∞ strictly plurisubharmonic function ϕ_1 such that the last two properties hold with $\widetilde{\phi_F}$ replaced by ϕ_1 and furthermore,

- for any curve $\gamma : [0,1] \to B_{2n}$ going from p to $q \in \Gamma_f$ with $\gamma((0,1)) \subset B_{2n} \backslash \Gamma_f$, γ has length at least $n+1$ with respect to $\partial\overline{\partial}\phi_1$ unless it satisfies the following condition $(\star)$:

$$\gamma(\tau) \notin A := \{z \mid |z_3 - F(x_1, x_2)| \geq \frac{\epsilon}{16}, 1 - \eta \leq |z_j| \leq 1 + \eta, j = 1, 2\}$$
$$\text{for all } \tau \geq t := \sup\{\tau \in [0,1] \mid \operatorname{dist}(\gamma(\tau), \Gamma_f) \geq \frac{\epsilon}{8}\},$$

  where η is independent of γ and small enough such that $Z_F \cap A \cap B_{2n} = \emptyset$.

The condition $(\star)$ means that γ approaches Γ_f along Z_F. It makes sense because due to the construction of ϕ, any curve going into Z_F transversely in B_{2n} has $+\infty$ length with respect to the metric induced by $\partial\overline{\partial}\phi_F$ and therefore $\partial\overline{\partial}\phi_1$.

The above property involving $(\star)$ can also be stated as:

- there exists $\delta_1 > 0, \delta_1 \ll \epsilon$, such that any curve $\gamma : [0,1] \to B_{2n} \setminus \Gamma_f$ going from p to q with $\operatorname{dist}(q, \Gamma_f) \leq \delta_1$ has length at least n with respect to the metric induced by $\partial\overline{\partial}\phi_1$ unless γ satisfies $(\star)$.

Next we will make some modifications on z_1 and z_2 directions. We can choose a polynomial $P(z_1)$ and let $G(z_1, z_2) := F(z_1, z_2) + P(z_1)$ such that

- $|P(z_1)| < \frac{\delta_1}{2}$ for all $|z_1| = 1$;
- $|P^{(k)}(z_1)| < \frac{\epsilon}{3}$ for all $|z_1| = 1, k \leq n$;
- $\operatorname{dist}((z_1, z_2, G(z_1, z_2)), \Gamma_f) > \epsilon$ for $|z_1| = 1 \pm \frac{\eta}{2}$.

Let $g := G|_{S^1 \times S^1}$ and repeat the above process to $\phi_G + |z|^2$ to obtain a smooth strictly plurisubharmonic function ϕ_2 such that

- $|\phi_2| < \frac{\epsilon}{16}$ on B_{2n};
- $|D^\alpha \phi_2| < \frac{\epsilon}{16}$ on $B_{2n} \cap \{z \mid \operatorname{dist}(z, \Gamma_f) \geq \frac{\epsilon}{2}\}$ for all $|\alpha| \leq n$;

- there exists $\delta_2 > 0, 0 < \delta_2 \ll \delta_1, \delta_2 < \frac{\eta}{2}$ such that any curve $\gamma : [0,1] \to B_{2n} \setminus \Gamma_g$ going from p to any q with $\mathrm{dist}(q, \Gamma_g) \leq \delta_2$ has length at least n with respect to the metric induced by $\partial\overline{\partial}(\phi_1 + \phi_2)$ unless γ satisfies $(\star)$ and $1 - \frac{\eta}{2} \leq |z_1| \leq 1 + \frac{\eta}{2}$ for all $\tau \geq t$.

Similarly, we can choose a polynomial $Q(z_2)$ and let $H(z_1, z_2) := G(z_1, z_2) + Q(z_2)$ such that

- $|Q(z_2)| < \delta_2$ for all $|z_2| = 1$;
- $|Q^{(k)}(z_2)| < \frac{\epsilon}{3}$ for all $|z_2| = 1, k \leq n$;
- $\mathrm{dist}((z_1, z_2, H(z_1, z_2)), \Gamma_f) > \epsilon$ for $|z_2| = 1 \pm \frac{\eta}{2}$.

