# Studies on Quotient Singularities via Cohen-Macaulay Representations

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# Contents

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# Chapter 1

# Introduction

The studies of invariant subrings under the action of linearly reductive groups have been investigated in terms of commutative algebra, algebraic geometry, homological algebra etc. The typical property of an invariant subring is the Cohen-Macaulayness. It comes from Hochster and Roberts's theorem.

**Theorem 1.0.1.** *([HR], see also [HE, HO]) Let $G$ be a linearly reductive group over an algebraically closed field $k$. Suppose $V$ is a $G$-module, and $S := k[V]$ is the symmetric algebra of $V$. Then the invariant subring $S^G$ is Cohen-Macaulay.*

The point is a linearly reductive group $G$ has the "Reynolds operator", and the invariant subring $R = S^G$ becomes a pure subring of $S$ via this operator. Also, Boutot's theorem is important.

**Theorem 1.0.2.** *([Bou]) Let $S$ and $R$ be essentially of finite type over a field of characteristic zero, and $R$ is a pure subring of $S$. If $S$ has only rational singularities, then $R$ also has only rational singularities.*

On the other hand, M. Hochster and C. Huneke introduced the notion of tight closure, and several classes of rings in positive characteristic. After that, commutative algebra in positive characteristic has developed rapidly. In particular, the class so-called "(strongly) $F$-regular" (see Section 2.2) behave well in the above context. Namely, if $S$ is an $F$-regular ring and $R$ is a pure subring of $S$, then $R$ is also $F$-regular (see Proposition 2.2.8 (3)). Moreover, the $F$-regularity implies Cohen-Macaulay under mild conditions. Thus, we can recover Hochster and Roberts's theorem in positive characteristic. We remark that the $F$-rationality (see Definition 2.2.9) is considered as the analogue of a rational singularity by [Har, Smi]. However, if we replace the condition "rational" in Theorem 1.0.2 by "$F$-rational", then it is no longer true [Wat2].

In this way, the methods used in positive characteristic commutative algebra give us other aspects of studies of invariant subrings. In this thesis, we will investigate invariant subrings in the context of positive characteristic.

For a Noetherian ring $R$ of positive characteristic $p > 0$, we can define the Frobenius morphism $F : R \to R\,(r \mapsto r^p)$. Also, we define the $e$-times iterated Frobenius morphism

$F^e : R \to R$ ($r \mapsto r^{p^e}$) for each $e \in \mathbb{N}$. By using this morphism $F^e$, we define the $R$-module ${}^eR$ (see Section 2.1). In positive characteristic commutative algebra, we investigate the properties of $R$ through the structure of ${}^eR$. However, it is difficult to describe such a structure explicitly. For example,

(Q1) What kind of $R$-module appears in ${}^eR$ as a direct summand ?
(Q2) Can we understand the asymptotic behavior of ${}^eR$ ?

These kinds of problems are difficult in general. Therefore we will consider these problems for the case of quotient singularities in this thesis. That is, let $G$ be a finite subgroup of $\mathrm{GL}(d, k)$ which contains no pseudo-reflections except the identity and $S := k[x_1, \cdots, x_d]$ be a polynomial ring or $S := k[[x_1, \cdots, x_d]]$ be a power series ring. We assume that the order of $G$ is coprime to $p = \operatorname{char} k$. We denote the invariant subring of $S$ under the action of $G$ by $R := S^G$. In the rest of this thesis, we mainly consider this quotient singularity $R$ (or $\operatorname{Spec} R$). In the process of investigating the structure of ${}^eR$ and determining some numerical invariants, the theory of Cohen-Macaulay representations (especially Auslander-Reiten theory) plays the crucial role.

Let $V_0 = k, V_1, \cdots, V_n$ be the complete set of irreducible representations of $G$, and we set $M_t := (S \otimes_k V_t)^G$ $(t = 0, 1, \cdots, n)$. Then each $M_i$ is an indecomposable maximal Cohen-Macaulay (= MCM) module and $M_i \ncong M_j$ if $i \neq j$.

K. Smith and M. Van den Bergh [SVdB] showed that ${}^eR$ decomposes as follows (see Proposition 3.2.1).

$$ {}^eR \cong R^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_n^{\oplus c_{n,e}}. $$

From these observations, we could understand (Q1). Thus, we will move to the problem (Q2). Namely, we will consider the asymptotic behavior of each multiplicity $c_{i,e}$. In this situation, it is known the limit $\lim_{e \to \infty} \dfrac{c_{i,e}}{p^{ed}}$ $(i = 0, 1, \cdots, n)$ exists and it is a positive rational number [SVdB, Yao1] (see Proposition 2.5.2). Especially, for the case where $i = 0$, this limit is also known as the $F$-signature of $R$ and is denoted by $s(R) = \lim_{e \to \infty} \dfrac{c_{0,e}}{p^{ed}}$ [HL]. Also, this numerical invariant characterizes some singularities (see Theorem 2.3.4). The explicit value of the $F$-signature of the invariant subring $R$ was determined by K. Watanabe and K. Yoshida [WY2], that is $s(R) = \dfrac{1}{|G|}$. One of the purpose in this thesis is to generalize this result for each non-free direct summand. Therefore, we will consider the limit of each multiplicity $c_{i,e}$ on the order of $p^{ed}$: $s(R, M_i) = \lim_{e \to \infty} \dfrac{c_{i,e}}{p^{ed}}$ and call it the generalized $F$-signature of $M_i$ with respect to $R$ (see Section 2.5). We can determine the explicit values as follows, and this is the answer for the problem (Q2).

**Theorem 1.0.3.** *(=Theorem 3.3.1 ) Let the notation be as above. Then for all $i = 0, \cdots, n$ one has*

$$ s(M_i, R) = \frac{\dim_k V_i}{|G|} = \frac{\operatorname{rank}_R M_i}{|G|}. $$

As a corollary, we can also consider the asymptotic behavior of decomposition of ${}^eM_i$. Since ${}^eM_i$ decomposes as

$$ {}^eM_j \cong M_0^{\oplus d_{0,j,e}} \oplus M_1^{\oplus d_{1,j,e}} \oplus \cdots \oplus M_n^{\oplus d_{n,j,e}}, $$

for $e \in \mathbb{N}$ (see Section 2.5), we consider the limit

$$s(M_i, M_j) := \lim_{e\to\infty} \frac{d_{i,j,e}}{p^{ed}} \quad (i, j = 0, 1, \cdots, n),$$

and call this limit the generalized $F$-signature of $M_i$ with respect to $M_j$. Then we have the following result.

**Corollary 1.0.4.** *(= Corollary 3.3.6) Let the notation be as above. Then for all $i, j = 0, \cdots, n$ one has*

$$s(M_i, M_j) = (\dim_k V_j) \cdot s(M_i, R) = \frac{(\dim_k V_i) \cdot (\dim_k V_j)}{|G|} = \frac{(\operatorname{rank}_R M_i) \cdot (\operatorname{rank}_R M_j)}{|G|}.$$

By this corollary, we see that each MCM $R$-module $M_i$ actually appears in ${}^eM_j$ as a direct summand for some MCM module $M_j$ and sufficiently large $e \gg 0$. In dimension two, it is known that an invariant subring $R$ is of finite CM representation type, that is, it has only finitely many non-isomorphic indecomposable MCM $R$-modules $\{R, M_1, \cdots, M_n\}$ (see Chapter 4). Thus, the additive closure $\operatorname{add}_R({}^eM_i)$ coincides with the category of MCM $R$-modules $\mathrm{CM}(R)$. So we use several results so-called Auslander-Reiten theory to $\operatorname{add}_R({}^eM_i)$. Especially, the Auslander-Reiten quiver of $R$ visualize the relationship between MCM modules. By using this idea, we can investigate some numerical invariants in positive characteristic.

In this thesis, we focus on the notion of the dual $F$-signature defined by A. Sannai [San] (see Definition 2.6.1). As this name shows, this is also a kind of generalization of the $F$-signature. This invariant is defined for each finitely generated $R$-module $M$, and we denote the dual $F$-signature of $M$ by $s(M)$. Notice that the dual $F$-signature of $R$ coincides with the $F$-signature of $R$ (see Remark 2.6.2). Thus, we use the same notation. Just like the $F$-signature, the value of the dual $F$-signature of the canonical module $\omega_R$ also characterizes some singularities (see Theorem 2.6.3). How about the value of other $R$-modules? Namely, let $M$ be a finitely generated $R$-module which may not be $R$ or $\omega_R$. Then

- Does the value of $s(M)$ have any information about singularities ?
- What does the explicit value of $s(M)$ mean ?
- Is there any connection between $s(M)$ and other numerical invariants ?

However, the computation of the dual $F$-signature is difficult for now, and we don't have effective method for determining it except in only a few cases. Thus, as the first step to understand this invariant, we will consider the dual $F$-signature for some MCM modules over quotient surface singularities. In particular, by paying attention to a certain MCM $R$-module so-called a special CM module (see subsection 5.2.1) and its Auslander-Reiten translation (see Section 4.2), we characterize the Gorensteiness.

**Theorem 1.0.5.** *(= Theorem 5.2.6) Let $R$ be a quotient surface singularity. Suppose $M$ is an indecomposable special CM $R$-module. Then we have*

$$s(M) \le s(\tau(M)).$$

*where $\tau(M)$ stands for the Auslander-Reiten translation of $M$. Moreover, $R$ is Gorenstein if and only if $s(M) = s(\tau(M))$.*

For a cyclic quotient surface singularity, a special CM module takes a simple form as follows, and we can determine the explicit value of the dual $F$-signature for each special CM module. (For more details on terminologies, see Section 5.3.)

Suppose $R$ is the invariant subring of $S = k[[x, y]]$ under the action of a cyclic group $\frac{1}{n}(1, a)$. In this situation, a non-free indecomposable special CM $R$-module is described as $M_{i_t} = Rx^{i_t} + Ry^{j_t}$ (i.e., it is minimally 2-generated). Then we have the value of the dual $F$-signature as follows.

**Theorem 1.0.6.** *(= Theorem 5.3.11) Let the notation be the same as above, then for any non-free special CM R-module $M_{i_t}$ one has*

$$s(M_{i_t}) = \begin{cases} \dfrac{\min(i_t, j_t) + 1}{n} & (\textit{if } \ i_t \neq j_t) \\ \\ \dfrac{2i_t + 1}{2n} & (\textit{if } \ i_t = j_t). \end{cases}$$

As the special case of this theorem, we have the value of the dual $F$-signature for all indecomposable MCM modules over the rational double point corresponding to Dynkin type $A_{n-1}$ (see Example 5.3.14). Also, we determine the dual $F$-signature of each indecomposable MCM module for other Dynkin types ($D_n, E_6, E_7$ and $E_8$) in Section 5.4, and collect their values in subsection 5.4.8.

On the process to determine the value of the dual $F$-signature, we also consider the number of minimal generators for each indecomposable MCM module. As an application, we also investigate the notion of Ulrich modules and the Hilbert-Kunz multiplicities (see Chapter 6).

Ulrich modules are a certain class of MCM modules, and their properties have been investigated in several contexts. However, even the existence of an Ulrich module for a given CM local ring is still not known. Also, even if a given ring $R$ has an Ulrich module, we don't know the shape of such modules for many cases. Thus, in this thesis, we investigate Ulrich modules over cyclic quotient surface singularities, and give the characterization of Ulrich modules. As we mentioned before, this singularity is of finite CM representation type. Therefore, the number of indecomposable Ulrich modules is finite. So we will also consider the number of them. In this problem, special CM modules play the crucial role again. Since the number of minimal generators of a special CM module is small, special CM modules are the opposite of Ulrich modules in that sense. However, those give us the simple description of Ulrich modules. In particular, the number of indecomposable special CM modules coincides with that of irreducible exceptional curves in the minimal resolution of a cyclic quotient surface singularity (see Theorem 5.2.4), and this geometric information determines boundaries of the number of Ulrich modules.

**Theorem 1.0.7** (= Theorem 6.1.23)**.** *Suppose $R$ is a cyclic quotient surface singularity whose number of irreducible exceptional curves (= that of non-free indecomposable special CM modules) is $r$. Then the number of Ulrich modules* $\mathsf{N}$ *satisfies* $r \leq \mathsf{N} \leq 2^{r-1}$.

This thesis is organized as follows. In Chapter 2, we prepare some basic facts on singularities and numerical invariants in positive characteristic. Especially, we see that quotient singularities are strongly $F$-regular and have finite $F$-representation type. For such rings, we can define the generalized $F$-signature, and this invariant is a positive rational number. In Chapter 3, we determine the value of the generalized $F$-signature explicitly for quotient singularities. By using this result and the Auslander-Reiten quiver, we can investigate the dual $F$-signature. Therefore, we review some results of Auslander-Reiten theory in Chapter 4. In particular, we define the Auslander-Reiten quiver. In Chapter 5, we will pay attention to special CM modules, and determine the value of the dual $F$-signature of such modules. Especially, we give the complete list of the dual $F$-signature of MCM modules over rational double points. Since the methods for determining the dual $F$-signature is also valid for investigating Ulrich modules and the Hilbert-Kunz multiplicities, we discuss them in Chapter 6.

This thesis is based on papers [HN, Nak1, Nak2, NY].

# Conventions and Notations

Throughout this thesis, we suppose that $k$ is an algebraically closed field and $R$ is a Noetherian ring unless otherwise noted. We denote the set of elements in $R$ which are not in any minimal prime of $R$ by $R^\circ$. For example, if $R$ is a domain, then $R^\circ = R \setminus \{0\}$.

For a Noetherian local ring $(R, \mathfrak{m}, k)$ and a finitely generated $R$-module $M$, $\mu_R(M)$ stands for the number of minimal generators (i.e. $\mu_R(M) = \dim_k M/\mathfrak{m}M$) and $\mathrm{e}^0_{\mathfrak{m}}(M)$ is the multiplicity of $M$ with respect to $\mathfrak{m}$. If situation is clear, we denote it by $\mathrm{e}(M)$. We denote the length of a finitely generated Artinian $R$-module $N$ by $\ell_R(N)$.

For a finitely generated $R$-module $M$, we define the depth of $M$ as

$$\operatorname{depth}_R M := \inf\{i \geq 0 \mid \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}.$$

We say $M$ is a maximal Cohen-Macaulay (= MCM) $R$-module if $\operatorname{depth}_R M = \dim R$. When $R$ is non-local, we say $M$ is an MCM module if $M_{\mathfrak{p}}$ is an MCM module for all $\mathfrak{p} \in \operatorname{Spec} R$. Furthermore, we say that $R$ is a Cohen-Macaulay (= CM) ring if $R$ is an MCM $R$-module.

For a Noetherian local ring $(R, \mathfrak{m}, k)$, we will denote the canonical module of $R$ by $\omega_R$. We denote the $R$-dual (resp. the canonical dual) functor by $(-)^* := \operatorname{Hom}_R(-, R)$ (resp. $(-)^\vee := \operatorname{Hom}_R(-, \omega_R)$). We say that a finitely generated $R$-module $M$ is reflexive if the natural morphism $M \to M^{**}$ is an isomorphism. Also, we denote the $n$-th syzygy functor by $\Omega^n(-)$. Namely, take the minimal free resolution of $R$-module $M$:

$$\cdots \to F_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0,$$

then $\Omega^n(M)$ is defined as $\operatorname{Ker} \varphi_n$.

We denote $\mathrm{CM}(R)$ to be the category of MCM modules and $\operatorname{add}_R(M)$ to be the full subcategory consisting of direct summands of finite direct sums of some copies of $M$. We say $M$ is a generator if $R \in \operatorname{add}_R(M)$.

When we discuss a composition of morphisms $fg$, it means $f$ is followed by $g$, that is, $fg = g \circ f$. Similarly, for quivers an arrow $ab$ means $a$ is followed by $b$. That is, $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$. (Although this notation seems to be opposite to the usual one, when we chase a path, this notation is convenient.)

Sometimes we use freely basic facts of commutative ring theory as in [Mat, BH].

# Chapter 2

# $F$-singularities and $F$-invariants

## 2.1 Frobenius morphism

Let $R$ be a Noetherian ring of prime characteristic $p > 0$, then we can define the Frobenius morphism $F : R \to R$ $(r \mapsto r^p)$. For $e \in \mathbb{N}$, we also define the $e$-times iterated Frobenius morphism $F^e : R \to R$ $(r \mapsto r^{p^e})$.

For any $R$-module $M$, we define the $R$-module ${}^eM$ (or $F^e_*M$) via $F^e$ as follows. That is, ${}^eM$ is just $M$ as an abelian group, and for $m \in M$ we denote the corresponding element of ${}^eM$ by ${}^em$ then its $R$-module structure is defined by $r({}^em) := {}^e(F^e(r)m) = {}^e(r^{p^e}m)$ for $r \in R$. Note that ${}^eR$ is isomorphic to $R$ as a ring and ${}^eM$ is naturally an ${}^eR$-module. In such a situation, we can view ${}^eR$ as an $R$-algebra via the morphism $F^e : R \to {}^eR$ $(r \mapsto {}^e(r^{p^e}))$, and this is an $R$-linear map. Moreover, if $R$ is reduced, we can identify ${}^eR$ with the $R$-module $R^{1/p^e}$ (the $R$-algebra consisting of $p^e$-th root of elements in $R$) by associating ${}^er$ and $r^{1/p^e}$ for any $r \in R$. From this viewpoint, the $e$-times iterated Frobenius morphism $F^e$ is identified with the inclusion $R \hookrightarrow R^{1/p^e}$. We will switch these notations from each other depending on the situation.

For an ideal $I$ of $R$, we set

$$I^{[p^e]} := (a^{p^e} \mid a \in I) \subset R.$$

**Definition 2.1.1.** *We say $R$ is $F$-finite if ${}^1R$ (and hence every ${}^eR$) is a finitely generated $R$-module.*

*Remark* 2.1.2. If $R$ is $F$-finite, then $R$ is excellent [Kun2] and has a dualizing complex [Gab].

For example, if $R$ is an essentially of finite type over a perfect field or complete Noetherian local ring with a perfect residue field $k$, then $R$ is $F$-finite. In this thesis, we only discuss such rings, thus the $F$-finiteness is always satisfied.

## 2.2 $F$-singularities

In positive characteristic commutative algebra, we investigate the properties of $R$ through the structure of ${}^eR$ or ${}^eM$. A typical result is the following.

**Theorem 2.2.1** ([Kun1])**.** *Let* $(R, \mathfrak{m}, k)$ *be a d-dimensional* $F$*-finite local ring. Then the following conditions are equivalent.*

(1) $R$ *is regular,*
(2) ${}^{e}R$ *is a free* $R$*-module of rank* $p^{ed}$ *for any* $e \in \mathbb{N}$*,*
(3) ${}^{e}R$ *is a free* $R$*-module of rank* $p^{ed}$ *for a natural number* $e \in \mathbb{N}$*,*
(4) $\ell_R(R/\mathfrak{m}^{[p^e]}) = p^{ed}$ *for any* $e \in \mathbb{N}$*,*
(5) $\ell_R(R/\mathfrak{m}^{[p^e]}) = p^{ed}$ *for a natural number* $e \in \mathbb{N}$*.*

In this subsection, we will introduce some classes of rings in positive characteristic and discuss their properties. For more details, we refer the reader to [Hoc, Sch, TW] etc. Originally, some of these classes were defined in terms of tight closure introduced in [HH1]. But we can also characterize them by using the Frobenius splittings. Thus, we don't refer to the tight closure theory in this thesis. For readers who are interested in such a theory, there are some good references e.g. [BH, Hoc, Hun1].

**Definition 2.2.2.** *Let* $R$ *be an* $F$*-finite ring of* $\operatorname{char} R = p > 0$*. We say* $R$ *is* $F$*-pure if the Frobenius morphism* $F : R \to {}^{1}R$ *splits as an* $R$*-linear map. Namely, there exists* $\varphi \in \operatorname{Hom}_R({}^{1}R, R)$ *such that* $\varphi \circ F = id_R$*. Thus,* $R$ *is a direct summand of* ${}^{1}R$*.*

*Remark* 2.2.3. More precisely, this kind of ring is called $F$-split. On the other hand, the $F$-purity is originally defined by the purity of the Frobenius morphism (i.e. $M \to {}^{1}R \otimes_R M$ is injective for every $R$-module $M$). These notions coincide with each other if $R$ is $F$-finite. Thus we will use the terminology "$F$-pure".

**Lemma 2.2.4.** *Let* $R$ *be an* $F$*-finite Noetherian ring. If* $R$ *is* $F$*-pure, then it is reduced.*

In order to investigate the $F$-purity, the following criterion is convenient.

**Lemma 2.2.5.** *([Fed, Theorem 1.12]) If* $(S, \mathfrak{n})$ *is a* $F$*-finite regular local ring and* $I$ *is an ideal of* $S$*, then* $R := S/I$ *is* $F$*-pure if and only if* $I^{[p]} : I \not\subset \mathfrak{m}^{[p]}$*.*

*In particular, if* $I = (f)$ *is a principal ideal, then* $I^{[p]} : I = (f^{p-1})$*. Thus,* $S/(f)$ *is* $F$*-pure if and only if* $f^{p-1} \notin m^{[p]}$*.*

Next, we define the strong $F$-regularity.

**Definition 2.2.6.** *Let* $R$ *be an* $F$*-finite ring of* $\operatorname{char} R = p > 0$*. We say* $R$ *is strongly* $F$*-regular if for every* $c \in R^{\circ}$ *there exists* $e \in \mathbb{N}$ *such that the following morphism splits as an* $R$*-linear map,*

$$R \xrightarrow{F^e} {}^{e}R \xrightarrow{\times^{e}c} {}^{e}R \quad (x \mapsto {}^{e}(x^{p^e}) \mapsto {}^{e}(cx^{p^e})).$$

Roughly, a strongly $F$-regular ring has many splittings. It is easy to see that the strongly $F$-regularity implies the $F$-purity if we take $c = 1$ in Definition 2.2.6.

*Remark* 2.2.7. Also, there are notions of "$F$-regular ring" and "weakly $F$-regular ring". They are defined in terms of tight closure theory [HH1] and as the name implies, we have the following (cf. [HH1, HH2]).

$$\text{strongly } F\text{-regular} \Rightarrow F\text{-regular} \Rightarrow \text{weakly } F\text{-regular}$$

These notions are equivalent if a ring is $\mathbb{Q}$-Gorenstein [AM] or a ring has finite $F$-representation type (see definition 2.4.1) [Yao1, Remark 4.3]. Thus, we will use just the terminology "$F$-regular" in such situations. In general, it is still open whether these notions coincide or not.

It is known that a weakly $F$-regular ring is normal. If a ring is excellent (e.g. an $F$-finite ring) then the weakly $F$-regularity implies Cohen-Macaulay [HH2].

Here, we collect some properties of strongly $F$-regular rings.

**Proposition 2.2.8.** *(cf. [HH2, Theorem 3.1]) Let $R$ be an $F$-finite Noetherian ring with* char $p > 0$.

*(1) $R$ is strongly $F$-regular if and only if $R_{\mathfrak{p}}$ is strongly $F$-regular for every prime (or for every maximal) ideal $\mathfrak{p}$ of $R$.*
*(2) If $R$ is regular, then it is strongly $F$-regular.*
*(3) If $S$ is strongly $F$-regular (e.g. regular ring) and $R$ is a direct summand of $S$ as an $R$-module, then $R$ is also strongly $F$-regular.*

By using these properties, we can recover the Hochster-Roberts theorem (see Theorem 1.0.1) in positive characteristic.

Next, we consider the Frobenius action on local cohomology and define some singularities via this action. We will use freely a basic knowledge about local cohomology from [BH, BS, Iye et al.]. In order to make a situation clear, we consider a general ring homomorphism $\varphi : R \to S$. For an ideal $\mathfrak{a} = (a_1, \cdots, a_n)$, the local cohomology $H^i_{\mathfrak{a}}(R)$ is obtained as the cohomology of the Čech complex $\check{C}^\bullet(\underline{a}\,; R)$. We consider the morphism of complexes

$$\check{C}^\bullet(\underline{a}\,; R) \to S \otimes_R \check{C}^\bullet(\underline{a}\,; R) = \check{C}^\bullet(\varphi(\underline{a})\,; S)$$

induced via $\varphi$. Then we have a morphism of $R$-modules $H^i_{\mathfrak{a}}(R) \to H^i_{\varphi(\mathfrak{a})S}(S)$ for each $i$. Here, we consider $H^i_{\varphi(\mathfrak{a})S}(S)$ as an $R$-module via $\varphi$.

We switch the situation to a $d$-dimensional $F$-finite Noetherian local ring $(R, \mathfrak{m})$. Then the Frobenius morphism $F : R \to R$ induces $H^i_{\mathfrak{m}}(R) \to H^i_{F(\mathfrak{m})R}(R) \cong H^i_{\mathfrak{m}}(R)$. Note that the last isomorphism follows from $F(\mathfrak{m})R = \mathfrak{m}^{[p]}$, $\sqrt{\mathfrak{m}^{[p]}} = \mathfrak{m}$. By abuse of notation, we use the same letter $F$ to express this morphism. In particular, in the case of $i = d$, we have $H^d_{\mathfrak{m}}(R) \cong \varinjlim R/(x_1^n, \cdots, x_d^n)$ where $x_1, \cdots, x_n$ is a system of parameters of $R$. Thus, we can describe $F$ as

$$F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R) \quad \left( \xi = [z \bmod (x_1^n, \cdots, x_d^n)] \mapsto \xi^p = [z^p \bmod (x_1^{np}, \cdots, x_d^{np})] \right).$$

By using this action on the local cohomology, we will introduce some classes of singularity in positive characteristic.

**Definition 2.2.9** ($F$-rationality)**.** *Let $(R, \mathfrak{m})$ be a $d$-dimensional $F$-finite local ring, we say $R$ is $F$-rational if $R$ is CM and if for any $c \in R^\circ$, there is a natural number $e \in \mathbb{N}$ such that*

$$H^d_{\mathfrak{m}}(R) \xrightarrow{F^e} H^d_{\mathfrak{m}}(R) \xrightarrow{\cdot c} H^d_{\mathfrak{m}}(R) \quad (\xi \mapsto c\xi^{p^e})$$

*is injective.*

*In the case when $R$ is not local, we say $R$ is $F$-rational if the local ring $R_{\mathfrak{p}}$ is $F$-rational for any* $\mathfrak{p} \in \operatorname{Spec}(R)$.

*Remark* 2.2.10. Originally, the $F$-rationality was defined via tight closure. Since we will not refer to it, in this thesis, we consider the above condition as the definition of the $F$-rationality.

**Definition 2.2.11** ($F$-injectivity)**.** *Let $(R, \mathfrak{m})$ be a $d$-dimensional $F$-finite local ring, we say $R$ is $F$-injective if $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for all $i$.*

*When $R$ is not local, we say $R$ is $F$-injective if the local ring $R_{\mathfrak{p}}$ is $F$-injective for any* $\mathfrak{p} \in \operatorname{Spec}(R)$.

From this definition, we see that the $F$-rationality implies the $F$-injectivity.

Next, we will give another description of the $F$-rationality and the $F$-injectivity by using the trace map. When $R$ is CM, we have the following isomorphism from the local duality.

$$\operatorname{Ext}^{d-i}_R(R, \omega_R) \cong \mathsf{D}H^i_{\mathfrak{m}}(R),$$

where $\mathsf{D}(-)$ stands for the Matlis dual, and we remark that $\mathsf{D}H^d_{\mathfrak{m}}(R) \cong \omega_R$. By applying the canonical dual $(-)^\vee$ to the Frobenius morphism $F : R \to {}^1R$, we obtain the following.

$$\operatorname{Tr} : {}^1\omega_R \cong ({}^1R)^\vee \longrightarrow R^\vee \cong \omega_R.$$

This morphism is called the trace map of $R$. For each $e \in \mathbb{N}$ we also define the $e$-times iterated trace map $\operatorname{Tr}^e : {}^e\omega_R \to \omega_R$ as the canonical dual of the $e$-times iterated Frobenius morphism $F^e$. The next proposition immediately follows from the local duality. That is, $F$-rationality and $F$-injectivity are characterized by the surjectivity of the trace map. In the future, this viewpoint leads us to the definition of the dual $F$-signature (see Section 2.6).

**Proposition 2.2.12.** *Let $R$ be an $F$-finite local ring. Then*

*(1) $R$ is $F$-rational if and only if $R$ is CM and for $c \in R^\circ$, there is $e \in \mathbb{N}$ such that*

$${}^e\omega_R \xrightarrow{\times^e c} {}^e\omega_R \xrightarrow{\operatorname{Tr}^e} \omega_R \quad ({}^e r \mapsto {}^e(cr) \mapsto \operatorname{Tr}^e({}^e(cr))).$$

*is surjective.*

*(2) If $R$ is $F$-injective, then* $\operatorname{Tr} : {}^1\omega_R \to \omega_R$ *is surjective. The converse holds if $R$ is CM.*

In this way, these classes are defined by using the Frobenius morphism. Surprisingly, these singularities are closely related with singularities in minimal model program (in characteristic 0 ) via the reduction modulo $p > 0$. Although, that is one of the important reason to study singularities in positive characteristic, we entrust details to other literatures.

For $F$-finite Noetherian rings, we collect the relationship between each class of singularities in positive characteristic:

$$
\begin{tikzcd}
\text{Regular} \arrow[r, Rightarrow] & \text{strongly } F\text{-regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & F\text{-rational} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[l, bend right, dashed, Rightarrow, "\text{if quasi-Gorenstein (i.e. } R \cong \omega_R)"'] & \text{Cohen-Macaulay, normal} \\
& F\text{-pure} \arrow[r, Rightarrow] & F\text{-injective} \arrow[l, bend left, dashed, Rightarrow, "\text{if quasi-Gorenstein}"] &
\end{tikzcd}
$$

## 2.3 $F$-invariants

In this section, we suppose that $(R, \mathfrak{m}, k)$ is a $d$-dimensional $F$-finite Noetherian local ring with char $R = p > 0$ and $k$ is an algebraically closed field. We will introduce some numerical invariants in positive characteristic.

### 2.3.1 $F$-signature

As we showed in 2.2.1, 2.2.2, 2.2.6, the number of free direct summands in ${}^eR$ is very important. In order to measure such a number, we introduce the notion of $F$-signature defined by C. Huneke and G. Leuschke. This numerical invariant is defined as the asymptotic behavior of free direct summands in ${}^eR$ on the order of $\mathrm{rank}_R\, {}^eR = p^{ed}$.

**Definition 2.3.1** ([HL]). *Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional reduced $F$-finite Noetherian local ring of prime characteristic $p > 0$. For each $e \in \mathbb{N}$, we decompose ${}^eR$ as follows*

$$ {}^eR \cong R^{\oplus a_e} \oplus M_e, $$

*where $M_e$ has no free direct summands. We call $a_e$ the $e$-th $F$-splitting number of $R$. Then, we call the limit*

$$ s(R) := \lim_{e \to \infty} \frac{a_e}{p^{ed}}, $$

*the $F$-signature of $R$.*

*Remark* 2.3.2. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of $R$. Since ${}^e(\widehat{R}) \cong \widehat{R} \otimes_R {}^eR$, the $e$-th $F$-splitting number of $R$ coincides with that of $\widehat{R}$. Since the Krull-Schmidt condition holds for $\widehat{R}$, the decomposition of ${}^e(\widehat{R})$ as in Definition 2.3.1 is unique up to isomorphism.

Also, we may drop the condition "$R$ is reduced ". In that case, $R$ is not $F$-pure (see Lemma 2.2.4). Thus, we have $a_e = 0$.

*Remark* 2.3.3. Even if $k$ is not an algebraically closed field, we can obtain similar results appearing in the following section, after an appropriate modification. That is, since the rank of ${}^eR$ is $p^{e(d+\alpha(R))}$ where $\alpha(R) = \log_p[k : k^p]$, we replace $p^{ed}$ by $p^{e(d+\alpha(R))}$. Note that if $k$ is a perfect field (e.g. an algebraically closed field), then $\alpha(R) = 0$.

The existence of the $F$-signature was shown by K. Tucker [Tuc]. Roughly speaking, the $F$-signature $s(R)$ measures the deviation from regularity by Kunz's theorem 2.2.1. The next theorem confirms this intuition.

**Theorem 2.3.4** ([HL], [Yao2], [AL])**.** *Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional reduced $F$-finite Noetherian local ring with* $\operatorname{char} R = p > 0$. *Then we have*

*(1) $R$ is regular if and only if $s(R) = 1$,*
*(2) $R$ is strongly $F$-regular if and only if $s(R) > 0$.*

Also, there are some computations of the value of the $F$-signature. For example, the following result is important in Chapter 3. For more computations, see the survey article [Hun2] and the references therein.

**Theorem 2.3.5.** *([WY2, Theorem 4.2]) Let $G$ be a finite subgroup of* $\mathrm{GL}(d, k)$ *which contains no pseudo-reflections and assume that the order of $G$ is coprime to $p$ =* char $k$. *Suppose that $S$ is the power series ring $k[[x_1, \cdots, x_d]]$. We denote the invariant subring of $S$ under the action of $G$ by $R := S^G$. Then*

$$s(R) = \frac{1}{|G|}.$$

Next we consider the decomposition of ${}^eM$ and the asymptotic behavior.

**Theorem 2.3.6.** *([Tuc, Theorem 4.11]) Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional reduced $F$-finite Noetherian local ring of prime characteristic $p > 0$ and $M$ be a finitely generated $R$-module. We denote the maximal rank of a free direct summand appearing in the decomposition of ${}^eM$ by $a_e(M)$. Then we have the following:*

$$s(R, M) := \lim_{e\to\infty} \frac{a_e(M)}{p^{ed}} = \operatorname{rank}_R(M)s(R). \tag{2.3.1}$$

*Remark* 2.3.7. With this terminology, the $F$-signature is nothing but $s(R, R)$. For simplicity, we will denote it by $s(R)$ unless it causes confusion.

**Proposition 2.3.8.** *Suppose $R$ is a strongly $F$-regular ring, and $R$ is not quasi Gorenstein (i.e. $R \not\cong \omega_R$). Then*

*(1) We have $\omega_R \in \operatorname{add}_R({}^eR)$ for sufficiently large $e \gg 0$.*
*(2) Suppose ${}^eR$ decomposes as*

$${}^eR \cong R^{\oplus a_e} \oplus \omega_R^{\oplus b_e} \oplus M_e \tag{2.3.2}$$

*where $R, \omega_R \notin \operatorname{add}_R(M_e)$. Then $\displaystyle\lim_{e\to\infty} \frac{b_e}{p^{ed}} = \lim_{e\to\infty} \frac{a_e}{p^{ed}}$ (= $s(R)$).*

*Proof.* (1) By Theorem 2.3.4 and 2.3.6, we have $s(R, \omega_R) > 0$. Especially, we have $a_e(\omega_R) > 0$. Thus, $R$ appears in ${}^e\omega_R$ as a direct summand for sufficiently large $e \gg 0$. Taking the canonical dual $(-)^\vee$,

$$ {}^eR \cong ({}^e\omega_R)^\vee \cong (R^{\oplus a_e(\omega_R)} \oplus N_e)^\vee \cong \omega_R^{\oplus a_e(\omega_R)} \oplus N_e^\vee, $$

where $N_e$ has no free direct summands. So we come to the conclusion.

(2) Suppose ${}^e\omega_R$ decomposes as

$$ {}^e\omega_R \cong R^{\oplus c_e} \oplus \omega_R^{\oplus d_e} \oplus N_e, \tag{2.3.3} $$

where $R, \omega_R \notin \mathrm{add}_R(N_e)$. By applying $(-)^\vee$ to (2.3.2), we have

$$ {}^e\omega_R \cong R^{\oplus b_e} \oplus \omega_R^{\oplus a_e} \oplus M_e^\vee. $$

Thus, this implies $c_e \geq b_e$. Similarly, we apply $(-)^\vee$ to (2.3.3) and have $b_e \geq c_e$. Thus,

$$ \lim_{e\to\infty} \frac{b_e}{p^{ed}} = \lim_{e\to\infty} \frac{c_e}{p^{ed}} = \mathrm{rank}_R(\omega_R)s(R) = s(R). $$

The second equation follows from Theorem 2.3.6.

□

*Remark* 2.3.9. As the following example shows, $a_e \neq b_e$ in general. Suppose $G = \langle \mathrm{diag}(-1,-1,-1) \rangle$ is a cyclic group of order 2 and consider the natural action on $S = k[[x,y,z]]$ where $\mathrm{char}\, k > 2$. Then the invariant subring $R = S^G$ is strongly $F$-regular (see Proposition 2.2.8 (3)) and ${}^eR$ is decomposed as ${}^eR \cong R^{\oplus a_e} \oplus \omega_R^{\oplus b_e}$ where $a_e = \frac{p^{3e}+1}{2}$ and $b_e = \frac{p^{3e}-1}{2}$ (see [Sei, Example 5.2]). Thus, $\lim_{e\to\infty} \frac{a_e}{p^{3e}} = \lim_{e\to\infty} \frac{b_e}{p^{3e}} = \frac{1}{2}$ but $a_e \neq b_e$.

The next proposition is convenient to understand the structure of ${}^eR$.

**Proposition 2.3.10.** *Let $R$ be a strongly $F$-regular ring. For a finitely generated $R$-module $M$, we have*

$$ M \in \mathrm{add}_R({}^eR) \Leftrightarrow M^\vee \in \mathrm{add}_R({}^{e'}R), $$

*for some $e, e' \in \mathbb{N}$.*

*Proof.* By the duality, we may show the "only if" part. Thus, we assume $M \in \mathrm{add}_R({}^eR)$. By applying the canonical dual, we have $M^\vee \in \mathrm{add}_R({}^e\omega_R)$. From Proposition 2.3.8 (1), $\omega_R \in \mathrm{add}_R({}^fR)$ for some $f \in \mathbb{N}$. Therefore we have $M^\vee \in \mathrm{add}_R({}^{e+f}R)$ immediately. □

### 2.3.2 Hilbert-Kunz multiplicity

Next, we will review the Hilbert-Kunz multiplicity. The study of this numerical invariant in positive characteristic was started in [Kun2] and its existence was shown by P. Monsky [Mon].

