COTORSION PAIRS ON TRIANGULATED AND EXACT CATEGORIES

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1. Introduction

There are two important notions in triangulated categories which are deeply studied in the representation theory.

The first one is t-structure, which is introduced by Beilinson, Bernstein and Deligne [BBD] in their study of perverse sheaves on an algebraic varieties. One of the important properties is that

(*) The heart of a t-structure is an abelian category.

A t-structure also provides a homological functor H with values in the heart from the original triangulated category. A typical example of a t-structure, which we call the standard t-structure, is a pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ for the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$, where $\mathcal{T}^{\leq 0}$ consists of complexes with vanishing cohomologies in positive degrees, and $\mathcal{T}^{\geq 0}$ consists of complexes with vanishing cohomologies in negative degrees. Moreover, if we have a derived equivalence between two algebra A and B, the we have a t-structure in standard t-structure of A by this equivalence. Therefore, t-structure is important to study the derived equivalence.

The second one are cluster tilting subcategories. They were introduced in [BMRRT] as a generalization of tilting theory for hereditary algebras, in order to categorify Fomin-Zelevinsky's cluster algebras [FZ]. It was proved that cluster tilting subcategories always exist in certain triangulated categories called cluster categories. Also cluster tilting subcategories of module categories were studied by Iyama in [I1] By Koenig and Zhu [KZ]

(**) The quotient of a triangulated category by a cluster tilting subcategory is abelian.

Moreover, this quotient is Iwanaga-Gorenstein of dimension at most one (see [KZ, Theorem 4.3]).

These two structures can be unified to the notion of *torsion pairs* on triangulated categories, which is a pair on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{U}, \mathcal{V})$ of full subcategories such that

- $\mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V}) = 0$.
- Any object $D \in \mathcal{D}$ admits a triangle $U \to D \to V \to U[1]$ such that $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

The notion is classical, going back to the example of torsion and torsion free abelian groups. Now the concept has been widely used in the representation theory, since it is also important in the study of the algebraic structure of triangulated categories.

By a technical reason, we consider a cotorsion pair instead of torsion pair. The notion of cotorsion pair is just an analog of torsion pair on triangulated category: a pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{T}$ is called a *cotorsion pair* if $(\mathcal{U}, \mathcal{V}[1])$ is a torsion pair. Nakaoka introduced the notion of *hearts* of cotorsion pairs on triangulated categories , as a generalizatoin of the heart of t-structure, and showed that the hearts are abelian categories [N]. His construction of hearts generalizes the above results (*) and (**) for t-structure and cluster tilting subcategory. Moreover, he generalized these results to a more general setting called twin cotorsion pair [N1].

Motivated by Nakaoka's results of cotorsion pairs on triangulated category, in this paper, we consider cotorsion pairs on Quillen's exact category, which is a generalization of abelian categories and there are many important examples of it. The cotorsion pairs on abelian categories goes back to Salce in [S], and it has been deeply studied in the representation theory during these years, especially in tilting theory and Cohen-Macaulay modules [AR] and [AB] (see [EJ, GT, HuI, Ri] for more examples).

By Happel [H, Theorem 2.6], the stable category of a Frobenius category (which is a specail case of exact category) has a structure of a triangulated category. Most triangulated categories appearing in representation theory turn out to be in fact algebraic (i.e. stable categories of Frobenius categories). Moreover, if we have a cotorsion pair on a Frobenius category, then it is still a cotorsion pair on the stable category of this Frobenius category.

In this article, we introduce the heart $\underline{\mathcal{H}}$ of a cotorsion pair $(\mathcal{U}, \mathcal{V})$ on the exact category $\mathcal{B}$ with enough projectives enough injectives (see subsection 2.1 for more details). We first prove that $\underline{\mathcal{H}}$ is abelian. We will apply this result to the case of cluster tilting subcategory. A more general setting, which is called twin cotorsion pair, is also discussed. We show several results for the hearts of twin cotorsion pairs. Then we construct a half exact functor from $\mathcal{B}$ to $\underline{\mathcal{H}}$, and as an application, we give a sufficient condition when two hearts are equivalent to each other. At last, by using this functor, we show that the heart is equivalent to functor category over the coheart of $(\mathcal{U},\mathcal{V})$.

1.1. **Hearts of twin cotorsion pairs.** We begin with the central concept of our results: a *cotorsion pair* in an Krull-Schmidt exact category $\mathcal{B}$ with enough projectives and enough injectives (see for example [KS, A.1]).

Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{B}$ which are closed under direct summands. We call $(\mathcal{U},\mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U},\mathcal{V})=0$.
(b) For any object $B\in\mathcal{B}$, there exits two short exact sequences

$$V_B \rightarrowtail U_B \twoheadrightarrow B, \quad B \rightarrowtail V^B \twoheadrightarrow U^B$$

satisfying $U_B, U^B\in\mathcal{U}$ and $V_B, V^B\in\mathcal{V}$.

Since $\mathcal{B}$ has enough projectives and injectives, we always have two cotorsion pairs $(\mathcal{P},\mathcal{B})$ and $(\mathcal{B},\mathcal{I})$.

Now let us define the heart of a cotorsion pair.

Definition 1.1. Let

$$\mathcal{B}^+ := \{B\in\mathcal{B} \mid U_B\in\mathcal{V}\}, \quad \mathcal{B}^- := \{B\in\mathcal{B} \mid V^B\in\mathcal{U}\}.$$

Let $\mathcal{H}=\mathcal{B}^+\cap\mathcal{B}^-$, we define the *heart* of $(\mathcal{U},\mathcal{V})$ as the quotient category

$$\underline{\mathcal{H}} := (\mathcal{B}^+\cap\mathcal{B}^-)/(\mathcal{U}\cap\mathcal{V}).$$

Now we introduce the following main theorem in Section 2.

Theorem 1.2. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on an exact category $\mathcal{B}$ with enough projectives and injectives. Then $\underline{\mathcal{H}}$ is abelian.*

Now we apply this theorem to the cluster tilting subcategory $\mathcal{M}$ of $\mathcal{B}$. In our words, $\mathcal{M}$ is cluster tilting if and only if $(\mathcal{M},\mathcal{M})$ is a cotorsion pair. In this case, the heart of $(\mathcal{M},\mathcal{M})$ is $\mathcal{B}/\mathcal{M}$. Therefore, we have the following corollary which is an analog of the result in [KZ] for triangulated category (see Proposition 2.56 for details)..

Corollary 1.3. [DL] *Let $\mathcal{M}$ be a cluster tiling subcategory on $\mathcal{B}$, The quotient category $\mathcal{B}/\mathcal{M}$ is abelian.*

We also prove more general results for twin cotorsion pairs defined as follows.

Definition 1.4. A pair of cotorsion pairs $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ is called a *twin cotorsion pair* if $\mathcal{S}\subseteq\mathcal{U}$.

The notion of semi-abelian category (see Definition 2.27) was introduced by Rump [R], as a special class of preabelian categories. In this setting, we still have the following results.

Theorem 1.5. *Let $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ be a twin cotorsion pair on $\mathcal{B}$. Then $\underline{\mathcal{H}}$ is semi-abelian.*

There are two nice classes of semi-abelian categories called *integral* (see [R, §2] for examples) and *almost abelian* (any torsion class associated with a tilting module is almost abelian [CF]). We give sufficient conditions for hearts to be integral (see Theorem 2.34) or almost abelian (see Theorem 2.38).

Finally, we consider a special twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{T},\mathcal{V})$, note that this is an analog of TTF theory and recollement. Then we have a theorem (see Theorem 2.49) which gives a more explicit description of the heart and can be regarded as an analog of [BM, Theorem 5.7].

1.2. **Associated half exact functors.** It is natural to ask whether we can find the relationship between the hearts and the original exact categories. For t-structures, there is a natural cohomological functor from triangulated category to the hearts. For cluster tilting subcategories, we have natural functors from triangulated category to the quotient category over them. Abe and Nakaoka unified these two functor by constructing a cohomological functor from triangulated categories to the hearts of cotorsion pairs [AN]. The main result of Section 3 is to answer this question by constructing an associated half exact functor H from the exact category $\mathcal{B}$ to the heart $\underline{\mathcal{H}}$.

we recall the definition of the half exact functor on $\mathcal{B}$ (also see [O, p.24]).

Definition 1.6. A covariant functor F from $\mathcal{B}$ to an abelian category $\mathcal{A}$ is called *half exact* if for any short exact sequence $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ in $\mathcal{B}$, the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact in $\mathcal{A}$.

Denote $\mathrm{add}(\mathcal{U} * \mathcal{V})$ by $\mathcal{K}$, We will prove the following theorem (see Theorem 3.11 and Proposition 3.12 for details).