Let $h := H|_{S^1 \times S^1}$ and repeat the same process to $\phi_H + |z|^2$ to get a smooth strictly plurisubharmonic function ϕ_3 such that

- $|\phi_3| < \frac{\epsilon}{16}$ on B_{2n};
- $|D^\alpha \phi_3| < \frac{\epsilon}{16}$ on $B_{2n} \cap \{z \mid \mathrm{dist}(z, \Gamma_f) \geq \frac{\epsilon}{2}\}$ for all $|\alpha| \leq n$;
- any curve $\gamma : [0,1] \to B_{2n} \setminus \Gamma_h$ going from p to any q with $\mathrm{dist}(q, \Gamma_h) \leq \delta_2$ has length at least n with respect to the metric induced by $\partial\overline{\partial}(\phi_1 + \phi_2 + \phi_3)$ unless it satisfies $(\star)$ and $1 - \frac{\eta}{2} \leq |z_j| \leq 1 + \frac{\eta}{2}, j = 1, 2$ for all $\tau \geq t$.

However, the last property is impossible since A defined in $(\star)$ and $\{|z_j| < 1 - \frac{\eta}{2}$ or $|z_j| > 1 + \frac{\eta}{2}\}, j = 1, 2$ wrap up Γ_h, which means any curve going to Γ_h has to intersect either of these three sets. □

Remark. In fact, since any periodic function on $\mathbb{R}$ can be considered as a function defined on S^1, if we allow the variable to take complex values, we get $\mathbb{C}^*$ as the complexification of S^1 and a function defined on $\mathbb{C}^*$. Therefore, in the statement of the lemma, we can take F as such extension of f from $S^1 \times S^1$ to $\mathbb{C}^{*2}$ conversely, where f can be chosen to be rational functions on $\mathbb{C}^2$ with a permissible pole at the origin.

Proposition 4.2.2. *There exists $A \subset \mathbb{P}^3$ given by the graph of a C^∞ function over $S^1 \times S^1$ such that $\mathbb{P}^3 \setminus A$ admits a complete Kähler metric.*

Proof. Let $f_1 \equiv 0, p = (\frac{3}{2}i, 0, 0) \in \mathbb{C}^3$. By Lemma 4.2.1 there exists a smooth strictly plurisubharmonic ϕ_1 and a polynomial f_2 such that $|\mathrm{D}^\alpha(f_2 - f_1)| < \frac{1}{2}$ on $S^1 \times S^1, |\alpha| \leq 1$; $|\phi_1| < \frac{1}{2}$ on B_2; $|\mathrm{D}^\alpha \phi_1| < \frac{1}{2}$ on $B_2 \cap \{z \mid \mathrm{dist}(z, \Gamma_{f_1}) \geq \frac{1}{4}\}, |\alpha| \leq 1$; the distance from p to $\Gamma_{f_2} \cap B_2$ with respect to $\partial\overline{\partial}\phi_1 \geq 1$.

Assume that smooth strictly plurisubharmonic functions $\phi_1, ..., \phi_{k-1}$ and polynomials $f_1, ..., f_k$ have been chosen satisfying the following conditions:

(A) $|\mathrm{D}^\alpha(f_j - f_{j-1})| < \frac{1}{2^{j-1}}$ on $S^1 \times S^1, |\alpha| \leq j - 1, j = 2, ..., k$;

(B) $|\phi_j| < \frac{1}{2^j}$ on $B_{2^j}, j = 1, ..., k - 1$;

(C) $|\mathrm{D}^\alpha \phi_j| < \frac{1}{2^j}$ on $B_{2^j} \cap \{z \mid \mathrm{dist}(z, \Gamma_{f_j}) \geq \frac{1}{2^{j+1}}\}, |\alpha| \leq j, j = 1, ..., k - 1$;

(D) the distance from p to $\Gamma_{f_k} \cap B_{2^{\alpha(j,k)}}$ in $B_{2^{\alpha(j,k)}}$ with respect to $\partial\overline{\partial}\phi_j \geq \alpha(j,k) := j - 1 + \frac{1}{2^{k-1}}, j = 1, ..., k-1$.