**Theorem-Definition 2.3.11** (Hilbert-Kunz multiplicity)**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of characteristic $p > 0$ and $I$ be an $\mathfrak{m}$-primary ideal of $R$. Then the limit*

$$\mathrm{e}_{HK}(I, R) \coloneqq \lim_{e\to\infty} \frac{1}{p^{ed}} \ell_R(R/I^{[p^e]})$$

*exists [Mon]. Sometimes, we simply denote $\mathrm{e}_{HK}(I, R)$ by $\mathrm{e}_{HK}(I)$. We call this limit the Hilbert-Kunz multiplicity of $R$ with respect to $I$. In particular, $\mathrm{e}_{HK}(\mathfrak{m}, R) \coloneqq \mathrm{e}_{HK}(R)$ is called the Hilbert-Kunz multiplicity of $R$.*

The Hilbert-Kunz multiplicity is an analogue of the usual multiplicity in positive characteristic, and we have the inequality

$$\frac{\mathrm{e}(I)}{d!} \leq \mathrm{e}_{\mathrm{HK}}(I) \leq \mathrm{e}(I)$$

where $I$ is an $\mathfrak{m}$-primary ideal of $R$, $\dim R = d$ (cf. [Hun1, Chapter 6]). In particular, if $R$ is a one dimensional ring, we have $\mathrm{e}_{\mathrm{HK}}(I) = \mathrm{e}(I)$. Also, if $I$ is a parameter ideal, then $\mathrm{e}_{\mathrm{HK}}(I) = \mathrm{e}(I)$. This invariant plays an important role to investigate singularities in positive characteristic. For example, Kunz proved the inequality $\ell_R(R/\mathfrak{m}^{[p^e]}) \geq p^{ed}$ holds for any local ring $R$ and for all $e \in \mathbb{N}$ [Kun1]. Therefore, we have $\mathrm{e}_{\mathrm{HK}}(R) \geq 1$. Especially, if $R$ is regular, then $\mathrm{e}_{\mathrm{HK}}(R) = 1$ (see Thorem 2.2.1). Under mild conditions, we have the converse.

**Theorem 2.3.12.** *([WY1], see also [HY]) Let $R$ be an unmixed local ring with* $\operatorname{char} R = p > 0$. *If* $\mathrm{e}_{HK}(R) = 1$*, then $R$ is regular.*

Also, the following is an improved version.

**Theorem 2.3.13.** *([BE]) Let $R$ be an unmixed local ring with* $\operatorname{char} R = p > 0$.

*(1) If* $\mathrm{e}_{HK}(R) < 1 + \frac{1}{p^d d!}$*, then $R$ is regular.*
*(2) If* $\mathrm{e}_{HK}(R) < 1 + \frac{1}{d!}$*, then $R$ is $F$-rational and Cohen-Macaulay.*

In this way, we can check the properties of a ring with positive characteristic via this numerical invariant. Therefore this invariant was observed in many articles (for more details, see the survey article [Hun2] and the references contained therein). However, in the spite of its importance, it is difficult to determine the explicit value of $\mathrm{e}_{\mathrm{HK}}(R)$ in general.

In Chapter 6, we will give some computation of this invariant as the application of series of our results.

## 2.4 Finite $F$-representation type

Throughout this section, we assume that the Krull-Schmidt condition holds for $R$, that is, every $R$-module decomposes into the direct sum of indecomposable modules uniquely up to isomorphism. For example, this condition holds for a complete local ring (cf. [LW, Chapter 1], [CYZ, Appendix]).

For understanding the structure of ${}^eR$, we introduce the notion of finite $F$-representation type defined by K. Smith and M. Van den Bergh [SVdB] as follows (see also [Yao1]).

**Definition 2.4.1** ([SVdB, Yao1]). *We say $R$ has finite $F$-representation type (= FFRT for short) by $\mathcal{S}$ if there is a finite set $\mathcal{S} := \{M_0, M_1, \cdots, M_n\}$ of isomorphism classes of indecomposable finitely generated $R$-modules such that for any $e \in \mathbb{N}$, the $R$-module ${}^eR$ is isomorphic to a finite direct sum of these modules:*

$$ {}^eR \cong M_0^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_n^{\oplus c_{n,e}} \quad (\textit{for some } c_{i,e} \geq 0). $$

*Moreover, we say a finite set $\mathcal{S} = \{M_0, \cdots, M_n\}$ is the FFRT system of $R$ if every $R$-module $M_i$ appears non-trivially in ${}^eR$ as a direct summand for some $e \in \mathbb{N}$.*

In addition, when $R$ has FFRT by the FFRT system $\{M_0, M_1, \cdots, M_n\}$, we call $M := M_0 \oplus M_1 \oplus \cdots \oplus M_n$ the FFRT module of $R$. Especially the FFRT module is basic (i.e. $M_i$'s are mutually non-isomorphic). Sometimes we say $R$ has FFRT by the FFRT module $M$ in such a situation. In particular, we say $M = M_0 \oplus M_1 \oplus \cdots \oplus M_n$ is the FFRT generator of $R$ if $M$ is the FFRT module and $R$ is a member of the FFRT system.

**Lemma 2.4.2.** *We suppose that $R$ has FFRT by the FFRT module $M$. Then*

*(1) $R$ is $F$-pure if and only if $M$ is the FFRT generator of $R$.*
*(2) If $M$ is the FFRT generator of $R$, then we have $\mathrm{add}_R({}^eR) = \mathrm{add}(M)$ for sufficiently large $e \gg 0$.*
*(3) $\mathrm{End}_R({}^eR)$ and $\mathrm{End}_R(M)$ are Morita equivalent for $e \gg 0$.*

*Proof.* (1) If $R$ is $F$-pure, we have $R \in \mathrm{add}_R(M)$. Conversely if $R \in \mathrm{add}_R(M)$, then we have a split morphism $R \to {}^eR$ for some $e \in \mathbb{N}$, and it factors through $R \to {}^1R \to {}^eR$. Thus, we have the assertion.
(2) Since $M$ is the FFRT generator (equivalently $R$ is $F$-pure), ${}^fR$ is a direct summand of ${}^eR$ for $e \geq f$. Thus, by the definition of FFRT, there exists sufficiently large $e \gg 0$ such that $\mathrm{add}_R({}^eR) = \mathrm{add}_R(M)$.
(3) The statement (2) induces the Morita equivalence via the progenerator $\mathrm{Hom}_R({}^eR, M)$. □

In the case where $R$ is (strongly) $F$-regular, we obtain better consequences.

**Lemma 2.4.3.** *We suppose that $R$ is (strongly) $F$-regular and has FFRT by the FFRT module $M$. (Especially $M$ is a generator.) Then*

*(1) For any $R$-module $N \in \mathrm{add}_R(M)$, we have $\mathrm{add}_R({}^eN) = \mathrm{add}(M)$ for sufficiently large $e \gg 0$.*
*(2) $\mathrm{End}_R({}^eN)$ and $\mathrm{End}_R(M)$ are Morita equivalent for $e \gg 0$.*

*Proof.* (1) From Lemma 2.4.2 (2), there exists sufficiently large $e \gg 0$ such that $N \in \mathrm{add}_R({}^eR) = \mathrm{add}_R(M)$. (In particular, $N \in \mathrm{add}_R({}^fR) = \mathrm{add}_R(M)$ for $f \geq e$.) Thus, ${}^{e'}N \in \mathrm{add}_R({}^{e+e'}R) = \mathrm{add}_R(M)$. On the other hand, we have $R \in \mathrm{add}_R({}^eN)$ for $e \gg 0$ by Theorem 2.3.6. So if we take sufficiently large $e' \in \mathbb{N}$, then $M \in \mathrm{add}_R({}^{e'}R) \subseteq \mathrm{add}_R({}^{e+e'}N)$.
(2) This is the same as Lemma 2.4.2 (3).

□

**Example 2.4.4.** *We collect some examples about a ring with FFRT.*

1. *(1) From Kunz's theorem (see Theorem 2.2.1), we see that an $F$-finite regular local ring $R$ has FFRT by the FFRT module $R$.*
2. *(2) Suppose $R$ is a CM local ring with* $\mathrm{char}\, R = p > 0$. *Then ${}^eR$ is an MCM $R$-module. Thus, if $R$ is of finite CM representation type (i.e. it has only finitely many non-isomorphic indecomposable MCM modules), then $R$ has FFRT. Here, we remark that even if $R$ is of finite CM representation type, every MCM $R$-module doesn't necessarily appear in ${}^eR$ as a direct summand. For example, let $G = \langle \mathrm{diag}(-1,-1,-1) \rangle$ be a cyclic group of order $2$. We consider the natural action of $G$ on $S = k[[x,y,z]]$ where* $\mathrm{char}\, k > 2$. *Then the invariant subring $R = S^G$ is of finite CM representation type and finitely many MCM modules are $R, \omega_R$ and $\Omega\omega_R$ (cf. [LW, Yos]). However $\Omega\omega_R$ never appears in ${}^eR$ (see Remark 2.3.9 or Proposition 3.2.1).*
3. *(3) (cf. [SVdB, Proposition 3.1.4]) Let $R \hookrightarrow S$ be an inclusion of rings of characteristic $p > 0$ such that $S$ is a finite $R$-module and $R$ is a direct summand of $S$ as an $R$-module. Then if $S$ has FFRT, $R$ also has FFRT.*
   *For example, the invariant subring of a regular ring under the action of a finite group $G$ such that $(|G|, p) = 1$ has FFRT.*
4. *(4) (cf. [SVdB, Proposition 3.1.6]) Let $R = k \oplus \bigoplus_{n \geq 1} R_n \hookrightarrow S = k \oplus \bigoplus_{n \geq 1} S_n$ be an inclusion of graded rings of characteristic $p > 0$ such that $R$ is an $R$-module direct summand of $S$. Then if $S$ has FFRT, $R$ also has FFRT.*
   *Especially, normal semigroup rings (or toric rings) and ring of invariants of regular ring under the action of linearly reductive groups have FFRT.*
5. *(5) Every one dimensional complete local or $\mathbb{N}$-graded domain with algebraically closed or finite residue field has FFRT [Shi].*
6. *(6) For now, the relation between a ring with FFRT and singularities introduced in Section 2.2 is unknown. Indeed, let $R := k[[x,y,z]]/(x^3+y^5+x^2y^3+z^2)$ be the simple singularity of type $E_8^1$ where $k$ is an algebraically closed field of characteristic three. Then we can see that $R$ is not $F$-pure by using Fedder's lemma 2.2.5. However, $R$ is of finite CM representation type [GK, Theorem 1.4], so $R$ has FFRT.*
   *On the other hand, by combining [TT, Corollary 3.3] and [SS, Theorem 5.1], we can see the following hypersurface is strongly $F$-regular but doesn't have FFRT:*
   $$k[[s,t,u,v,w,x,y,z]]/(su^2x^2 + sv^2y^2 + tuxvy + tw^2z^2)$$
   *where $k$ is a field of positive characteristic.*
7. *(7) For more examples, see [TT, Example 1.3].*

## 2.5 Generalized $F$-signature

Let $R$ be a ring which has FFRT by the FFRT system $\{M_0, M_1, \cdots, M_n\}$. Next, we consider the decomposition of ${}^eM_j$. Since each MCM $R$-module $M_j$ appears in ${}^{e'}R$ for some $e' \in \mathbb{N}$ as a direct summand, we suppose that ${}^eM_j$ decomposes as

$$ {}^eM_j \cong M_0^{\oplus d_{0,j,e}} \oplus M_1^{\oplus d_{1,j,e}} \oplus \cdots \oplus M_n^{\oplus d_{n,j,e}}. \tag{2.5.1}$$

for $e \in \mathbb{N}$. In order to grasp the asymptotic behavior of multiplicities of each direct summand, we will extend the notion of the $F$-signature and name it the generalized $F$-signature after [HN]. As we will show in Proposition 2.5.2 and 2.5.4 , this numerical invariant makes sense and has good properties.

**Definition 2.5.1.** *For each multiplicity $d_{i,j,e}$, we consider the limit*

$$s(M_i, M_j) \coloneqq \lim_{e\to\infty} \frac{d_{i,j,e}}{p^{ed}} \quad (i, j = 0, 1, \cdots, n).$$

*We call this limit the generalized $F$-signature of $M_i$ with respect to $M_j$.*

Clearly, the generalized $F$-signature of $R$ with respect to $R$ is the same as the $F$-signature (see Definition 2.3.1). As the next proposition shows, this limit exists and has good properties.

**Proposition 2.5.2.** *([SVdB, Proposition 3.3.1], [Yao1, Theorem 3.11]) If $R$ has FFRT by the FFRT system $\{M_0, M_1, \cdots, M_n\}$, then the generalized $F$-signature $s(M_i, M_j)$ exists for $i, j = 0, 1, \cdots, n$. In addition, if $R$ is (strongly) $F$-regular, then $s(M_i, M_j)$ is a positive rational number.*

*Remark* 2.5.3. In [SVdB], this proposition is proved under the assumption "$R$ is strongly $F$-regular and has FFRT". After that, Y. Yao showed the condition of strongly $F$-regular is unnecessary for proving the existence of it [Yao1]. Note that the existence of the limit for free direct summands (i.e. $F$-signature) is proved under more general settings as we mentioned before.

More precisely, we establish the following formula.

**Theorem 2.5.4.** *Let $R$ be a (strongly) $F$-regular ring which has FFRT by the FFRT system $\{M_0 \cong R, M_1, \cdots, M_n\}$. For $e \in \mathbb{N}$ and $j = 0, 1, \cdots, n$, we suppose that ${}^eM_j$ decomposes as (2.5.1). Then we obtain*

$$s(M_i, M_j) = (\operatorname{rank}_R M_j) s(M_i, R)$$

*for $i, j = 0, 1, \cdots, n$.*

*Proof.* Although the idea is the same as [SVdB, Proposition 3.3.1 and Lemma 3.3.2], we provide a proof for the sake of completeness.

Set the $(n+1)\times(n+1)$ matrix $D \coloneqq (d_{i,j,1})_{1\le i,j\le n}$. We can see that $d_{i,j,e} = (D^e)_{i,j}$ by the induction on $e$. Indeed, the case of $e = 1$ is trivial. We suppose $e > 1$ and have the following.

$$\begin{aligned}{}^eM_j &\cong {}^{e-1}\left(\bigoplus_{l=0}^{n} M_l^{\oplus d_{l,j,1}}\right) \cong \bigoplus_{l=0}^{n} ({}^{e-1}M_l)^{\oplus d_{l,j,1}} \cong \bigoplus_{l=0}^{n}\left(\bigoplus_{i=0}^{n} M_i^{\oplus d_{i,l,e-1}}\right)^{\oplus d_{l,j,1}} \\ &\overset{(\clubsuit)}{\cong} \bigoplus_{i=0}^{n} M_i^{\oplus \sum_{l=0}^{n}(D^{e-1})_{i,l}\cdot d_{l,j,1}} = \bigoplus_{i=0}^{n} M_i^{\oplus (D^e)_{i,j}}.\end{aligned}$$

Here, we used the hypothesis of induction at (♣).

Set the diagonal matrix $Q := \operatorname{diag}(\operatorname{rank}_R M_0, \cdots, \operatorname{rank}_R M_n)$, then we see the following.

$$\begin{aligned}(1,1,\cdots,1)QD &= \left(\sum_{i=0}^{n}(\operatorname{rank}_R M_i)d_{i,0,1}, \sum_{i=0}^{n}(\operatorname{rank}_R M_i)d_{i,1,1}, \cdots, \sum_{i=0}^{n}(\operatorname{rank}_R M_i)d_{i,n,1}\right)\\ &= (\operatorname{rank}_R {}^1M_0, \operatorname{rank}_R {}^1M_1, \cdots, \operatorname{rank}_R {}^1M_n)\\ &= p^d(\operatorname{rank}_R M_0, \operatorname{rank}_R M_1, \cdots, \operatorname{rank}_R M_n) = p^d(1,1,\cdots,1)Q.\end{aligned}$$

Thus, the matrix $E := \frac{1}{p^d}QDQ^{-1}$ satisfies $(1,1,\cdots,1)E = (1,1,\cdots,1)$ and each entry of $E$ is contained in $\mathbb{R}_{\geq 0}$. This means that $E$ is a stochastic matrix. From Lemma 2.4.3, there exists sufficiently large $e \gg 0$ such that every entry of $E^e$ is strictly positive. In this case, $E^e$ is also a stochastic matrix. From the Perron-Frobenius theorem, there is the unique eigenvector $v_0$ of $E^e$ whose eigenvalue is 1 such that

$$\lim_{e\to\infty} E^e v = v_0$$

for any stochastic vector $v$. Since we can consider the vector $e_j = {}^t(0,\cdots,0,1,0,\cdots,0)$ as a stochastic vector, we have $v_0 = \lim_{e\to\infty} E^e e_j = \lim_{e\to\infty} E^e e_0$ for $j = 0,1,\cdots,n$, Finally, we obtain the following formula.

$$\begin{aligned}{}^t(s(M_0,M_j),\cdots,s(M_n,M_j)) &= \lim_{e\to\infty}\frac{1}{p^{ed}}D^e e_j = \lim_{e\to\infty} Q^{-1}E^e Q e_j = \lim_{e\to\infty} Q^{-1}E^e(\operatorname{rank}_R M_j)e_j\\ &= (\operatorname{rank}_R M_j)\lim_{e\to\infty} Q^{-1}E^e e_j = (\operatorname{rank}_R M_j)\lim_{e\to\infty} Q^{-1}E^e e_0\\ &= (\operatorname{rank}_R M_j)(\operatorname{rank}_R M_0)^{-1}\lim_{e\to\infty}\frac{1}{p^{ed}}D^e e_0 = (\operatorname{rank}_R M_j)\cdot {}^t(s(M_0,M_0),\cdots,s(M_n,M_0)).\end{aligned}$$

□

By Proposition 2.2.8 and Example 2.4.4, a certain invariant subring (quotient singularity) is strongly $F$-regular and has FFRT. Thus, we will consider the explicit value of the generalized $F$-signature for such a singularity in Chapter 3.

## 2.6 Dual $F$-signature

In this subsection, we introduce another generalization of the $F$-signature. As we mentioned, the $F$-signature $s(R)$ characterizes some singularities. In particular, its positivity characterizes the strong $F$-regularity. Therefore, $s(R) = 0$ whenever $R$ is not strongly $F$-regular. Therefore, this invariant can't grasp worse singularities. Is there a good invariant to characterize the $F$-rationality ? Recall that a strongly $F$-regular ring is defined via a splitting of a certain map $R \to R^{1/p^e}$ (see Definition 2.2.6). On the other hand, an $F$-rational ring is characterized by the surjectivity of a certain map $\omega_R^{1/p^e} \to \omega_R$ (see Proposition 2.2.12). From these observations, A. Sannai formulated the notion of the dual $F$-signature as follows.

**Definition 2.6.1** ([San]). *Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional reduced $F$-finite Noetherian local ring with* $\operatorname{char} R = p > 0$*. For a finitely generated $R$-module $M$ and $e \in \mathbb{N}$, set*

$$b_e(M) := \max\{n \mid \exists \varphi : {}^e M \twoheadrightarrow M^{\oplus n}\},$$

*and call it the $e$-th $F$-surjective number of $M$. Then we call the limit*

$$s(M) := \lim_{e \to \infty} \frac{b_e(M)}{p^{ed}}$$

*the dual $F$-signature of $M$ if it exists.*

*Remark* 2.6.2. Since the morphism ${}^e R \twoheadrightarrow R^{\oplus b_e(R)}$ splits, if $M$ is isomorphic to the base ring $R$, then the dual $F$-signature of $R$ in sense of Definition 2.6.1 coincides with the $F$-signature of $R$. Thus, we use the same notation unless it causes confusion.

Just like the $F$-signature, the dual $F$-signature also characterizes some singularities.

**Theorem 2.6.3** ([San]). *Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional reduced $F$-finite Cohen-Macaulay local ring with* $\operatorname{char} R = p > 0$*. Then we have*

*(1) $R$ is $F$-rational if and only if $s(\omega_R) > 0$,*
*(2) $s(R) \leq s(\omega_R)$,*
*(3) $s(R) = s(\omega_R)$ if and only if $R$ is Gorenstein.*

In this way, the value of $s(R)$ and $s(\omega_R)$ characterize some singularities. However, the value of the dual $F$-signature is not known except in only a few cases. For example, the case of two-dimensional Veronese subrings is studied in [San, Example 3.17]. We don't have an effective method for determining it for now. Thus, we will consider this numerical invariant for the case where quotient surface singularities in Chapter 5 as the first step to understand it.

# Chapter 3

# Generalized $F$-signature of invariant subrings

In Section 2.5, we introduced the notion of the generalized $F$-signature as a kind of generalization of the $F$-signature. In this section, we will determine the explicit value of this invariant for quotient singularities. This chapter is based on [HN].

Therefore, in the rest of this chapter, let $G$ be a finite subgroup of $\mathrm{GL}(d, k)$ which contains no pseudo-reflections (see Remark 3.0.4) except the identity, and assume that the order of $G$ is coprime to $p = \mathrm{char}\, k$. Suppose that $S$ is the polynomial ring $k[x_1, \cdots, x_d]$ or the power series ring $k[[x_1, \cdots, x_d]]$. We denote the invariant subring of $S$ under the action of $G$ by $R \coloneqq S^G$.

*Remark* 3.0.4. We say $g \in G$ is a pseudo-reflection if it has an eigenvalue 1 of multiplicity $d-1$ and another eigenvalue $\alpha$ of multiplicity 1. In this thesis, we don't consider the identity as a pseudo-reflection. If $G$ contains no pseudo-reflections, sometimes we say $G$ is small.

## 3.1 Skew group algebras

Let $S * G$ be the skew group algebra of $S$ and $G$. That is, as an $S$-module, $S * G = \bigoplus_{g \in G} S \cdot g$ is free whose basis is elements in $G$. The multiplication is given by

$$(s \cdot g)(s' \cdot g') = s(gs') \cdot (gg')$$

for $s, s' \in S$ and $g, g' \in G$. Note that an $S * G$-module $M$ is an $S$-module with a compatible $G$-action. Namely, $g(sm) = g(s)g(m)$ for $s \in S, m \in M, g \in G$. So a $(G, S)$-module and an $S * G$-module are the same thing.

Also, $f : M \to N$ is an $S * G$-linear map if and only if $f$ is an $S$-homomorphism as well as $G$-homomorphism (i.e. $f(gm) = g(f(m))$). For $S * G$-modules $M$ and $N$, $\mathrm{Hom}_S(M, N)$ has an $S * G$-module structure as follows.

$$(gf)(m) = gf(g^{-1}m), \quad g \in G,\ m \in M,\ f \in \mathrm{Hom}_S(M, N).$$

Note that $f \in \mathrm{Hom}_S(M, N)$ is $G$-invariant if and only if it is an $S * G$-homomorphism. It follows that

$$\mathrm{Hom}_{S*G}(M, N) = \mathrm{Hom}_S(M, N)^G.$$

Since $|G|$ is coprime to char $k$, the functor $(-)^G$ is exact. Therefore, the derived functors of $\mathrm{Hom}_{S*G}(-,-)$ are given by

$$\mathrm{Ext}^i_{S*G}(M,N) = \mathrm{Ext}^i_S(M,N)^G \quad (i \geq 0).$$

Thus, we have the following.

**Proposition 3.1.1.** *An $S*G$-module $M$ is projective if and only if it is projective $S$-module.*

Furthermore, the multiplication on $S * G$ gives a ring homomorphism

$$\phi : S * G \to \mathrm{End}_R(S) \quad (s \cdot g \mapsto (s' \mapsto sg(s'))).$$

The next theorem plays a crucial role in the future. This was first shown in [Aus1]. The precise proof is in [IT, Theorem 4.2], [LW, Theorem 5.12] or [Yos, (for $d = 2$)].

**Theorem 3.1.2.** *Suppose that $G \subset \mathrm{GL}(d,k)$ contains no pseudo-reflections. Then the morphism $\phi : S * G \to \mathrm{End}_R(S)$ is an isomorphism.*

The next theorem plays an important role. This was proved by M. Auslander in [Aus2] for the two dimensional case. This kind of equivalence holds in more general situation.

**Theorem 3.1.3.** *If $G$ contains no pseudo-reflections, then the functor $\mathrm{Ref}(G,S) \to \mathrm{Ref}(R)$ $(M \mapsto M^G)$ is an equivalence, where $\mathrm{Ref}(G,S)$ is the category of reflexive $(G,S)$-modules and $\mathrm{Ref}(R)$ is the category of reflexive $R$-modules. The quasi-inverse is $N \mapsto (S \otimes_R N)^{**}$.*

*The same functors give an equivalence ${}^*\mathrm{Ref}(G,S) \to {}^*\mathrm{Ref}(R)$, where ${}^*\mathrm{Ref}(G,S)$ is the category of $\mathbb{Z}[1/p]$-graded reflexive $(G,S)$-modules and ${}^*\mathrm{Ref}(R)$ is the category of $\mathbb{Z}[1/p]$-graded reflexive $R$-modules.*

*Proof.* A $(G,S)$-module and an $S*G$-module are one and the same thing. As a $(G,S)$-module, $S * G$ and $S \otimes_k kG$ are the same thing, where $kG$ is the group algebra (the left regular representation) of $G$ over $k$. So $\mathrm{Hom}_S(S * G, S) \cong S \otimes_k k[G] \cong S \otimes_k kG \cong S * G$, where $k[G] = (kG)^*$ is the $k$-dual of $kG$ (the left regular representation).

Let us denote by $S'$ the $R$-module $S$ with the trivial $G$-module structure. Note that $S' \to (S \otimes_k k[G])^G$ given by $s \mapsto \sum_{g\in G} gs \otimes e_g$ is an isomorphism, where $\{e_g \mid g \in G\}$ is the dual basis of $k[G]$, dual to $G$, which is a basis of $kG$. Note that $g' e_g = e_{g'g}$.

For $M \in \mathrm{Ref}(G,S)$, $M^G$ is certainly reflexive. Indeed, there is a presentation

$$(S * G)^u \to (S * G)^v \to M^* \to 0. \tag{3.1.1}$$

Applying $(-)^G \circ \mathrm{Hom}_S(-,S)$,

$$0 \to M^G \to (S')^v \to (S')^u$$

is exact. As it is easy to see that $S'$ satisfies the $(S_2)$-condition as an $R$-module (that is, for $P \in \mathrm{Spec}\, R$, if $\mathrm{depth}_{R_P}(S'_P) < 2$, then $S'_P$ is a maximal Cohen–Macaulay $R_P$-module), so is $M^G$, and it is reflexive.

On the other hand, it is obvious that $(S \otimes_R N)^{**}$ is a reflexive $(G,S)$-module, since it is a dual of some $S$-finite $(G,S)$-module.

Let $u : N \to ((S \otimes_R N)^{**})^G$ be the map given by $u(n) = \lambda(1 \otimes n)$, where $\lambda : S \otimes_R N \to (S \otimes_R N)^{**}$ is the canonical map. We show that $u$ is an isomorphism. To verify this, since both $N$ and $((S \otimes_R N)^{**})^G$ are reflexive, it suffices to show that

$$u_P : N_P \to (((S \otimes_R N)^{**})^G)_P \cong ((S_P \otimes_{R_P} N_P)^{**})^G$$

is an isomorphism for $P \in \operatorname{Spec} R$ with $\dim R_P \le 1$ (cf. [LW, Lemma 5.11]). Then $N_P$ is a free module, and we may assume that $N_P = R_P$ by additivity. This case is trivial.

Let $\varepsilon : (S \otimes_R M^G)^{**} \to M$ be the composite

$$(S \otimes_R M^G)^{**} \xrightarrow{a^{**}} M^{**} \xrightarrow{\lambda^{-1}} M,$$

where $a : S \otimes_R M^G \to M$ is given by $a(s \otimes m) = sm$. We show that $\varepsilon$ is an isomorphism. Since $(S \otimes_R M^G)^*$ and $M$ are reflexive, it suffices to show that $a^* : M^* \to (S \otimes_R M^G)^*$ is an isomorphism. By the five lemma and the existence of the presentation of the form (3.1.1), we may assume that $M = S \otimes_k k[G]$. Then $a^*$ is identified with the map

$$S * G \cong (S \otimes_k k[G])^* \xrightarrow{a^*} (S \otimes_R (S \otimes_k k[G])^G)^* \cong (S \otimes_R S')^* \cong \operatorname{Hom}_R(S', S).$$

It is easy to see that this map is given by $sg \mapsto (s' \mapsto s(gs'))$. This is an isomorphism by Theorem 3.1.2.

As $u$ and $\varepsilon$ are isomorphisms, $M \mapsto M^G$ and $N \mapsto (S \otimes_R N)^{**}$ are quasi-inverse each other, and hence they are category equivalences.

The graded version is proved similarly.

□

## 3.2 Decomposition of Frobenius push-forward

As we saw in Example 2.4.4, an invariant subring $R$ has FFRT. More precisely, we have the following proposition.

**Proposition 3.2.1** ([SVdB], Proposition 3.2.1)**.** *Let $V_0 = k, V_1, \cdots, V_n$ be the full set of non-isomorphic irreducible representations of $G$. We set $M_i := (S \otimes_k V_i)^G \quad (i = 0, 1, \cdots, n)$. Then we see that $R$ has finite $F$-representation type by the finite set $\{M_0 \cong R, M_1, \cdots, M_n\}$.*

From this proposition, we can describe ${}^eR$ as follows.

$${}^eR \cong R^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_n^{\oplus c_{n,e}}. \tag{3.2.1}$$

In this section, we show the uniqueness of the multiplicities. Firstly, we introduce the notion of Frobenius twist (e.g. [Jan]).

**Definition 3.2.2.** *For $k$-vector space $V$ and $e \in \mathbb{Z}$, we define $k$-vector space ${}^eV$ as follows*

- *${}^eV$ is the same as $V$ as an additive group;*

- *the action of $\alpha \in k$ on ${}^eV$ is $\alpha \cdot v = \alpha^{p^e} v$.*

*An element $v \in V$, viewed as an element of ${}^eV$, is sometimes denoted by ${}^ev$. Thus $\alpha \cdot {}^ev = {}^e(\alpha^{p^e} v)$. By the composition $G \hookrightarrow \mathrm{GL}(V) \xrightarrow{\phi} \mathrm{GL}({}^eV)$, ${}^eV$ is also a representation of $G$, where $\phi$ is given by $\phi(g)({}^ev) = {}^e(gv)$ for $g \in G$ and $v \in V$. We call this representation the Frobenius twist of $V$. Sometimes we denote this representation by $V^{(-e)}$.*

Let $v_1, \cdots, v_d$ be a basis of $V$. For this basis, we suppose that a representation of $G$ is defined by

$$g \cdot v_j = \sum_{i=1}^{d} f_{ij}(g) v_i \quad (g \in G,\ f_{ij} : G \to k).$$

Namely, a matrix representation of $V$ is described by $(f_{ij}(g))$. Since $k$ is an algebraically closed field, the basis $v_1, \cdots, v_d$ also form a basis of ${}^eV$, and the action of $G$ on ${}^eV$ is described as follows

$$g \cdot {}^ev_j = {}^e(g \cdot v_j) = {}^e(\sum_{i=1}^{d} f_{ij}(g) v_i) = \sum_{i=1}^{d} f_{ij}(g)^{p^{-e}} ({}^ev_i).$$

From this observation, a matrix representation of the Frobenius twist ${}^eV$ is described by $((f_{ij}(g))^{p^{-e}})$, that is, each component of the matrix representation of ${}^eV$ is the $p^{-e}$-th power of the original one.

In order to show the uniqueness of the multiplicities, we prove the following.

**Proposition 3.2.3.** *For $e \geq 1$, $c_{0,e}, \cdots, c_{n,e} \geq 0$, the following decompositions are equivalent*

*(1)* ${}^eR \cong M_0^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_n^{\oplus c_{n,e}}$ *as $R$-modules;*

*(2)* ${}^eS \cong (S \otimes_k V_0)^{\oplus c_{0,e}} \oplus (S \otimes_k V_1)^{\oplus c_{1,e}} \oplus \cdots \oplus (S \otimes_k V_n)^{\oplus c_{n,e}}$ *as $(G, S)$-modules;*

*(3)* ${}^eS/\mathfrak{m}^eS \cong V_0^{\oplus c_{0,e}} \oplus V_1^{\oplus c_{1,e}} \oplus \cdots \oplus V_n^{\oplus c_{n,e}}$ *as $G$-modules;*

*(4) there exist $\alpha_{ij} \in \frac{1}{q}\mathbb{Z}_{\geq 0}$ such that* ${}^eS \cong \bigoplus_{i=0}^{n} \bigoplus_{j=1}^{c_{i,e}} (S \otimes_k V_i)(-\alpha_{ij})$ *as $\frac{1}{q}\mathbb{Z}$-graded $(G, S)$-modules;*

*(5) there exist $\alpha_{ij} \in \frac{1}{q}\mathbb{Z}_{\geq 0}$ such that* ${}^eR \cong \bigoplus_{i=0}^{n} \bigoplus_{j=1}^{c_{i,e}} M_i(-\alpha_{ij})$ *as $\frac{1}{q}\mathbb{Z}$-graded $R$-modules.*

*Remark* 3.2.4. A similar correspondence holds for more general situation up to the action of the $e$-th Frobenius kernel of a group scheme [Has]. For the case of a finite group $G$, the $e$-th Frobenius kernel of $G$ is trivial. Thus, we may ignore it in our context.

*Proof of Proposition 3.2.3.* The equivalence of (1) and (2), (4) and (5) follow from Theorem 3.1.3, and (3) is obtained by applying $(- \otimes_S k)$ to (2). If we forget the grading from (4), then we obtain (2).

$$\begin{tikzcd}
(1) \arrow[r, Leftrightarrow, "3.1.3"] & (2) \arrow[r, Rightarrow, "\otimes_S k"] & (3) \\
 & (4) \arrow[u, Rightarrow, "\text{forget grading}"'] \arrow[r, Leftrightarrow, "3.1.3"'] & (5)
\end{tikzcd}$$

So we will show (3) $\Rightarrow$ (4). If we consider ${}^eS/\mathfrak{m}^eS$ as a $\frac{1}{q}\mathbb{Z}$-graded $G$-module, then we can write

$${}^eS/\mathfrak{m}^eS \cong \bigoplus_{i=0}^{n} \bigoplus_{j=1}^{c_{i,e}} V_i(-\alpha_{ij})$$

for some $\alpha_{ij} \in \frac{1}{q}\mathbb{Z}_{\geq 0}$. Then as in the proof of [SVdB, Proposition 3.2.1], we have ${}^eS \cong S \otimes_k ({}^eS/\mathfrak{m}^eS)$, and (4) follows. $\square$

Especially, the decomposition (3) appears in Proposition 3.2.3 is unique. Thus, we obtain the next statement as a corollary.

**Corollary 3.2.5.** *Each $M_i$ is indecomposable and the multiplicities $c_{i,e}$ are determined uniquely, and $M_i \ncong M_j$ if $i \neq j$.*

In Proposition 3.2.3 and Corollary 3.2.5, the condition "$G$ contains no pseudo-reflections" is essential. If $G$ contains a pseudo-reflection, then there is a counter-example as follows.

**Example 3.2.6.** *Let $S = k[x, y]$ be a polynomial ring, where $(\mathrm{char}\ k, |G|) = 1$. Set $G = \langle \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$, that is $G$ is a symmetric group $\mathfrak{S}_2$, and, $V_0 = k, V_1 = \mathrm{sgn}$ are irreducible representations of $G$. (Note that $\sigma$ is a pseudo-reflection.) Then, $R := S^G \cong k[x + y, xy]$. Since $R$ is a polynomial ring, ${}^eR \cong R^{\oplus p^{2e}}$. On the other hand,*

$$M_1 := (S \otimes_k V_1)^G = \{f \in S \mid \sigma \cdot f = (\mathrm{sgn}\ \sigma)f\} = (x - y)R \cong R.$$

*So ${}^eR$ also decompose as ${}^eR \cong M_1^{\oplus p^{2e}}$. Therefore, the uniqueness doesn't hold in this case.*

## 3.3 Generalized $F$-signature of invariant subrings

For now, we considered the decomposition

$${}^eR \cong R^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_n^{\oplus c_{n,e}},$$

and showed the uniqueness of the multiplicity $c_{i,e}$. Next we will consider the generalized $F$-signature of $M_i$ (with respect to $R$). Since an invariant subring $R$ is strongly $F$-regular and has FFRT, the limit $s(M_i, R) = \lim\limits_{e\to\infty} \dfrac{c_{i,e}}{p^{de}}$ exists and it is a positive rational number (see Proposition 2.5.2). Especially, we can determine the explicit value of it as follows.

**Theorem 3.3.1.** *Let the notation be as above. Then for all $i = 0, \cdots, n$ one has*

$$s(M_i, R) = \frac{\dim_k V_i}{|G|} = \frac{\operatorname{rank}_R M_i}{|G|}.$$

*Remark* 3.3.2. The second equation follows from $\dim_k V_i = \operatorname{rank}_R M_i$ clearly.

The case that $i = 0$ is due to K. Watanabe and K. Yoshida (see Theorem 2.3.5). A similar result holds for finite subgroup scheme of $\mathrm{SL}_2$ [HS, Lemma 4.10].

*Remark* 3.3.3. From this theorem, we can see that each indecomposable MCM $R$-modules in the finite set $\{R, M_1, \cdots, M_n\}$ actually appear in ${}^eR$ as a direct summand for sufficiently large $e$ (see also [TY, Proposition 2.5]). In particular, $\{R, M_1, \cdots, M_n\}$ is the FFRT system of $R$.

In order to prove this theorem, we introduce the notion of the Brauer character. In the representation theory of finite groups over $\mathbb{C}$, the character gives us very effective method to distinguish each representation. But now, we are in a positive characteristic field $k$, not in $\mathbb{C}$. So the character in the original sense doesn't work well. Therefore we have to modify it for applying to our context. For this purpose, we introduce the Brauer character (for more details, refer to some textbooks e.g. [CR], [Wei]).

As we assume that $m := |G|$ is not divisible by $p$, there is a primitive $m$th root of unity in $k$, and thus both $\mu_m(k) = \{\omega \in k^\times \mid \omega^m = 1\}$ and $\mu_m(\mathbb{C}) = \{\omega \in \mathbb{C}^\times \mid \omega^m = 1\}$ are the cyclic groups of order $m$. Fix a group isomorphism $\Phi : \mu_m(k) \to \mu_m(\mathbb{C})$.

**Definition 3.3.4.** *For a $kG$-module $V$, the Brauer character $\chi_V$ of $V$ is the function $\chi_V : G \to \mathbb{C}$ given by*

$$\chi_V(g) := \sum_{i=1}^{d} \Phi(\omega_i) \in \mathbb{C} \quad (g \in G),$$

*where $\omega_1, \cdots, \omega_d$ are the eigenvalues of $g$.*

The following proposition is well-known for the original character over $\mathbb{C}$. This kind of formula also holds for the Brauer character.

**Proposition 3.3.5.** *Let $V, W$ be $kG$-modules and $g \in G$, then*

*(1)* $\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$.
*(2)* $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$.
*(3)* $\chi_{V^*}(g) = \overline{\chi_V(g)}$*, where the bar denotes the conjugate of a complex number.*
*(4)* $\chi_V(1_G) = \dim_k V$.
*(5)* $\dim_k V^G = \dfrac{1}{|G|} \sum_{g \in G} \chi_V(g)$.
*(6)* $\dim_k \operatorname{Hom}_G(V, W) = \dfrac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \cdot \chi_W(g)$.