According to Lemma 4.2.1, one can take a polynomial f_{k+1} and a smooth strictly plurisubharmonic function ϕ_k such that

(A') $|\mathrm{D}^\alpha(f_{k+1} - f_k)| < \delta_k$ on $S^1 \times S^1, |\alpha| \leq j-1, j = 1, ..., k$ where δ_k will be determined later;

(B') $|\phi_k| < \frac{1}{2^k}$ on B_{2^k};

(C') $|\mathrm{D}^\alpha \phi_k| < \frac{1}{2^k}$ on $B_{2^k} \cap \{z \mid \mathrm{dist}(z, \Gamma_{f_k}) \geq \frac{1}{2^{k+1}}\}, |\alpha| \leq k$;

(D') the distance from p to $\Gamma_{f_{k+1}}$ in $B_{2^{k+1}}$ with respect to $\partial\overline{\partial}\phi_k \geq k$.

The number δ_k in (A') should be chosen small enough such that the distance of p to $\Gamma_{f_{k+1}}$ in $B_{2^{\alpha(j,k+1)}}$ with respect to $\partial\overline{\partial}\phi_j \geq \alpha(j, k+1), j = 1, ..., k$. Therefore, the conditions (A-D) hold for the new set of $f_1, ..., f_{k+1}$ and $\phi_1, ..., \phi_k$.

It follows from (A) that $f_k \to f_\infty$ in the C^∞-topology on $S^1 \times S^1$ and from (B) that $|z|^2 + \sum_{j=1}^k \phi_j \to \phi$ uniformly on every compact subset such that ϕ is continuous on $\mathbb{C}^3$ and C^∞ on $\mathbb{C}^3 \setminus \Gamma_{f_\infty}$ where $\partial\overline{\partial}\phi$ induces a complete Kähler metric. Combining with a Fubini–Study metric, we get the desired complete Kähler metric on $\mathbb{P}^3 \setminus \Gamma_{f_\infty}$.

□

Proof of Theorem 4.1.5. In the above construction, by restricting the function f_∞ to $\mathbb{C} \times \{0\}$, we get its graph as a smooth curve in $\mathbb{C} \times \{0\} \times \mathbb{C} \cong \mathbb{C}^2$ (the three real codimensional case). It is also easily seen that Lemma 4.2.1 and Proposition 4.2.2 can be generalized to higher dimensions, i.e., we can construct the graph $\Gamma_f \subset \mathbb{C}^n$ of a smooth function f over $\underbrace{S^1 \times ... \times S^1}_{n-1}$ and a continuous plurisubharmonic function ϕ, smooth outside Γ_f. In every case, $\partial\overline{\partial}\phi$ combined with a Fubini–Study metric will give complete Kähler metrics on $\mathbb{P}^n \setminus \Gamma_{f_\infty}$. □

4.3 Remarks

Sometimes, curvature conditions are considered when one studies the complements of complete Kähler domains. For example, Anchouche [2] used additional curvature conditions to reduce the compact complements of complete Kähler manifolds into finite point sets. It is shown here that in general there exist nontrivial examples.

In Chapter 3, we have shown that complete pluripolar sets are complements of some complete Kähler domains. Using similar techniques in [15], Diederich–Fornæss proved that complete pluripolar sets are not necessarily complex, even when they are closed C^∞ real submanifolds [16]. In $\mathbb{C}^2$, they gave a closed smooth curve, which is complete pluripolar, as the graph of a function over $\mathbb{R}$

This provides another approach to Theorem 4.1.5, if one can prove the existence of such compact pluripolar sets firstly. The problem was studied by Edlund and answered affirmatively.

Theorem 4.3.1 (Edlund [18]). *For any $k \in \mathbb{N}, k \geq 3$, there exists a compact C^∞ submanifold A of real codimension k in $\mathbb{C}^n$, such that A is complete pluripolar but not complex-analytic.*

A remaining problem is whether or not one can follow Diederich–Fornæss' method in [16] to provide an alternative (and simpler) proof of Theorem 4.3.1.