*Proof.* The statements (1)–(4) follow easily from the definition. (6) follows from (1), (3), and (5). So we only prove (5). If we show (5) for a particular choice of $\Phi$, then (5) is true for arbitrary choice, say $\Phi'$, because we can write $\Phi' = \alpha \circ \Phi$, where $\alpha$ is some automorphism of $\mathbb{Q}(\omega)$ over $\mathbb{Q}$, where $\omega$ is a primitive $m$th root of unity in $\mathbb{C}$. Let $R$ be the ring of Witt vectors over $k$. Note that $R$ is a complete DVR (discrete valuation ring). Let $t$ be its uniformizing parameter. We identify $R/tR$ with $k$. Let $\bar{\omega}$ be a fixed primitive $m$th root of unity in $k$. By Hensel's lemma, it is easy to see that $\bar{\omega}$ lifts to a primitive $m$th root of unity in $R$ uniquely, say to $\omega$. Note that $V$ is a $kG$-module, and hence is an $RG$-module. Let $V_R \to V$ be the projective cover as an $RG$-module, which exists (note that $RG$ is semiperfect). Note that $V_R/tV_R = V$, and $V_R$ is an $R$-free module of rank $\dim_k V$.

Let $R_0 = \mathbb{Z}[\omega]$ be the subring of $R$ generated by $\omega$. Then regarding $R_0$ as a subring of $\mathbb{C}$, we have that $\tilde{\chi}_V$ is a Brauer character of $V$, where $\tilde{\chi}_V(g) = \mathrm{trace}_{V_R}(g)$ (the trace makes sense, since $V_R$ is a finite free $R$-module). Let $\gamma = \dfrac{1}{|G|}\sum_{g\in G} g \in RG$. Then it is easy to see that $\gamma$ is a projector from any $RG$-module $M$ to $M^G$. In particular, the $G$-invariance $(-)^G$ is an exact functor on the category of $RG$-modules. It follows that $V^G = (V_R/tV_R)^G \cong V_R^G/tV_R^G = k\otimes_R V_R^G$. Let $U := (1-\gamma)V_R$. Then $V_R = V_R^G \oplus U$, and $\gamma$ is the identity map on $V_R^G$ and zero on $U$. So $\dfrac{1}{|G|}\sum_{g\in G}\tilde{\chi}_V(g) = \mathrm{trace}_{V_R}(\gamma) = \mathrm{rank}_R V_R^G = \dim_k V^G$. This is what we wanted to prove. □

So we are now ready to prove Theorem 3.3.1.

*Proof of Theorem 3.3.1.* Firstly, there is $e_0 \geq 1$ such that the group ring $\mathbb{F}_{q_0}G$ is isomorphic to the direct product of total matrix rings over $\mathbb{F}_{q_0}$, where $q_0 = p^{e_0}$. Namely,

$$\mathbb{F}_{q_0}G \cong \mathrm{Mat}_{r_1}(\mathbb{F}_{q_0}) \times \cdots \times \mathrm{Mat}_{r_m}(\mathbb{F}_{q_0}), \quad (r_1, \cdots, r_m \in \mathbb{N}).$$

Since the component of matrix representation of the Frobenius twist is $p^{-e}$-th power of the original one, so if we take an appropriate basis, then any component of matrix representation is in the finite field $\mathbb{F}_{q_0}$. Thus, if $e = e_0 t$, then we can consider ${}^eM \cong M$ for any $G$-module $M$.

Since we know the existence of the limit, it suffices to show the subsequence $\{\frac{c_{i,e_0t}}{p^{de_0t}}\}_{t\in\mathbb{N}}$ converge on $(\dim_k V_i)/|G|$. So we prove

$$\lim_{t\to\infty}\frac{c_{i,e_0t}}{p^{de_0t}} = \frac{\dim_k V_i}{|G|}.$$

For $e = e_0t$, we obtain ${}^eS/\mathfrak{m}^eS \cong {}^e(S/\mathfrak{m}^{[q]}) \cong S/\mathfrak{m}^{[q]}$, and ${}^eS/\mathfrak{m}^eS$ is also isomorphic to the finite direct sum of irreducible representations (cf. Proposition 3.2.3). By Proposition 3.3.5 (6), the multiplicity $c_{i,e}$ is described as follows.

$$c_{i,e} = \dim_k \mathrm{Hom}_G(V_i, S/\mathfrak{m}^{[q]}) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{V_i}(g)}\cdot\chi_{S/\mathfrak{m}^{[q]}}(g).$$

Set $g \in G$ and suppose that the order of $g$ is $m$. Then there is a basis $\{x_1, \cdots, x_d\}$ of $V$ such that each $x_i$ is an eigenvector of $g$ and we can write $g \cdot x_i = \omega_i x_i$ with $\omega_i = \omega^{\delta_i}$ for some $0 \le \delta_i < m$, where $\omega$ is a primitive $m$-th root of unity. In this situation

$$\{x_1^{\lambda_1}\cdots x_d^{\lambda_d} \mid 0 \le \lambda_1, \ldots, \lambda_d < q\} \subset \bigoplus_{l=0}^{(q-1)d} \mathrm{Sym}_l\, V$$

is a basis of $S/\mathfrak{m}^{[q]}$. As each $x_1^{\lambda_1}\cdots x_d^{\lambda_d}$ is an eigenvector of $g$ with the eigenvalue $\omega_1^{\lambda_1}\cdots\omega_d^{\lambda_d}$, we have

$$\chi_{S/\mathfrak{m}^{[q]}}(g) = \sum_{0\le\lambda_1,\ldots,\lambda_d<q} \Phi(\omega_1^{\lambda_1}\cdots\omega_d^{\lambda_d}) = \prod_{i=1}^{d}(1+\theta_i+\cdots+\theta_i^{q-1}),$$

where $\theta_i := \Phi(\omega_i)$.

(i) In case $g = 1$, by Proposition 3.3.5 (4),

$$\frac{\overline{\chi_{V_i}(g)}\cdot\chi_{S/\mathfrak{m}^{[q]}}(g)}{q^d} = \frac{\dim_k V_i \cdot q^d}{q^d} = \dim_k V_i.$$

(ii) In case $g \ne 1$, we may assume $\theta_d \ne 1$. Then

$$\left|\frac{\overline{\chi_{V_i}(g)}\cdot\chi_{S/\mathfrak{m}^{[q]}}(g)}{q^d}\right| \le \frac{\left|\overline{\chi_{V_i}(g)}\right|}{q^d}\prod_{i=1}^{d-1}(|1|+|\theta_i|+\cdots+|\theta_i|^{q-1})\cdot\left|\frac{1-\theta_d^q}{1-\theta_d}\right|$$

$$\le \frac{\dim_k V_i}{q}\cdot\frac{2}{|1-\theta_d|} \xrightarrow{t\to\infty} 0.$$

The first inequation is obtained by applying the triangle inequality. Since $|\theta_i| \le 1$, we can obtain the second inequation.

From previous arguments, we may only discuss in case $g = 1$. Thus, we conclude

$$\lim_{e\to\infty}\frac{c_{i,e}}{q^d} = \lim_{e\to\infty}\frac{1}{q^d}\cdot\frac{1}{|G|}\sum_{g\in G}\overline{\chi_{V_i}(g)}\cdot\chi_{S/\mathfrak{m}^{[q]}}(g) = \frac{\dim_k V_i}{|G|}.$$

□

The next corollary immediately follows from Theorem 2.5.4 and 3.3.1.

**Corollary 3.3.6.** *Let the notation be as above. Then for all $i, j = 0, \cdots, n$ one has*

$$s(M_i, M_j) = (\dim_k V_j)\cdot s(M_i, R) = \frac{(\dim_k V_i)\cdot(\dim_k V_j)}{|G|} = \frac{(\mathrm{rank}_R\, M_i)\cdot(\mathrm{rank}_R\, M_j)}{|G|}.$$

# Chapter 4

# Auslander-Reiten theory

In this chapter, we restrict the case to $d = 2$. Thus, in the rest of this chapter, $R$ is the invariant subring of $S = k[[x, y]]$ under the action of a finite subgroup $G \subset \mathrm{GL}(2, k)$ which contains no pseudo-reflections and $(|G|, \operatorname{char} k) = 1$. In this situation, $R$ has a typical property. Namely, $R$ is of finite CM representation type (i.e. it has only finitely many non-isomorphic indecomposable MCM modules) as we will see below. For more details, see original papers [Aus2, Aus3, AR1, AR2] or some textbooks (e.g. [LW], [Yos]).

## 4.1 McKay correspondence

We start this section with the following theorem.

**Theorem 4.1.1** ([Her]). *Every indecomposable MCM $R$-module is a direct summand of the $R$-module $S$. In particular, we have* $\mathrm{CM}(R) = \mathrm{add}_R(S)$ *and $R$ is of finite CM representation type.*

By combining this theorem and Theorem 3.1.3, we have the following equivalence. Note that a reflexive $R$-module is an MCM $R$-module because $R$ is a two dimensional normal domain.

**Corollary 4.1.2** ([Aus2]). *For an $S * G$-module (= $(G, S)$-module) $M$, we consider the functor* $\mathrm{proj}\, S * G \to \mathrm{CM}(R)$ $(M \mapsto M^G)$. *Then this functor gives an equivalence of categories:*

$$\mathrm{CM}(R) \cong \mathrm{proj}\, S * G.$$

Also, we note the relation between these objects and representations of $G$ (see also Proposition 3.2.3). Let $V$ be a $kG$-module. Then we can define the functor $\mathrm{mod}(kG) \to \mathrm{proj}\, S * G$ $(V \mapsto S \otimes_k V)$ and this one has the left adjoint functor $S/\mathfrak{n} \otimes_S -$ where $\mathfrak{n}$ is the maximal ideal of $S$. Moreover these functors give a one to one correspondence between the set of isomorphism classes of objects in $\mathrm{mod}(kG)$ and that of $\mathrm{proj}\, S * G$. From these results, we can see every indecomposable MCM $R$-module takes the form $M_t := (S \otimes_k V_t)^G$ where $V_t$ is an irreducible representation of $G$.

Collectively, we could obtain the following correspondence.

**Corollary 4.1.3.** *Let $G$, $S$ and $R$ be the same as above. Then we have one to one correspondences between*

- *indecomposable MCM $R$-modules;*
- *indecomposable projective $S * G$-modules;*
- *indecomposable projective $\mathrm{End}_R(S)$-modules;*
- *irreducible representations of $G$.*

Next, we restrict the case to a finite subgroup $G \subset \mathrm{SL}(2,k)$ and $|G|$ is invertible in $k$. Note that $G$ automatically contains no pseudo-reflections in this situation. It is well known that a finite subgroup of $\mathrm{SL}(2,k)$ is conjugate to one of the following finite groups (e.g. [Yos, Chapter 10]):

$$
\begin{array}{ll}
(A_n): & \text{the cyclic group of order } n+1 \ \ (n \geq 1) \\
& C_{n+1} := \langle \begin{pmatrix} \zeta_{n+1} & 0 \\ 0 & \zeta_{n+1}^{-1} \end{pmatrix} \rangle \\
(D_n): & \text{the binary dihedral group of order } 4(n-2) \ \ (n \geq 4) \\
& \mathcal{D}_{n-2} := \langle\, C_{2(n-2)}, \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix} \rangle \\
(E_6): & \text{the binary tetrahedral group of order } 24 \\
& \mathcal{T} := \langle\, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{pmatrix}, \mathcal{D}_2 \,\rangle \\
(E_7): & \text{the binary octahedral group of order } 48 \\
& \mathcal{O} := \langle \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}, \mathcal{T} \,\rangle \\
(E_8): & \text{the binary icosahedral group of order } 120 \\
& \mathcal{I} := \langle\, \frac{1}{\sqrt{5}} \begin{pmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} \zeta_5^2 - \zeta_5^4 & \zeta_5^4 - 1 \\ 1 - \zeta_5 & \zeta_5^3 - \zeta_5 \end{pmatrix} \rangle
\end{array}
\tag{4.1.1}
$$

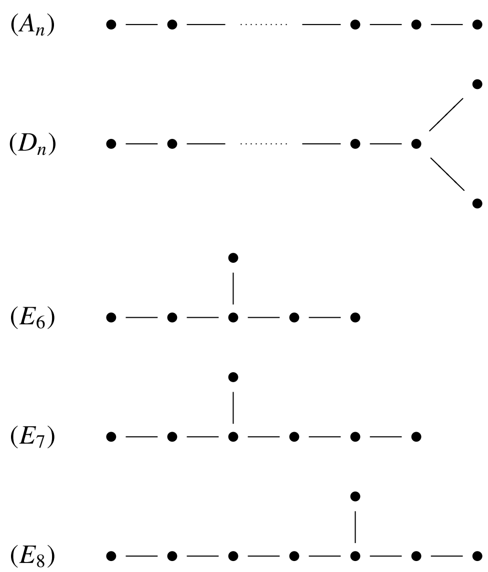
where $\zeta_n$ is a primitive $n$-th root of unity.

Then an invariant subring $R$ is Gorenstein [Wat1] and called a rational double point (or Du Val singularity, Kleinian singularity, ADE singularity and so on). In this situation, we can see the connection between the above objects and geometric objects. Namely, let $\pi : X \to \mathrm{Spec}\, R$ be the minimal resolution of singularities and $E := \pi^{-1}(\mathfrak{m})$ be the exceptional divisor where $\mathfrak{m}$ is the maximal ideal of $R$. We decompose $E = \bigcup_{i=1}^{r} E_i$ into irreducible components. Then there exists a one to one correspondence between irreducible exceptional curves and non-trivial irreducible representations. Thus, we also have a one to one correspondence between irreducible exceptional curves and non-free indecomposable objects in Corollary 4.1.3. This beautiful phenomenon was first observed in [McK] and after that many mathematician contributed to understand this kind of correspondence (e.g. [GSV, AV, Esn, Knö]).

In order to mention more precise connections, we introduce the notion of the McKay quiver.

**Definition 4.1.4** (McKay quiver)**.** *The McKay quiver of a finite subgroup $G \subset \mathrm{GL}(d,k)$ is an oriented graph whose vertices are irreducible representations of $G$: $\{V_0 \cong k, V_1, \cdots, V_n\}$ and draw $m_{ij}$ arrows from $V_i$ to $V_j$. Here, $m_{ij}$ is the multiplicity of $V_i$ in the decomposition of $V \otimes_k V_j$ into irreducible representations. Note that $V$ is a natural representation of $G$.*

**Theorem 4.1.5.** *Let $R = S^G$ be a rational double point. Then the McKay quiver of $G \subset \mathrm{SL}(2,k)$ coincides with the dual graph of the minimal resolution of singularity $\pi : X \to \mathrm{Spec}\ R$ after deleting the trivial vertex and replacing double arrows $\leftrightarrows$ by edges. Furthermore, it takes a form of Dynkin diagrams of type ADE. (That is, the McKay quiver of $G$ takes a form of extended Dynkin diagrams.)*




We remark that this correspondence is no longer true if we consider a finite subgroup $G \subset \mathrm{GL}(2,k)$, because the number of non-trivial irreducible representations (= that of indecomposable MCM modules) is greater than or equal to that of irreducible exceptional curves. But if we consider a part of irreducible representations (resp. indecomposable MCM modules) so-called irreducible special representations (resp. indecomposable special CM modules), then we again have a one to one correspondence. This one is called the special McKay correspondence and we will discuss it in subsection 5.2.1.

## 4.2 Auslander-Reiten quiver

Let $V_0 = k, V_1, \cdots, V_n$ be the full set of non-isomorphic irreducible representations of $G$. In the previous section, we showed that an invariant subring $R$ is of finite CM representation type and finitely many indecomposable MCM $R$-modules are $M_t = (S \otimes_k V_t)^G$ for $t = 0, 1, \cdots, n$. That is,

$$\mathrm{CM}(R) = \mathrm{add}_R(R \oplus M_1 \oplus \cdots \oplus M_n).$$

In this chapter, we investigate the structure of this category $\mathrm{CM}(R)$.

So keeping the above notations.

**Definition 4.2.1** (Auslander-Reiten sequence)**.** *Let $R$ be the same as above and $M, N$ be indecomposable MCM $R$-modules. We call a non split short exact sequence*

$$0 \to N \xrightarrow{f} L \xrightarrow{g} M \to 0$$

*the Auslander-Reiten (= AR) sequence (or almost split sequence) ending in $M$ (or starting at $N$) if for all MCM modules $X$ and for any morphism $\varphi : X \to M$ which is not a split surjection there exists $\phi : X \to L$ such that $\varphi = g \circ \phi$.*

Since $R$ is an isolated singularity, the AR sequence ending in $M_t = (S \otimes_k V_t)^G$ $(t \neq 0)$ actually exists where $M_t$ is a non-free indecomposable MCM $R$-module [Aus2]. It is unique up to isomorphism. In our situation, the AR sequence constructed by the Koszul complex over $S$ and a natural representation $V$ of $G$ as follows. Firstly, let

$$0 \longrightarrow S \otimes_k \wedge^2 V \longrightarrow S \otimes_k V \longrightarrow S \longrightarrow k \longrightarrow 0,$$

be the the Koszul complex over $S$. This is also an exact sequence of $S * G$-modules. By applying $- \otimes_k V_t$, we have

$$0 \longrightarrow S \otimes_k (\wedge^2 V \otimes_k V_t) \longrightarrow S \otimes_k (V \otimes_k V_t) \longrightarrow S \otimes_k V_t \longrightarrow V_t \longrightarrow 0.$$

Furthermore, we apply the functor $(-)^G$. (Note that this functor is exact in our situation.) Then we obtain the following sequence and this is just the AR sequence ending in $M_t$

In the case of $t \neq 0$, the AR sequence ending in $M_t$ is

$$0 \longrightarrow (S \otimes_k (\wedge^2 V \otimes_k V_t))^G \longrightarrow (S \otimes_k (V \otimes_k V_t))^G \longrightarrow M_t = (S \otimes_k V_t)^G \longrightarrow 0.$$

In the case of $t = 0$, there exists the following sequence

$$0 \longrightarrow \omega_R = (S \otimes_k \wedge^2 V)^G \longrightarrow (S \otimes_k V)^G \longrightarrow R = S^G \longrightarrow k \longrightarrow 0.$$

This sequence is called the fundamental sequence of $R$.

We call the left term of these sequences the Auslander-Reiten translation and denote by $\tau(M_t)$. On the other hand, we denote the right term of the AR sequence starting at $M_t$ by $\tau^{-1} M_t$. In general, the AR translation is obtained be the following fashion. Take a presentation of $M_t$ by free modules: $R^{\oplus b} \xrightarrow{f} R^{\oplus a} \to M_t \to 0$. We define $\operatorname{Tr} M_t :=$ $\operatorname{Coker} \operatorname{Hom}_R(f, R)$. Then we have

$$\tau(M_t) \cong \operatorname{Hom}_R(\Omega^d \operatorname{Tr} M_t, \omega_R).$$

Since $\dim R = 2$ in our situation, it is easy to see $M_t^* \cong \Omega^2 \operatorname{Tr} M_t$. Thus, the AR translation $\tau$ is obtained via the functors

$$\tau : \mathrm{CM}(R) \xrightarrow{(-)^*} \mathrm{CM}(R) \xrightarrow{(-)^\vee} \mathrm{CM}(R).$$

On the other hand, we also have $\tau(M_t) \cong (M_t \otimes_R \omega_R)^{**}$ (see [Aus2]).

Next, we introduce the notion of irreducible morphism.

**Definition 4.2.2** (Irreducible morphism)**.** *Suppose $M$ and $N$ are MCM $R$-modules. We decompose $M$ and $N$ into indecomposable modules as $M = \oplus_i M_i$, $N = \oplus_j N_j$. Also, we decompose $\psi \in \mathrm{Hom}_R(M,N)$ along the above decomposition as $\psi = (\psi_{ij} : M_i \to N_j)_{ij}$. Then we define submodule $\mathrm{rad}_R(M,N) \subset \mathrm{Hom}_R(M,N)$ as*

$$\psi \in \mathrm{rad}_R(M,N) \iff \textit{no } \psi_{ij} \textit{ is an isomorphism.}$$

*Furthermore, we define submodule $\mathrm{rad}_R^2(M,N) \subset \mathrm{Hom}_R(M,N)$. The submodule $\mathrm{rad}_R^2(M,N)$ consists of morphisms $\psi : M \to N$ such that $\psi$ decomposes as $\psi = fg$,*

$$\begin{tikzcd} M \arrow[rr, "\psi"] \arrow[dr, "f"'] & & N \\ & X \arrow[ur, "g"'] & \end{tikzcd}$$

*where $X$ is an MCM $R$-module, $f \in \mathrm{rad}_R(M,X)$, $g \in \mathrm{rad}_R(X,N)$. We say that a morphism $\psi : M \to N$ is irreducible if $\psi \in \mathrm{rad}_R(M,N) \setminus \mathrm{rad}_R^2(M,N)$. In this setting, we define the $k$-vector space $\mathrm{Irr}_R(M,N)$ as*

$$\mathrm{Irr}_R(M,N) := \mathrm{rad}_R(M,N)/\,\mathrm{rad}_R^2(M,N).$$

We are now ready to define the AR quiver.

**Definition 4.2.3** (Auslander-Reiten quiver)**.** *The AR quiver of $R$ is an oriented graph whose vertices are indecomposable MCM $R$-modules $\{R, M_1, \cdots, M_n\}$ and draw $\dim_k \mathrm{Irr}_R(M_s, M_t)$ arrows from $M_s$ to $M_t$ $(s,t = 0,1,\cdots,n)$.*

*Remark* 4.2.4. Sometimes we connect the vertex $M_t$ to $\tau(M_t)$ by a dotted line. In this thesis, we don't use this manner.

Let $E_{M_t}$ be the middle term of the AR sequence ending in $M_t$ for $t = 1,\cdots,n$. Then it is known that $\dim_k \mathrm{Irr}_R(M_s, M_t)$ is equal to the multiplicity of $M_s$ in the indecomposable decomposition of $E_{M_t}$. So we describe the AR quiver from the structure of AR sequences. From the construction of the AR sequence and a one to one correspondences in the previous section, we have the following.

**Theorem 4.2.5** ([Aus2])**.** *Let $G$ and $R$ be as above. Then the AR quiver of $R$ coincides with the McKay quiver of $G$.*

So we can describe the AR quiver of $R$ from representations of $G$. Note that finite subgroups of $\mathrm{GL}(2,k)$ which contain no pseudo-reflections are classified in [Bri] and their McKay quiver (equivalently AR quiver) are described in [AR1]. As we mentioned, if $G$ is a finite subgroup of $\mathrm{SL}(2,k)$, then the associated quivers take the form of extended Dynkin diagrams (see also the beginning of Section 5.4). In the next chapter, we will give many examples of these quivers.

# Chapter 5

# Dual $F$-signature of Cohen-Macaulay modules

From now, we investigate the notion of the dual $F$-signature (see Section 2.6). As we saw in Theorem 2.6.3, the value of $s(R)$ and $s(\omega_R)$ characterize some singularities. Now we have some questions. Let $M$ be a finitely generated $R$-module which may not be $R$ or $\omega_R$. Then

· Does the value of $s(M)$ contain any information about singularities ?
· What does the explicit value of $s(M)$ mean ?
· Is there any connection between $s(M)$ and other numerical invariants ?

It is difficult to answer these questions for now, because the value of the dual $F$-signature is not known and we don't have effective method for determining it except in only a few cases. In this chapter, we will determine the value of the dual $F$-signature for certain MCM modules over quotient surface singularity. Therefore, in the rest of this chapter, we suppose that $G$ is a finite subgroup of $\mathrm{GL}(2,k)$ which contains no pseudo-reflections and $S := k[[x,y]]$ be the power series ring. We assume that the order of $G$ is coprime to $p = \mathrm{char}\ k$. We denote the invariant subring of $S$ under the action of $G$ by $R := S^G$. Let $V_0 = k, V_1, \cdots, V_n$ be the complete set of irreducible representations of $G$ and set the indecomposable MCM $R$-modules $M_t := (S \otimes_k V_t)^G \quad (t = 0, 1, \cdots, n)$ as in Chapter 4. We will consider the dual $F$-signature of $M_t$.

By the definition of the dual $F$-signature, we should understand the following topics:

(1) The structure of ${}^eM_t$, namely

 · What kind of MCM appears in ${}^eM_t$ as a direct summand?
 · The asymptotic behavior of ${}^eM_t$ on the order of $p^{2e}$.

(2) How do we construct a surjection ${}^eM_t \twoheadrightarrow M_t^{\oplus b_e}$ efficiently ?

We could understand (1) by Theorem 3.3.1 and Corollary 3.3.6. As we saw in the previous chapter, it is well known that $R$ is of finite CM representation type, that is, it has only finitely many non-isomorphic indecomposable MCM $R$-modules $\{R, M_1, \cdots, M_n\}$. As Corollary 3.3.6 shows, every indecomposable MCM $R$-modules appear in ${}^eM_t$ as a direct summand for sufficiently large $e \gg 0$. Therefore, the additive closure $\mathrm{add}_R({}^eM_t)$ coincides with the category of MCM $R$-modules $\mathrm{CM}(R)$. So we can apply Auslander-Reiten theory to $\mathrm{add}_R({}^eM_t)$. Especially, we will use the AR quiver to construct a surjection ${}^eM_t \twoheadrightarrow M_t^{\oplus b_e}$. By using it, we visualize relations among MCM $R$-modules and construct a surjection efficiently and can determine the value of the dual $F$-signature for a certain MCM module. This chapter is based on [Nak1] and [Nak2].

## 5.1 Counting argument of Auslander-Reiten quiver

From Nakayama's lemma, when we discuss the surjectivity of ${}^eM_t \to M_t^{\oplus b}$, we may consider an MCM module $M_t$ as a vector space after tensoring by the residue field $k$. Thus, we investigate a basis of $M_t/\mathfrak{m}M_t$, equivalently a set of minimal generators of $M_t$.

Let $M$ be a non-free indecomposable MCM $R$-module. The number of minimal generator $\mu_R(M)$ is equal to $\dim_k M/\mathfrak{m}M$ and

$$\begin{array}{ccc} M & \cong & \mathrm{Hom}_R(R, M) \\ \cup & & \\ \mathfrak{m}M & \cong & \{R \overset{\text{non split}}{\to} R^{\oplus m} \to M\} \end{array}$$

for some $m \in \mathbb{N}$. From this observation, we identify a minimal generator of $M$ with a morphism from $R$ to $M$ which doesn't factor through free modules except the starting point. We will use this idea in sections below. We spend the rest of this section describing such morphisms.

In order to find such morphisms, we define the stable category $\underline{\mathrm{CM}}(R)$ as follows. The objects of $\underline{\mathrm{CM}}(R)$ are same as those of $\mathrm{CM}(R)$ and the morphism set is given by

$$\underline{\mathrm{Hom}}_R(X, Y) := \mathrm{Hom}_R(X, Y)/\mathcal{P}(X, Y), \quad X, Y \in \mathrm{CM}(R)$$

where $\mathcal{P}(X, Y)$ is the submodule of $\mathrm{Hom}_R(X, Y)$ consisting of morphisms which factor through a free $R$-module.

Assume that $R$ is not isomorphic to $\omega_R$ ($\cong \tau R$), that is, $R$ is not Gorenstein. Let

$$0 \longrightarrow R \xrightarrow{g} E \xrightarrow{f} \tau^{-1}R \longrightarrow 0 \tag{5.1.1}$$

be the AR sequence ending in $\tau^{-1}R$. For the morphism of functor category

$$\underline{\mathrm{Hom}}_R(\tau^{-1}R, -) \xrightarrow{f\cdot -} \underline{\mathrm{Hom}}_R(E, -),$$

we define the covariant additive functor $\mathbb{F} : \underline{\mathrm{CM}}(R) \to \mathcal{A}$ as the cokernel of $(f \cdot -)$

$$\underline{\mathrm{Hom}}_R(\tau^{-1}R, -) \xrightarrow{f\cdot -} \underline{\mathrm{Hom}}_R(E, -) \longrightarrow \mathbb{F} \longrightarrow 0,$$

where $\mathcal{A}$ is the category of abelian groups. It is easy to see $\mathrm{Ker}(f\cdot -)=0$. By properties of the AR sequence (5.3.1), any morphism $R\to M$ factors through $E$ and $gf=0$. Thus, on the short exact sequence

$$\underline{\mathrm{Hom}}_R(\tau^{-1}R, M) \xrightarrow{f\cdot -} \underline{\mathrm{Hom}}_R(E, M) \longrightarrow \mathbb{F}(M) \longrightarrow 0,$$

the composition morphisms of $R \xrightarrow{g} E$ and non-zero elements of $\mathbb{F}(M)$ are exactly what we wanted.

*Remark* 5.1.1. In the case when $R$ is Gorenstein, we use the fundamental sequence

$$0 \longrightarrow R \longrightarrow E \xrightarrow{f} R \longrightarrow k \longrightarrow 0$$

instead of the AR sequence (5.3.1), and we obtain $\mathbb{F}(M)\cong \underline{\mathrm{Hom}}_R(E, M)$ by similar arguments.

In order to find non-zero elements of $\mathbb{F}(M)$, we compute $\dim_k \mathbb{F}(M) = \dim_k \underline{\mathrm{Hom}}_R(E, M) - \dim_k \underline{\mathrm{Hom}}_R(\tau^{-1}R, M)$. More precisely, we will find a $k$-basis of $\mathbb{F}(M)$. For this purpose, the counting arguments of AR quiver plays a crucial role. This method first appeared in the work of Gabriel [Gab] and it was also used for classifying special CM modules over quotient surface singularities [IW]. For more details about the counting arguments of AR quiver, see e.g. [Gab, Iya, IW, Wem2]. For simplicity, we give a brief review of this kind of arguments in the form of algorithm as follows (cf. [Wem2, Section 4]).

1. In the AR quiver $Q$, we write a 1 (resp. $-1$) at the position corresponding to $E$ (resp. $\tau^{-1}R$). For every MCM $R$-module $N$, we define the following number

$$\nu_N^{(0)} := \lambda_N^{(0)} := \begin{cases} 1 & \text{if } N = E, \\ -1 & \text{if } N = \tau^{-1}R, \\ 0 & \text{otherwise.} \end{cases}$$

2. Next, we consider all arrows out of $E$ in $Q$ and call the head of these arrows the first-step vertices of $E$. For every MCM $R$-module $N$, we set

$$\lambda_N^{(1)} := \begin{cases} 1+\nu_N^{(0)} & \text{if } N \text{ is a first-step vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

   Then we define

$$\nu_N^{(1)} := \begin{cases} 0 & \text{if } N = R, \\ \lambda_N^{(1)} & \text{otherwise.} \end{cases}$$

   For every first-step vertex $N_1$, we put the number $\lambda_{N_1}^{(1)}$ on the corresponding vertex.

3. We consider all arrows out of the first-step vertices and call the head of these arrows the second-step vertices. For every MCM $R$-module $N$, we set

$$\lambda_N^{(2)} := \begin{cases} -\nu_{\tau(N)}^{(0)} + \sum_{L_1\to N} \nu_{L_1}^{(1)} & \text{if } N \text{ is a second-step vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

where $L_1$ runs over all first-step vertices. Then we define

$$\nu_N^{(2)} := \begin{cases} 0 & \text{if } N = R, \\ \lambda_N^{(2)} & \text{otherwise.} \end{cases}$$

For every second-step vertex $N_2$, we write the corresponding number $\lambda_{N_2}^{(2)}$.

4. Then we consider all arrows out of the second-step vertices, and we call the head of these arrows the third-step vertices. For every MCM $R$-module $N$, we set

$$\lambda_N^{(3)} := \begin{cases} -\nu_{\tau(N)}^{(1)} + \sum\limits_{L_2 \to N} \nu_{L_2}^{(2)} & \text{if } N \text{ is a third-step vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

where $L_2$ runs over all second-step vertices. We set

$$\nu_N^{(3)} := \begin{cases} 0 & \text{if } N = R, \\ \lambda_N^{(3)} & \text{otherwise.} \end{cases}$$

For every third-step vertex $N_3$, we write the corresponding number $\lambda_{N_3}^{(3)}$.

5. Continuing with this process, we record the number $\lambda_N^{(i)}$ on each vertex $N$. Since $R$ is of finite CM representation type, we have $\lambda_N^{(i)} = 0$ for some $i \in \mathbb{N}$ sooner or later. Thus, we will stop there.

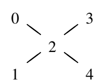
   The number $\lambda_N^{(i)}$ means that there are $\lambda_N^{(i)}$ non-zero morphisms in $\mathbb{F}(N)$ for each corresponding vertex $N$, and such morphisms consist a $k$-basis of $\mathbb{F}(N)$. Note that we have $\dim_k \mathbb{F}(N) = \sum_{i \geq 0} \lambda_N^{(i)}$.

**Example 5.1.2.** *Let $G$ be the following finite group*

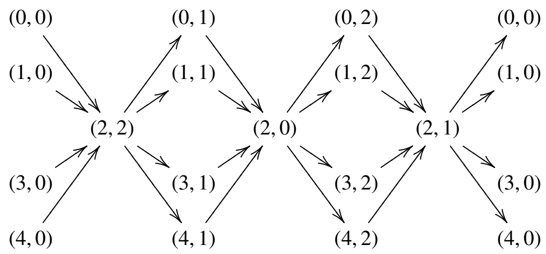
$$G := \langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6 \end{pmatrix} \rangle \subset \mathrm{GL}(2,k),$$

*where $\zeta_6$ is a primitive 6-th root of unity. This group is isomorphic to $\mathcal{D}_2 \times Z_3$ where $Z_3$ is generated by the scalar matrix* $\mathrm{diag}(\zeta_3, \zeta_3)$ *and $\mathcal{D}_2$ is the binary dihedral group of order* 8 (*see also the beginning of Section 5.4*). *Note that this group is denoted by $D_{5,2}$ in [Rie]. Then we have finitely many irreducible representations*

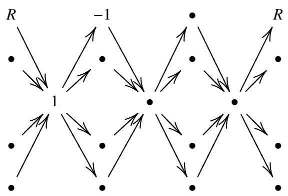
$$V_{i,j} := V_i \otimes W_j \quad (i = 0, 1, \cdots, 4, \ \ j = 0, 1, 2)$$

*where $W_j$ is a irreducible representation of $Z_3$ which represents* $\mathrm{diag}(\zeta_3, \zeta_3) \mapsto \zeta_3^j$ *and $V_i$ is a that of $\mathcal{D}_2$ associated to the extended Dynkin diagram* 

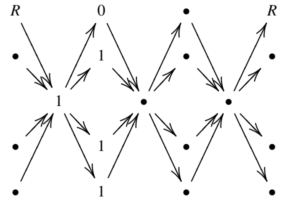
*and set the indecomposable MCM module $M_{i,j} := (S \otimes_k V_{i,j})^G$. The AR quiver of $k[[x,y]]^G$ is the following* (*for simplicity we only describe subscripts as vertices*);



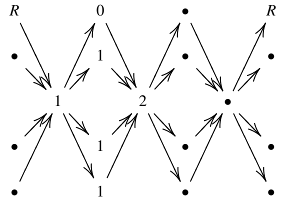

*where the left and right hand sides are identified and the vertex* $(0, 0)$ *represents* $R$ (*for more details, see [AR1]*). *In this case, we can see that* $E = M_{2,2}$, $\tau^{-1}R = M_{0,1}$. (*Check the notation used in the above algorithm.*) *By applying the counting argument to this quiver, we have the following.*



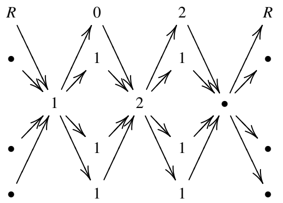
*Step 1*



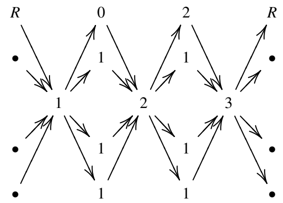
*Step 2*



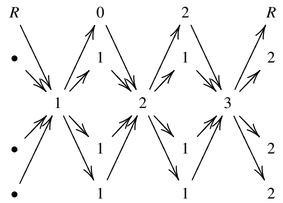
*Step 3*



*Step 4*

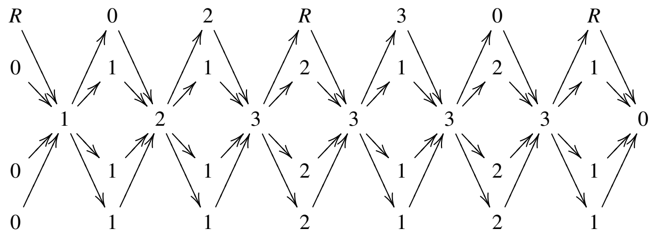


*Step 5*



*Step 6*

*Continuing with this process, finally we get to the following picture.*



*By extracting non-zero paths from the above quiver, we have the Figure 5.1 where the exponent of each vertex implies the multiplicity.*

Thus, we can identify minimal generators of $M_{i,j}$ with non-zero paths from $R$ to $M_{i,j}$ on Figure5.1. For example, $M_{1,1}$ has two minimal generators associated to the following two paths.

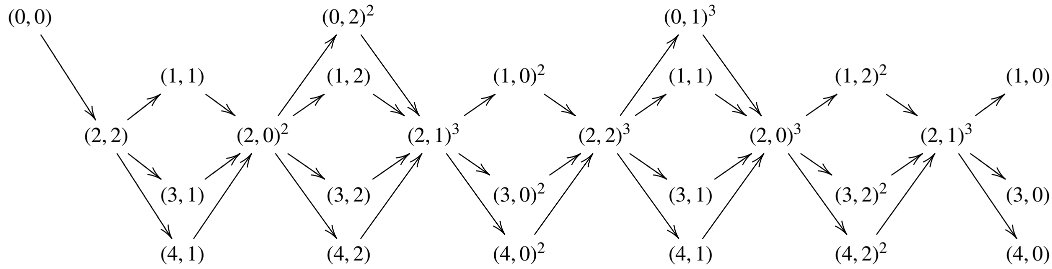
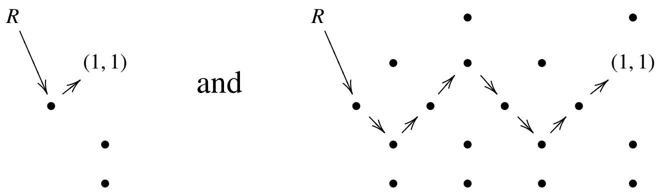



Figure 5.1: The composition $R \to M_{2,2}$ and non-zero elements of $\underline{\mathrm{Hom}}(M_{2,2}, -)$




In other words, a minimal generator of $R$ (i.e. a unit of $R$) generates those of $M_1$ by chasing the above paths. Of course, there are several paths from $R$ to $M_{1,1}$ not only the above ones. Since the AR quiver has relations originated from AR sequences, they generate the same minimal generator up to modulo radical. Furthermore, composing the irreducible morphism $M_{1,1} \to M_{2,0}$ and the above paths, we have a part of minimal generators of $M_{2,0}$.