4.4 Nishino's problem

It is mentioned that analytic varieties are complete pluripolar in Example 2.1.9. For the converse, T. Nishino [28] posed the following question in 1962:

Question. *Assume $f : \mathbb{D} \to \mathbb{C}$ is a continuous function whose graph Γ_f is pluripolar in $\mathbb{C}^2$. Does it follow that f is holomorphic?*

As a corollary of Theorem 4.1.2, Ohsawa gave a partial solution under the additional assumption that the graph Γ_f is complete pluripolar and of C^1 smoothness [29]. It was Shcherbina who finally solved the problem affirmatively in 2005.

Theorem 4.4.1 (Shcherbina [38]). *Assume $\Omega \subset \mathbb{C}^n$ is a domain and $f : \Omega \to \mathbb{C}$ is a continuous function. The graph Γ_f of f is a pluripolar set if and only if f is holomorphic.*

The proof used polynomial convex hulls mentioned in the remark after Definition 2.3.1. Besides methods from the potential theory, it contained deep results from algebraic topology.

Compared with Edlund's result, it is known that Theorem 4.3.1 does not follow for $k = 2$ by constructing such pluripolar set as the graph of a continuous function over a domain in $\mathbb{C}^n$. Again, the two real codimensional case is proved to be special.

Chapter 5

Conclusion

5.1 Proof of Main Theorem

Now we are in the position to prove the main theorem of this article. It is divided into two parts: First we consider the case all Kähler potentials are bounded from below. Next we show that if there is one potential unbounded from below, then after some steps, it can be modified to be bounded.

Proposition 5.1.1. *Assume M is a Stein manifold and $A \subset M$ is a closed subset. If $M \setminus A$ locally admits complete Kähler metrics induced by bounded plurisubharmonic functions, i.e.,*

- *$\{U_i\}_{i\in\mathbb{N}}$ is a locally finite open covering of M with $U_i \Subset M, i \in \mathbb{N}$;*
- *on each U_i, there exists $\varphi_i \in \mathrm{PSH}(U_i) \cap C^\infty(U_i \setminus A)$ such that $\varphi_i \geq 0$ and $\partial\overline{\partial}\varphi_i$ gives a complete Kähler metric on $U_i \setminus A$ along $A \cap U_i$,*

then there exists a complete Kähler metric on $M\setminus A$ induced by a globally defined plurisubharmonic function on M. Moreover, this potential can be constructed to be bounded from below. In particular, if every φ_i is continuous, a continuous global potential can be chosen so that it is smooth outside A.

Proof. It is known that plurisubharmonic function is always locally bounded from above. Since for each $i \in \mathbb{N}$, φ_i is bounded, a linear transformation $t \mapsto a_i t + b_i$ with $a_i > 0$ can be used to modify each φ_i such that $1 \leq \varphi_i \leq 2$. Note that such modifications keep the completeness of the metrics.

Set $u_i := \varphi_i^2$. It follows that

$$\partial\overline{\partial}u_i = 2(\partial\varphi_i \wedge \overline{\partial}\varphi_i + \varphi_i\partial\overline{\partial}\varphi_i).$$

So we know that $u_i \in \mathrm{PSH}(U_i)$ also induces a complete Kähler metric on $U_i \setminus A$ along $A \cap U_i$. Moreover, we have the following estimate:

$$\partial\overline{\partial}u_i \geq 2\partial\varphi_i \wedge \overline{\partial}\varphi_i \geq \frac{1}{8}\partial\varphi_i^2 \wedge \overline{\partial}\varphi_i^2 = \frac{1}{8}\partial u_i \wedge \overline{\partial}u_i.$$

Note that on A, $\partial\overline{\partial}u_i$ should be understood in the sense of current. Since u_i is bounded, we can choose a decreasing sequence of smooth plurisubharmonic functions v_j which tends to u_i and define $\partial\overline{\partial}u_i = \lim \partial\overline{\partial}v_j$. $\partial\overline{\partial}\varphi_i$ is defined in the same way. Then $\partial\varphi_i \wedge \overline{\partial}\varphi_i$ and $\partial u_i \wedge \overline{\partial}u_i$ are also defined and the same estimates hold.