We will use this technique in Section 5.3 and 5.4.

## 5.2 Dual $F$-signature of special Cohen-Macaulay modules

In order to investigate properties of the dual $F$-signature for the case where quotient surface singularities. We will introduce a certain class of MCM modules "so-called special CM modules". As we will mention below, special CM modules are compatible with geometry as the special McKay correspondence. In the this section, we will compare the dual $F$-signature of special CM modules with its AR translation. It will give us a characterization of Gorensteiness (see Proposition 5.2.6). This is an analogue of Theorem 2.6.3 (2), (3).

### 5.2.1 Special McKay correspondence

Special CM modules appear when we try to extend the classical McKay correspondence to a finite subgroup $G \subset \mathrm{GL}(2, k)$. For a finite subgroup $G \subset \mathrm{SL}(2, k)$, the original McKay correspondence says that there is a one-to-one correspondence between non-free indecomposable MCM $R$-modules (equivalently, non-trivial irreducible representations of $G$) and irreducible exceptional curves on the minimal resolution of Spec $R$ (see Chapter 4). This brilliant correspondence collapse if we consider a finite subgroup $G \subset \mathrm{GL}(2, k)$. Indeed, we have more indecomposable MCM modules than exceptional curves. So J. Wunram introduced the notion of special CM modules. By choosing indecomposable special ones from all MCM modules, we again have a one-to-one correspondence between non-free indecomposable "special" MCM $R$-modules and irreducible exceptional curves [Wun2] (see Theorem 5.2.4). For more details, see also [Wun1, Wun2, Ish, Ito, Rie] etc. Let us recall the definition of special CM modules.

**Definition 5.2.1** ([Wun2])**.** *For an MCM $R$-module $M$, we say $M$ is special if* $(M \otimes_R \omega_R)/\operatorname{tor}$ *is also an MCM $R$-module.*

*Remark* 5.2.2. From the definition, if $R$ is Gorenstein (i.e. $G \subset \mathrm{SL}(2, k)$ [Wat1]), then every MCM module is special. Thus, the original McKay correspondence is recovered from the special one.

Definition 5.2.1 is the original one. There are now several characterizations of special CM modules. For example, the following conditions are manageable in our context.

**Proposition 5.2.3.** *([IW, 2.7 and 3.6]) Suppose that $M$ is an MCM $R$-module. Then the following are equivalent.*

- *(a) $M$ is a special CM module,*
- *(b)* $\mathrm{Ext}^1_R(M, R) = 0$,
- *(c)* $(\Omega M)^* \cong M$.

Suppose $M$ is a special CM $R$-module, then we have the following exact sequence by the condition (c).

$$0 \to M^* \cong \Omega M \to R^{\oplus \mu_R(M)} \to M \to 0.$$

Thus, we have $\mu_R(M) = 2 \operatorname{rank}_R M$. The converse is true if $\operatorname{rank}_R M = 1$ (cf. [Wun2, Theorem 2.1] [GOTWY2, Lemma 4.6]). If $\operatorname{rank}_R M > 1$, the converse is no longer true (cf. Example 6.2.5 and [IW]). As we will see later, every MCM modules over cyclic quotient surface singularities has rank one. Thus, the structure of a special CM module is quite simple. (More precise description is given in Theorem 5.3.3.)

As we mentioned, special CM modules are compatible with the geometry. Thus we will introduce terminologies in the geometric side and show the relationship between special CM modules and geometrical objects.

Let $\pi : X \to \operatorname{Spec} R$ be the minimal resolution of singularities and $E := \pi^{-1}(\mathfrak{m})$ be the exceptional divisor. We decompose $E = \bigcup_{i=1}^{r} E_i$ into irreducible components and define the set of cycles supported on $E$:

$$C = \left\{ \sum_{i=1}^{r} a_i E_i \mid a_i \in \mathbb{Z} \right\}.$$

Also, we can impose a partial order $\leq$ on $C$. That is, $Z \leq Z'$ if every coefficient of $E_i$ in $Z' - Z$ is non-negative ($Z, Z' \in C$). We say a cycle $Z = \sum_{i=1}^{r} a_i E_i$ is positive if $Z \geq 0$

and $Z \neq 0$. (So we denote it by $Z > 0$.) We call a positive cycle $Z = \sum_{i=1}^{r} a_i E_i$ anti-nef if $Z \cdot E_i \leq 0$ for all $i = 1, \cdots, r$. Here, $Z \cdot Z'$ means the intersection number of $Z$ and $Z'$ $(Z, Z' \in C)$. If $Z = Z'$, the self-intersection number of $Z$ is denoted by $Z^2$. We define the fundamental cycle $Z_0$ as the unique smallest element of anti-nef cycles. There is an algorithm to determine $Z_0$ by [Lau].

The following is famous as the special McKay correspondence.

**Theorem 5.2.4** ([Wun2])**.** *For any i, there is a unique indecomposable MCM R-module $M_i$ (up to isomorphism) such that $\mathrm{H}^1(\widetilde{M_i}^{\dagger}) = 0$ and $\mathrm{c}_1(\widetilde{M_i}) \cdot E_j = \delta_{ij}$ for $1 \leq i, j \leq r$ where $\widetilde{M_i} = \pi^*(M_i)/\operatorname{tor}$ and $\mathrm{c}_1(\widetilde{M_i})$ stands for the first Chern class of $\widetilde{M_i}$ and $(-)^{\dagger} = \mathcal{H}om_{O_X}(-, O_X)$. These MCM modules $M_1, \cdots, M_r$ are precisely indecomposable non-free special CM modules in the sense of Definition 5.2.1 and* $\operatorname{rank}_R M_i = \mathrm{c}_1(\widetilde{M_i}) \cdot Z_0$.

### 5.2.2 Comparing with Auslander-Reiten translation

Before moving to the comparison between the dual $F$-signature of a special CM module and that of its AR translation, we prepare the following lemma.

**Lemma 5.2.5.** *Let $M_t$ be an MCM R-module as in the beginning of this chapter. Then we have*

$$ {}^{e}M_t \approx (R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}})^{\oplus \frac{p^{2e}}{n}} \approx {}^{e}\tau(M_t) \tag{5.2.1}$$

*on the order of $p^{2e}$, where $d_{i,t} = (\operatorname{rank}_R M_t)\cdot(\operatorname{rank}_R M_i)$ and $\tau$ stands for the AR translation.*

*Furthermore, we have*

$$R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}} \cong \tau(R)^{\oplus d_{0,t}} \oplus \tau(M_1)^{\oplus d_{1,t}} \oplus \cdots \oplus \tau(M_n)^{\oplus d_{n,t}}.$$

*Proof.* From Corollary 3.3.6, we may consider as

$$ {}^{e}M_t \approx (R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}})^{\oplus \frac{p^{2e}}{n}},$$

$$ {}^{e}\tau(M_t) \approx (R^{\oplus d'_{0,t}} \oplus M_1^{\oplus d'_{1,t}} \oplus \cdots \oplus M_n^{\oplus d'_{n,t}})^{\oplus \frac{p^{2e}}{n}},$$

where $d'_{i,t} = (\operatorname{rank}_R \tau(M_t)) \cdot (\operatorname{rank}_R M_i)$. Since $\operatorname{rank}_R M_t = \operatorname{rank}_R \tau(M_t)$, it follows that $d_{i,t} = d'_{i,t}$ $(i = 0, 1, \cdots, n)$. This implies (5.2.1).

Since the AR translation $\tau$ gives a bijection from the set of finitely many indecomposable MCM $R$-modules to itself, we set $\tau(M_i) = M_{\sigma(i)}$ $(i = 0, 1, \cdots, n)$, where $\sigma$ is an element of symmetric group $\mathfrak{S}_{n+1}$. Then we have

$$R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}} \cong M_{\sigma(0)}^{\oplus d_{\sigma(0),t}} \oplus M_{\sigma(1)}^{\oplus d_{\sigma(1),t}} \oplus \cdots \oplus M_{\sigma(n)}^{\oplus d_{\sigma(n),t}},$$

and

$$\begin{aligned} d_{\sigma(i),t} &= (\operatorname{rank}_R M_t) \cdot (\operatorname{rank}_R M_{\sigma(i)}) = (\operatorname{rank}_R M_t) \cdot (\operatorname{rank}_R \tau(M_i)) \\ &= (\operatorname{rank}_R M_t) \cdot (\operatorname{rank}_R M_i) = d_{i,t}. \end{aligned}$$

Thus,

$$M_{\sigma(0)}^{\oplus d_{\sigma(0),t}} \oplus M_{\sigma(1)}^{\oplus d_{\sigma(1),t}} \oplus \cdots \oplus M_{\sigma(n)}^{\oplus d_{\sigma(n),t}} = \tau(R)^{\oplus d_{0,t}} \oplus \tau(M_1)^{\oplus d_{1,t}} \oplus \cdots \oplus \tau(M_n)^{\oplus d_{n,t}}.$$

□

**Theorem 5.2.6.** *Suppose $M_t$ is an indecomposable special CM R-module. Then we have*

$$s(M_t) \leq s(\tau(M_t)).$$

*Moreover, R is Gorenstein if and only if $s(M_t) = s(\tau(M_t))$.*

*Remark* 5.2.7. Since $\tau(R) \cong \omega_R$ in our situation, this theorem is an analogue of Theorem 2.6.3 (2), (3). But it says that this characterization is obtained by not only the comparison between $R$ and $\omega_R$ but also the comparison between a special CM module and its AR translation.

*Proof.* From Lemma 5.2.5, we may consider as

$${}^eM_t \approx {}^e\tau(M_t) \approx (R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}})^{\oplus \frac{p^{2e}}{n}}$$

when we discuss the asymptotic behavior on the order of $p^{2e}$, where $d_{i,t} = (\operatorname{rank}_R M_t) \cdot (\operatorname{rank}_R M_i)$. In the rest of this proof, we discuss on this setting and for simplicity we identify ${}^eM_t \approx {}^e\tau(M_t)$ with $R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}}$.

Since $M_t$ is special, the morphism $\varphi : M_t \otimes_R \omega_R \to (M_t \otimes_R \omega_R)^{**}$ is surjective. Let $b_e := b_e(M_t)$ be the $e$-th $F$-surjective number of $M_t$. Then there exists the surjection ${}^eM_t \twoheadrightarrow M_t^{\oplus b_e}$. Applying the functor $(- \otimes_R \omega_R)$ and combining with $\varphi$, we obtain the surjection

$${}^eM_t \otimes_R \omega_R \twoheadrightarrow (M_t \otimes_R \omega_R)^{\oplus b_e} \xrightarrow{\varphi^{\oplus b_e}} ((M_t \otimes_R \omega_R)^{**})^{\oplus b_e} \cong \tau(M_t)^{\oplus b_e}. \tag{5.2.2}$$

Since we consider as ${}^eM_t \cong R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}}$, it follows that ${}^eM_t \otimes_R \omega_R \cong \bigoplus_{i=0}^n (M_i \otimes_R \omega_R)^{\oplus d_{i,t}}$ and the surjection (5.2.2) induces the following commutative diagram.

$$\begin{tikzcd}
\bigoplus_{i=0}^n (M_i \otimes_R \omega_R)^{\oplus d_{i,t}} \arrow[r, two heads] \arrow[d] & \tau(M_t)^{\oplus b_e} \arrow[d, "\cong"] \\
\left(\bigoplus_{i=0}^n (M_i \otimes_R \omega_R)^{\oplus d_{i,t}}\right)^{**} \arrow[r] & (\tau(M_t)^{\oplus b_e})^{**}
\end{tikzcd}$$

Thus, the morphism

$$\left(\bigoplus_{i=0}^n (M_i \otimes_R \omega_R)^{\oplus d_{i,t}}\right)^{**} \cong \bigoplus_{i=0}^n \tau(M_i)^{\oplus d_{i,t}} \twoheadrightarrow \tau(M_t)^{\oplus b_e}$$

is also surjective. From Lemma 5.2.5, we obtain ${}^e\tau(M_t) \approx \bigoplus_{i=0}^n \tau(M_i)^{\oplus d_{i,t}}$. Thus, there exists the surjection ${}^e\tau(M_t) \twoheadrightarrow \tau(M_t)^{\oplus b_e}$. This implies $s(M_t) \leq s(\tau(M_t))$.

If $R$ is Gorenstein, then $M_t \cong \tau(M_t)$. Thus $s(M_t) = s(\tau(M_t))$ holds. So we shall show the opposite direction. Assume that $R$ is not Gorenstein. Since $M_t$ is special, the number of minimal generators of $M_t$ is equal to $u := 2 \operatorname{rank}_R M_t$. Thus, there exists the surjection $R^{\oplus b_e u} \twoheadrightarrow M_t^{\oplus b_e}$ and induces the following commutative diagram.

$$\begin{tikzcd}
{}^eM_t \arrow[r, two heads] & M_t^{\oplus b_e} \\
R^{\oplus b_e u} \arrow[u, dotted] \arrow[ur]
\end{tikzcd}$$

Applying the functor $(- \otimes_R \omega_R)^{**}$ to this commutative diagram, then we obtain the commutative diagram.

$$\begin{tikzcd} {}^e\tau(M_t) \approx ({}^eM_t \otimes_R \omega_R)^{**} \arrow[r, "\psi_2"] & \tau(M_t)^{\oplus b_e} \\ \omega_R^{\oplus b_e u} \arrow[u] \arrow[ur, "\psi_1"'] & \end{tikzcd}$$

Note that the morphism $\psi_1$ is surjective because the surjection $R^{\oplus b_e u} \twoheadrightarrow M_t^{\oplus b_e}$ induces

$$\begin{tikzcd} \omega_R^{\oplus b_e u} \arrow[r, two heads] \arrow[d, "\cong"'] & (M_t \otimes_R \omega_R)^{\oplus b_e} \arrow[d, "\oplus\varphi"] \\ ((\omega_R)^{**})^{\oplus b_e u} \arrow[r, "\psi_1"'] & ((M_t \otimes_R \omega_R)^{**})^{\oplus b_e} \end{tikzcd}$$

and $\varphi : M_t \otimes_R \omega_R \to (M_t \otimes_R \omega_R)^{**}$ is surjective, and this implies $\psi_2$ is also surjective. On the surjection

$$\omega_R^{\oplus b_e u} \longrightarrow {}^e\tau(M_t) \cong \bigoplus_{i=0}^{n} \tau(M_i)^{\oplus d_{i,t}} \xrightarrow{\psi_2} \tau(M_t)^{\oplus b_e},$$

the morphisms which go through $R$ don't contribute to construct a surjection by Nakayama's lemma. Thus,

$$\bigoplus_{i=0}^{n} \tau(M_i)^{\oplus d_{i,t}} \Big/ R^{\oplus d_{0,t}} \twoheadrightarrow \tau(M_t)^{\oplus b_e}$$

is also surjective. In addition to this surjection, we can construct the surjection

$$R^{\oplus d_{0,t}} \twoheadrightarrow \tau(M_t)^{\oplus \frac{d_{0,t}}{v}},$$

where $v$ is the number of minimal generators of $\tau(M_t)$. This implies

$$b_e(\tau(M_t)) \geq b_e + \frac{d_{0,t}}{v} > b_e,$$

where $b_e(\tau(M_t))$ is the $e$-th $F$-surjective number of $\tau(M_t)$. Thus, $s(\tau(M_t)) > s(M_t)$. $\square$

## 5.3 Dual $F$-signature for cyclic quotient singularities

Until now, we considered a special CM module in general situation and showed that the dual $F$-signature of special CM modules has a typical property (see Theorem 5.2.6). In this section, we focus on the case of cyclic quotient surface singularities. Especially we will determine the explicit value of the dual $F$-signature for special CM modules.

Thus, we suppose that $G$ is a cyclic group as follows.

$$G := \langle\ \sigma = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^a \end{pmatrix} \rangle,$$

where $\zeta_n$ is a primitive $n$-th root of unity, $1 \leq a \leq n-1$, and $\gcd(a,n) = 1$ and assume that $n$ is invertible in $k$. This cyclic group $G$ is denoted by $\frac{1}{n}(1,a)$. We will consider the invariant subring $R = k[[x,y]]^G$ under the action of this cyclic group $G$. Since $G$ is an abelian group, every irreducible representation of $G$ is one dimensional and described as

$$V_t : \sigma \mapsto \zeta_n^{-t} \quad (t = 0, 1, \cdots, n-1).$$

We set,

$$M_t := (S \otimes_k V_t)^G = \left\langle x^i y^j \;\middle|\; i + ja \equiv t \pmod{n} \right\rangle, \quad (t = 0, 1, \cdots, n-1).$$

Then, these $M_t$ are MCM modules over $R$ and $\operatorname{rank} M_t = 1$. Note that $R$ is of finite CM representation type and $\mathrm{CM}(R) = \mathrm{add}_R(R \oplus \bigoplus_{t=1}^{n-1} M_t)$ (see Chapter 4).

For a cyclic group $G = \frac{1}{n}(1, a)$, we can determine special CM modules by using the following combinatorial data. As the first step, we consider the Hirzebruch-Jung continued fraction expansion of $n/a$, that is,

$$\frac{n}{a} = \alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\cdots - \frac{1}{\alpha_r}}} := [\alpha_1, \alpha_2, \cdots, \alpha_r],$$

and then we define the notion of $i$-series and $j$-series (cf. [Wem1], [Wun1]).

**Definition 5.3.1.** *For $n/a = [\alpha_1, \alpha_2, \cdots, \alpha_r]$, the $i$-series and the $j$-series are defined as follows.*

$$\begin{aligned} &i_0 = n, \quad i_1 = a, \qquad i_t = \alpha_{t-1} i_{t-1} - i_{t-2} \qquad (t = 2, \cdots, r+1), \\ &j_0 = 0, \quad j_1 = 1, \qquad j_t = \alpha_{t-1} j_{t-1} - j_{t-2} \qquad (t = 2, \cdots, r+1). \end{aligned}$$

*Remark* 5.3.2. From the construction method, it is easy to see

- $i_t \equiv j_t a \pmod{n}$,
- $i_0 = n > i_1 = a > i_2 > \cdots > i_r = 1 > i_{r+1} = 0$,
- $j_0 = 0 < j_1 = 1 < j_2 = \alpha_1 < \cdots < j_r < j_{r+1} = n$.

By using the $i$-series and the $j$-series, we can characterize special CM $R$-modules.

**Theorem 5.3.3** ([Wun1])**.** *For a cyclic group $G = \frac{1}{n}(1, a)$ with $n/a = [\alpha_1, \alpha_2, \cdots, \alpha_r]$, $M_{i_t}$ $(t = 1, \cdots, r)$ and $R$ are precisely special CM modules over $R$. Furthermore, the minimal generators of $M_{i_t}$ are $x^{i_t}$ and $y^{j_t}$ for $t = 1, \cdots, r$.*

From Theorem 5.2.4, there is a one-to-one correspondence between non-free indecomposable special CM modules and irreducible exceptional curves. The dual graph of the minimal resolution of singularity $X \to \mathrm{Spec}(R)$ is also obtained by the Hirzebruch-Jung continued fraction expansion:

$$\overset{E_{i_1}}{(-\alpha_1)} \text{ --- } \overset{E_{i_2}}{(-\alpha_2)} \text{ --- } \cdots\cdots \text{ --- } \overset{E_{i_r}}{(-\alpha_r)}$$

Here, an including number in each circle is the self-intersection number of the corresponding exceptional curve. The fundamental cycle is $Z_0 = \sum_{t=1}^{r} E_{i_t}$.

**Example 5.3.4.** *Let $G = \frac{1}{7}(1,3)$ be a cyclic group of order 7. The Hirzebruch-Jung continued fraction expansion of $7/3$ is*

$$\frac{7}{3} = 3 - \frac{1}{2-1/2} = [3,2,2],$$

*and the i-series and the j-series are described as follows.*

$$\begin{array}{lllll} i_0 = 7, & i_1 = 3, & i_2 = 2, & i_3 = 1, & i_4 = 0, \\ j_0 = 0, & j_1 = 1, & j_2 = 3, & j_3 = 5, & j_4 = 7. \end{array}$$

*Thus, the special CM modules are $R, M_1, M_2, M_3$ and these are described explicitly*

$$R = k[[x^7, x^4y, xy^2, y^7]], \quad M_1 = Rx + Ry^5, \quad M_2 = Rx^2 + Ry^3, \quad \text{and} \quad M_3 = Rx^3 + Ry.$$

**Example 5.3.5.** *Suppose $G = \frac{1}{n}(1, n-1) \subset \mathrm{SL}(2,k)$ is a cyclic group of order $n$ (= Dynkin type $A_{n-1}$). The Hirzebruch-Jung continued fraction expansion of $n/(n-1)$ is*

$$\frac{n}{n-1} = 2 - \cfrac{1}{2 - \cfrac{1}{\cdots - 1/2}} = [\underbrace{2, 2, \cdots, 2}_{n-1}],$$

*and the i-series and the j-series are*

$$\begin{array}{llllll} i_0 = n, & i_1 = n-1, & i_2 = n-2, & \cdots, & i_{n-1} = 1, & i_n = 0, \\ j_0 = 0, & j_1 = 1, & j_2 = 2, & \cdots, & j_{n-1} = n-1, & j_n = n. \end{array}$$

*Therefore, every MCM module is special (cf. Remark 5.2.2).*

Also, we consider the AR quiver for the cyclic cases. For a cyclic quotient surface singularity $R$, the AR sequence ending in $M_t$ ($t \neq 0$) is

$$0 \longrightarrow M_{t-a-1} \longrightarrow M_{t-1} \oplus M_{t-a} \longrightarrow M_t \longrightarrow 0. \tag{5.3.1}$$

For the case where $t = 0$, we have the fundamental sequence of $R$;

$$0 \longrightarrow \omega_R \longrightarrow M_{-1} \oplus M_{-a} \longrightarrow R \longrightarrow k \longrightarrow 0. \tag{5.3.2}$$

Thus, $E_{M_t} = M_{t-1} \oplus M_{t-a}$ and $\tau(M_t) = M_{t-a-1}$ for $t = 0, 1, \cdots, n-1$.

*Remark* 5.3.6. It is known that $\dim_k \mathrm{Irr}_R(M_s, M_t)$ is equal to the multiplicity of $M_s$ in the decomposition of $E_{M_t}$. From (5.3.1) and (5.3.2), we have $\dim_k \mathrm{Irr}_R(M_{t-1}, M_t) = 1$ and $\dim_k \mathrm{Irr}_R(M_{t-a}, M_t) = 1$ for $t = 0, 1, \cdots, n-1$. We can take a morphism $\cdot x$ (resp. $\cdot y$) as a basis of $\mathrm{Irr}_R(M_{t-1}, M_t)$ (resp. $\mathrm{Irr}_R(M_{t-a}, M_t)$).

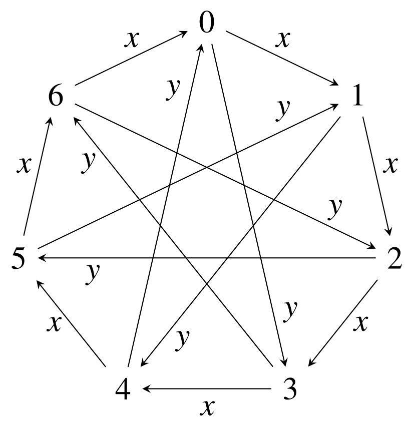
$$M_{t-1} = \{ f \in S \mid \sigma \cdot f = \zeta_n^{t-1} f \} \xrightarrow{\cdot x} M_t = \{ f \in S \mid \sigma \cdot f = \zeta_n^t f \}$$

$$M_{t-a} = \{ f \in S \mid \sigma \cdot f = \zeta_n^{t-a} f \} \xrightarrow{\cdot y} M_t = \{ f \in S \mid \sigma \cdot f = \zeta_n^t f \}$$

**Example 5.3.7.** *Let $G = \frac{1}{7}(1,3)$ be a cyclic group of order 7. The irreducible representations of $G$ are*

$$V_t : \sigma \mapsto \zeta_7^{-t} \quad (t = 0, \cdots, 6),$$

*where $\zeta_7$ is a primitive 7-th root of unity. Then the AR quiver of $R = S^G$ is the left figure below. For simplicity, we only describe subscripts as vertices. We can rewrite the left one as the form of the right one. (Here, the left and right columns are identified, moreover the top and bottom row are identified.)*




$$\begin{array}{ccccccccccccc}
0 & \to & 3 & \to & 6 & \to & \cdots & \to & 1 & \to & 4 & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
6 & \to & 2 & \to & 5 & \to & \cdots & \to & 0 & \to & 3 & \to & 6 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
5 & \to & 1 & \to & 4 & \to & \cdots & \to & 6 & \to & 2 & \to & 5 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & \iddots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
2 & \to & 5 & \to & 1 & \to & \cdots & \to & 3 & \to & 6 & \to & 2 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
1 & \to & 4 & \to & 0 & \to & \cdots & \to & 2 & \to & 5 & \to & 1 \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & 3 & \to & 6 & \to & \cdots & \to & 1 & \to & 4 & \to & 0
\end{array}$$

*Remark* 5.3.8. For each diagram $\begin{smallmatrix} a & \to & b \\ \uparrow & & \uparrow \\ c & \to & d \end{smallmatrix}$, if $b \neq 0$ then $0 \to M_c \to M_a \oplus M_d \to M_b \to 0$ is the AR sequence ending in $M_b$, and any diagram commutes $\begin{smallmatrix} a & \xrightarrow{y} & b \\ {\scriptstyle x}\uparrow & \circlearrowleft & \uparrow{\scriptstyle x} \\ c & \xrightarrow[y]{} & d \end{smallmatrix}$ from Remark 5.3.6.

Next, we will determine the explicit value of the dual $F$-signature for a special CM module $M_{i_t}$. As we mentioned in the beginning of this chapter, we have to construct a surjection ${}^eM_{i_t} \twoheadrightarrow M_{i_t}^{\oplus b_e}$ efficiently. For this purpose, we will pay attention to minimal generators of each MCM modules. As we saw in Section 5.1, we identify a minimal generator of $M_{i_t}$ with a morphism from $R$ to $M_{i_t}$ which doesn't factor through free modules except the starting point. We can find such a path through the counting argument of the AR quiver. Since the number of minimal generators of special CM $R$-module $M_{i_t}$ is two and minimal generators take a form like $x^{i_t}$, $y^{j_t}$ (cf. Theorem 5.3.3), we can see the corresponding paths are of the form as in Figure 5.2. Here, there is no "0" in dotted vertices area. By the above arguments, in order to construct the surjection ${}^eM_{i_t} \twoheadrightarrow M_{i_t}^{\oplus b_e}$, we may only discuss horizontal direction arrows from $R$ to $M_{i_t}$ and vertical direction arrows from $R$ to $M_{i_t}$. We consider sets of subscripts of vertices $\mathcal{F}_t = \{0, 1, \cdots, i_t - 1\}$ and $\mathcal{G}_t = \{i_t - a, \cdots, i_t - j_t a \equiv 0\}$. It is easy to see that $|\mathcal{F}_t| = i_t$, $|\mathcal{G}_t| = j_t$.

To determine the dual $F$-signature of special CM $R$-modules, we prepare some notations and lemmas.

For the $i$-series $(i_1, \cdots, i_r)$ associated with $\frac{1}{n}(1, a)$ and any $t \in \mathbb{Z}_{\geq 0}$ with $0 \leq t \leq n-1$,

$$
\begin{tikzcd}
\arrow[rrrrr, phantom, "\overbrace{\hspace{12em}}^{\mathcal{G}_t}"] & & & & & & \\
0 \equiv i_t - j_t a \arrow[r, "y"] & i_t - (j_t - 1)a \arrow[r, "y"] & \cdots \arrow[r, "y"] & i_t - 2a \arrow[r, "y"] & i_t - a \arrow[r, "y"] & i_t & \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \cdots \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & i_t - 1 \arrow[u, "x"'] & \\
\vdots \arrow[u] & \vdots \arrow[u] & \iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u, "x"'] & \mathcal{F}_t \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \cdots \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & 1 \arrow[u, "x"'] & \\
\bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & \cdots \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \arrow[r] & 0 \arrow[u, "x"'] &
\end{tikzcd}
$$

Figure 5.2

there are unique non-negative integers $d_{1,t}, \cdots, d_{r,t} \in \mathbb{Z}_{\geq 0}$ such that

$$
\begin{gathered}
t = d_{1,t} i_1 + h_1, \qquad h_1 \in \mathbb{Z}_{\geq 0}, \qquad 0 \leq h_1 < i_1; \\
h_u = d_{u+1} i_{u+1} + h_{u+1}, \qquad h_{u+1} \in \mathbb{Z}_{\geq 0}, \qquad 0 \leq h_{u+1} < i_{u+1}, \quad (u = 1, \cdots, r-1); \\
h_r = 0.
\end{gathered}
$$

Thus, we can describe $t$ as follows,

$$t = d_{1,t} i_1 + d_{2,t} i_2 + \cdots + d_{r,t} i_r,$$

and if a situation is clear, then we simply denote $d_{u,t}$ by $d_u$. For such $t$, there is the unique integer $\widetilde{t} \in \mathbb{Z}_{\geq 0}$ such that $a\widetilde{t} \equiv t \pmod n$, $0 \leq \widetilde{t} \leq n-1$.

**Lemma 5.3.9** ([Wun1]). *Let $\widetilde{t}$ be same as above. Then $\widetilde{t}$ is described as*

$$\widetilde{t} = d_{1,t} j_1 + d_{2,t} j_2 + \cdots + d_{r,t} j_r,$$

*where $(j_1, \cdots, j_r)$ is the j-series associated with $\frac{1}{n}(1, a)$.*

**Lemma 5.3.10.** *Let the notation be same as above, then $\mathcal{F}_t \cap \mathcal{G}_t = \{0\}$ as a set of subscripts of vertices.*

*Proof.* It is trivial that $0 \in \mathcal{F}_t \cap \mathcal{G}_t$ by the definition of $\mathcal{F}_t$ and $\mathcal{G}_t$. Thus, it suffices to show there is no pair $(m_1, m_2) \in \mathbb{Z}^2_{>0}$ such that $m_1 \equiv m_2 a \pmod n$, where $1 \leq m_1 \leq i_t - 1$ and $1 \leq m_2 \leq j_t - 1$. Assume that there exists such a pair $(m_1, m_2)$. Then there are non-negative integers $d_1, \cdots, d_r$ such that $m_1 = d_1 i_1 + d_2 i_2 + \cdots + d_r i_r$. Since $1 \leq m_1 \leq i_t - 1$ and $i_t > i_{t+1}$ (cf. Remark 5.3.2), $d_1 = \cdots = d_t = 0$ and there exists $\lambda$ such that $t + 1 \leq \lambda \leq r$ and $d_\lambda \neq 0$. From Lemma 5.3.9 we obtain $m_2 = d_1 j_1 + d_2 j_2 + \cdots + d_r j_r$. Thus,

$$m_2 = d_{t+1} j_{t+1} + \cdots + d_r j_r \geq j_\lambda > j_t.$$

This contradicts $m_2 \leq j_t - 1$. $\square$

So we are now ready to state the theorem.

**Theorem 5.3.11.** *Let the notation be the same as above, then for any non-free special CM R-module $M_{i_t}$ one has*

$$s(M_{i_t}) = \begin{cases} \dfrac{\min(i_t, j_t) + 1}{n} & (\text{if } i_t \neq j_t) \\[2ex] \dfrac{2i_t + 1}{2n} & (\text{if } i_t = j_t). \end{cases}$$

*Proof.* From Corollary 3.3.6, we may consider as

$$ {}^{e}M_{i_t} \approx (R \oplus M_1 \oplus \cdots \oplus M_{n-1})^{\oplus \frac{p^{2e}}{n}}.$$

Firstly, we shall show in the case of $i_t > j_t$. If $\mathcal{G}_t \setminus \{0\} \neq \emptyset$, then we choose an element from $\mathcal{F}_t \setminus \{0\}$ (named it $f$) and also choose an element from $\mathcal{G}_t \setminus \{0\}$ (named it $g$). Note that $f \neq g$, from Lemma 5.3.10. By using the corresponding indecomposable MCM $R$-modules $M_f$ and $M_g$, we construct a surjection $M_f \oplus M_g \twoheadrightarrow M_{i_t}$.

$$\begin{tikzcd}
0 \arrow[r, two heads] & \cdots \arrow[r, two heads] & g \arrow[r, two heads] & \cdots \arrow[r, two heads] & i_t \\
& & & & \vdots \arrow[u, two heads] \\
& & & & f \arrow[u, two heads] \\
& & & & \vdots \arrow[u, two heads] \\
& & & & 0 \arrow[u, two heads]
\end{tikzcd}$$

Then we replace the set $\mathcal{F}_t \setminus \{0, f\}$ (resp. $\mathcal{G}_t \setminus \{0, g\}$) by the set $\mathcal{F}_t \setminus \{0\}$ (resp. $\mathcal{G}_t \setminus \{0\}$).

If $\mathcal{G}_t \setminus \{0\} \neq \emptyset$, then we repeat a similar process for the sets $\mathcal{F}_t \setminus \{0\}$ and $\mathcal{G}_t \setminus \{0\}$.

If $\mathcal{G}_t \setminus \{0\} = \emptyset$, then we construct a surjection by combining $0 \in \mathcal{G}_t$ and an element of $\mathcal{F}_t \setminus \{0\}$. Thus, we can obtain the total of $|\mathcal{G}_t| = j_t$ surjections through these processes, and there is the trivial surjection $M_{i_t} \twoheadrightarrow M_{i_t}$. So the dual $F$-signature of $M_{i_t}$ is $s(M_{i_t}) = \dfrac{j_t}{n} + \dfrac{1}{n}$.

Similarly, we obtain $s(M_{i_t}) = \dfrac{i_t}{n} + \dfrac{1}{n}$ in the case of $i_t < j_t$.

In the case of $i_t = j_t$, we can obtain the total of $i_t - 1$ surjections by using a similar process as above. We also obtain $\mathcal{F}_t \setminus \{0\} = \emptyset$ and $\mathcal{G}_t \setminus \{0\} = \emptyset$ at the same time. In addition to these surjections, we construct

$$M_{i_t} \twoheadrightarrow M_{i_t} \quad \text{and} \quad R^{1/2} \oplus R^{1/2} \twoheadrightarrow M_{i_t}^{1/2}$$

Thus, the dual $F$-signature of $M_{i_t}$ is

$$s(M_{i_t}) = \frac{i_t - 1}{n} + \frac{1}{n} + \frac{1}{2n} = \frac{2i_t + 1}{2n}.$$

□

**Example 5.3.12.** *Let the notation be as in Example 5.3.4. Then, the dual F-signature of special CM modules are*

$$s(M_1) = \frac{2}{7},\ s(M_2) = \frac{3}{7},\ s(M_3) = \frac{2}{7}.$$

Next, we give an example in the case $i_t = j_t$.

**Example 5.3.13.** *Let $G = \frac{1}{8}(1,5)$ be a cyclic group of order 8. The Hirzebruch-Jung continued fraction expansion of 8/5 is*

$$\frac{8}{5} = 2 - \frac{1}{3 - 1/2} = [2,3,2],$$

*and the i-series and the j-series are described as follows.*

$$\begin{array}{lllll} i_0 = 8, & i_1 = 5, & i_2 = 2, & i_3 = 1, & i_4 = 0, \\ j_0 = 0, & j_1 = 1, & j_2 = 2, & j_3 = 5, & j_4 = 8. \end{array}$$

*Thus, special CM modules are $R, M_1, M_2, M_5$. In this case, we have $i_2 = j_2$ and there exists the surjection as follows.*

$$\begin{tikzcd} 0 \arrow[r, "y"] & 5 \arrow[r, "y"] & 2 \\ & & 1 \arrow[u, "x"] \\ & & 0 \arrow[u, "x"] \end{tikzcd} \qquad \begin{array}{lll} M_2 & \twoheadrightarrow & M_2 \\ M_1 \oplus M_5 & \twoheadrightarrow & M_2 \\ R^{\oplus 1/2} \oplus R^{\oplus 1/2} & \twoheadrightarrow & M_2^{\oplus 1/2} \end{array}$$

*Thus, the dual F-signature of $M_2$ is*

$$s(M_2) = \frac{1}{8} + \frac{1}{8} + \frac{1}{16} = \frac{5}{16}.$$

**Example 5.3.14.** *Let $G = \frac{1}{n}(1, n-1) \subset \mathrm{SL}(2,k)$ be a cyclic group of order n, that is, Dynkin type $A_{n-1}$. The Hirzebruch-Jung continued fraction expansion of $n/(n-1)$ is*

$$\frac{n}{n-1} = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \cdots}} = [\underbrace{2, 2, \cdots, 2}_{n-1}],$$

*and the i-series and the j-series are described as follows,*

$$\begin{array}{llllll} i_0 = n, & i_1 = n-1, & i_2 = n-2, & \cdots, & i_{n-1} = 1, & i_n = 0, \\ j_0 = 0, & j_1 = 1, & j_2 = 2, & \cdots, & j_{n-1} = n-1, & j_n = n. \end{array}$$

*Namely, $i_t = n - t,\ j_t = t\ (t = 1, 2, \cdots, n-1)$. As we mentioned in Remark 5.2.2, any $M_t$ is a special CM module and the dual F-signature of $M_t$ is obtained by Theorem 5.3.11.*

$$s(M_{i_t}) = \begin{cases} \dfrac{1}{n} + \dfrac{j_t}{n} = \dfrac{t+1}{n} & (\textit{if } t < \frac{n}{2}) \\[2ex] \dfrac{1}{n} + \dfrac{t-1}{n} + \dfrac{1}{2n} = \dfrac{2t+1}{2n} & (\textit{if } t = \frac{n}{2}) \\[2ex] \dfrac{1}{n} + \dfrac{i_t}{n} = \dfrac{n-t+1}{n} & (\textit{if } t > \frac{n}{2}). \end{cases}$$

*About other Dynkin types (i.e. $D_n, E_6, E_7, E_8$), see the next section.*

## 5.4 Dual $F$-signature for rational double points

In the previous section, we could obtain the value of the dual $F$-signature of each MCM module over the rational double point corresponding to Dynkin type $A_{n-1}$ (Example 5.3.14). In this section, we will determine the dual $F$-signature for other Dynkin types. Firstly, we recall some well-known facts about two-dimensional rational double points (see Chapter 4). We suppose that $G$ is a finite subgroup of $\mathrm{SL}(2,k)$ and the order of $G$ is coprime to $p = \operatorname{char} k$. Moreover, we can see that $G$ contains no pseudo-reflections in this situation. As before, we denote the invariant subring of $S := k[[x,y]]$ under the action of $G$ by $R := S^G$ and the maximal ideal of $R$ by $\mathfrak{m}$. In this situation, an invariant subring $R$ is Gorenstein by [Wat1]. We call $R$ (or equivalently $\operatorname{Spec} R$) rational double points (or Du Val singularities, Kleinian singularities, ADE singularities in the literature).