Take $U_i' \Subset U_i$ such that $\{U_i'\}$ still forms an open covering of M. Choose $\rho_i \in C^\infty(M)$ such that $\rho_i \geq 0, \mathrm{Supp}\ \rho_i \subset U_i, \rho_i \equiv 1$ on U_i'. Then $\rho_i u_i$ extends to a function defined on M and is smooth outside A. If $\rho_i \neq 0$, consider $\partial\overline{\partial}\rho_i u_i$ on $U_i \setminus A$:

$$\partial\overline{\partial}\rho_i u_i = u_i\partial\overline{\partial}\rho_i + \partial\rho_i \wedge \overline{\partial}u_i + \partial u_i \wedge \overline{\partial}\rho_i + \rho_i\partial\overline{\partial}u_i.$$

Since

$$\begin{aligned} 0 &\leq (\frac{4}{\sqrt{\rho_i}}\partial\rho_i + \frac{\sqrt{\rho_i}}{4}\partial u_i) \wedge (\frac{4}{\sqrt{\rho_i}}\overline{\partial}\rho_i + \frac{\sqrt{\rho_i}}{4}\overline{\partial}u_i) \\ &= \frac{16}{\rho_i}\partial\rho_i \wedge \overline{\partial}\rho_i + \partial\rho_i \wedge \overline{\partial}u_i + \partial u_i \wedge \overline{\partial}\rho_i + \frac{\rho_i}{16}\partial u_i \wedge \overline{\partial}u_i, \end{aligned}$$

it follows that

$$\begin{aligned} \partial\rho_i \wedge \overline{\partial}u_i + \partial u_i \wedge \overline{\partial}\rho_i &\geq -\frac{16}{\rho_i}\partial\rho_i \wedge \overline{\partial}\rho_i - \frac{\rho_i}{16}\partial u_i \wedge \overline{\partial}u_i \\ &\geq -\frac{16}{\rho_i}\partial\rho_i \wedge \overline{\partial}\rho_i - \frac{\rho_i}{2}\partial\overline{\partial}u_i. \end{aligned}$$

Therefore,

$$\partial\overline{\partial}\rho_i u_i \geq u_i\partial\overline{\partial}\rho_i - \frac{16}{\rho_i}\partial\rho_i \wedge \overline{\partial}\rho_i + \frac{\rho_i}{2}\partial\overline{\partial}u_i.$$

Note that

$$\frac{16}{\rho_i}\partial\rho_i \wedge \overline{\partial}\rho_i \to 0 \text{ as } \rho_i \to 0.$$

Since M is Stein, there exists a smooth strictly plurisubharmonic exhaustion function ψ on M. Let

$$\varphi := r(\psi(z)) + \sum_i \rho_i u_i(z),$$

where $r : \mathbb{R} \to \mathbb{R}$ is an increasing convex function. Locally there are only finite non-zero terms in the sum, so φ is well defined.

It is known that for each $c \in \mathbb{R}$, $\{\psi < c\} \Subset X$. Therefore, a large enough coefficient C_c can be chosen such that $C_c\partial\overline{\partial}\psi$ removes the negativity brought by $\sum_i u_i\partial\overline{\partial}\rho_i - \frac{16}{\rho_i}\partial\rho_i \wedge \overline{\partial}\rho_i$ in $\{\psi < c\}$. If r is chosen to increase rapidly enough at $+\infty$, then φ is plurisubharmonic on M such that $\mathrm{H}\varphi \geq \frac{1}{2}\mathrm{H}\varphi_i$ on $U_i' \setminus A$. Therefore, φ induces a complete Kähler metric on $M \setminus A$.

If every φ_i is continuous, φ constructed as above is also continuous. By setting μ a small positive constant and $\gamma = \mathrm{H}\varphi$, Richberg's regularization can be applied to φ on $M \setminus A$ to obtain a smooth strictly plurisubharmonic function $\widetilde{\varphi}$. The estimate $\mathrm{H}\widetilde{\varphi} \geq (1-\mu)\mathrm{H}\varphi$ implies that $\widetilde{\varphi}$ induces a complete Kähler metric on $M \setminus A$. □

For the case not every φ_i is bounded from below, we need the following lemma to reduce it into the previous case.

Lemma 5.1.2. *Under the same hypothesis as above, if on some open $U \subset M$, $\varphi \in \mathrm{PSH}(U) \cap C^\infty(U \setminus A)$ is unbounded and $\partial\overline{\partial}\varphi$ gives a complete Kähler metric on $U \setminus A$ along $A \cap U$, then there exists another potential $\widetilde{\varphi} \in \mathrm{PSH}(U) \cap C^\infty(U \setminus A)$ such that $\widetilde{\varphi}$ is bounded from below and $\partial\overline{\partial}\widetilde{\varphi}$ gives a complete Kähler metric on $U \setminus A$ along A.*

Proof. Consider

$$\Phi_1 := e^{\varphi}, \Phi_2 := h(\varphi)$$

where $h(t) := \frac{1}{\log(-t)}\chi(t+3) + K\alpha(t)$, $\chi(t) \in C^\infty(\mathbb{R}, [0,1])$ with $\chi \equiv 1$ on $(-\infty, 0]$ and $\chi \equiv 0$ on $[1, +\infty)$, $\alpha(t) \in C^\infty(\mathbb{R}, [0, +\infty))$ with $\alpha \equiv 0$ on $(-\infty, -4]$ and $\alpha''(t) > 0$ on $(-4, +\infty)$, $K > 0$ is chosen large enough such that $h(t)$ is increasing and convex. It is clear that Φ_1, Φ_2 are plurisubharmonic on U and nonnegative.

Choose any differential curve $\gamma : [0, 1) \to U \setminus A$ which is non relatively compact with respect to $M \setminus A$. If $\varphi \circ \gamma([0, 1)) > C$ for some constant C, then the computation

$$\partial\overline{\partial}\Phi_1 = e^{\varphi}(\partial\varphi \wedge \overline{\partial}\varphi + \partial\overline{\partial}\varphi) \geq e^C \partial\overline{\partial}\varphi$$

implies that

$$\ell_{\partial\overline{\partial}\Phi_1}(\gamma) \geq e^{\frac{C}{2}} \ell_{\partial\overline{\partial}\varphi}(\gamma),$$

where $\ell_{\partial\overline{\partial}\cdot}$ stands for the length with respect to $\partial\overline{\partial}\cdot$. Since $\partial\overline{\partial}\varphi$ is a complete Kähler metric on $U \setminus A$ along A, which means the latter is $+\infty$, it follows that the former is also $+\infty$.

If $\varphi \circ \gamma([0, 1))$ is unbounded from below, when $\varphi \circ \gamma(t) < -1$, the following computation

$$\begin{aligned}
& \partial\overline{\partial}\Big(\frac{1}{\log(-\varphi)}\Big) \\
= \; & \frac{2}{\log^3(-\varphi)} \frac{1}{\varphi^2} \partial\varphi \wedge \overline{\partial}\varphi + \frac{1}{\log^2(-\varphi)} \frac{1}{\varphi^2} \partial\varphi \wedge \overline{\partial}\varphi - \frac{1}{\log^2(-\varphi)} \frac{1}{\varphi} \partial\overline{\partial}\varphi \\
\geq \; & \frac{1}{\log^2(-\varphi)} \frac{1}{\varphi^2} \partial\varphi \wedge \overline{\partial}\varphi
\end{aligned}$$

implies that if $t_0 \in (0, 1)$ is chosen such that $\varphi \circ \gamma(t_0) < -3$, then $\ell_{\partial\overline{\partial}\Phi_2}(\gamma)$ has the estimate

$$\begin{aligned}
\ell_{\partial\overline{\partial}\Phi_2}(\gamma) & \geq \int_{t_0}^{1} \sqrt{\sum \frac{1}{\log^2(-\varphi)} \frac{1}{\varphi^2} \frac{\partial\varphi}{\partial z_i} \frac{\mathrm{d}z_i}{\mathrm{d}t} \overline{\frac{\partial\varphi}{\partial z_j} \frac{\mathrm{d}z_j}{\mathrm{d}t}}} \mathrm{d}t \\
& = \int_{t_0}^{1} \frac{1}{|\varphi \log(-\varphi)|} \Big|\frac{\mathrm{d}\varphi \circ \gamma(t)}{\mathrm{d}t}\Big| \mathrm{d}t \\
& \geq \liminf_{t \to 1} \int_{t_0}^{t} \frac{1}{-\varphi \log(-\varphi)} \mathrm{d}(-\varphi \circ \gamma(t)) \\
& = +\infty.
\end{aligned}$$