Then the AR quiver of $R$ (= the McKay quiver of $G$) coincides with the extended Dynkin diagram corresponding to the types of classification of $G \subset \mathrm{SL}(2,k)$ after replacing each edges " $-$ " by arrows " $\leftrightarrows$ ". Therefore the AR quiver of $R$ is the left hand side of the following:

$(A_n)$
$$\begin{tikzcd}[column sep=small]
 & & & 0 \arrow[llld, shift left] \arrow[rrrd, shift left] & & & \\
1 \arrow[rrru, shift left] \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n-2 \arrow[l, shift left] \arrow[r, shift left] & n-1 \arrow[l, shift left] \arrow[r, shift left] & n \arrow[l, shift left] \arrow[lllu, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & & 1 \arrow[llld, dash] \arrow[rrrd, dash] & & & \\
1 \arrow[r, dash] & 1 \arrow[r, dash] & 1 \arrow[r, dash] & \cdots \arrow[r, dash] & 1 \arrow[r, dash] & 1 \arrow[r, dash] & 1
\end{tikzcd}$$

$(D_n)$
$$\begin{tikzcd}[column sep=small]
0 \arrow[dr, shift left] & & & & & n-1 \arrow[dl, shift left] \\
 & 2 \arrow[ul, shift left] \arrow[dl, shift left] \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & n-2 \arrow[l, shift left] \arrow[ur, shift left] \arrow[dr, shift left] & \\
1 \arrow[ur, shift left] & & & & & n \arrow[ul, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
1 \arrow[dr, dash] & & & & 1 \arrow[dl, dash] \\
 & 2 \arrow[r, dash] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & 2 \\
1 \arrow[ur, dash] & & & & 1 \arrow[ul, dash]
\end{tikzcd}$$

$(E_6)$
$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[d, shift left] & & \\
 & & 1 \arrow[u, shift left] \arrow[d, shift left] & & \\
5 \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[r, shift left] & 6 \arrow[l, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & 1 \arrow[d, dash] & & \\
 & & 2 \arrow[d, dash] & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1
\end{tikzcd}$$

$(E_7)$
$$\begin{tikzcd}[column sep=small]
 & & & 7 \arrow[d, shift left] & & & \\
0 \arrow[r, shift left] & 1 \arrow[l, shift left] \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[r, shift left] & 5 \arrow[l, shift left] \arrow[r, shift left] & 6 \arrow[l, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & & 2 \arrow[d, dash] & & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 3 \arrow[r, dash] & 2 \arrow[r, dash] & 1
\end{tikzcd}$$

$(E_8)$
$$\begin{tikzcd}[column sep=small]
 & & & & & 8 \arrow[d, shift left] & & \\
0 \arrow[r, shift left] & 1 \arrow[l, shift left] \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[r, shift left] & 5 \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & 6 \arrow[l, shift left] \arrow[r, shift left] & 7 \arrow[l, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & & & & 3 \arrow[d, dash] & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 4 \arrow[r, dash] & 2
\end{tikzcd}$$

where a vertex $t$ corresponds the MCM $R$-module $M_t$ and the right hand side of the figure means $\operatorname{rank}_R M_t$. In this chapter, we will determine the value of the dual $F$-signature for each MCM $R$-module $M_t$.

### 5.4.1 Key lemma for determining the dual $F$-signature

In order to investigate a surjection from ${}^eM_t$ to a finite direct sum of some copies of $M_t$, we will prepare a technical lemma. As we noted in the beginning of subsection 5.1, we may consider an MCM $R$-module $M_t$ as a vector space. More precisely, let $M_i, M_j$ be indecomposable MCM modules and suppose a morphism $\varphi_i : M_i \to M_j$ is a non-zero path appearing in the AR quiver after applying the counting argument. Then, $\operatorname{Im}\varphi_i$ constructs part of minimal generators of $M_j$. Therefore in the commutative diagram

$$\begin{tikzcd} M_i \arrow[r, "\varphi_i"] \arrow[d] & M_j \arrow[d, "\pi"] \\ V_i := \operatorname{Im}(\varphi_i\pi) \arrow[r, hook] & V_j := M_j/\mathfrak{m}M_j, \end{tikzcd}$$

we may consider $V_i$ as a vector subspace of $V_j$ and take a injective morphism

$$X_i \cdot 1_{V_i} : V_i \hookrightarrow V_j \quad (X_i \in k).$$

Now we prove the key lemma related to these vector spaces in more general settings (Lemma 5.4.1). Let $V$ be a $d$-dimensional $k$-vector space and fix a basis $\{v_1, \cdots, v_d\}$. Suppose $W_1, \cdots, W_r$ are subspaces of $V$ (admit repetition) where $\dim_k W_i = d_i \leq d$ and the basis of $W_i$ is the part of $\{v_1, \cdots, v_d\}$. Namely, we choose $d_i$ elements from $\{v_1, \cdots, v_d\}$ as the basis of $W_i$. Define the $d \times r$ table $[a_{ij}]$ associated with $W_i$'s as follows,

$$a_{ij} = \begin{cases} 1 & (\text{if } v_i \text{ is a basis of } W_j) \\ 0 & (\text{if } v_i \text{ is not a basis of } W_j) \end{cases}$$

where $i = 1, \cdots, d$ and $j = 1, \cdots, r$.

**Lemma 5.4.1.** *Set $n := \min\{\sum_{j=1}^{r} a_{ij} \mid i = 1, \cdots, d\} \leq r$, then there exists a surjection*

$$W_1 \oplus \cdots \oplus W_r \twoheadrightarrow V^{\oplus n}.$$

*Proof.* Firstly, we define a $(dn) \times (dr)$ matrix

$$C = \begin{pmatrix} A_1^{(1)} & A_2^{(1)} & \cdots & A_r^{(1)} \\ A_1^{(2)} & A_2^{(2)} & \cdots & A_r^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ A_1^{(n)} & A_2^{(n)} & \cdots & A_r^{(n)} \end{pmatrix}$$

where $A_j^{(\alpha)}$ $(1 \leq \alpha \leq n,\ 1 \leq j \leq n)$ is a $d \times d$ diagonal matrix as follows.

$$A_j^{(\alpha)} = X_j^{(\alpha)} \begin{pmatrix} a_{1j} & & & \\ & a_{2j} & & \\ & & \ddots & \\ & & & a_{nj} \end{pmatrix}, \quad \text{where } X_j^{(\alpha)} \in k.$$

Especially, we can take $X_j^{(\alpha)}$ $(1 \leq \alpha \leq n,\ 1 \leq j \leq n)$ as algebraically independent variables. Note that for the vector space $V^{\oplus n} = V^{(1)} \oplus \cdots \oplus V^{(n)}$ where $V^{(\alpha)} \cong V$ and linear morphisms $W_j \overset{\cdot X_j^{(\alpha)}}{\hookrightarrow} V^{(\alpha)}$, the matrix $C = (c_{st})$ is a representation matrix of $\varphi$ :

$W_1 \oplus \cdots \oplus W_r \longrightarrow V^{\oplus n}$. Thus, $\varphi$ is surjective if and only if there exists a nonzero $dn$-minor of $C$. From now on, we construct such a $dn$-minor.

For this purpose, we choose $dn$ columns which are distinct from each other from $C$ and consider a sequence $(t_1, t_2, \cdots, t_{dn})$ where $1 \leq t_1, \cdots, t_{dn} \leq dn$ are column numbers. From a sequence $(t_1, t_2, \cdots, t_{dn})$ of $C$, we obtain the monomial in a natural fashion,

$$(t_1, t_2, \cdots, t_{dn}) \mapsto \prod_{s=1}^{dn} c_{s,t_s} \in \mathrm{Mon}(X_j^{(\alpha)} \mid 1 \leq j \leq r,\ 1 \leq \alpha \leq n)$$

where $\mathrm{Mon}(X_j^{(\alpha)})$ is the monomial set of $k[X_j^{(\alpha)}]$. We say a sequence $(t_1, t_2, \cdots, t_{dn})$ is chain if the corresponding monomial is not zero, and we impose the lexicographic order

$$X_1^{(1)} > \cdots > X_r^{(1)} > X_1^{(2)} > \cdots > X_r^{(2)} > \cdots > X_1^{(n)} > \cdots > X_r^{(n)} \tag{5.4.1}$$

on $\mathrm{Mon}(X_j^{(\alpha)})$.

From now on, we consider the chain of $C$ constructed from the following algorithm.

**(Step1)** For the $d \times r$ table $[a_{ij}]$, if there is a number $i$ such that $a_{ij} = 0$ for all $j = 1, \cdots, r$, then we stop this operation (Namely, if $n = 0$ then we stop here). Otherwise, we set

$$j_1^{(1)} := \min\{j \mid a_{1j} = 1\} \quad \text{and} \quad t_1 := 1 + (j_1^{(1)} - 1)d.$$

After that, we replace the number $a_{1,j_1^{(1)}} = 1$ by 0.

Similarly, we set

$$j_2^{(1)} := \min\{j \mid a_{2j} = 1\} \quad \text{and} \quad t_2 := 2 + (j_2^{(1)} - 1)d,$$

and replace $a_{2,j_2^{(1)}} = 1$ by 0.

$$\vdots$$

We set

$$j_d^{(1)} := \min\{j \mid a_{dj} = 1\} \quad \text{and} \quad t_d := d + (j_d^{(1)} - 1)d,$$

and replace $a_{d,j_d^{(1)}} = 1$ by 0, then we stop (Step1) here.

•
•
•

**(Step $\alpha$)** For the $d \times r$ table $[a_{ij}]$, if there is a number $i$ such that $a_{ij} = 0$ for all $j = 1, \cdots, r$, then we stop this operation (Namely, if $\alpha > n$, then we stop here). Otherwise, we set

$$j_1^{(\alpha)} := \min\{j \mid a_{1j} = 1\} \quad \text{and} \quad t_{d(\alpha-1)+1} := 1 + (j_1^{(\alpha)} - 1)d.$$

replace $a_{1,j_1^{(\alpha)}} = 1$ by 0.

$$\vdots$$

We set

$$j_d^{(\alpha)} := \min\{j \mid a_{dj} = 1\} \quad \text{and} \quad t_{d(\alpha-1)+d} := d + (j_d^{(\alpha)} - 1)d,$$

and replace $a_{d,j_d^{(\alpha)}} = 1$ by 0, then we stop (Step $\alpha$) here.

•
•
•

(repeat this process up to Step $n$)

By the definition of the number $n$, we can repeat this process up to (Step $n$). After that, we have $a_{i1} = \cdots = a_{ir} = 0$ for some $i$. Therefore, we stop this algorithm.

From the above operation, we obtain the sequence $(t_1, t_2, \cdots, t_{dn})$ and this sequence is clearly a chain by the construction method. Finally, we prove the following Claim 5.4.2 and complete the proof of Lemma 5.4.1. □

**Claim 5.4.2.** *The $dn$-minor $[t_1, t_2, \cdots, t_{dn}]$ of $C$ is non-zero.*

*Proof.* Define $dn \times dn$-matrix $D = (d_{st})$ by choosing the columns $t_1, t_2, \cdots, t_{dn}$ from $C$. By the definition of determinant

$$[t_1, t_2, \cdots, t_{dn}] = \det D = \sum_{\sigma \in \mathfrak{S}_{dn}} (\operatorname{sgn} \sigma) d_{1,\sigma(1)} \cdots d_{dn,\sigma(dn)},$$

where $\mathfrak{S}_{dn}$ is a symmetric group of degree $dn$. From the selecting method of $t_1, \cdots, t_{dn}$, the monomial $0 \neq \prod_{s=1}^{dn} c_{s,t_s}$ appears in the monomial set $\{d_{1,\sigma(1)} \cdots d_{dn,\sigma(dn)}\}_{\sigma \in \mathfrak{S}_{dn}}$ and it is the unique maximal element with respect to the lexicographic order (5.4.1). Furthermore, the algebraically independence of $X_j^{(\alpha)}$s implies $\det D \neq 0$. □

**Example 5.4.3.** *Let $V$ be a 3-dimensional vector space over $k$ and fix a basis $\{v_1, v_2, v_3\}$. Consider subspaces of $V$ ;*

$$W_1 =< v_1, v_2 >, \quad W_2 =< v_2, v_3 >, \quad W_3 =< v_1 >, \quad W_4 =< v_1, v_3 > .$$

$$[a_{ij}] = \begin{array}{c|cccc} & W_1 & W_2 & W_3 & W_4 \\ \hline v_1 & 1 & 0 & 1 & 1 \\ v_2 & 1 & 1 & 0 & 0 \\ v_3 & 0 & 1 & 0 & 1 \end{array}$$

*By Lemma 5.4.1, we have a surjection $W_1 \oplus W_2 \oplus W_3 \oplus W_4 \twoheadrightarrow V^{\oplus 2}$.*

*Note that $(t_1, \cdots, t_6) = (1, 2, 6, 7, 5, 12)$ and $\prod_{s=1}^{6} c_{s,t_s}$ is just the product of underlined entries of $C$.*

$$C = \left(\begin{array}{ccc|ccc|ccc|ccc} \underline{X_1^{(1)}} & & & 0 & & & X_3^{(1)} & & & X_4^{(1)} & & \\ & \underline{X_1^{(1)}} & & & X_2^{(1)} & & & 0 & & & 0 & \\ & & 0 & & & \underline{X_2^{(1)}} & & & 0 & & & X_4^{(1)} \\ \hline X_1^{(2)} & & & 0 & & & \underline{X_3^{(2)}} & & & X_4^{(2)} & & \\ & X_1^{(2)} & & & \underline{X_2^{(2)}} & & & 0 & & & 0 & \\ & & 0 & & & X_2^{(2)} & & & 0 & & & \underline{X_4^{(2)}} \end{array}\right)$$

### 5.4.2 Computations of the dual $F$-signature

From Corollary 3.3.6, we may consider as

$$ {}^{e}M_t \approx (R^{\oplus d_{0,t}} \oplus M_1^{\oplus d_{1,t}} \oplus \cdots \oplus M_n^{\oplus d_{n,t}})^{\oplus \frac{p^{2e}}{|G|}}, $$

where $d_{i,t} = (\operatorname{rank}_R M_t) \cdot (\operatorname{rank}_R M_i)$. When we try to determine the dual $F$-signature, the part of $o(p^{2e})$ is harmless. Therefore, we identify ${}^{e}M_t$ with $R^{\oplus d_{0,t}/|G|} \oplus M_1^{\oplus d_{1,t}/|G|} \oplus \cdots \oplus M_n^{\oplus d_{n,t}/|G|}$ and sometimes omit $|G|^{-1}$ for simplicity.

For reasons of showing the ratio of $s(M_t)$ to $|G|$ clearly, we don't reduce a fraction.

In order to determine the value of the dual $F$-signature, we need understand the paths which generate minimal generators by applying the counting argument of AR quiver. As the counting argument written below shows, the number of minimal generators of $M_t$ is equal to $m_t := 2 \operatorname{rank}_R M_t$ (see also [Wun2, Theorem 1.2]). We denote minimal generators of $M_t$ by $g_{t,1}, g_{t,2}, \cdots, g_{t,m_t}$ and assume $\deg g_{t,1} \le \deg g_{t,2} \le \cdots \le \deg g_{t,m_t}$.

### 5.4.3 Type $A_n$

We saw the dual $F$-signature of $A_n$ type in Example 5.3.14 as follows.

$$ s(M_t) = \begin{cases} \dfrac{t+1}{n+1} & (\text{if } \ t < \frac{n+1}{2}) \\[2ex] \dfrac{2t+1}{2(n+1)} & (\text{if } \ t = \frac{n+1}{2}) \\[2ex] \dfrac{n-t+2}{n+1} & (\text{if } \ t > \frac{n+1}{2}). \end{cases} $$

### 5.4.4 Type $D_n$

Firstly, we show a method for determining the dual $F$-signatures in the case of type $D_5$ as an example. This method also applied to other cases afterward.

**Example 5.4.4.** *The binary dihedral group* $G := \mathcal{D}_3 = \langle \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix} \rangle$ *is the type* $D_5$ *in the list* (4.1.1) *and* $|G| = 12$. *For the invariant subring under the action of* $G$, *the AR quiver takes the form as follows:*

$$ \begin{tikzcd} 0 \arrow[dr, shift left, "a"] & & & 4 \arrow[dl, shift left, "D"] \\ & 2 \arrow[ul, shift left, "A"] \arrow[dl, shift left, "B"] \arrow[r, shift left, "C"] & 3 \arrow[l, shift left, "c"] \arrow[ur, shift left, "d"] \arrow[dr, shift left, "e"] & \\ 1 \arrow[ur, shift left, "b"] & & & 5 \arrow[ul, shift left, "E"] \end{tikzcd} $$

*and it has the relations:*

$$ \begin{cases} aA = 0, & cC + dD + eE = 0, \\ bB = 0, & Dd = 0, \\ Aa + Bb + Cc = 0, & Ee = 0. \end{cases} \tag{5.4.2} $$

*Rewriting this quiver as a repetition of the original one shown in dotted areas. Namely, we associate the translation quiver* $\mathbb{Z}D_5$. (*The meaning of* $\mathbb{Z}D_5$, *see [Gab].*)

$$
\begin{tikzcd}
0 \arrow[dr, "a"] & & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 \\
1 \arrow[r, "b"] & 2 \arrow[r, "B"] \arrow[ur, "A"] \arrow[dr, "C"] & 1 \arrow[r] & 2 \arrow[r] \arrow[ur] \arrow[dr] & 1 \arrow[r] & 2 \arrow[r] \arrow[ur] \arrow[dr] & 1 \arrow[r] & 2 \arrow[r] \arrow[ur] \arrow[dr] & 1 \\
3 \arrow[r, "d"'] \arrow[ur, "c"] \arrow[dr, "e"'] & 4 \arrow[r, "D"'] & 3 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3 \\
& 5 \arrow[ur, "E"'] & & 5 \arrow[ur] & & 5 \arrow[ur] & & 5 \arrow[ur] &
\end{tikzcd}
$$

*After applying the counting argument* (*cf. subsection 5.1*), *we have*

$$
\begin{tikzcd}
0 \arrow[dr] & & & & & & & \\
& 2 \arrow[r] \arrow[dr] & 1 \arrow[r] & 2 \arrow[dr] & & 2 \arrow[r] \arrow[dr] & 1 \arrow[r] & 2 \\
& & 3 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3^2 \arrow[r] \arrow[ur] \arrow[dr] & 4 \arrow[r] & 3 \arrow[ur] & \\
& & & 5 \arrow[ur] & & 5 \arrow[ur] & &
\end{tikzcd}
\tag{5.4.3}
$$

*Thus, we identify paths on this quiver with minimal generators of each MCM module* $M_t$.

*By using this one, we will determine the dual F-signature of* $M_1$ *and* $M_3$ *as an example.*

**· *the case of*** $M_1$ ***in*** $D_5$

*Since* $\operatorname{rank}_R M_1 = 1$, *the multiplicity* $d_{t,1}$ *of* $M_t$ *in* ${}^eM_1$ *is the following. Note that we consider them on the order of* $p^{2e}$ *and omit* $p^{2e}/|G|$ *times for simplicity.*

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ |
|---|---|---|---|---|---|---|
| $d_{t,1}$ | 1 | 1 | 2 | 2 | 1 | 1 |

(5.4.4)

*Firstly, R generates a minimal generator* $g_{1,1}$ *through the path* ($R \xrightarrow{aB} M_1$) *on the quiver* (5.4.3). *Similarly, the paths* ($M_1 \xrightarrow{1_{M_1}} M_1$) *and* ($M_2 \xrightarrow{B} M_1$) $\times 2$ *also generate* $g_{1,1}$ (*Since* $d_{2,1} = 2$, *we double the last one*) *and we have no other such a path. Thus, the dual F-signature of* $M_1$ *can take* $s(M_1) \leq \frac{1}{12} + \frac{1}{12} + \frac{2}{12} = \frac{4}{12}$. *So we obtain the upper bound of* $s(M_1)$. *Next, we will show that we can actually construct a surjection*

$$R \oplus M_1 \oplus M_2^{\oplus 2} \oplus M_3^{\oplus 2} \oplus M_4 \oplus M_5 \twoheadrightarrow M_1^{\oplus 4}.$$

*So if there exists such a surjection, then we can conclude* $s(M_1) = \dfrac{4}{12}$.

*From the quiver* (*5.4.3*), *we read off that* ($M_2 \xrightarrow{B} M_1$) $\times 2$ *and* ($M_3 \xrightarrow{cB} M_1$) $\times 2$ *generate* $g_{1,2}$. *Thus, we have the following table. As a consequence, we have the above* *surjection by Lemma 5.4.1 and conclude* $s(M_1) = \dfrac{4}{12}$. *Note that a construction method of a surjection is not unique. It depends on a choice of paths.*

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | *Total* |
|---|---|---|---|---|---|---|---|
| *Path* | $a$ | $1_{M_1}$ | $B$ | $cB$ | 0 | 0 | |
| $g_{1,1}$ | 1 | 1 | 2 | 0 | 0 | 0 | 4 |
| $g_{1,2}$ | 0 | 1 | 2 | 2 | 0 | 0 | 5 |

· ***the case of*** $M_3$ ***in*** $D_5$

*The strategy for determining* $s(M_3)$ *is the same as the case of* $M_1$. *But we need to pay attention to the central vertex "*$3^2$*". As the multiplicity 2 shows, paths from R to this vertex could generate two kinds of minimal generator. Suppose that* $\alpha$ (*resp.* $\beta, \gamma$) *is a minimal generator of* $M_3$ *generated by a path which factor through* $2 \xrightarrow{C} 3^2$ (*resp.* $4 \xrightarrow{D} 3^2, 5 \xrightarrow{E} 3^2$). *By the relations (5.4.2), they satisfy* $\alpha + \beta + \gamma \in \mathfrak{m}$ *and we can take two of them as minimal generators associated to the vertex* $3^2$. *Thus, we fix* $g_{3,2} := \alpha,\ g_{3,3} := \beta$. *Since* $\gamma$ *is equivalent to* $\alpha + \beta$ *up to modulo radical, we use it freely as one of* $\{\alpha, \beta\}$. *Note that when we continue chasing a path after this vertex, we must not choose the following three paths, because the relations (5.4.2) force them to be zero.*

$$\searrow^{C}\ 3^2\ \nearrow^{c} \qquad \xrightarrow{D} 3^2 \xrightarrow{d} \qquad \nearrow^{E}\ 3^2\ \searrow^{e}$$

*Since* $\mathrm{rank}_R\, M_3 = 2$, *the multiplicity* $d_{t,3}$ *of* $M_t$ *in* ${}^eM_3$ *is the following.*

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ |
|---|---|---|---|---|---|---|
| $d_{t,3}$ | 2 | 2 | 4 | 4 | 2 | 2 |

(5.4.5)

*In order to estimate the upper bounds of* $s(M_3)$, *we consider paths which can be identified with* $g_{3,1}$ *or* $g_{3,3}$. *Then we classify each MCM modules as follows.*

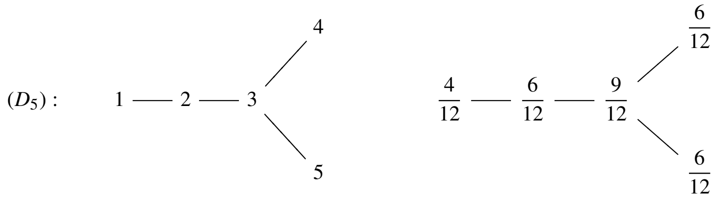
$$\text{(I) } \{ M_3 \times 4 \} \quad \text{(II) } \{ R \times 2, M_2 \times 4 \} \quad \text{(III) } \{ M_4 \times 2, M_5 \times 2 \}$$

*The MCM modules in the class of* (I) *generate the both* $g_{3,1}$ *and* $g_{3,3}$ *at the same time by* ($M_3 \xrightarrow{1_{M_3}} M_3$) *and those of* (II) *generate either* $g_{3,1}$ *or* $g_{3,3}$. *Also, those of* (III) *only generate* $g_{3,3}$. *For constructing a surjection as many as possible, we should combine MCM modules in* (II) *and* (III), *that is, we use* (II)*'s for* $g_{3,1}$ *and* (III)*'s for* $g_{3,3}$. *After making an appropriate pair of them (we can make four pairs), we have two remaining MCM modules in* (II). *We can use a one of remainders for* $g_{3,1}$ *and the other for* $g_{3,3}$.

*Thus, the dual F-signature of* $M_3$ *can take* $s(M_3) \leq \frac{4}{12} + \frac{4}{12} + \frac{1}{12} = \frac{9}{12}$ *and the following table and Lemma 5.4.1 asserts equality (in this table, we use* $5 \xrightarrow{E} 3^2$ *for generating* $g_{3,3}$ *).*

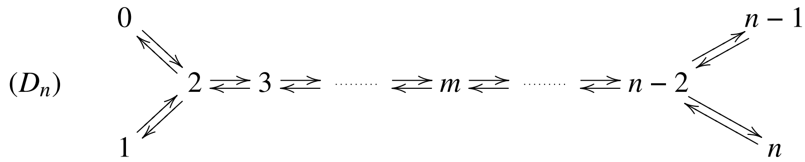
| $M_t$ | $R$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | *Total* |
|---|---|---|---|---|---|---|---|---|
| *Path* | $aC$ | $aCdD$ | $bC$ | $C$ | $1_{M_3}$ | $D$ | $E$ | |
| $g_{3,1}$ | 1 | 0 | 0 | 4 | 4 | 0 | 0 | 9 |
| $g_{3,2}$ | 0 | 0 | 2 | 4 | 4 | 0 | 0 | 10 |
| $g_{3,3}$ | 0 | 1 | 0 | 0 | 4 | 2 | 2 | 9 |
| $g_{3,4}$ | 0 | 0 | 0 | 4 | 4 | 2 | 2 | 12 |

*By using a similar method, we have the dual $F$-signature of other MCM modules. The following is the value of the dual $F$-signature corresponding to the Dynkin diagram $D_5$.*



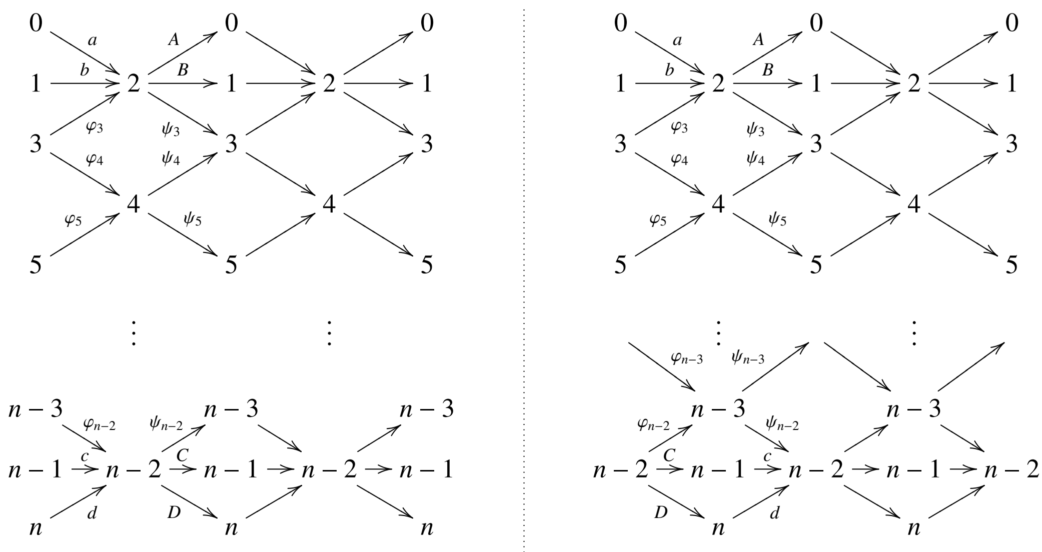

Now, we move to the case of type $D_n$ while referring to the type $D_5$. Since the basic idea of determining the dual $F$-signature is the same as above, we only mention an outline for the case of $D_n$ and also for $E_6$, $E_7$ and $E_8$ (see subsection 5.4.5, 5.4.6 and 5.4.7).

The AR quiver of type $D_n$ is the following.




Rewriting it as a repetition of the original one (i.e. a translation quiver $\mathbb{Z}D_n$), we have

$n$ : even ($n = 2r$) $\qquad\qquad$ $n$ : odd ($n = 2r-1$)



Also, this quiver has the relations.

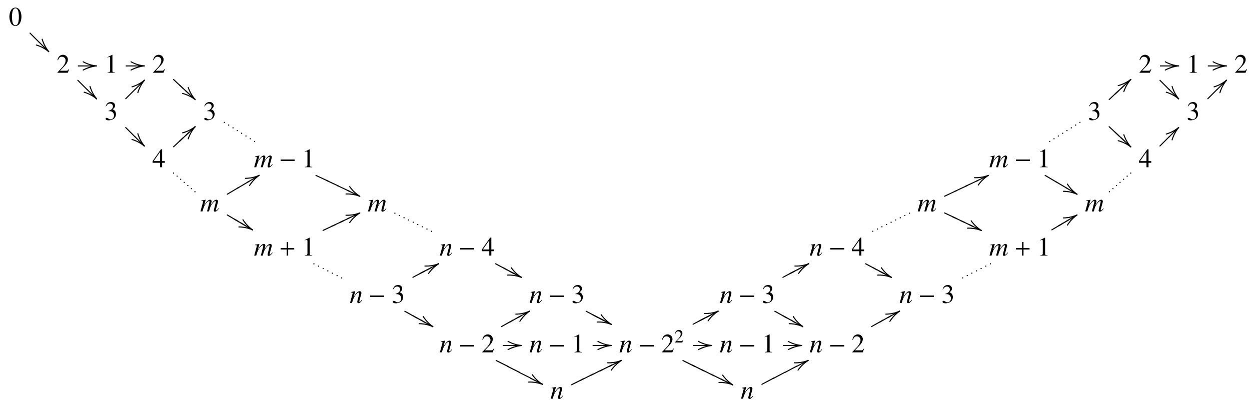
$n$ : even ($n = 2r$)

$$\begin{cases} aA = 0, & bB = 0, \\ Aa + Bb + \psi_3\varphi_3 = 0, & cC = 0, \\ \psi_{n-2}\varphi_{n-2} + Cc + Dd = 0, & dD = 0, \\ \varphi_{2l-1}\psi_{2l-1} + \varphi_{2l}\psi_{2l} = 0 & (l = 2, \cdots, r-1), \\ \psi_{2l}\varphi_{2l} + \psi_{2l+1}\varphi_{2l+1} = 0 & (l = 2, \cdots, r-2). \end{cases}$$

$n$ : odd ($n = 2r-1$)

$$\begin{cases} aA = 0, & bB = 0, \\ Aa + Bb + \psi_3\varphi_3 = 0, & cC = 0, \\ \varphi_{n-2}\psi_{n-2} + Cc + Dd = 0, & dD = 0, \\ \varphi_{2l-1}\psi_{2l-1} + \varphi_{2l}\psi_{2l} = 0 & (l = 2, \cdots, r-2), \\ \psi_{2l}\varphi_{2l} + \psi_{2l+1}\varphi_{2l+1} = 0 & (l = 2, \cdots, r-2). \end{cases}$$

Applying the counting argument, we have the following picture.



Since there is no big differences between an even number case and an odd number case, we will explain the former case. Thus, in the rest of this subsection, we suppose $n = 2r$.

**$\cdot$ the case of $M_1$ in $D_n$**

Since $\mathrm{rank}_R\, M_1 = 1$, the multiplicity $d_{t,1}$ of $M_t$ in ${}^eM_1$ is the following.

$$\begin{array}{c|cccccccc} M_t & R & M_1 & M_2 & \cdots & M_m & \cdots & M_{n-2} & M_{n-1} & M_n \\ \hline d_{t,1} & 1 & 1 & 2 & \cdots & 2 & \cdots & 2 & 1 & 1 \end{array} \tag{5.4.6}$$

The paths which generate $g_{1,1}$ are only $(R \xrightarrow{aB} M_1)$, $(M_1 \xrightarrow{1_{M_1}} M_1)$ and $(M_2 \xrightarrow{B} M_1) \times 2$. Thus, the dual $F$-signature of $M_1$ can take $s(M_1) \le \frac{1}{4(n-2)} + \frac{1}{4(n-2)} + \frac{2}{4(n-2)} = \frac{4}{4(n-2)}$. The following table and Lemma 5.4.1 shows $s(M_1) = \dfrac{4}{4(n-2)}$.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $\cdots$ | $M_m$ | $\cdots$ | $M_{n-2}$ | $M_{n-1}$ | $M_n$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aB$ | $1_{M_1}$ | $B$ | $\varphi_3 B$ | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | 0 | |
| $g_{1,1}$ | 1 | 1 | 2 | 0 | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | 0 | 4 |
| $g_{1,2}$ | 0 | 1 | 2 | 2 | 0 | $\cdots$ | 0 | $\cdots$ | 0 | 0 | 0 | 5 |

**$\cdot$ the case of $M_m$ ( $2 \le m \le n/2$ ) in $D_n$**

Since $\mathrm{rank}_R\, M_m = 2$, the multiplicity $d_{t,m}$ of $M_t$ in ${}^eM_m$ is the following.

$$\begin{array}{c|cccccccc} M_t & R & M_1 & M_2 & \cdots & M_m & \cdots & M_{n-2} & M_{n-1} & M_n \\ \hline d_{t,m} & 2 & 2 & 4 & \cdots & 4 & \cdots & 4 & 2 & 2 \end{array} \tag{5.4.7}$$

In the same way as the previous example, we can see that the MCM $R$-modules $R \times 2, M_2 \times 4, \cdots, M_m \times 4$ can generate $g_{m,1}$, and we have no other such MCMs.

Thus, the dual $F$-signature of $M_m$ can take $s(M_m) \leq \frac{2}{4(n-2)} + \frac{4(m-1)}{4(n-2)} = \frac{4m-2}{4(n-2)}$. By Lemma 5.4.1 and the following table, we conclude $s(M_m) = \dfrac{4m-2}{4(n-2)}$.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $\cdots$ | $M_{m-1}$ | $M_m$ | $M_{m+1}$ | $\cdots$ | $M_{n-2}$ | $M_{n-1}$ | $M_n$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Path | $a\Gamma^2_m$ | $b\Gamma^2_m$ | $\Gamma^2_m$ | | $\varphi_m$ | $1_{M_m}$ | $\varphi_{m+1}$ | | $\Lambda^{n-2}_m$ | $c\Lambda^{n-2}_m$ | $0$ | |
| $g_{m,1}$ | 2 | 0 | 4 | $\cdots$ | 4 | 4 | 0 | $\cdots$ | 0 | 0 | 0 | $4m-2$ |
| $g_{m,2}$ | 0 | 2 | 4 | $\cdots$ | 4 | 4 | 0 | $\cdots$ | 0 | 0 | 0 | $4m-2$ |
| $g_{m,3}$ | 0 | 0 | 0 | $\cdots$ | 0 | 4 | 4 | $\cdots$ | 4 | 2 | 0 | $4n-4m-2$ |
| $g_{m,4}$ | 0 | 0 | 0 | $\cdots$ | 0 | 4 | 4 | $\cdots$ | 4 | 2 | 0 | $4n-4m-2$ |

Here, we set $\Gamma^i_m := \psi_{i+1}\varphi_{i+2}\cdots\psi_{m-1}\varphi_m,\ \Lambda^i_m := \psi_i\varphi_{i-1}\cdots\psi_{m+2}\varphi_{m+1}$. In this table, we suppose that $m$ is an even number. Although the notation is slightly different, we obtain a similar table for an odd number case.

**· the case of $M_m$ ( $n/2 < m \leq n-2$ ) in $D_n$**

The multiplicity $d_{t,m}$ of $M_t$ in ${}^eM_m$ is the same as the table (5.4.7). In order to obtain the upper bounds of $s(M_m)$, we classify the MCM $R$-modules in ${}^eM_m$ as follows;

$$\text{(I) } \{\ M_m \times 4\ \} \quad \text{(II) } \{\ R \times 2, M_2 \times 4, \cdots, M_{m-1} \times 4\ \}$$

$$\text{(III) } \{\ M_{m+1} \times 4, \cdots, M_{n-2} \times 4, M_{n-1} \times 2, M_n \times 2\ \}$$

where the class (I) (resp. (II), (III)) is the set of MCM $R$-modules which generate $g_{m,1}$ and $g_{m,3}$ at the same time (resp. either $g_{m,1}$ or $g_{m,3}$, only $g_{m,3}$ ). For constructing a surjection as many as possible, we should combine MCM modules in (II) and (III), that is, we use (II)'s for $g_{m,1}$ and (III)'s for $g_{m,3}$. After making an appropriate pair of them (we obtain $4(n-m-1)pairs$), we have $2(4m-2n-1)$ remaining MCM modules in (II). We can use a half of remainders for $g_{m,1}$ and the others for $g_{m,3}$. Thus, we have the upper bounds $s(M_m) \leq \frac{4}{4(n-2)} + \frac{4(n-m-1)}{4(n-2)} + \frac{4m-2n-1}{4(n-2)} = \frac{2n-1}{4(n-2)}$. We have the following table. (In this table, we suppose that $m$ is an even number. Although the notation is slightly different, we obtain a similar table for an odd number case.)

| $M_t$ | $R$ | $R$ | $M_1$ | $M_1$ | $M_2$ | $\cdots$ | $M_{2m-n}$ | $M_2$ | $\cdots$ | $M_{2m-n}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $a\Gamma^2_m$ | $a\Gamma^2_{n-2}Dd\Lambda^{n-2}_m$ | $b\Gamma^2_m$ | $b\Gamma^2_{n-2}Dd\Lambda^{n-2}_m$ | $\Gamma^2_m$ | | $\Gamma^{2m-n}_m$ | $\Gamma^2_{n-2}Dd\Lambda^{n-2}_m$ | | $\Gamma^{2m-n}_{n-2}Dd\Lambda^{n-2}_m$ |
| $g_{m,1}$ | 1 | 0 | 0 | 0 | 2 | $\cdots$ | 2 | 0 | $\cdots$ | 0 |
| $g_{m,2}$ | 0 | 0 | 1 | 0 | 2 | $\cdots$ | 2 | 0 | $\cdots$ | 0 |
| $g_{m,3}$ | 0 | 1 | 0 | 0 | 0 | $\cdots$ | 0 | 2 | $\cdots$ | 2 |
| $g_{m,4}$ | 0 | 0 | 0 | 1 | 0 | $\cdots$ | 0 | 2 | $\cdots$ | 2 |

| $M_{2m-n+1}$ | $\cdots$ | $M_{m-1}$ | $M_m$ | $M_{m+1}$ | $\cdots$ | $M_{n-2}$ | $M_{n-1}$ | $M_n$ | Total |
|---|---|---|---|---|---|---|---|---|---|
| $\varphi_{2m-n+2}\Gamma^{2m-n+2}_m$ | | $\varphi_m$ | $1_{M_m}$ | $\varphi_{m+2}\Gamma^{m+2}_{n-2}Dd\Lambda^{n-2}_m$ | | $\Lambda^{n-2}_m$ | $c\Lambda^{n-2}_m$ | $d\Lambda^{n-2}_m$ | |
| 4 | $\cdots$ | 4 | 4 | 0 | $\cdots$ | 0 | 0 | 0 | $2n-1$ |
| 4 | $\cdots$ | 4 | 4 | 0 | $\cdots$ | 0 | 0 | 0 | $2n-1$ |
| 0 | $\cdots$ | 0 | 4 | 4 | $\cdots$ | 4 | 2 | 2 | $2n-1$ |
| 0 | $\cdots$ | 0 | 4 | 4 | $\cdots$ | 4 | 2 | 2 | $2n-1$ |

Thus, we conclude $s(M_m) = \dfrac{2n-1}{4(n-2)}$ by Lemma 5.4.1.

**$\cdot$ the case of $M_{n-1}$ in $D_n$**

The multiplicity $d_{t,n-1}$ of $M_t$ in ${}^eM_{n-1}$ is the same as the table (5.4.6). Similarly, we have the upper bounds $s(M_{n-1}) \leq \frac{2(n-2)}{4(n-2)}$ by selecting paths which generate $g_{n-1,1}$, and we have the following table. (In this table, we suppose that $m$ is an even number, the same as the previous case.)

| $M_t$ | $R$ | $M_1$ | $M_2$ | $\cdots$ | $M_m$ | $\cdots$ | $M_{n-2}$ | $M_{n-1}$ | $M_n$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $a\Gamma_{n-2}^2 C$ | $b\Gamma_{n-2}^2 C$ | $\Gamma_{n-2}^2 C$ | | $\Gamma_{n-2}^m C$ | | $C$ | $1_{M_{n-1}}$ | 0 | |
| $g_{n-1,1}$ | 1 | 0 | 2 | $\cdots$ | 2 | $\cdots$ | 2 | 1 | 0 | 2(n-2) |
| $g_{n-1,2}$ | 0 | 1 | 2 | $\cdots$ | 2 | $\cdots$ | 2 | 1 | 0 | 2(n-2) |

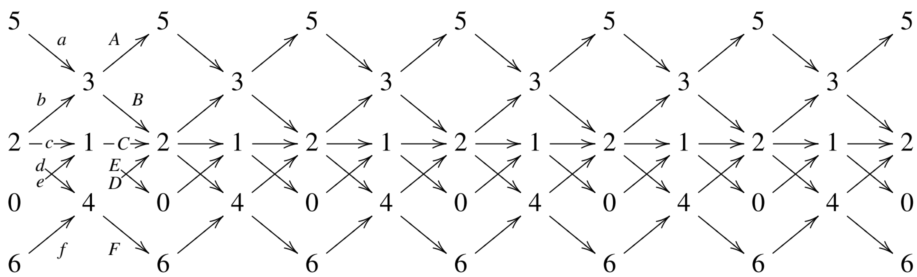
Thus, we conclude $s(M_{n-1}) = \dfrac{2(n-2)}{4(n-2)}$ by Lemma 5.4.1.

**$\cdot$ the case of $M_n$ in $D_n$**

The AR quiver of $D_n$ is symmetric with respect to $M_{n-1}$ and $M_n$, and $\mathrm{rank}_R\, M_{n-1} = \mathrm{rank}_R\, M_n$. So we have $s(M_n) = \dfrac{2(n-2)}{4(n-2)}$ in the same way.

### 5.4.5 Type $E_6$

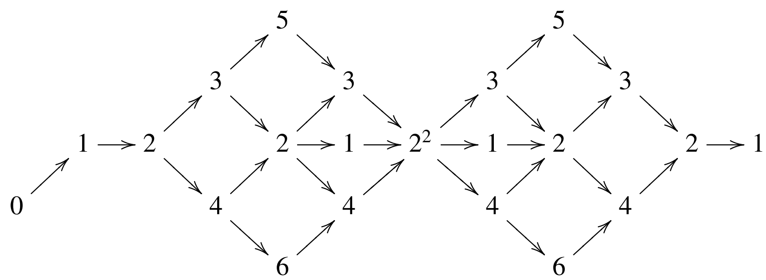
The AR quiver of type $E_6$ (as the form of $\mathbb{Z}E_6$) is



with relations

$$\begin{cases} aA = 0, & bB + cC + dD = 0, \\ eE = 0, & fF = 0, \\ Aa + Bb = 0, & Cc + Ee = 0, \\ Dd + Ff = 0. \end{cases}$$

After applying the counting argument, we have the following quiver.



· **the case of $M_1$ in $E_6$**

Since $\operatorname{rank}_R M_1 = 2$, the multiplicity $d_{t,1}$ of $M_t$ in ${}^eM_1$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ |
|---|---|---|---|---|---|---|---|
| $d_{t,1}$ | 2 | 4 | 6 | 4 | 4 | 2 | 2 |

(5.4.8)

Since the paths which generate $g_{1,1}$ are only $(R \xrightarrow{e} M_1)\times 2$ and $(M_1 \xrightarrow{1_{M_1}} M_1)\times 4$, the dual $F$-signature of $M_1$ can take $s(M_1) \leq \frac{6}{24}$. The following table and Lemma 5.4.1 assert $s(M_1) = \dfrac{6}{24}$.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | Total |
|---|---|---|---|---|---|---|---|---|
| Path | $e$ | $1_{M_1}$ | c | 0 | 0 | 0 | 0 | |
| $g_{1,1}$ | 2 | 4 | 0 | 0 | 0 | 0 | 0 | 6 |
| $g_{1,2}$ | 0 | 4 | 6 | 0 | 0 | 0 | 0 | 10 |
| $g_{1,3}$ | 0 | 4 | 6 | 0 | 0 | 0 | 0 | 10 |
| $g_{1,4}$ | 0 | 4 | 6 | 0 | 0 | 0 | 0 | 10 |

· **the case of $M_2$ in $E_6$**

Since $\operatorname{rank}_R M_2 = 3$, the multiplicity $d_{t,2}$ of $M_t$ in ${}^eM_2$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ |
|---|---|---|---|---|---|---|---|
| $d_{t,2}$ | 3 | 6 | 9 | 6 | 6 | 3 | 3 |

(5.4.9)

Similarly, we have the upper bounds $s(M_2) \leq \frac{18}{24}$ by selecting paths which generate $g_{2,1}$. Suppose that $g_{2,3}$ (resp. $g_{2,4}$) is generated through a path which factor through $1 \xrightarrow{C} 2^2$ (resp. $4 \xrightarrow{D} 2^2$). So we can use paths which factor through $3 \xrightarrow{B} 2^2$ for either $g_{2,3}$ or $g_{2,4}$ (see the arguments in the case of $M_3$ in $D_5$). Then we have the following table.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | Total |
|---|---|---|---|---|---|---|---|---|
| Path | $eC$ | $C$ | $1_{M_2}$ | $B$ | $D$ | 0 | $fD$ | |
| $g_{2,1}$ | 3 | 6 | 9 | 0 | 0 | 0 | 0 | 18 |
| $g_{2,2}$ | 0 | 0 | 9 | 6 | 6 | 0 | 0 | 21 |
| $g_{2,3}$ | 0 | 6 | 9 | 6 | 0 | 0 | 0 | 21 |
| $g_{2,4}$ | 0 | 0 | 9 | 0 | 6 | 0 | 3 | 18 |
| $g_{2,5}$ | 0 | 6 | 9 | 6 | 6 | 0 | 0 | 27 |
| $g_{2,6}$ | 0 | 0 | 9 | 6 | 6 | 0 | 3 | 24 |

Thus, we conclude $s(M_2) = \dfrac{18}{24}$ by Lemma 5.4.1.

· **the case of $M_3$ in $E_6$**

Since $\mathrm{rank}_R M_3 = 2$, the multiplicity $d_{t,3}$ of $M_t$ in ${}^e M_3$ is the same as the table (5.4.8) and we have the upper bounds $s(M_3) \leq \frac{16}{24}$ by selecting paths which generate $g_{3,1}$. The equality follows from the following table and Lemma 5.4.1.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | Total |
|---|---|---|---|---|---|---|---|---|
| Path | $eCb$ | $Cb$ | $b$ | $1_{M_3}$ | $Db$ | $a$ | $fDb$ | |
| $g_{3,1}$ | 2 | 4 | 6 | 4 | 0 | 0 | 0 | 16 |
| $g_{3,2}$ | 0 | 0 | 6 | 4 | 4 | 2 | 0 | 16 |
| $g_{3,3}$ | 0 | 4 | 6 | 4 | 4 | 0 | 2 | 20 |
| $g_{3,4}$ | 0 | 4 | 6 | 4 | 4 | 2 | 0 | 20 |

· **the case of $M_4$ in $E_6$**

The AR quiver of $E_6$ is symmetric with respect to $M_3$ and $M_4$, and $\mathrm{rank}_R M_3 = \mathrm{rank}_R M_4$. So we have $s(M_4) = \dfrac{16}{24}$ in the same way.

· **the case of $M_5$ in $E_6$**

Since $\mathrm{rank}_R M_5 = 1$, the multiplicity $d_{t,5}$ of $M_t$ in ${}^e M_5$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ |
|---|---|---|---|---|---|---|---|
| $d_{t,5}$ | 1 | 2 | 3 | 2 | 2 | 1 | 1 |

(5.4.10)

We have the upper bounds $s(M_5) \leq \frac{9}{24}$ by selecting paths which generate $g_{5,1}$. The following table and Lemma 5.4.1 assert the equality.

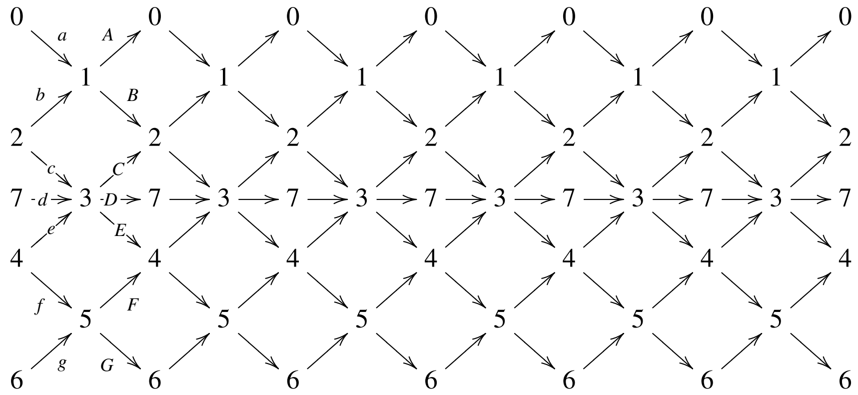
| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | Total |
|---|---|---|---|---|---|---|---|---|
| Path | $eCbA$ | $CbA$ | $bA$ | $A$ | $DbA$ | $1_{M_5}$ | 0 | |
| $g_{5,1}$ | 1 | 2 | 3 | 2 | 0 | 1 | 0 | 9 |
| $g_{5,2}$ | 0 | 2 | 3 | 2 | 2 | 1 | 0 | 10 |

· **the case of $M_6$ in $E_6$**

The AR quiver of $E_6$ is symmetric with respect to $M_5$ and $M_6$, and $\mathrm{rank}_R M_5 = \mathrm{rank}_R M_6$. So we have $s(M_6) = \dfrac{9}{24}$ in the same way.

### 5.4.6 Type $E_7$

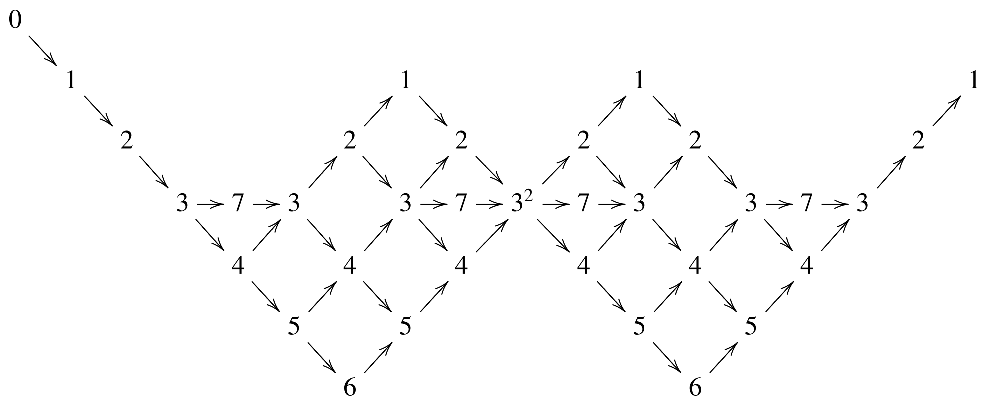
The AR quiver of type $E_7$ (as the form of $\mathbb{Z}E_7$) is



with relations

$$\begin{cases} aA = 0, & bB + cC = 0, \\ dD = 0, & eE + fF = 0, \\ gG = 0, & Aa + Bb = 0, \\ Cc + Dd + Ee = 0, & Ff + Gg = 0. \end{cases}$$

After applying the counting argument, we have the following quiver.



**$\cdot$ the case of $M_1$ in $E_7$**

Since $\mathrm{rank}_R M_1 = 2$, the multiplicity $d_{t,1}$ of $M_t$ in ${}^e M_1$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ |
|---|---|---|---|---|---|---|---|---|
| $d_{t,1}$ | 2 | 4 | 6 | 8 | 6 | 4 | 2 | 4 |

(5.4.11)

Since the paths which generate $g_{1,1}$ are only $(R \xrightarrow{a} M_1) \times 2, (M_1 \xrightarrow{1_{M_1}} M_1) \times 4$, we have the upper bounds $s(M_1) \leq \frac{6}{48}$.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | Total |
|---|---|---|---|---|---|---|---|---|---|
| Path | $a$ | $1_{M_1}$ | $b$ | 0 | 0 | 0 | 0 | 0 | |
| $g_{1,1}$ | 2 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 6 |
| $g_{1,2}$ | 0 | 4 | 6 | 0 | 0 | 0 | 0 | 0 | 10 |
| $g_{1,3}$ | 0 | 4 | 6 | 0 | 0 | 0 | 0 | 0 | 10 |
| $g_{1,4}$ | 0 | 4 | 6 | 0 | 0 | 0 | 0 | 0 | 10 |

We conclude $s(M_1) = \dfrac{6}{48}$ by Lemma 5.4.1 and the above table.

**· the case of $M_2$ in $E_7$**

Since $\mathrm{rank}_R\, M_2 = 3$, the multiplicity $d_{t,2}$ of $M_t$ in ${}^eM_2$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ |
|---|---|---|---|---|---|---|---|---|
| $d_{t,2}$ | 3 | 6 | 9 | 12 | 9 | 6 | 3 | 6 |

(5.4.12)

Similarly, we have the upper bounds $s(M_2) \leq \frac{18}{48}$ by selecting paths which generate $g_{2,1}$. The following table and Lemma 5.4.1 assert the equality.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | Total |
|---|---|---|---|---|---|---|---|---|---|
| Path | $aB$ | $B$ | $1_{M_2}$ | 0 | $eC$ | 0 | 0 | 0 | |
| $g_{2,1}$ | 3 | 6 | 9 | 0 | 0 | 0 | 0 | 0 | 18 |
| $g_{2,2}$ | 0 | 0 | 9 | 0 | 9 | 0 | 0 | 0 | 18 |
| $g_{2,3}$ | 0 | 6 | 9 | 0 | 9 | 0 | 0 | 0 | 24 |
| $g_{2,4}$ | 0 | 0 | 9 | 0 | 9 | 0 | 0 | 0 | 18 |
| $g_{2,5}$ | 0 | 6 | 9 | 0 | 9 | 0 | 0 | 0 | 24 |
| $g_{2,6}$ | 0 | 0 | 9 | 0 | 9 | 0 | 0 | 0 | 18 |

**· the case of $M_3$ in $E_7$**

Since $\mathrm{rank}_R\, M_3 = 4$, the multiplicity $d_{t,3}$ of $M_t$ in ${}^eM_3$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ |
|---|---|---|---|---|---|---|---|---|
| $d_{t,3}$ | 4 | 8 | 12 | 16 | 12 | 8 | 4 | 8 |

(5.4.13)

In this case, we need to pay attention for determining the upper bounds. The MCM $R$-module $M_3$ can generate both $g_{3,1}$ and $g_{3,2}$ through the path ($M_3 \xrightarrow{1_{M_3}} M_3$). Similarly, we can read off that $R, M_1, M_2$ generate either $g_{3,1}$ or $g_{3,2}$ and $M_4, M_7$ generate $g_{3,2}$ but don't generate $g_{3,1}$. Collectively, we classify the MCM $R$-modules in ${}^eM_3$ as follows;

$$\text{(I) } \{\, M_3 \times 16 \,\} \quad \text{(II) } \{\, R \times 4, M_1 \times 8, M_2 \times 12 \,\} \quad \text{(III) } \{\, M_4 \times 12, M_7 \times 8 \,\}$$

where the class(I) (resp. (II), (III)) is the set of MCM $R$-modules which generate $g_{3,1}$ and $g_{3,2}$ at the same time (resp. either $g_{3,1}$ or $g_{3,2}$, only $g_{3,2}$ ). For constructing a surjection as many as possible, we should combine MCM modules in (II) and (III), that is, we use (II)'s for $g_{3,1}$ and (III)'s for $g_{3,2}$. After making an appropriate pair of them, we have four remaining MCM modules in (II). We can use half of remainders for $g_{3,1}$ and others for $g_{3,2}$. Thus, we have the upper bounds $s(M_3) \leq \frac{16}{48}+\frac{20}{48}+\frac{2}{48} = \frac{38}{48}$.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aBc$ | $Bc$ | $c$ | $cDd$ | $1_{M_3}$ | $e$ | $Fe$ | $gFe$ | $d$ | |
| $g_{3,1}$ | 4 | 8 | 10 | 0 | 16 | 0 | 0 | 0 | 0 | 38 |
| $g_{3,2}$ | 0 | 0 | 0 | 2 | 16 | 12 | 0 | 0 | 8 | 38 |
| $g_{3,3}$ | 0 | 0 | 10 | 0 | 16 | 12 | 8 | 0 | 0 | 46 |
| $g_{3,4}$ | 0 | 8 | 10 | 0 | 16 | 0 | 8 | 0 | 0 | 42 |
| $g_{3,5}$ | 0 | 0 | 0 | 2 | 16 | 12 | 0 | 4 | 8 | 42 |
| $g_{3,6}$ | 0 | 0 | 10 | 2 | 16 | 12 | 0 | 0 | 8 | 48 |
| $g_{3,7}$ | 0 | 8 | 10 | 0 | 16 | 12 | 8 | 0 | 0 | 54 |
| $g_{3,8}$ | 0 | 0 | 0 | 2 | 16 | 12 | 8 | 4 | 8 | 50 |

By the above table and Lemma 5.4.1, we conclude $s(M_3) = \dfrac{38}{48}$. In this table, we fix that $g_{3,4}$ (resp. $g_{3,5}$) is a minimal generator identified with a path which factor through $2 \xrightarrow{c} 3^2$ (resp. $7 \xrightarrow{d} 3^2$). We can use paths which factor through $4 \xrightarrow{e} 3^2$ for generating either $g_{3,4}$ or $g_{3,5}$.

**· the case of $M_4$ in $E_7$**

Since $\mathrm{rank}_R\, M_4 = 3$, the multiplicity $d_{t,4}$ of $M_t$ in ${}^eM_4$ is the same as the table (5.4.12). In the same way as $M_3$, we classify the MCM $R$-modules in ${}^eM_4$ as

(I) $\{\, M_3 \times 12, M_4 \times 9 \,\}$ (II) $\{\, R \times 3, M_1 \times 6, M_2 \times 9 \,\}$ (III) $\{\, M_5 \times 6, M_7 \times 6 \,\}$

where the class(I) (resp. (II), (III)) is the set of MCM $R$-modules which generate $g_{4,1}$ and $g_{4,2}$ at the same time (resp. either $g_{4,1}$ or $g_{4,2}$, only $g_{4,2}$ ) and obtain the upper bound $s(M_4) \leq \frac{21}{48} + \frac{12}{48} + \frac{3}{48} = \frac{36}{48}$ in the same way as the case of $M_3$.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | Total |
|---|---|---|---|---|---|---|---|---|---|
| Path | $aBcDdE$ | $BcE$ | $cE$ | $E$ | $1_{M_4}$ | $F$ | $0$ | $dE$ | |
| $g_{4,1}$ | 0 | 6 | 9 | 12 | 9 | 0 | 0 | 0 | 36 |
| $g_{4,2}$ | 3 | 0 | 0 | 12 | 9 | 6 | 0 | 6 | 36 |
| $g_{4,3}$ | 0 | 0 | 9 | 12 | 9 | 6 | 0 | 0 | 36 |
| $g_{4,4}$ | 0 | 6 | 9 | 12 | 9 | 0 | 0 | 6 | 42 |
| $g_{4,5}$ | 0 | 0 | 9 | 12 | 9 | 6 | 0 | 6 | 42 |
| $g_{4,6}$ | 0 | 6 | 9 | 12 | 9 | 6 | 0 | 0 | 42 |

Thus, we conclude $s(M_4) = \dfrac{36}{48}$ by Lemma 5.4.1 and the above table.

**· the case of $M_5$ in $E_7$**

Since $\mathrm{rank}_R\, M_5 = 2$, the multiplicity $d_{t,5}$ of $M_t$ in ${}^eM_5$ is the same as the table (5.4.11). In the same as before, we can classify the MCM $R$-modules in ${}^eM_5$ as

(I) $\{\, M_3 \times 8, M_4 \times 6, M_5 \times 4 \,\}$ (II) $\{\, R \times 2, M_1 \times 4, M_2 \times 6 \,\}$ (III) $\{\, M_6 \times 2, M_7 \times 4 \,\}$

where the class(I) (resp. (II), (III)) is the set of MCM $R$-modules which generate $g_{5,1}$ and $g_{5,2}$ at the same time (resp. either $g_{5,1}$ or $g_{5,2}$, only $g_{5,2}$ ) and obtain the upper bound $s(M_5) \leq \frac{18}{48} + \frac{6}{48} + \frac{3}{48} = \frac{27}{48}$. By Lemma 5.4.1 and the following table, we have $s(M_5) = \dfrac{27}{48}$.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aBcEf$ | $BcEf$ | $cEf$ | $cDdEf$ | $Ef$ | $f$ | $1_{M_5}$ | $g$ | $dEf$ | |
| $g_{5,1}$ | 2 | 4 | 3 | 0 | 8 | 6 | 4 | 0 | 0 | 27 |
| $g_{5,2}$ | 0 | 0 | 0 | 3 | 8 | 6 | 4 | 2 | 4 | 27 |
| $g_{5,3}$ | 0 | 4 | 3 | 3 | 8 | 6 | 4 | 0 | 4 | 32 |
| $g_{5,4}$ | 0 | 0 | 3 | 3 | 8 | 6 | 4 | 2 | 4 | 30 |

### · the case of $M_6$ in $E_7$

Since $\mathrm{rank}_R\, M_6 = 1$, the multiplicity $d_{t,6}$ of $M_t$ in ${}^eM_6$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ |
|---|---|---|---|---|---|---|---|---|
| $d_{t,6}$ | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 2 |

(5.4.14)

We have $s(M_6) \leq \frac{16}{48}$ by selecting paths which generate $g_{6,1}$. We conclude $s(M_6) = \dfrac{16}{48}$ by Lemma 5.4.1 and the following table.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | Total |
|---|---|---|---|---|---|---|---|---|---|
| Path | $aBcEfG$ | $BcEfG$ | $cEfG$ | $EfG$ | $fG$ | $G$ | $1_{M_6}$ | $dEfG$ | |
| $g_{6,1}$ | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 | 16 |
| $g_{6,2}$ | 0 | 2 | 3 | 4 | 3 | 2 | 1 | 2 | 17 |

### · the case of $M_7$ in $E_7$

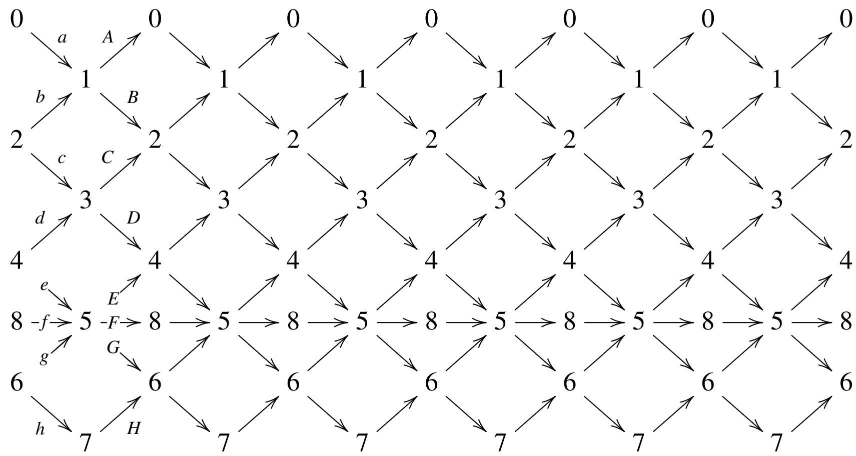
Since $\mathrm{rank}_R\, M_7 = 2$, the multiplicity $d_{t,7}$ of $M_t$ in ${}^eM_7$ is the same as the table (5.4.11). By selecting paths which generate $g_{7,1}$, we have $s(M_7) \leq \frac{24}{48}$.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | Total |
|---|---|---|---|---|---|---|---|---|---|
| Path | $aBcD$ | $BcD$ | $cD$ | $D$ | $eD$ | $FeD$ | 0 | $1_{M_7}$ | |
| $g_{7,1}$ | 2 | 4 | 6 | 8 | 0 | 0 | 0 | 4 | 24 |
| $g_{7,2}$ | 0 | 0 | 6 | 8 | 6 | 4 | 0 | 4 | 28 |
| $g_{7,3}$ | 0 | 4 | 6 | 8 | 6 | 4 | 0 | 4 | 32 |
| $g_{7,4}$ | 0 | 4 | 6 | 8 | 6 | 4 | 0 | 4 | 32 |

Similarly, we conclude $s(M_7) = \dfrac{24}{48}$.

### 5.4.7 Type $E_8$

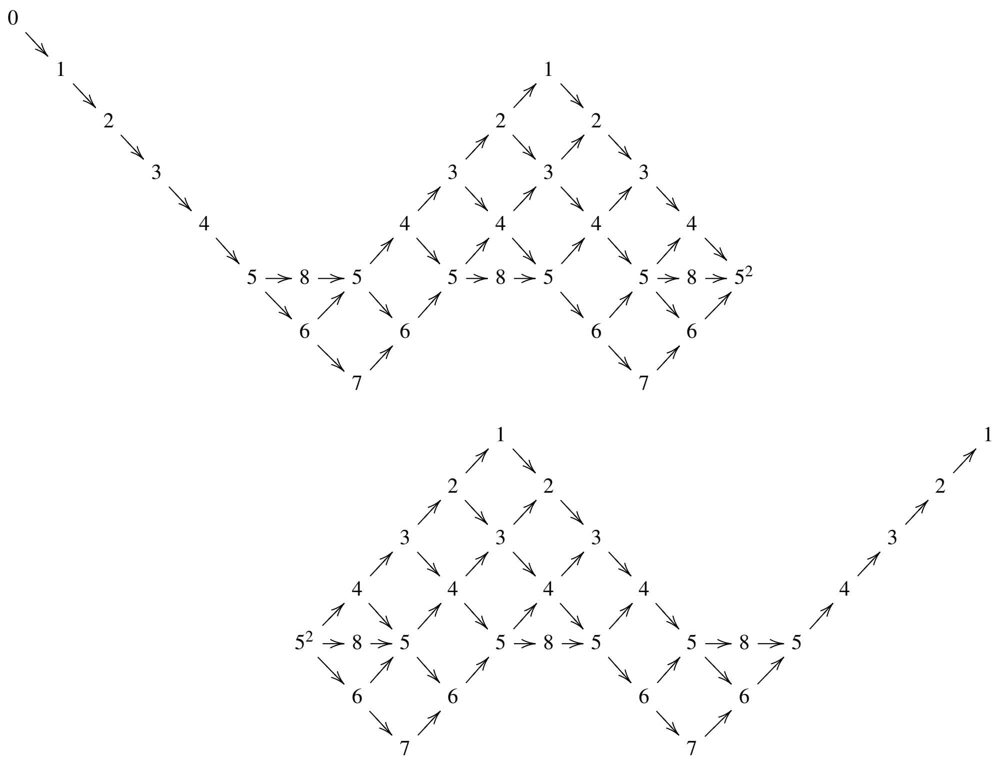
The AR quiver of type $E_8$ (as the form of $\mathbb{Z}E_8$) is




with relations

$$\begin{cases} aA = 0, & bB + cC = 0, \\ dD + eE = 0, & fF = 0, \\ gG + hH = 0, & Aa + Bb = 0, \\ Cc + Dd = 0, & Ee + Ff + Gg = 0, \\ Hh = 0. \end{cases}$$

After applying the counting argument, we have the following quiver



where the right side of upper part and the left side of lower part are identified.

**· the case of $M_1$ in $E_8$**

Since $\mathrm{rank}_R\, M_1 = 2$, the multiplicity $d_{t,1}$ of $M_t$ in ${}^eM_1$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $d_{t,1}$ | 2 | 4 | 6 | 8 | 10 | 12 | 8 | 4 | 6 |

(5.4.15)

In a similar way to the other cases, we have the upper bounds $s(M_1) \leq \frac{6}{120}$ by selecting paths which generate $g_{1,1}$. The following table and Lemma 5.4.1 assert the equality.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $a$ | $1_{M_1}$ | $b$ | 0 | 0 | 0 | 0 | 0 | 0 | |
| $g_{1,1}$ | 2 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 |
| $g_{1,2}$ | 0 | 4 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 10 |
| $g_{1,3}$ | 0 | 4 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 10 |
| $g_{1,4}$ | 0 | 4 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 10 |

**· the case of $M_2$ in $E_8$**

Since $\mathrm{rank}_R\, M_2 = 3$, the multiplicity $d_{t,2}$ of $M_t$ in ${}^eM_2$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $d_{t,2}$ | 3 | 6 | 9 | 12 | 15 | 18 | 12 | 6 | 9 |

(5.4.16)

We have the upper bounds $s(M_2) \leq \frac{18}{120}$ by selecting paths which generate $g_{2,1}$. Thus, we conclude $s(M_2) = \dfrac{18}{120}$ by the following table and Lemma 5.4.1.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aB$ | $B$ | $1_{M_2}$ | $C$ | 0 | 0 | 0 | 0 | 0 | |
| $g_{2,1}$ | 3 | 6 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 18 |
| $g_{2,2}$ | 0 | 0 | 9 | 12 | 0 | 0 | 0 | 0 | 0 | 21 |
| $g_{2,3}$ | 0 | 6 | 9 | 12 | 0 | 0 | 0 | 0 | 0 | 27 |
| $g_{2,4}$ | 0 | 0 | 9 | 12 | 0 | 0 | 0 | 0 | 0 | 21 |
| $g_{2,5}$ | 0 | 6 | 9 | 12 | 0 | 0 | 0 | 0 | 0 | 27 |
| $g_{2,6}$ | 0 | 0 | 9 | 12 | 0 | 0 | 0 | 0 | 0 | 21 |

**· the case of $M_3$ in $E_8$**

Since $\mathrm{rank}_R\, M_3 = 4$, the multiplicity $d_{t,3}$ of $M_t$ in ${}^eM_3$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $d_{t,3}$ | 4 | 8 | 12 | 16 | 20 | 24 | 16 | 8 | 12 |

(5.4.17)

We have the upper bounds $s(M_3) \leq \frac{40}{120}$ by selecting paths which generate $g_{3,1}$. Thus, we conclude $s(M_3) = \dfrac{40}{120}$ by the following table and Lemma 5.4.1.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aBc$ | $Bc$ | $c$ | $1_{M_3}$ | 0 | $Ed$ | 0 | 0 | 0 | |
| $g_{3,1}$ | 4 | 8 | 12 | 16 | 0 | 0 | 0 | 0 | 0 | 40 |
| $g_{3,2}$ | 0 | 0 | 0 | 16 | 0 | 24 | 0 | 0 | 0 | 40 |
| $g_{3,3}$ | 0 | 0 | 12 | 16 | 0 | 24 | 0 | 0 | 0 | 52 |
| $g_{3,4}$ | 0 | 8 | 12 | 16 | 0 | 24 | 0 | 0 | 0 | 60 |
| $g_{3,5}$ | 0 | 0 | 0 | 16 | 0 | 24 | 0 | 0 | 0 | 40 |
| $g_{3,6}$ | 0 | 0 | 12 | 16 | 0 | 24 | 0 | 0 | 0 | 52 |
| $g_{3,7}$ | 0 | 8 | 12 | 16 | 0 | 24 | 0 | 0 | 0 | 60 |
| $g_{3,8}$ | 0 | 0 | 0 | 16 | 0 | 24 | 0 | 0 | 0 | 40 |

**· the case of $M_4$ in $E_8$**

Since $\mathrm{rank}_R\, M_4 = 5$, the multiplicity $d_{t,4}$ of $M_t$ in ${}^eM_4$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $d_{t,4}$ | 5 | 10 | 15 | 20 | 25 | 30 | 20 | 10 | 15 |

(5.4.18)

Similarly, we have $s(M_4) \leq \frac{75}{120}$. The following table and Lemma 5.4.1 assert the equality.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aBcD$ | $BcD$ | $cD$ | $D$ | $1_{M_4}$ | $E$ | $gE$ | 0 | 0 | |
| $g_{4,1}$ | 5 | 10 | 15 | 20 | 25 | 0 | 0 | 0 | 0 | 75 |
| $g_{4,2}$ | 0 | 0 | 0 | 0 | 25 | 30 | 20 | 0 | 0 | 75 |
| $g_{4,3}$ | 0 | 0 | 0 | 20 | 25 | 30 | 20 | 0 | 0 | 95 |
| $g_{4,4}$ | 0 | 0 | 15 | 20 | 25 | 30 | 0 | 0 | 0 | 90 |
| $g_{4,5}$ | 0 | 10 | 15 | 20 | 25 | 30 | 20 | 0 | 0 | 120 |
| $g_{4,6}$ | 0 | 0 | 0 | 0 | 25 | 30 | 20 | 0 | 0 | 75 |
| $g_{4,7}$ | 0 | 0 | 0 | 20 | 25 | 30 | 20 | 0 | 0 | 95 |
| $g_{4,8}$ | 0 | 0 | 15 | 20 | 25 | 30 | 20 | 0 | 0 | 110 |
| $g_{4,9}$ | 0 | 10 | 15 | 20 | 25 | 30 | 0 | 0 | 0 | 100 |
| $g_{4,10}$ | 0 | 0 | 0 | 0 | 25 | 30 | 20 | 0 | 0 | 75 |

**· the case of $M_5$ in $E_8$**

Since $\mathrm{rank}_R\, M_5 = 6$, the multiplicity $d_{t,5}$ of $M_t$ in ${}^eM_5$ is the following.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $d_{t,5}$ | 6 | 12 | 18 | 24 | 30 | 36 | 24 | 12 | 18 |

(5.4.19)

In the same way as before, we can classify the MCM $R$-modules appear in ${}^eM_5$ as

(I) $\{ M_5{\times}36 \}$ (II) $\{ R{\times}6, M_1{\times}12, M_2{\times}18, M_3{\times}24, M_4{\times}30 \}$ (III) $\{ M_6{\times}24, M_8{\times}18 \}$

where the class(I) (resp. (II), (III)) is the set of MCM $R$-modules which generate $g_{5,1}$ and $g_{5,2}$ at the same time (resp. either $g_{5,1}$ or $g_{5,2}$, only $g_{5,2}$ ) and obtain the upper bound $s(M_5) \leq \frac{36}{120} + \frac{42}{120} + \frac{24}{120} = \frac{102}{120}$. By Lemma 5.4.1 and the following table, we have $s(M_5) = \dfrac{102}{120}$. In this table, we fix that $g_{5,6}$ (resp. $g_{5,7}$) is a minimal generator identified with a path which factor through $4 \xrightarrow{e} 5^2$ (resp. $8 \xrightarrow{f} 5^2$). We can use paths which factor through $6 \xrightarrow{g} 5^2$ for generating either $g_{5,6}$ or $g_{5,7}$.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aBcDeFf$ | $BcDe$ | $cDeFf$ | $De$ | $e$ | $1_{M_5}$ | $g$ | $Hg$ | $f$ | |
| $g_{5,1}$ | 0 | 12 | 0 | 24 | 30 | 36 | 0 | 0 | 0 | 102 |
| $g_{5,2}$ | 6 | 0 | 18 | 0 | 0 | 36 | 24 | 0 | 18 | 102 |
| $g_{5,3}$ | 0 | 0 | 0 | 0 | 30 | 36 | 24 | 12 | 0 | 102 |
| $g_{5,4}$ | 0 | 0 | 0 | 24 | 30 | 36 | 0 | 0 | 18 | 108 |
| $g_{5,5}$ | 0 | 0 | 0 | 24 | 20 | 36 | 24 | 0 | 0 | 114 |
| $g_{5,6}$ | 0 | 12 | 0 | 24 | 30 | 36 | 0 | 0 | 0 | 102 |
| $g_{5,7}$ | 0 | 0 | 18 | 0 | 0 | 36 | 24 | 12 | 18 | 108 |
| $g_{5,8}$ | 0 | 0 | 18 | 0 | 30 | 36 | 24 | 0 | 18 | 126 |
| $g_{5,9}$ | 0 | 0 | 0 | 24 | 30 | 36 | 24 | 12 | 0 | 126 |
| $g_{5,10}$ | 0 | 0 | 0 | 24 | 30 | 36 | 0 | 0 | 18 | 108 |
| $g_{5,11}$ | 0 | 12 | 18 | 24 | 30 | 36 | 24 | 0 | 0 | 144 |
| $g_{5,12}$ | 0 | 0 | 18 | 0 | 0 | 36 | 24 | 12 | 18 | 108 |

· **the case of $M_6$ in $E_8$**

Since $\operatorname{rank}_R M_6 = 4$, the multiplicity $d_{t,6}$ of $M_t$ in ${}^eM_6$ is the same as the table (5.4.17). In the same way as before, we can classify the MCM $R$-modules appear in ${}^eM_6$ as