Therefore, $\widetilde{\varphi} := \Phi_1 + \Phi_2 \geq 0$ serves as the potential of a complete Kähler metric on $U \setminus A$ along A. $\square$

The main theorem follows immediately from Proposition 5.1.1 and Lemma 5.1.2.

5.2 Further questions

Complete pluripolar sets serve as important examples of complements of complete Kähler domains. We want to consider what kind of set satisfies this condition and give more examples.

And at the same time, we plan to consider what kind of set contains complements of complete Kähler domains as a subclass and whether or not the similar local and global equivalence holds there.

Comparing the results of Ohsawa and Shcherbina, one natural question is whether or not the C^1-regularity assumption in Theorem 4.1.2 can be weakened to C^0 as in Theorem 4.4.1, to obtain a strong version of Nishino's problem.

Among the related topics in pluripotential theory, we are interested in the conjecture pluripolar hulls of type G_δ must be complete pluripolar. This will lead the powerful tools developed by complex Monge–Ampère operators to the study of complete pluripolar sets.

Appendix A

A.1 Proof of Colţoiu's theorem

Here we will summarize the proof of Theorem 3.4.2. Following the same assumption, first we show that A can be written as follows:

Proposition A.1.1. *If M and A are defined as in Theorem 3.4.2, then there exists $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ such that $\{U_i\}$ is a locally finite open covering of M with $U_i \Subset M$, $\varphi_i \in \mathrm{PSH}(U_i)$ with $A \cap U_i = \{\varphi_i = -\infty\}$,* $\exp\varphi_i$ *is continuous, φ_i is C^∞ outside A (*cf. [19], p.19), $\varphi_i - \varphi_j$ is bounded on $U_i \cap U_j \setminus A, \forall i, j \in \mathbb{N}$.

In order to get the boundedness of differences between φ_i, we need to compose them with a suitably chosen increasing convex function.

Lemma A.1.2. *Assume $\{a_i\}_{i\in\mathbb{N}}$ is a sequence of negative numbers such that a_i decreases to $-\infty$ as $i \to \infty$. Then there exists a smooth increasing convex function $\tau : (-\infty, 0) \to (-\infty, 0)$ such that*

$$\lim_{i\to\infty} \tau(a_i) = -\infty, \tau(a_i) - \tau(a_{i+1}) < 1, \forall i \in \mathbb{N}.$$

Proof. Let

$$\tau(t) := \begin{cases} (\frac{a_1}{a_2} + \cdots \frac{a_i}{a_{i+1}}) - i - \frac{t}{a_{i+1}} & a_{i+1} \le t \le a_i, i \ge 1 \\ \frac{a_1}{a_2} - \frac{t}{a_2} - 1 & a_1 \le t < 0 \end{cases}.$$

Then we can check that $\tau(a_i) - \tau(a_{i+1}) = -\frac{a_i}{a_{i+1}} + 1 < 1$ and

$$\tau(a_i) - \tau(a_{i+p}) = \frac{a_{i+1} - a_i}{a_{i+1}} + \cdots + \frac{a_{i+p} - a_{i+p-1}}{a_{i+p}} \ge \frac{a_{i+p} - a_i}{a_{i+p}}.$$

p can be chosen large enough (depending on i) such that $\tau(a_i) - \tau(a_{i+p}) \ge \frac{1}{2}$, therefore, $\lim_{i\to\infty} \tau(a_i) = -\infty$. At last, τ can be made smooth easily. □

Lemma A.1.3. *Assume $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of increasing function with $f_n : (-\infty, 0) \to (-\infty, 0)$ satisfying* $\lim_{t\to-\infty} f_n(t) = -\infty, \forall n \in \mathbb{N}$. *Then there exists a smooth increasing convex function $\tau : (-\infty, 0) \to (-\infty, 0)$ such that* $\lim_{t\to-\infty} \tau(t) = -\infty$, *$\tau \circ f_n - \tau \circ f_m$ is bounded for* $\forall n, m \in \mathbb{N}$.