(I) $\{ M_5{\times}24, M_6{\times}16 \}$ (II) $\{ R{\times}4, M_1{\times}8, M_2{\times}12, M_3{\times}16, M_4{\times}20 \}$ (III) $\{ M_7{\times}8, M_8{\times}12 \}$

where the class(I) (resp. (II), (III)) is the set of MCM $R$-modules which generate $g_{6,1}$ and $g_{6,2}$ at the same time (resp. either $g_{6,1}$ or $g_{6,2}$, only $g_{6,2}$ ) and obtain the upper bound $s(M_6) \leq \frac{40}{120} + \frac{20}{120} + \frac{20}{120} = \frac{80}{120}$. The following table and Lemma 5.4.1 assert the equality.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aBcDeFfG$ | $BcDeFfG$ | $cDeG$ | $DeG$ | $eG$ | $eFfG$ | $G$ | $1_{M_6}$ | $H$ | $fG$ | |
| $g_{6,1}$ | 0 | 0 | 12 | 16 | 12 | 0 | 24 | 16 | 0 | 0 | 80 |
| $g_{6,2}$ | 4 | 8 | 0 | 0 | 0 | 8 | 24 | 16 | 8 | 12 | 80 |
| $g_{6,3}$ | 0 | 0 | 0 | 16 | 12 | 8 | 24 | 16 | 0 | 12 | 88 |
| $g_{6,4}$ | 0 | 0 | 12 | 16 | 12 | 0 | 24 | 16 | 8 | 0 | 88 |
| $g_{6,5}$ | 0 | 0 | 12 | 16 | 12 | 8 | 24 | 16 | 0 | 12 | 100 |
| $g_{6,6}$ | 0 | 8 | 0 | 0 | 12 | 8 | 24 | 16 | 8 | 12 | 88 |
| $g_{6,7}$ | 0 | 0 | 12 | 16 | 12 | 8 | 24 | 16 | 0 | 12 | 100 |
| $g_{6,8}$ | 0 | 0 | 12 | 16 | 12 | 0 | 24 | 16 | 8 | 0 | 88 |

· **the case of $M_7$ in $E_8$**

Since $\mathrm{rank}_R M_7 = 2$, the multiplicity $d_{t,7}$ of $M_t$ in ${}^eM_7$ is the same as the table (5.4.15). In the same way as before, we can classify the MCM $R$-modules appear in ${}^eM_7$ as

(I) $\{ M_3{\times}8, M_4{\times}10, M_5{\times}12, M_6{\times}8, M_7{\times}4 \}$ (II) $\{ R{\times}2, M_1{\times}4, M_2{\times}6 \}$ (III) $\{ M_8{\times}6 \}$

where the class(I) (resp. (II), (III)) is the set of MCM $R$-modules which generate $g_{7,1}$ and $g_{7,2}$ at the same time (resp. either $g_{7,1}$ or $g_{7,2}$, only $g_{7,2}$ ) and obtain the upper bound $s(M_7) \leq \frac{42}{120} + \frac{6}{120} + \frac{3}{120} = \frac{51}{120}$. The following table and Lemma 5.4.1 assert the equality.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aBcDeGh$ | $BcDeGh$ | $cDeGh$ | $cDeF\cdots Gh$ | $DeGh$ | $eGh$ | $Gh$ | $h$ | $1_{M_7}$ | $fGh$ | |
| $g_{7,1}$ | 2 | 4 | 3 | 0 | 8 | 10 | 12 | 8 | 4 | 0 | 51 |
| $g_{7,2}$ | 0 | 0 | 0 | 3 | 8 | 10 | 12 | 8 | 4 | 6 | 51 |
| $g_{7,3}$ | 0 | 4 | 3 | 0 | 8 | 10 | 12 | 8 | 4 | 6 | 55 |
| $g_{7,4}$ | 0 | 0 | 3 | 3 | 8 | 10 | 12 | 8 | 4 | 6 | 54 |

· **the case of $M_8$ in $E_8$**

Since $\mathrm{rank}_R M_8 = 3$, the multiplicity $d_{t,8}$ of $M_t$ in ${}^eM_8$ is the same as the table (5.4.16). In the same way as before, we can classify the MCM $R$-modules appear in ${}^eM_8$ as

(I) $\{ M_4{\times}15, M_5{\times}18, M_8{\times}9 \}$ (II) $\{ R{\times}3, M_1{\times}6, M_2{\times}9, M_3{\times}12 \}$ (III) $\{ M_6{\times}12, M_7{\times}6 \}$

where the class(I) (resp. (II), (III)) is the set of MCM $R$-modules which generate $g_{8,1}$ and $g_{8,2}$ at the same time (resp. either $g_{8,1}$ or $g_{8,2}$, only $g_{8,2}$ ) and obtain the upper bound $s(M_8) \leq \frac{42}{120} + \frac{18}{120} + \frac{6}{120} = \frac{66}{120}$. The following table and Lemma 5.4.1 assert the equality.

| $M_t$ | $R$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ | $M_7$ | $M_8$ | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| Path | $aBcDeF$ | $aBcDeFfEeF$ | $cDeF$ | $DeF$ | $eF$ | $F$ | $gF$ | $HgF$ | $1_{M_8}$ | |
| $g_{8,1}$ | 3 | 0 | 9 | 12 | 15 | 18 | 0 | 0 | 9 | 66 |
| $g_{8,2}$ | 0 | 6 | 0 | 0 | 15 | 18 | 12 | 6 | 9 | 66 |
| $g_{8,3}$ | 0 | 0 | 9 | 12 | 15 | 18 | 12 | 0 | 9 | 75 |
| $g_{8,4}$ | 0 | 0 | 9 | 12 | 15 | 18 | 12 | 6 | 9 | 81 |
| $g_{8,5}$ | 0 | 6 | 0 | 12 | 15 | 18 | 12 | 6 | 9 | 78 |
| $g_{8,6}$ | 0 | 0 | 9 | 12 | 15 | 18 | 12 | 0 | 9 | 75 |

### 5.4.8 Summary of the value of the dual $F$-signature

**Theorem 5.4.5.** *The following is the Dynkin diagram $Q$ and corresponding values of the dual $F$-signature* (*In order to show the ratio of dual $F$-signature to the order of $G$ clearly, we don't reduce fractions*).

*(1) Type $A_n$*

- *$n$ is an even number (i.e. $n = 2r$)*

$$A_n : \quad 1 \;\text{———}\; 2 \;\text{——}\; \cdots \;\text{——}\; r \;\text{——}\; r+1 \;\text{—}\; \cdots \;\text{—}\; n-1 \;\text{——}\; n$$

$$\frac{2}{n+1} - \frac{3}{n+1} - \cdots - \frac{r+1}{n+1} - \frac{r+1}{n+1} - \cdots - \frac{3}{n+1} - \frac{2}{n+1}$$

- *$n$ is an odd number (i.e. $n = 2r-1$)*

$$A_n : \quad 1 \;\text{———}\; 2 \;\text{——}\; \cdots \;\text{—}\; r-1 \;\text{———}\; r \;\text{———}\; r+1 \;\text{—}\; \cdots \;\text{—}\; n-1 \;\text{——}\; n$$

$$\frac{2}{n+1} - \frac{3}{n+1} - \cdots - \frac{r}{n+1} - \frac{2r+1}{2(n+1)} - \frac{r}{n+1} - \cdots - \frac{3}{n+1} - \frac{2}{n+1}$$

*(2) Type $D_n$*

- *$n$ is an even number (i.e. $n = 2r$)*

$$\begin{tikzcd}[column sep=small] & & & & & & & & & n-1 \\ D_n : & 1 \arrow[r, dash] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & m \arrow[r, dash] & \cdots \arrow[r, dash] & r \arrow[r, dash] & r+1 \arrow[r, dash] & \cdots \arrow[r, dash] & n-2 \arrow[u, dash] \arrow[d, dash] \\ & & & & & & & & & n \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & & & & & & & & \frac{2(n-2)}{4(n-2)} \\ \frac{4}{4(n-2)} \arrow[r, dash] & \frac{6}{4(n-2)} \arrow[r, dash] & \cdots \arrow[r, dash] & \frac{4m-2}{4(n-2)} \arrow[r, dash] & \cdots \arrow[r, dash] & \frac{4r-2}{4(n-2)} \arrow[r, dash] & \frac{4r-1}{4(n-2)} \arrow[r, dash] & \cdots \arrow[r, dash] & \frac{4r-1}{4(n-2)} \arrow[u, dash] \arrow[d, dash] \\ & & & & & & & & \frac{2(n-2)}{4(n-2)} \end{tikzcd}$$

- *$n$ is an odd number (i.e. $n = 2r-1$)*

$$\begin{tikzcd}[column sep=small] & & & & & & & & & n-1 \\ D_n : & 1 \arrow[r, dash] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & m \arrow[r, dash] & \cdots \arrow[r, dash] & r-1 \arrow[r, dash] & r \arrow[r, dash] & \cdots \arrow[r, dash] & n-2 \arrow[u, dash] \arrow[d, dash] \\ & & & & & & & & & n \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & & & & & & & & \frac{2(n-2)}{4(n-2)} \\ \frac{4}{4(n-2)} \arrow[r, dash] & \frac{6}{4(n-2)} \arrow[r, dash] & \cdots \arrow[r, dash] & \frac{4m-2}{4(n-2)} \arrow[r, dash] & \cdots \arrow[r, dash] & \frac{4r-6}{4(n-2)} \arrow[r, dash] & \frac{4r-3}{4(n-2)} \arrow[r, dash] & \cdots \arrow[r, dash] & \frac{4r-3}{4(n-2)} \arrow[u, dash] \arrow[d, dash] \\ & & & & & & & & \frac{2(n-2)}{4(n-2)} \end{tikzcd}$$

*(3) Type $E_6$*

$$
E_6 : \quad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & | & & & & \\ 5 & — & 3 & — & 2 & — & 4 & — & 6 \end{array}
\qquad\qquad
\begin{array}{ccccccccc} & & & & \frac{6}{24} & & & & \\ & & & & | & & & & \\ \frac{9}{24} & — & \frac{16}{24} & — & \frac{18}{24} & — & \frac{16}{24} & — & \frac{9}{24} \end{array}
$$

*(4) Type $E_7$*

$$
E_7 : \quad \begin{array}{ccccccccccc} & & & & 7 & & & & & & \\ & & & & | & & & & & & \\ 1 & — & 2 & — & 3 & — & 4 & — & 5 & — & 6 \end{array}
\qquad\qquad
\begin{array}{ccccccccccc} & & & & \frac{24}{48} & & & & & & \\ & & & & | & & & & & & \\ \frac{6}{48} & — & \frac{18}{48} & — & \frac{38}{48} & — & \frac{36}{48} & — & \frac{27}{48} & — & \frac{16}{48} \end{array}
$$

*(5) Type $E_8$*

$$
E_8 : \quad \begin{array}{ccccccccccccc} & & & & & & & & 8 & & & & \\ & & & & & & & & | & & & & \\ 1 & — & 2 & — & 3 & — & 4 & — & 5 & — & 6 & — & 7 \end{array}
\qquad\qquad
\begin{array}{ccccccccccccc} & & & & & & & & \frac{66}{120} & & & & \\ & & & & & & & & | & & & & \\ \frac{6}{120} & — & \frac{18}{120} & — & \frac{40}{120} & — & \frac{75}{120} & — & \frac{102}{120} & — & \frac{80}{120} & — & \frac{51}{120} \end{array}
$$

*Remark* 5.4.6. As these lists show, we have $s(M_t) = s(M_t^*)$. Indeed, each AR quiver is symmetric with respect to $M_t$ and $M_t^*$, and $\operatorname{rank}_R M_t = \operatorname{rank}_R M_t^*$. Thus, it follows from arguments used in this chapter.

# Chapter 6

# Further topics

## 6.1 Ulrich modules over cyclic quotient surface singularities

In the previous chapter, we explained counting arguments of the AR quiver and by using such a technique we could determine the number of minimal generators of each MCM module over quotient surface singularities. By applying this idea, we also investigate Ulrich modules. This is a certain class of MCM module. In the study of Ulrich modules, spacial CM modules play the curtail role again. This section is based on [NY].

### 6.1.1 Ulrich modules

Let $(R, \mathfrak{m}, k)$ be a CM local ring. For each MCM $R$-module $M$, we have $\mu_R(M) \leq \mathrm{e}^0_{\mathfrak{m}}(M)$. Note that if $R$ is a domain, then we have $\mathrm{e}^0_{\mathfrak{m}}(M) = (\mathrm{rank}_R\, M)\mathrm{e}^0_{\mathfrak{m}}(R)$.

An Ulrich module is defined as a module which has the maximum number of generators with respect to the above inequality. So we sometimes call it a maximally generated maximal Cohen-Macaulay (= MGMCM) module after the original terminology [Ulr, BHU]. The name "Ulrich modules" was introduced in [HK].

**Definition 6.1.1** ([Ulr, BHU])**.** *Let $M$ be an MCM $R$-module. We say $M$ is an Ulrich module if it satisfies $\mu_R(M) = \mathrm{e}^0_{\mathfrak{m}}(M)$.*

We remark that the above conditions are inherited by direct summands and direct sums. So Ulrich modules are closed under direct summands and direct sums.

The properties of these modules were investigated in the aforementioned references. More geometrically, they are also studied as Ulrich bundles e.g. [ESW, CH1, CH2, CKM]. Recently, this notion is generalized for each non-parameter $\mathfrak{m}$-primary ideal $I$ as follows [GOTWY1] and it is studied actively (cf. [GOTWY2, GOTWY3]). Namely, we say an MCM $R$-module $M$ is an Ulrich module " with respect to $I$ " if it satisfies the following conditions:

$$(1)\, \mathrm{e}^0_I(M) = \ell_R(M/IM), \quad (2)\, M/IM \text{ is an } R/I\text{-free module}$$

where $\mathrm{e}^0_I(M)$ is the multiplicity of $M$ with respect to $I$ and $\ell_R(M/IM)$ stands for the length of the $R$-module $M/IM$. Thus, an Ulrich module with respect to $\mathfrak{m}$ is nothing else but an Ulrich module in the sense of Definition 6.1.1. (The condition (2) is automatically if $I = \mathfrak{m}$.) In this thesis, we only discuss Ulrich modules with respect to $\mathfrak{m}$. Thus, we simply denote the multiplicity of $M$ by $\mathrm{e}(M)$. Also, Ulrich modules appear in an attempt to formulate the notion of "almost Gorenstein rings" [GTT]. Therefore, the importance to understand this module has increased. However, even the existence of an Ulrich module for a given CM local ring is still not known in general. Another important problem is to characterize (and to classify) Ulrich modules when a given ring $R$ has an Ulrich module. For example, we know the existence of such a module for the case where

- a two dimensional domain with the infinite field [BHU],
- a CM local ring which has maximal embedding dimension [BHU],
- a strict complete intersection [HUB],
- a Veronese subring of polynomial ring over field of characteristic 0 [ESW] etc.

But the characterization problem is also not known for many cases. Therefore, we will consider Ulrich modules over quotient surface singularities. We remark that this singularity is of finite CM representation type. Since the number of indecomposable Ulrich modules is finite, we will also consider the number of them. The point is to consider special CM modules (see Definition 5.2.1). Roughly, the number of minimal generators of a special CM module is small (see subsection 5.2.1). So special CM modules are the opposite of Ulrich modules in that sense. However, those give us the simple description of Ulrich modules. For example, by applying several functors to special CM modules, we have some Ulrich modules.

**Proposition 6.1.2.** *Let $R$ be a quotient surface singularity as in Chapter 5. Suppose $M$ is a non-free special CM module over $R$. Then we have*

*(1) $M^*$ is an Ulrich $R$-module,*
*(2) $\tau(M)$ is also an Ulrich $R$-module where $\tau$ is the AR translation.*

*Proof.* (1) By Proposition 5.2.3, $M^*$ is the syzygy of an MCM $R$-module. Thus, it is an Ulrich $R$-module by similar arguments as in [GOTWY1, Lemma 4.2].

(2) By [GOTWY1, Theorem 5.1], the canonical dual of an Ulrich module is also an Ulrich module. Combining with (1), we have the conclusion.

□

In this way, we can obtain some Ulrich modules from special ones. However, there exists Ulrich modules which don't come from this operation. In the next section, we focus on the case of cyclic quotient surface singularities and characterize Ulrich modules.

We finish this subsection with the following remark. When we consider Ulrich modules, the multiplicity $\mathrm{e}(M) = (\mathrm{rank}_R\, M)\mathrm{e}(R)$ is important. It is known that the multiplicity $\mathrm{e}(R)$ is computed by the self-intersection number of the fundamental cycle $Z_0$ as follows [Art].

**Proposition 6.1.3.** *Let the notations be the same as in subsection 5.2.1. Then we have* $\mathrm{e}(R) = -Z_0^2$.

### 6.1.2 Characterizations of Ulrich modules for cyclic case

In this subsection, we will characterize Ulrich modules for the case of cyclic quotient surface singularities. In order to make the settings clear, we again note that $G$ is a cyclic group as follows.

$$G := \langle\ \sigma = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^a \end{pmatrix}\ \rangle,$$

where $\zeta_n$ is a primitive $n$-th root of unity, $1 \le a \le n-1$, and $\gcd(a,n) = 1$ and assume that $n$ is invertible in $k$. We will denote this cyclic group by $G = \frac{1}{n}(1,a)$ and consider the invariant subring $R$ of $S = k[[x,y]]$ under the action of $G$. Since $G$ is an abelian group, every irreducible representation of $G$ is one dimensional and described as

$$V_t : \sigma \mapsto \zeta_n^{-t} \quad (t = 0, 1, \cdots, n-1).$$

Then we set,

$$M_t := (S \otimes_k V_t)^G = \left\langle x^i y^j \;\middle|\; i + ja \equiv t \pmod n \right\rangle, \quad (t = 0, 1, \cdots, n-1).$$

Then, these $M_t$'s are only indecomposable MCM modules over $R$ and rank $M_t = 1$. Since special CM module play the crucial role to characterize Ulrich modules, we recall some facts mentioned in Section 5.3. Firstly, we consider the Hirzebruch-Jung continued fraction expansion of $n/a$:

$$\frac{n}{a} = \alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\cdots - \frac{1}{\alpha_r}}} := [\alpha_1, \alpha_2, \cdots, \alpha_r],$$

and consider the notion of $i$-series and $j$-series as follows.

$$\begin{aligned} &i_0 = n, \;\; i_1 = a, && i_t = \alpha_{t-1} i_{t-1} - i_{t-2} && (t = 2, \cdots, r+1), \\ &j_0 = 0, \;\; j_1 = 1, && j_t = \alpha_{t-1} j_{t-1} - j_{t-2} && (t = 2, \cdots, r+1). \end{aligned}$$

In this situation, special CM $R$-modules are $M_{i_t} = Rx^{i_t} + Ry^{j_t}$ for $t = 1, \cdots, r$ and $R$ (see Theorem 5.3.3). Furthermore, there is a one-to-one correspondence between non-free indecomposable special CM modules and irreducible exceptional curves (see Theorem 5.2.4). The dual graph of the minimal resolution of singularity $X \to \mathrm{Spec}(R)$ is obtained by the Hirzebruch-Jung continued fraction expansion:

$$\begin{array}{ccccccc} E_{i_1} & & E_{i_2} & & & & E_{i_r} \\ (-\alpha_1) & — & (-\alpha_2) & — & \cdots\cdots & — & (-\alpha_r) \end{array}$$

and the fundamental cycle is $Z_0 = \sum_{t=1}^{r} E_{i_t}$. Moreover, we have $\mathrm{e}(R) = \alpha_1 + \cdots + \alpha_r - 2(r-1)$ by Proposition 6.1.3.

**Example 6.1.4.** *Let $G = \frac{1}{12}(1,7)$ be a cyclic group of order 12. The Hirzebruch-Jung continued fraction expansion of 12/7 is*

$$\frac{12}{7} = 2 - \frac{1}{4 - 1/2} = [2, 4, 2],$$

*and the i-series and the j-series are obtained as follows.*

$$\begin{array}{lllll} i_0 = 12, & i_1 = 7, & i_2 = 2, & i_3 = 1, & i_4 = 0, \\ j_0 = 0, & j_1 = 1, & j_2 = 2, & j_3 = 7, & j_4 = 12. \end{array}$$

*Thus, the special CM modules are* $M_7$, $M_2$, $M_1$, $R$ *and they take the form*

$$M_7 = Rx^7 + Ry, \quad M_2 = Rx^2 + Ry^2, \quad M_1 = Rx + Ry^7.$$

*In this case, the dual graph is*

$$\begin{array}{ccccc} E_7 & & E_2 & & E_1 \\ -2 & — & -4 & — & -2 \end{array}$$

*and the fundamental cycle is* $Z_0 = E_7 + E_2 + E_1$. *Thus, we have the multiplicity* $\mathrm{e}(R) = -Z_0^2 = 4$.

**Example 6.1.5.** *(= Example 5.3.5) Suppose* $G = \frac{1}{n}(1, n-1) \subset \mathrm{SL}(2, k)$ *is a cyclic group of order n (= Dynkin type* $A_{n-1}$*). The Hirzebruch-Jung continued fraction expansion of* $n/(n-1)$ *is*

$$\frac{n}{n-1} = 2 - \cfrac{1}{2 - \cfrac{1}{\cdots - 1/2}} = [\underbrace{2, 2, \cdots, 2}_{n-1}],$$

*and the i-series and the j-series are*

$$\begin{array}{llllll} i_0 = n, & i_1 = n-1, & i_2 = n-2, & \cdots, & i_{n-1} = 1, & i_n = 0, \\ j_0 = 0, & j_1 = 1, & j_2 = 2, & \cdots, & j_{n-1} = n-1, & j_n = n. \end{array}$$

*Therefore, every MCM module is special (cf. Remark 5.2.2). Since* $\mathrm{e}(R) = 2$, *every non-free MCM R-module is also an Ulrich module. This kind of property holds in more general situation (cf. [GOTWY2, Theorem 5.2], [HK, Corollary 1.4]).*

By applying Proposition 6.1.2, we have the following.

**Proposition 6.1.6.** *Let the notation be the same as above. For a non-free special MCM R-module* $M_{i_t}$, *MCM modules* $M_{n-i_t}$ *and* $M_{i_t-a-1}$ *are Ulrich modules.*

*Proof.* Since $M_{i_t}^* \cong M_{n-i_t}$ and $\tau(M_{i_t}) \cong M_{i_t-a-1}$, it follows from Proposition 6.1.2. □

From this proposition, we can obtain some Ulrich modules. However, there exists Ulrich modules which don't take the form as in Proposition 6.1.6. In order to determine all of them, we will show the relationship between the multiplicity $\mathrm{e}(M_t) = \mathrm{e}(R)$ and the number of minimal generators $\mu_R(M_t)$ in terms of the $i$-series. As a conclusion, we characterize Ulrich $R$-modules. To state the theorem, we again use the sequence $(d_{1,t}, \cdots, d_{r,t})$ defined in Section 5.3. Namely, for the $i$-series $(i_1, \cdots, i_r)$ associated with $\frac{1}{n}(1, a)$ and for any $t \in [0, n-1]$, there are unique non-negative integers $d_{1,t}, \cdots, d_{r,t} \in \mathbb{Z}_{\geq 0}$ such that

$$\begin{array}{rlll} t = d_{1,t}i_1 + h_{1,t}, & h_{1,t} \in \mathbb{Z}_{\geq 0}, & 0 \leq h_{1,t} < i_1, & \\ h_{u,t} = d_{u+1,t}i_{u+1} + h_{u+1,t}, & h_{u+1,t} \in \mathbb{Z}_{\geq 0}, & 0 \leq h_{u+1,t} < i_{u+1}, & (u = 1, \cdots, r-1), \\ & h_{r,t} = 0. & & \end{array}$$

Thus, we describe $t$ as follows.

$$\begin{aligned} t &= d_{1,t}i_1 + d_{2,t}i_2 + \cdots + d_{r,t}i_r \\ &= \underbrace{(i_1 + \cdots + i_1)}_{d_{1,t}} + \underbrace{(i_2 + \cdots + i_2)}_{d_{2,t}} + \cdots + \underbrace{(i_r + \cdots + i_r)}_{d_{r,t}}. \end{aligned}$$

If a situation is clear, we will denote simply $d_{u,t}$ by $d_u$. This sequence $(d_{1,t}, \cdots, d_{r,t}) \in (\mathbb{Z}_{\geq 0})^r$ is characterized as follows. We will use this lemma heavily in the future.

**Lemma 6.1.7.** *([Wun1, Lemma 1]) A sequence $(d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ is obtained from the description*

$$t = d_1 i_1 + d_2 i_2 + \cdots + d_r i_r$$

*for some subscript $t = 0, 1, \cdots, n-1$ if and only if a sequence satisfies the following two condition.*

- $\cdot$ $0 \leq d_u \leq \alpha_u - 1$ *for every $u = 1, \cdots, r$.*
- $\cdot$ *If $d_u = \alpha_u - 1$ and $d_v = \alpha_v - 1$ $(u < v)$, then there exists $w$ such that $u < w < v$ and $d_w \leq \alpha_w - 3$.*

So we are now in a position to state the theorem.

**Theorem 6.1.8.** *Let the notation be the same as above. Then we have*

$$\mu_R(M_t) = d_1 + d_2 + \cdots + d_r + 1.$$

To this theorem, we will give two kinds of proofs (geometric one and representation theoretic one). The geometric proof is quite simple and it says the above formula is a reinterpretation of special McKay correspondence from the viewpoint of Ulrich modules. However, the author believe that the method used in another one will give us a new aspect for this subject (e.g. Remark 6.1.11). Therefore, we note both of them.

*Geometric proof of Theorem 6.1.8.* From Kato's Riemann-Roch formula [Kat], we have

$$\mu_R(M_t) = 1 + \mathrm{c}_1(\widetilde{M_t}) \cdot (E_{i_1} + \cdots + E_{i_r}).$$

Also, $\mathrm{c}_1(\widetilde{M_t}) \cdot E_{i_u} = d_{u,t}$ [Wun2]. So we have the conclusion. □

*Representation theoretic proof of Theorem 6.1.8.* We recall that a minimal generator of $M_t$ is identified with a path from $R$ to $M_t$ which doesn't factor through a free module except the starting point. Thus, we will count such paths on the AR quiver. Firstly, we write the AR quiver Q as the repetition of the AR quiver, see Figure 6.1 (it is the translation quiver $\mathbb{Z}$Q). Note that each diagram $\begin{smallmatrix} a & \to & b \\ \uparrow & & \uparrow \\ c & \to & d \end{smallmatrix}$ corresponds to the AR sequence ending in $M_b$ $(b \neq 0)$:

$$0 \to M_c \to M_a \oplus M_d \to M_b \to 0,$$

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & a \arrow[r] & \cdots \arrow[r] & t-2a \arrow[r] & t-a \arrow[r] & t \\
\cdots \arrow[r] & n-1 \arrow[r] \arrow[u] & a-1 \arrow[r] \arrow[u] & \cdots \arrow[r] & t-2a-1 \arrow[r] \arrow[u] & t-a-1 \arrow[r] \arrow[u] & t-1 \arrow[u] \\
\cdots \arrow[r] & n-2 \arrow[r] \arrow[u] & a-2 \arrow[r] \arrow[u] & \cdots \arrow[r] & t-2a-2 \arrow[r] \arrow[u] & t-a-2 \arrow[r] \arrow[u] & t-2 \arrow[u] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
 & & \iddots & \cdots \arrow[r] & 2-2a \arrow[r] \arrow[u] & 2-a \arrow[r] \arrow[u] & 2 \arrow[u] \\
 & & & \cdots \arrow[r] & 1-2a \arrow[r] \arrow[u] & 1-a \arrow[r] \arrow[u] & 1 \arrow[u] \\
 & & & \cdots \arrow[r] & -2a \arrow[r] \arrow[u] & -a \arrow[r] \arrow[u] & 0 \arrow[u] \\
 & & & \iddots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}
$$

Figure 6.1

and the fundamental sequence of $R$. These diagrams commutes $\begin{tikzcd} a \arrow[r, "y"] & b \\ c \arrow[u, "x"] \arrow[r, "y"'] \arrow[ur, phantom, "\circlearrowleft"] & d \arrow[u, "x"'] \end{tikzcd}$ from Remark 5.3.6 and 5.3.8.

From this quiver, we extract an appropriate part which implies paths from $R$ to $M_t$ corresponding to minimal generators of $M_t$. Such paths takes the form like Figure 6.2. Here, we assume grayed areas don't contain $R(=0)$ (otherwise we can divide those areas into smaller ones). Indeed, a vertex 0 which is located at outside of Figure 6.2 certainly go through free modules on the way to $M_t$. Thus, we may only consider the paths from $R(=0)$ to $M_t(=t)$ appearing in Figure 6.2. Furthermore, the number of vertex 0 appearing in Figure 6.2 coincides with $\mu_R(M_t)$ and we see that the rightmost vertical arrows are divided into $\mu_R(M_t)-1$ blocks. We have to remark that vertices described by $\bigstar$ and $\bigstar_1$ are special CM modules because the number of minimal generators of them is two (see Theorem 5.3.3 and the discussion following Proposition 5.2.3). From now on, we will show this division corresponds to the integers $(d_1, d_2, \cdots, d_r)$.

We set $i_s = \max\{ i_u \text{ in } i\text{-series} \,|\, d_u \neq 0\}$, then we can find the vertex $i_s$ on the rightmost vertical column in Figure 6.2. From this position, we will follow vertices to the left direction and if we arrive at a vertex 0, then we stop there (see Figure 6.3). Since $M_{i_s}$ is a special CM module, the length of the vertical (resp. horizontal) path from 0 to $i_s$ in Figure 6.3 is $i_s$ (resp. $j_s$) by Theorem 5.3.3. From the selecting method of $i_s$, we have $\bigstar_1 \leq i_s$. If $\bigstar_1 < i_s$, we have Figure 6.4 by Remark 5.3.2 and see that a path from the lower vertex 0 go through the upper one. Thus, this contradicts the choice of $\bigstar_1$. It follows that $i_s$ coincides with $\bigstar_1$. After that, we replace $d_s$ by $d_s - 1$. If $d_s \neq 0$, then we

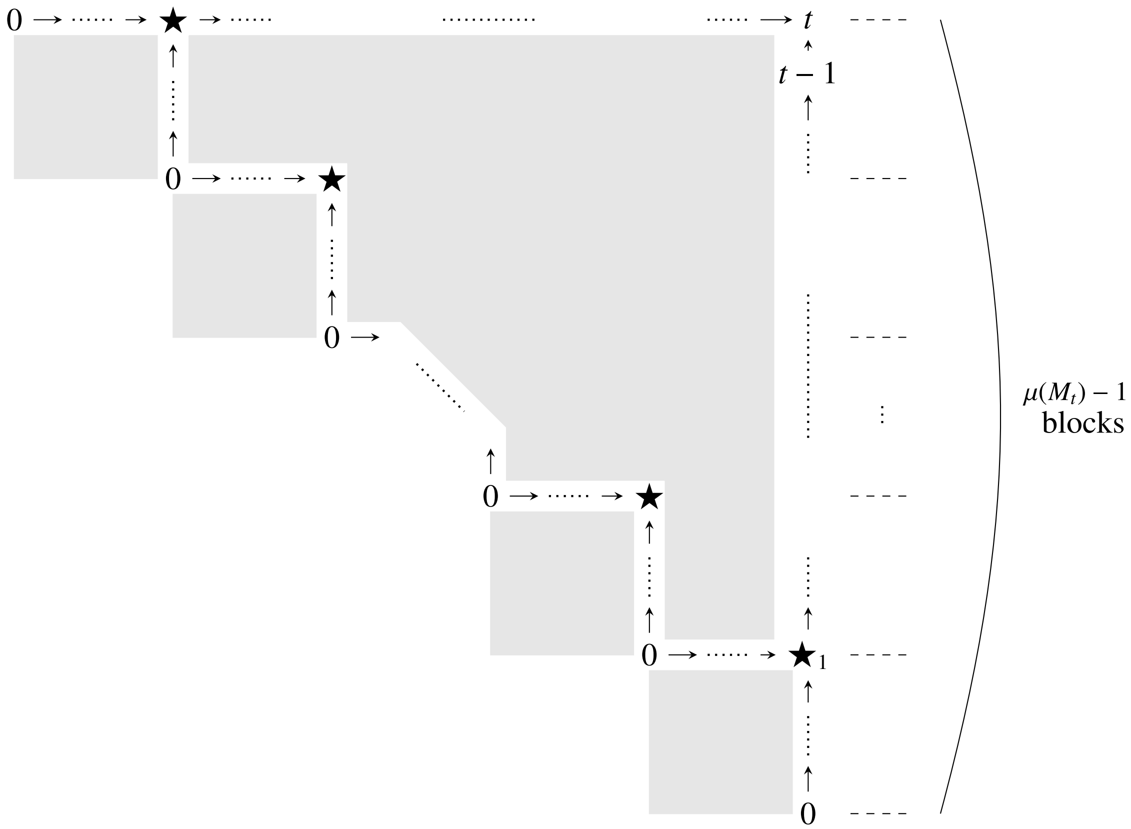



Figure 6.2

repeat the same operation to the vertical column starting at the second rightmost vertex 0. Repeating these processes to the other vertical columns in order until $d_s$ become 0, we have $d_s$ blocks of length $i_s$.

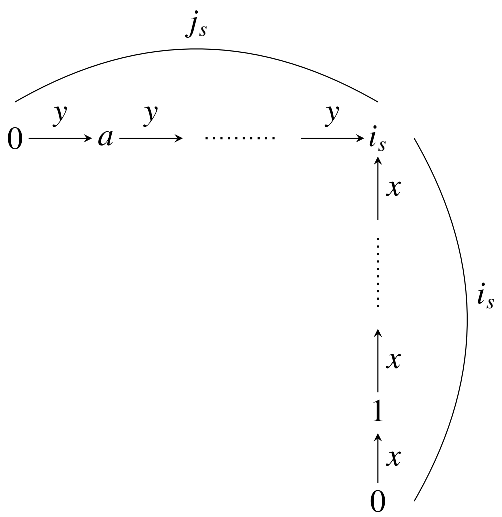



Figure 6.3

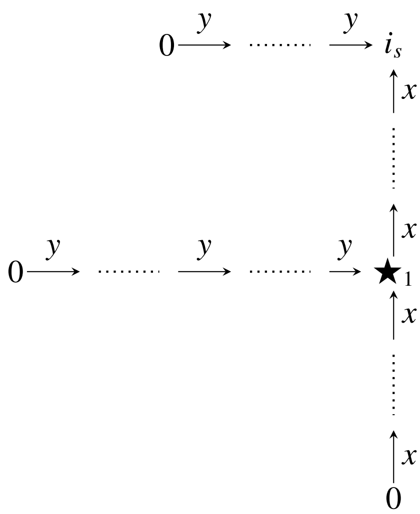



Figure 6.4

Next, we set $i'_s = \max\{\, i_u \text{ in } i\text{-series} \,|\, d_u \neq 0,\ i_u < i_s\}$ and apply the same process to $i'_s$. Repeating the above processes until we arrive at the top row, we can see the number of divided blocks in Figure 6.2 is equal to $d_1 + d_2 + \cdots + d_r$. □

Since $\mathrm{e}(R) = \mathrm{e}(M_t)$ and $\mu_R(M_t) \leq \mathrm{e}(M_t)$, we may set $\mu_R(M_t) = \mathrm{e}(R) - s$ where $0 \leq s \leq \mathrm{e}(R) - 1$. The next corollary immediately follows from the theorem. By this corollary, we can determine which $M_t$ is Ulrich module for a given cyclic quotient surface singularity.

**Corollary 6.1.9.** *Let the notation be the same as above. Then*

$$\mu_R(M_t) = \mathrm{e}(R) - s \Longleftrightarrow d_1 + d_2 + \cdots + d_r = \mathrm{e}(R) - (s+1)$$

*for* $s = 0, 1, \cdots, \mathrm{e}(R) - 1$.

*In particular, an MCM R-module $M_t$ is Ulrich if and only if* $d_1 + d_2 + \cdots + d_r = \mathrm{e}(R) - 1$.

**Example 6.1.10.** *Let $G = \frac{1}{12}(1,7)$ be a cyclic group of order* 12 *(cf. Example 6.1.4). In this case, non-free special CM modules are $M_7, M_2, M_1$ and* $\mathrm{e}(R) = 4$.

*So we obtain the following division of each subscript into integers appearing in the i-series.*

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
11 &=& 7+2+2 & 7 &=& 7 & 3 &=& 2+1 \\
10 &=& 7+2+1 & 6 &=& 2+2+2 & 2 &=& 2 \\
9 &=& 7+2 & 5 &=& 2+2+1 & 1 &=& 1 \\
8 &=& 7+1 & 4 &=& 2+2 & & &
\end{array}$$

*Therefore, Ulrich modules are $M_{11}$, $M_{10}$, $M_6$, and $M_5$. For example, paths in the AR quiver which correspond to minimal generators of $M_{10}$ are described as follows:*

$$\begin{tikzcd}
0 \arrow[r] & 7 \arrow[r] & \cdots \arrow[r] & 6 \arrow[r] & 1 \arrow[r] & 8 \arrow[r] & 3 \arrow[r] & 10 \\
& & & & 0 \arrow[u] \arrow[r] & 7 \arrow[u] \arrow[r] & 2 \arrow[u] \arrow[r] & 9 \arrow[u] \\
& & & & & & 1 \arrow[u] \arrow[r] & 8 \arrow[u] \\
& & & & & & 0 \arrow[u] \arrow[r] & 7 \arrow[u] \\
& & & & & & & 6 \arrow[u] \\
& & & & & & & \vdots \arrow[u] \\
& & & & & & & 0 \arrow[u]
\end{tikzcd}$$

*Remark* 6.1.11. The method used in the representation theoretic proof enables us to determine Ulrich modules for other quotient surface singularities. For example, see Example 5.1.2 and 6.2.5.

In this way, we can check which MCM $R$-module $M_t$ is an Ulrich one. But if the order of $G$ is large enough, then a process to obtain the sequence $(d_{1,t}, \cdots, d_{r,t})$ for every $t = 0, 1, \cdots, n-1$ will be tough (although it is not difficult). Therefore we will show another characterization of Ulrich modules in terms of the $i$-series. Firstly, for each subscript $t = 0, 1, \cdots, n-1$, we decompose it as in Lemma 6.1.7,

$$t = d_{1,t} i_1 + d_{2,t} i_2 + \cdots + d_{r,t} i_r. \tag{6.1.1}$$

Then, for each subscript $t = 0, 1, \cdots, n-1$, we define a subset of the $i$-series as follows.

$$\mathtt{I}_t := \{i_s \mid d_{s,t} \neq 0 \text{ in the decomposition (6.1.1)}\}.$$

In order to characterize Ulrich modules, we need $\mathtt{I}_{n-1}$. Since we can decompose $n-1$ as

$$\begin{aligned}
n-1 &= \alpha_1 i_1 - i_2 - 1 \\
&= (\alpha_1 - 1)i_1 + (i_1 - i_2) - 1 \\
&= (\alpha_1 - 1)i_1 + (\alpha_2 - 1)i_2 - i_3 - 1 \\
&= (\alpha_1 - 1)i_1 + (\alpha_2 - 2)i_2 + (i_2 - i_3) - 1 \\
&\vdots \\
&= (\alpha_1 - 1)i_1 + (\alpha_2 - 2)i_2 + \cdots + (\alpha_{r-1} - 2)i_{r-1} + (\alpha_r - 1)i_r - i_{r+1} - 1 \\
&= (\alpha_1 - 1)i_1 + (\alpha_2 - 2)i_2 + \cdots + (\alpha_{r-1} - 2)i_{r-1} + (\alpha_r - 2)i_r,
\end{aligned}$$

we have

$$\mathtt{I}_{n-1} = \{i_1\} \cup \{i_s \mid \alpha_s > 2 \text{ and } 2 \leq s \leq r\}.$$

Here, since the sum of coefficient is $(\alpha_1 - 1) + \sum_{u=2}^{r}(\alpha_u - 2) = \alpha_1 + \cdots + \alpha_r - 2r + 1 = \mathrm{e}(R) - 1$, $M_{n-1}$ is an Ulrich module. (This also come from Proposition 6.1.6 because from the definition of the $i$-series, we have $i_r = 1$.) Then we define pairs of integers appearing in the $i$-series:

$$\mathsf{U} := \{(i_s, i_u) \mid i_s \in \mathtt{I}_{n-1} \text{ and } i_s > i_u \text{ (equivalently } u > s)\}. \tag{6.1.2}$$

We emphasize that the determination method of $\mathsf{U}$ is the core of the characterization of Ulrich modules.

We are now ready to state the theorem.

**Theorem 6.1.12.** *Consider any sequences of pairs $(i_{k(1)}, i_{k(1)'}), \cdots, (i_{k(b)}, i_{k(b)'}) \in \mathsf{U}$ which satisfy $i_{k(c)'} > i_{k(c+1)}$ for any $c = 1, 2, \cdots, b-1$. If $t = n - 1 - \sum_{c=1}^{b}(i_{k(c)} - i_{k(c)'})$ or $t = n-1$, then $M_t$ is an Ulrich module.*

*Proof.* We already know that $M_{n-1}$ is an Ulrich module. Thus, we will consider the other case.

We take a pair $(i_{k(1)}, i_{k(1)'}) \in \mathsf{U}$, then $n - 1 - (i_{k(1)} - i_{k(1)'})$ is deformed as follows.

$$\begin{aligned}
(\text{ if } k(1) = 1\ ) \quad &= \sum_{v=1}^{k(1)'-1} (\alpha_v - 2)i_v + (\alpha_{k(1)'} - 1)i_{k(1)'} + \sum_{v=k(1)'+1}^{r} (\alpha_v - 2)i_v. \\
(\text{ if } k(1) \neq 1\ ) \quad &= (\alpha_1 - 1)i_1 + \sum_{v=2}^{k(1)-1} (\alpha_v - 2)i_v + (\alpha_{k(1)} - 3)i_{k(1)} \\
&\quad + \sum_{v=k(1)+1}^{k(1)'-1} (\alpha_v - 2)i_v + (\alpha_{k(1)'} - 1)i_{k(1)'} + \sum_{v=k(1)'+1}^{r} (\alpha_v - 2)i_v.
\end{aligned}$$

In this decomposition, the coefficients of the $i$-series satisfy the conditions as in Lemma 6.1.7 and the sum of them is equal to $\alpha_1+\cdots+\alpha_r-2r+1 = \mathrm{e}(R)-1$. Therefore, $M_{n-1-(i_{k(1)}-i_{k(1)'})}$ is an Ulrich module by Corollary 6.1.9. Then we take a pair $(i_{k(2)}, i_{k(2)'}) \in \mathsf{U}$ with $i_{k(1)'} > i_{k(2)}$. By the same argument, we can show that $M_t$ is an Ulrich module for $t = n-1-(i_{k(1)} - i_{k(1)'}) - (i_{k(2)} - i_{k(2)'})$. Repeating these processes, we have the conclusion. $\square$

**Theorem 6.1.13.** *Conversely, if $M_t$ is an Ulrich module ($t \neq n-1$), then we can take a sequence of pairs $(i_{k(1)}, i_{k(1)'}), \cdots, (i_{k(b)}, i_{k(b)'}) \in \mathsf{U}$ with $i_{k(c)'} > i_{k(c+1)}$ for any $c = 1, 2, \cdots, b-1$ and*

$$t = n-1-\sum_{c=1}^{b}(i_{k(c)} - i_{k(c)'}).$$

*Proof.* Suppose $M_t$ is an Ulrich module and describe $t = d_1 i_1 + d_2 i_2 + \cdots + d_r i_r$ as in Lemma 6.1.7. Recall that $n-1 = (\alpha_1-1)i_1 + (\alpha_2-2)i_2 + \cdots + (\alpha_r-2)i_r$. Then we set the integers

$$(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r) := (d_1-(\alpha_1-1), d_2-(\alpha_2-2), \cdots, d_r-(\alpha_r-2)).$$

Since $t \neq n-1$, $(\varepsilon_1, \cdots, \varepsilon_r) \neq (0, \cdots, 0)$. By Lemma 6.1.14 and 6.1.15, we can take elements of $\mathsf{U}$ as in the statement. $\square$

So we have to show the following two lemmas.

**Lemma 6.1.14.** *One has $\varepsilon_1 \in \{-1, 0\}$ and $\varepsilon_u \in \{-1, 0, 1\}$ for $u = 2, \cdots, r$.*

*Proof.* By Lemma 6.1.7, we have $0 \leq d_u \leq \alpha_u - 1$ for $u \in [1, r]$. So $\varepsilon_1 \leq 0$ and $\varepsilon_u \leq 1$ for $u \in [2, r]$. Since $M_{n-1}$ and $M_t$ are Ulrich modules, we have $\varepsilon_1 + \cdots + \varepsilon_r = 0$ by Corollary 6.1.9.

Case 1. Assume $\varepsilon_1 \leq -2$. Since $\varepsilon_1 + \cdots + \varepsilon_r = 0$, there are $-\varepsilon_1 (\geq 2)$ components which satisfy $\varepsilon_u = 1$, $u \in [2, r]$. Set $k := -\varepsilon_1$ and suppose such components are

$$\varepsilon_{u_1} = \varepsilon_{u_2} = \cdots = \varepsilon_{u_k} = 1 \quad (u_1 < u_2 < \cdots < u_k).$$

Here, we may assume that for any $j \in [1, k-1]$ there is no component $\varepsilon_{u'} = 1$ such that $u_j < u' < u_{j+1}$. Then, by Lemma 6.1.7, there exists a subscript $v$ such that $u_1 < v < u_2$ and $\varepsilon_v \leq -1$. So there is a subscript $u_{k+1}$ with $u_k < u_{k+1}$ such that $\varepsilon_{u_{k+1}} = 1$ because $\varepsilon_1 + \cdots + \varepsilon_r = 0$. We again use Lemma 6.1.7 and there is a subscript $v'$ such that $u_k < v' < u_{k+1}$ and $\varepsilon_{v'} \leq -1$. These processes will continue infinitely, but the sequence $(\varepsilon_1, \cdots, \varepsilon_r)$ is finite. Thus, we have $\varepsilon_1 \in \{-1, 0\}$.

Case 2. Assume that there exists $u \in [2, r]$ such that $\varepsilon_u \leq -2$. As we did in Case 1, set $k = -\varepsilon_u (\geq 2)$ and $\varepsilon_{u_1} = \varepsilon_{u_2} = \cdots = \varepsilon_{u_k} = 1$.

If $\varepsilon_1 = 0$, then $d_1 = \alpha_1 - 1$. Thus, the sequence $(d_1, \cdots, d_r)$ is of the form

$$(\alpha_1 - 1, \boxed{\quad A \quad}, \alpha_{u_1} - 1, \boxed{\quad B \quad}, \alpha_{u_2} - 1, \cdots).$$

By Lemma 6.1.7, we can find $d_v$'s which satisfy $d_v \leq \alpha_v - 3$ in the both part of $A$ and $B$. Even if one is the above $d_u$ with $\varepsilon_u = d_u - (\alpha_u - 2) \leq -2$, the other one leads us to the conclusion that there is a subscript $u_{k+1}$ with $u_k < u_{k+1}$ such that $\varepsilon_{u_{k+1}} = 1$. In the same way as Case 1, we have the contradiction.

If $\varepsilon_1 = -1$, then there is a subscript $u_{k+1}$ with $u_k < u_{k+1}$ such that $\varepsilon_{u_{k+1}} = 1$ as well because $\varepsilon_1 + \cdots + \varepsilon_r = 0$. Similarly, we have the contradiction.

As the consequence, $\varepsilon_u \in \{-1, 0, 1\}$ for $u \in [2, r]$.

□

**Lemma 6.1.15.** *Let* $(\varepsilon'_1, \varepsilon'_2, \cdots, \varepsilon'_\ell) \in \{-1, 1\}^\ell$ *be the subsequence of* $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r)$ *removing every* 0 *components from* $(\varepsilon_1, \cdots, \varepsilon_r)$. *Then* $(\varepsilon'_1, \cdots, \varepsilon'_\ell)$ *takes the alternate form as*

$$(-1, +1, -1, +1, \cdots, -1, +1).$$

*Proof.* By the definition, we have $\varepsilon'_1 + \cdots + \varepsilon'_\ell = 0$ and the number of $+1$ appearing in $(\varepsilon'_1, \cdots, \varepsilon'_\ell)$ coincides with that of $-1$.

Case 1. If $\varepsilon_1 = -1$, then $\varepsilon_1 = \varepsilon'_1 = -1$. Assume the sequence $(\varepsilon'_1, \cdots, \varepsilon'_\ell)$ is not alternate. Then we can find a part appearing $+1$ continuously. This contradicts Lemma 6.1.7.

Case 2. If $\varepsilon_1 = 0$, then $\varepsilon_1 \neq \varepsilon'_1$ and $d_1 = \alpha_1 - 1$. So we set $\varepsilon_p = \varepsilon'_1 \in \{-1, 1\}$.

If $\varepsilon_p = \varepsilon'_1 = 1$, then there exists a subscript $q$ with $1 < q < p$ such that $\varepsilon_q = -1$ by Lemma 6.1.7. This contradicts the definition of $\varepsilon_p$. Therefore we have $\varepsilon_p = \varepsilon'_1 = -1$.

Assume the sequence $(\varepsilon'_1, \cdots, \varepsilon'_\ell)$ is not alternate. Then we will run into the contradiction by the same reason as in Case 1.

□

**Corollary 6.1.16.** *We suppose $M_t$ is an Ulrich module. Then we have $n - a \leq t \leq n - 1$. Furthermore, $M_{n-1}$ and $M_{n-a}$ are actually Ulrich modules.*

*Proof.* By Theorem 6.1.13, we describe $t$ as $t = n - 1 - \sum_{c=1}^{b} (i_{k(c)} - i_{k(c)'})$ where $(i_{k(1)}, i_{k(1)'}), \cdots, (i_{k(b)}, i_{k(b)'}) \in \mathsf{U}$ with $i_{k(c)'} > i_{k(c+1)}$ for any $c = 1, 2, \cdots, b - 1$. So we have

$$\sum_{c=1}^{b} (i_{k(c)} - i_{k(c)'}) = i_{k(1)} - (i_{k(1)'} - i_{k(2)}) - \cdots - (i_{k(b-1)'} - i_{k(b)}) - i_{k(b)'} \leq i_1 - i_r = a - 1.$$

Also, $M_{n-1}$ and $M_{n-a}$ are Ulrich modules because $M_1$ and $M_a$ are special CM modules (see Proposition 6.1.6). □

**Example 6.1.17.** *Suppose* $G = \frac{1}{158}(1, 57)$*. Then we have* $\frac{158}{57} = [3, 5, 2, 3, 3]$ *and* $i_1 = 57, i_2 = 13, i_3 = 8, i_4 = 3, i_5 = 1$*. Since* $\mathtt{I}_{n-1} = \mathtt{I}_{157} = \{i_1, i_2, i_4, i_5\}$*, we obtain*

$$\mathsf{U} = \{(i_1, i_2), (i_1, i_3), (i_1, i_4), (i_1, i_5), (i_2, i_3), (i_2, i_4), (i_2, i_5), (i_4, i_5)\}.$$

*By the following table and Theorem 6.1.12 and 6.1.13, we see that Ulrich modules are only*

$$M_{101}, M_{103}, M_{106}, M_{108}, M_{111}, M_{113}, M_{145}, M_{147}, M_{150}, M_{152}, M_{155} \text{ and } M_{157}.$$

| *pairs* $(i_{k(1)}, i_{k(1)'}), \cdots, (i_{k(b)}, i_{k(b)'})$ | $t = n - 1 - \sum_{c=1}^{b} (i_{k(c)} - i_{k(c)'})$ |
|---|---|
| $(i_1, i_2) = (57, 13)$ | $157 - (57 - 13) = 113$ |
| $(i_1, i_3) = (57, 8)$ | $157 - (57 - 8) = 108$ |
| $(i_1, i_4) = (57, 3)$ | $157 - (57 - 3) = 103$ |
| $(i_1, i_5) = (57, 1)$ | $157 - (57 - 1) = 101$ |
| $(i_2, i_3) = (13, 8)$ | $157 - (13 - 8) = 152$ |
| $(i_2, i_4) = (13, 3)$ | $157 - (13 - 3) = 147$ |
| $(i_2, i_5) = (13, 1)$ | $157 - (13 - 1) = 145$ |
| $(i_4, i_5) = (3, 1)$ | $157 - (3 - 1) = 155$ |
| $\{(i_1, i_2), (i_4, i_5)\}$ | $157 - (57 - 13) - (3 - 1) = 111$ |
| $\{(i_1, i_3), (i_4, i_5)\}$ | $157 - (57 - 8) - (3 - 1) = 106$ |
| $\{(i_2, i_3), (i_4, i_5)\}$ | $157 - (13 - 8) - (3 - 1) = 150$ |

### 6.1.3 The number of minimal generators for each MCM modules

In this subsection, we will consider the following question.

**Question 6.1.18.** *For a cyclic quotient surface singularity* $R$*, fix the integer* $1 \leq m \leq \mathrm{e}(R)$*. How many indecomposable MCM modules which satisfy* $\mu_R(M_t) = m$ *are there? In particular, how many indecomposable Ulrich modules are there ??*

For simplicity, we denote the number of indecomposable MCM modules $M_t$ which satisfies $\mu_R(M_t) = m$ by $\mathsf{N}_m$. Namely,

$$\mathsf{N}_m = \#\{M_t \in \mathrm{CM}(R) \mid \mu_R(M_t) = m\}.$$

Firstly, we show that there actually exists an MCM $R$-module which satisfies $\mu_R(M_t) = m$ for any $m = 1, \cdots, \mathrm{e}(R)$. That is, $\mathsf{N}_m \geq 1$ for any $m = 1, \cdots, \mathrm{e}(R)$ (see Proposition 6.1.20).

**Proposition 6.1.19.** *Fix an integer* $m = 1, \cdots, \mathrm{e}(R)$*. Assume there exists an MCM* $R$*-module* $M_t$ *such that* $\mu_R(M_t) = m$ *and* $t$ *is described as* $t = d_{1,t}i_1 + d_{2,t}i_2 + \cdots + d_{r,t}i_r$*. Then for every* $\ell = m, m-1, \cdots, 1$*, there is an MCM* $R$*-module* $M_{t'}$ *such that* $\mu_R(M_{t'}) = \ell$

*Proof.* Taking $i_u \in \mathtt{I}_t$, we have $\mu_R(M_{t-i_u}) = m - 1$ by Theorem 6.1.8. Similarly we take $i_v \in \mathtt{I}_{t-i_u}$ and have $\mu_R(M_{t-i_u-i_v}) = m-2$. Since $d_{1,t}+d_{2,t}+\cdots+d_{r,t} = m-1$ by the hypothesis, we can repeat the above process $m - 1$ times. $\square$

**Proposition 6.1.20.** *For every integer $m = 1, \cdots, \mathrm{e}(R)$, there is an MCM R-module $M_t$ such that $\mu_R(M_t) = m$.*

*Proof.* Since there exists an Ulrich module, we apply Proposition 6.1.19 to $m = \mathrm{e}(R)$ and have the conclusion. $\square$

From these results we have the following relation among some classes of MCM $R$-modules.

**Corollary 6.1.21.** *Let R be a cyclic quotient surface singularity. Then*

(1) *If* $\mathrm{e}(R) = 2$, $\mathrm{CM}(R) = \mathrm{SCM}(R) = \mathrm{add}(R) \sqcup \mathrm{UCM}(R)$ *(cf. Example 5.3.14),*
(2) *If* $\mathrm{e}(R) = 3$, $\mathrm{CM}(R) = \mathrm{SCM}(R) \sqcup \mathrm{UCM}(R)$,
(3) *If* $\mathrm{e}(R) > 3$, $\mathrm{CM}(R) \supsetneq \mathrm{SCM}(R) \sqcup \mathrm{UCM}(R)$.

*where* $\mathrm{SCM}(R)$ *(resp.* $\mathrm{UCM}(R)$ *) is the full subcategory of* $\mathrm{CM}(R)$ *consisting of special (resp. Ulrich) CM R-modules.*

*Remark* 6.1.22. These are typical results for cyclic quotient surface singularities.

(1) Proposition 6.1.20 doesn't hold in a higher dimension. For example, we consider the action of $G = \langle \mathrm{diag}(-1,-1,-1) \rangle$ on $S = k[[x,y,z]]$. Then the invariant subring $R = S^G$ is of finite CM representation type and finitely many indecomposable MCMs are $R, \omega_R$ and $\Omega\omega_R$ (cf. [Yos, LW]). Also, we have $\mathrm{e}(R) = 4$ but $\mu_R(\omega_R) = 3$ and $\mu_R(\Omega\omega_R) = 8$.

(2) Corollary 6.1.21 (2) doesn't hold for non-cyclic cases. For example, let $R$ be the invariant subring as in Example 5.1.2. Note that $\mathrm{e}(R) = 3$. We can find some indecomposable MCM $R$-modules which are neither special CM modules nor Ulrich modules (see Example 6.2.5 and [IW]).

### 6.1.4 The number of Ulrich modules

In the previous subsection, we investigated the number $\mathsf{N}_m$ and showed $\mathsf{N}_m \geq 1$ for any $m = 1, \cdots, \mathrm{e}(R)$. In this subsection, we will focus on $\mathsf{N}_{\mathrm{e}(R)}$, that is, the number of Ulrich modules.

Firstly, we should remark that Corollary 6.1.16 gives an upper bound of $\mathsf{N}_{\mathrm{e}(R)}$. Namely, we have $\mathsf{N}_{\mathrm{e}(R)} \leq a$. Next, we will give other bounds in terms of the number of irreducible exceptional curves.

**Theorem 6.1.23.** *Suppose R is a cyclic quotient surface singularity whose number of irreducible exceptional curves (= that of non-free indecomposable special CM modules) is r:*

$$(-\alpha_1) — (-\alpha_2) — \cdots\cdots — (-\alpha_r)$$

*Then we have* $r \leq \mathsf{N}_{\mathrm{e}(R)} \leq 2^{r-1}$. *Especially,* $\mathsf{N}_{\mathrm{e}(R)} = 2^{r-1}$ *holds only if* $\alpha_u > 2$ *for every* $u = 2, \cdots, r-1$, *and* $\mathsf{N}_{\mathrm{e}(R)} = r$ *holds only if* $\alpha_2 = \cdots = \alpha_{r-1} = 2$.

*Proof.* By Theorem 6.1.12 and 6.1.13, $M_t$ is an Ulrich module if and only if $t = n-1$ or $t$ is described by a sequence of elements which satisfy

$$(\clubsuit) \quad (i_{k(1)}, i_{k(1)'}), \cdots, (i_{k(b)}, i_{k(b)'}) \in \mathsf{U} \text{ with } i_{k(c)'} > i_{k(c+1)} \text{ for any } c = 1, 2, \cdots, b-1.$$

Note that if we take different sequences of elements in $\mathsf{U}$ which satisfy ($\clubsuit$), then corresponding subscripts are also different, because the sequence $(d_{1,t}, \cdots, d_{r,t})$ as in Lemma 6.1.7 is unique for each subscript $t$. Thus, $\mathsf{N}_{\mathrm{e}(R)} - 1$ is equal to the number of sequences satisfying the condition ($\clubsuit$). Therefore we may show the maximal (resp. minimal) number of such sequences is equal to $2^{r-1} - 1$ (resp. $r-1$). Clearly, we should consider the case where $\mathtt{I}_{n-1} = \{i_1, \cdots, i_r\}$ to obtain the upper bound of $\mathsf{N}_{\mathrm{e}(R)}$. (Notice that the element $i_r$ doesn't influence the number of elements in $\mathsf{U}$.)

To make the situation clear, we set

$$\mathtt{I}^k := \{i_k, \cdots, i_r\}, \quad \mathsf{U}^k := \{(i_s, i_u) \mid i_s \in \mathtt{I}^k \text{ and } i_s > i_u\}.$$

Furthermore, we denote the set of sequences of $\mathsf{U}^k$ which satisfies ($\clubsuit$) by $\mathcal{U}^{r-k}$ and the number of elements in $\mathcal{U}^{r-k}$ by ${}^\#\mathcal{U}^{r-k}$. In this situation, we may show ${}^\#\mathcal{U}^{r-1} = 2^{r-1} - 1$ and the following inductive argument asserts the conclusion.

The case where $k = r$ is easy. ($R$ is a Veronese subring and $\mathsf{U}^r = \emptyset$.) Assume we have ${}^\#\mathcal{U}^{r-k} = 2^{r-k} - 1$ for $k = 2, \cdots, r$. Then we can obtain elements in $\mathcal{U}^{r-1}$ as follows.

- $(i_1, i_2), (i_1, i_3), \cdots, (i_1, i_r)$,
- elements in $\mathcal{U}^{r-2}$,
- combine $(i_1, i_2)$ and elements in $\mathcal{U}^{r-3}$,

$\vdots$

- combine $(i_1, i_{r-3})$ and elements in $\mathcal{U}^2$,
- combine $(i_1, i_{r-2})$ and elements in $\mathcal{U}^1 = \{(i_{r-1}, i_r)\}$.

By the hypothesis, the number of these elements is less than or equal to

$$(r-1) + (2^{r-2} - 1) + (2^{r-3} - 1) + \cdots + (2^1 - 1) = 2^{r-2} + 2^{r-3} + \cdots + 2 + 1 = 2^{r-1} - 1.$$

In order to obtain the lower bound, we consider the case where $\mathtt{I}_{n-1} = \{i_1\}$, and it is easy to see $\mathsf{N}_{\mathrm{e}(R)} = r$. $\square$

*Remark* 6.1.24. We could obtain two upper bounds $\mathsf{N}_{\mathrm{e}(R)} \leq a$ or $2^{r-1}$. But it depends on a case whether which one is a better bound.

For the case where $r$ is small, we can compute $\mathsf{N}_{\mathrm{e}(R)}$ explicitly. (Check the bounds of $\mathsf{N}_{\mathrm{e}(R)}$ for the following examples.)

**Example 6.1.25.** *Suppose $R$ is a cyclic quotient surface singularity whose dual graph of the following form $\mathsf{C}$.*

*(1) The case where*

$$\mathsf{C}: \ (-\alpha) \text{———} (-\beta) \qquad (\alpha, \beta \geq 2).$$

*Then we have $\mathsf{N}_{\mathrm{e}(R)} = 2$.*

*(2) The case where*

$$\mathsf{C}:\ \bigcirc\!\!\!\!\!\!-\alpha \;—\; -\beta \;—\; -\gamma \qquad (\alpha, \beta, \gamma \geq 2)\,.$$

*(2-1) If* $\beta = 2$, *then we have* $\mathsf{N}_{\mathrm{e}(R)} = 3$ .

*(2-2) If* $\beta > 2$, *then we have* $\mathsf{N}_{\mathrm{e}(R)} = 4$ .

*(3) The case where*

$$\mathsf{C}:\ -\alpha \;—\; -\beta \;—\; -\gamma \;—\; -\delta \qquad (\alpha, \beta, \gamma, \delta \geq 2)\,.$$

*(3-1) If* $\beta = 2, \gamma = 2$, *then we have* $\mathsf{N}_{\mathrm{e}(R)} = 4$ .

*(3-2) If* $\beta = 2, \gamma > 2$, *then we have* $\mathsf{N}_{\mathrm{e}(R)} = 6$ .

*(3-3) If* $\beta > 2, \gamma = 2$, *then we have* $\mathsf{N}_{\mathrm{e}(R)} = 6$ .

*(3-4) If* $\beta > 2, \gamma > 2$, *then we have* $\mathsf{N}_{\mathrm{e}(R)} = 8$ .

*Proof.* We only show the case (3-2). The other cases are similar.

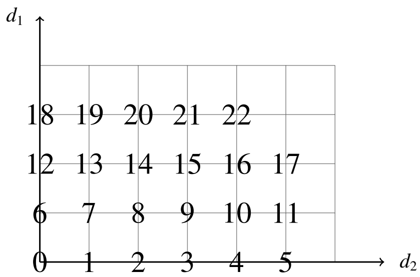
Let $i_1, \cdots, i_4$ be the $i$-series corresponding to each exceptional curve. Since $\beta = 2$ and $\gamma > 2$, we have $\mathtt{I}_{n-1} = \{i_1, i_3\}$. The statement follows from Theorem 6.1.12 and 6.1.13 because we can take the following pairs: $\{(i_1, i_2)\}, \{(i_1, i_3)\}, \{(i_1, i_4)\}, \{(i_3, i_4)\}, \{(i_1, i_2), (i_3, i_4)\}$. □

### 6.1.5 Examples

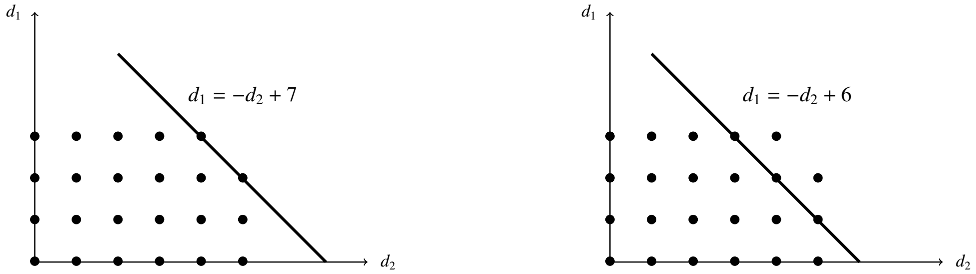
We finish this section with some special examples. In particular, we determine the number $\mathsf{N}_m$ completely. From Theorem 6.1.8, special CM modules behave like a "basis". Thus, we can identify each MCM $R$-module $M_t$ with the lattice point $(d_{1,t}, \cdots, d_{r,t}) \in \mathbb{Z}^r$.

**Example 6.1.26.** *Suppose* $G = \frac{1}{23}(1, 6)$ *and* $R = k[[x, y]]^G$. *Then* $\frac{23}{6} = 4 - \frac{1}{6} = [4, 6]$ *and* $\mathrm{e}(R) = 8$. *The i-series are* $i_1 = 6, i_2 = 1$.

*In this situation, we identify each subscript* $t = 0, 1, \cdots, 22$ *with the lattice point* $(d_{1,t}, d_{2,t})$. *For example, since* $20 = (1 + 1) + (6 + 6 + 6)$, *it corresponds to the lattice point* $(2, 3)$.



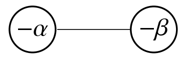

*By the theorem, we have* $d_{1,t} = -d_{2,t} + \mu(M_t) - 1$ *and suppose* $\mu_R(M_t) = \mathrm{e}(R) = 8$, *then an MCM module whose corresponding lattice point is on the line* $d_{1,t} = -d_{2,t} + 7$ *is an Ulrich module. In this case, there are two Ulrich modules* ($M_{17}$ *and* $M_{22}$).




*Similarly we suppose $\mu_R(M_t) = 7$ then the above figure implies* $\mathsf{N}_7 = 3$.

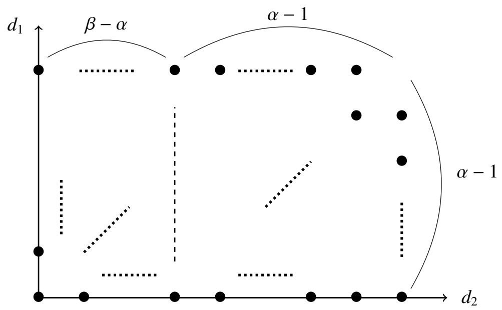
In this way, we can compute the number $\mathsf{N}_m$. In general, we have the following.

**Example 6.1.27.** *Take integers $\alpha, \beta \geq 2$. Suppose $G = \frac{1}{n}(1, a)$ which satisfies $n/a = \alpha - \frac{1}{\beta}$. Then $R = S^G$ is the cyclic quotient surface singularity whose dual graph is*




*and* $\mathrm{e}(R) = \alpha + \beta - 2$. *The i-series are $i_1 = \beta$, $i_2 = 1$.*

*If $\alpha \leq \beta$, we have*



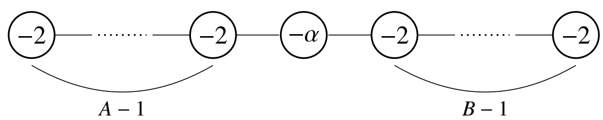

$$\begin{array}{lcllcllcl}
\mathsf{N}_{\alpha+\beta-2} & = & 2, & \mathsf{N}_{\beta-1} & = & \alpha, & \mathsf{N}_{\alpha-2} & = & \alpha - 2, \\
\mathsf{N}_{\alpha+\beta-3} & = & 3, & & \vdots & & & \vdots & \\
& \vdots & & \mathsf{N}_{\alpha} & = & \alpha, & \mathsf{N}_2 & = & 2, \\
\mathsf{N}_{\beta} & = & \alpha, & \mathsf{N}_{\alpha-1} & = & \alpha - 1, & \mathsf{N}_1 & = & 1.
\end{array}$$

*The case $\alpha \geq \beta$ is similar. (replace $\alpha$ by $\beta$ and vice versa.)*

*Proof.* By the above figure of lattice points, we easily count the number of desired MCM modules. □

We can also determine $\mathsf{N}_m$ for the following situation.

**Example 6.1.28.** *Consider a cyclic quotient surface singularity R whose dual graph is*



*where $\alpha \geq 2$ and $A, B \geq 1$. Then* $\mathrm{e}(R) = \alpha$ *and we have the following.*

$$\begin{array}{rclcrcl} \mathsf{N}_\alpha &=& AB, & & \mathsf{N}_3 &=& AB, \\ \mathsf{N}_{\alpha-1} &=& AB, & & \mathsf{N}_2 &=& A+B-1, \\ &\vdots& & & \mathsf{N}_1 &=& 1. \\ \mathsf{N}_4 &=& AB, & & & & \end{array}$$

*Proof.* Let $i_1, \cdots, i_{A-1}, i_A, i_{A+1}, \cdots, i_{A+B-1}$ be the $i$-series corresponding to each exceptional curve. Especially, $i_A$ corresponds to the exceptional curve whose self-intersection number is $-\alpha$. Thus, we have $\mathtt{I}_{n-1} = \{i_1, i_A\}$ and $\mathsf{U} = \{(i_1, i_2), \cdots, (i_1, i_{A+B-1}), (i_A, i_{A+1}), \cdots, (i_A, i_{A+B-1})\}$. It is easy to see $i_A = B, i_{A+1} = B-1, \cdots, i_{A+B-1} = 1$. Therefore, we can see an MCM $R$-module $M_t$ whose subscript is appearing in the following table is an Ulrich module by Theorem 6.1.12 and 6.1.13.

$$\begin{array}{llcll} n-1, & n-1-(i_1-i_2), & \cdots & n-1-(i_1-i_{A-1}), & n-1-(i_1-i_A), \\ n-2 = n-1-(i_A-i_{A+1}), & n-2-(i_1-i_2), & \cdots & n-2-(i_1-i_{A-1}), & n-1-(i_1-i_{A+1}), \\ \vdots & \vdots & & \vdots & \vdots \\ n-B = n-1-(i_A-i_{A+B-1}), & n-B-(i_1-i_2), & \cdots & n-B-(i_1-i_{A-1}), & n-1-(i_1-i_{A+B-1}). \end{array}$$

Thus, we obtain $\mathsf{N}_\alpha = AB$. By the same arguments as in Proposition 6.1.19, we can determine $\mathsf{N}_m$ for $m = 3, 4, \cdots, \alpha-1$. The value of $\mathsf{N}_2$ follows from the special McKay correspondence. □

## 6.2 Hilbert-Kunz multiplicities for quotient surface singularities

By using arguments similar to those in Chapter 5, we can investigate the Hilbert-Kunz multiplicity for quotient surface singularities. Again, we suppose $G$ is a finite subgroup of $\mathrm{GL}(2, k)$ which contains no pseudo-reflections and $S := k[[x, y]]$ be the power series ring. We assume that $(|G|, \mathrm{char}\ k) = 1$. We will consider an invariant subring $R := S^G$. Let $V_0 = k, V_1, \cdots, V_n$ be the complete set of irreducible representations of $G$ and set the indecomposable MCM $R$-modules $M_t := (S \otimes_k V_t)^G \quad (t = 0, 1, \cdots, n)$. In this situation, the Hilbert-Kunz multiplicity is determined by the next formula.

**Theorem 6.2.1.** (*cf. [WY1, Theorem 2.7]*) *Let the notation be same as above. Then*

$$e_{HK}(R) = \frac{1}{|G|} \ell_S(S/\mathfrak{m}S).$$

We can deform it as follows. Since $S \cong R^{\oplus d_0} \oplus M_1^{\oplus d_1} \oplus \cdots \oplus M_n^{\oplus d_n}$ ($d_t = \mathrm{rank}_R\, M_t = \dim_k V_t$),

$$\ell_s(S/\mathfrak{m}S) = \dim_k(S \otimes R/\mathfrak{m}) = \mu_R(S) = \sum_{t=0}^{n} d_t\, \mu_R(M_t),$$

where $\mu_R(M)$ stands for the number of minimal generator of a finitely generated $R$-module $M$. Thus,

$$e_{\mathrm{HK}}(R) = \frac{1}{|G|}\sum_{t=0}^{n} d_t\, \mu_R(M_t).$$

Note that this formula is also obtained by the isomorphism (3.2.1) and Theorem 3.3.1. As we showed in subsection 5.1, we can calculate $\mu_R(M_t)$ by using the AR quiver (or the McKay quiver). Thus, we can determine the value of the Hilbert-Kunz multiplicity by a relatively easy process.

**Example 6.2.2.** (*[WY1, Theorem 5.4], see also [HL, Corollary 20], [Tuc, Corollary 4.15]*) *Suppose $G$ is a finite subgroup of $\subset \mathrm{SL}(2,k)$. Then $R$ is a two-dimensional rational double point as in Section 5.4. For an indecomposable $R$-module $M_t$, we have $\mu_R(M_t) = 2d_t$ for $t \neq 0$ and clearly $\mu_R(R) = d_0 = 1$. Thus, we have*

$$e_{HK}(R) = \frac{1}{|G|}(2\sum_{t=0}^{n} d_t^2 - 1) = \frac{1}{|G|}(2|G| - 1) = 2 - \frac{1}{|G|}.$$

**Example 6.2.3.** *Let $G := \langle\ \sigma = \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}\ \rangle$ be a cyclic group of order 8 where $\zeta_8$ is a primitive 8-th root of unity. We consider irreducible representations of $G$;*

$$V_t : \sigma \mapsto \zeta_8^{-t} \quad (t = 0, 1, \cdots, 7).$$

*We consider the invariant subring $R := S^G$ and its indecomposable MCM module $M_t := (S \otimes_k V_t)^G$. By the counting argument of AR quiver, we obtain the following (the meaning of this picture, see Section 5.3).*

$$\begin{tikzcd}
0 \\
7 \arrow[u] \\
6 \arrow[u] \\
5 \arrow[u] \\
4 \arrow[u] \\
3 \arrow[u] \\
2 \arrow[u] \arrow[r] & 7 \arrow[r] & 4 \\
1 \arrow[u] \arrow[r] & 6 \arrow[u] \arrow[r] & 3 \arrow[u] \\
0 \arrow[u] \arrow[r] & 5 \arrow[u] \arrow[r] & 2 \arrow[u] \arrow[r] & 7 \arrow[r] & 4 \arrow[r] & 1 \arrow[r] & 6 \arrow[r] & 3 \arrow[r] & 0
\end{tikzcd}$$

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\mu(M_t)$ | 1 | 2 | 2 | 3 | 3 | 2 | 3 | 3 |
| rank $M_t$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

*So we have $e_{HK}(R) = \dfrac{19}{8}$.*

**Example 6.2.4.** *In Example 6.1.27 and 6.1.28, we could obtain $\mathsf{N}_m$ for $m = 1, \cdots, \mathrm{e}(R)$. Thus, we can compute the Hilbert-Kunz multiplicity $\mathrm{e}_{HK}(R)$.*

*That is, let $R$ be as in Example 6.1.27 (resp. Example 6.1.28). Then we have $\mathrm{e}_{HK}(R) = \frac{1}{2n}\{(\alpha\beta - 2)(\alpha + \beta) + 2\}$ (resp. $\mathrm{e}_{HK}(R) = \frac{1}{2n}\{AB(\alpha - 2)(\alpha + 3) + 4(A + B) - 2\}$).*

**Example 6.2.5.** *Let the notation be same as Example 5.1.2. By the counting argument of AR quiver, we have the number of minimal generators as follows.*

| $(i, j)$ | $(0,0)$ | $(1,0)$ | $(3,0)$ | $(4,0)$ | $(2,2)$ | $(0,1)$ | $(1,1)$ | $(3,1)$ |
|---|---|---|---|---|---|---|---|---|
| $\mu(M_{i,j})$ | 1 | 3 | 3 | 3 | 4 | 3 | 2 | 2 |
| rank $M_{i,j}$ | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |

| | $(4,1)$ | $(2,0)$ | $(0,2)$ | $(1,2)$ | $(3,2)$ | $(4,2)$ | $(2,1)$ |
|---|---|---|---|---|---|---|---|
| | 2 | 5 | 2 | 3 | 3 | 3 | 6 |
| | 1 | 2 | 1 | 1 | 1 | 1 | 2 |

*Thus, we have* $e_{HK}(R) = \dfrac{60}{24} = \dfrac{5}{2}$.

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