Proof. We can take $\{\lambda_i\}_{i\in\mathbb{N}}$ a sequence of negative numbers with $\lambda_i \to -\infty$ as $i \to \infty$ and

$$\min\{f_n(\lambda_i) \mid n \leq i\} > \max\{f_n(\lambda_{i+1}) \mid n \leq i+1\}, \forall i \in \mathbb{N}.$$

Let

$$\begin{aligned} a_{2i-1} &:= \min\{f_n(\lambda_i) \mid n \leq i\}, \\ a_{2i} &:= \max\{f_n(\lambda_{i+1}) \mid n \leq i+1\}. \end{aligned}$$

We can check that $\{a_i\}_{i\in\mathbb{N}}$ satisfies the assumption of Lemma A.1.2. Since $a_{2i-1} \leq \min\{f_n(t) \mid n \leq i\} \leq \max\{f_n(t) \mid n \leq i+1\} \leq a_{2i+2}$, we can find τ satisfying our requires and moreover,

$$\tau \circ f_n - \tau \circ f_m < 3.$$

□

Proof of Proposition A.1.1. We can assume that $\{U_i\}_{i\in\mathbb{N}}, \{V_i\}_{i\in\mathbb{N}}$ are locally finite open coverings of M with $U_i \Subset V_i \Subset M$ and $\varphi_i : V_i \to [-\infty, 0)$ is plurisubharmonic with $A \cap V_i = \{\varphi_i = -\infty\}, \exp\varphi_i$ is continuous and φ_i is smooth outside A.

For $\forall i, j$ such that $U_i \cap U_j \neq \emptyset$, let

$$B_{ij}(t) := \sup\{\varphi_i(x) \mid x \in U_i \cap U_j, \varphi(x) \leq t\}.$$

It is easy to see that B_{ij} is increasing and satisfies $\lim_{t\to-\infty} B_{ij}(t) = -\infty$.

According to Lemma A.1.3, there exists a smooth increasing convex function τ such that $\lim_{t\to-\infty}\tau(t) = -\infty$, $\tau \circ B_{ij} - \tau$ is bounded for $\forall i, j \in \mathbb{N}$.

If $x \in U_i \cap U_j \setminus A$, then $B_{ij}(\varphi_j(x)) \geq \varphi_i(x)$ and

$$\tau(\varphi_i(x)) - \tau(\varphi_j(x)) \leq \tau(B_{ij}(\varphi_j(x)) - \tau(\varphi_j(x)) < +\infty.$$

□

Proof of Theorem 3.4.2. Take $U_i'' \Subset U_i' \Subset U_i$ such that $\{U_i''\}$ still forms an open covering of M. Since $\varphi_i - \varphi_j$ is bounded on $U_i' \cap U_j' \setminus A$, we can choose $p_i \in C_0^\infty(M)$ such that $p_i \geq 0$, Supp $p_i \subset U_i'$ and

$$\varphi_i + p_i < \varphi_j + p_j \text{ on } \partial U_i' \cap U_j'' \setminus A.$$

Since M is Stein, there exists a strictly plurisubharmonic exhaustion function ψ. Let

$$\varphi_0 := r(\psi(z)) + \max\{\varphi_i(z) + p_i(z) \mid z \in U_i'\}.$$

φ_0 is well-defined and continuous on $M \setminus A$. Note that $\varphi_0(z) \to -\infty$ as $z \to z' \in A$. r is an increasing convex function and should be chosen to increase rapidly enough at $+\infty$ such that φ_0 is strictly plurisubharmonic on $M \setminus A$.

Then Richberg's regularization can be applied to φ_0 on $M \setminus A$ to obtain a smooth strictly plurisubharmonic function φ on $M \setminus A$. Moreover, the fact φ can be chosen arbitrarily close to φ_0 implies that $\varphi(z) \to -\infty$ as $z \to z' \in A$. □

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