Theorem 1.7. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{B}$, there exists an associated half exact functor $H : \mathcal{B} \to \underline{\mathcal{H}}$ such that*

(a) *$H|_{\mathcal{H}} = \pi|_{\mathcal{H}}$ where $\pi : \mathcal{B} \to \mathcal{B}/(\mathcal{U} \cap \mathcal{V})$ is the natural functor.*
(b) *$H(B) = 0$ if and only if $B \in \mathcal{K}$.*

We denote by $\Omega : \mathcal{B}/\mathcal{P} \to \mathcal{B}/\mathcal{P}$ the syzygy functor and by $\Omega^- : \mathcal{B}/\mathcal{I} \to \mathcal{B}/\mathcal{I}$ the cosyzygy functor. We will prove that any half exact functor F which satisfies $F(\mathcal{P}) = 0$ and $F(\mathcal{I}) = 0$ has a similar property as cohomological functors on triangulated categories.

In particular, as an application, we have the following corollary (see Corollary 3.16 for details).

Corollary 1.8. *For any short exact sequence*

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

in $\mathcal{B}$, there exist morphisms $h : C \to \Omega^- A$ and $h' : \Omega C \to A$ such that the sequence

$$\cdots \xrightarrow{H(\Omega h')} H(\Omega A) \xrightarrow{H(\Omega f)} H(\Omega B) \xrightarrow{H(\Omega g)} H(\Omega C) \xrightarrow{H(h')} H(A) \xrightarrow{H(f)} H(B)$$
$$\xrightarrow{H(g)} H(C) \xrightarrow{H(h)} H(\Omega^- A) \xrightarrow{H(\Omega^- f)} H(\Omega^- B) \xrightarrow{H(\Omega^- g)} H(\Omega^- C) \xrightarrow{H(\Omega^- h)} \cdots$$

is exact in $\underline{\mathcal{H}}$.

The half exact functor we construct gives us a way to find out the relationship between different hearts. Let $k \in \{1, 2\}$, $(\mathcal{U}_k, \mathcal{V}_k)$ be a cotorsion pair on $\mathcal{B}$ and $\mathcal{W}_k = \mathcal{U}_k \cap \mathcal{V}_k$. Let $\mathcal{H}_k/\mathcal{W}_k$ be the heart of $(\mathcal{U}_k, \mathcal{V}_k)$ and H_k be the associated half exact functor. If $\mathcal{W}_1 \subseteq \mathcal{K}_2$, then H_2 induces a functor $\beta_{12} : \mathcal{H}_1/\mathcal{W}_1 \to \mathcal{H}_2/\mathcal{W}_2$, and we have the following proposition (see Proposition 3.21, 3.20 and Theorem 3.23 for details).

Theorem 1.9. *Let $(\mathcal{U}_1, \mathcal{V}_1)$, $(\mathcal{U}_2, \mathcal{V}_2)$ be cotorsion pairs on $\mathcal{B}$. If $\mathcal{W}_1 \subseteq \mathcal{K}_2 \subseteq \mathcal{K}_1$, then*

(a) *We have a natural isomorphism $\beta_{21}\beta_{12} \simeq \mathrm{id}_{\mathcal{H}_1/\mathcal{W}_1}$ of functors.*
(b) *$(\mathcal{H}_2 \cap \mathcal{K}_1)/\mathcal{W}_2$ is a Serre subcategory.*
(c) *Let $\overline{\mathcal{H}}_2$ be the localization of $\mathcal{H}_2/\mathcal{W}_2$ by $(\mathcal{H}_2 \cap \mathcal{K}_1)/\mathcal{W}_2$, then we have an equivalence $\mathcal{H}_1/\mathcal{W}_1 \simeq \overline{\mathcal{H}}_2$.*

This implies the following corollary which gives a sufficient condition when two different hearts (see Corollary 3.22).

Corollary 1.10. *If $\mathcal{K}_1 = \mathcal{K}_2$, then we have an equivalence $\mathcal{H}_1/\mathcal{W}_1 \simeq \mathcal{H}_2/\mathcal{W}_2$ between two hearts.*

1.3. **Hearts are equivalent to functor categories.** By using the half exact functor, we give an equivalence between hearts and the functor categories over cohearts. For the details of functor category, see [A] and also [IY, Definition 2.9].

Let $\mathcal{T}$ be a triangulated category. For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{T}$, We denote by ${}^{\perp}\mathcal{U}$ the subcategory such that $X \in {}^{\perp}\mathcal{U}$ if $\mathrm{Hom}_{\mathcal{T}}(X, \mathcal{U}) = 0$. We introduce the notion of *cohearts* of a cotorsion pair, denote by

$$\mathcal{C} = \mathcal{U}[-1] \cap {}^{\perp}\mathcal{U}.$$

This is a generalization of coheart of a co-t-structure, which plays an important role in [KY]. We have the following theorem in triangulated category.

Theorem 1.11. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on a triangulated category $\mathcal{T}$. If $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, then the heart of $(\mathcal{U}, \mathcal{V})$ has enough projectives $H(\mathcal{C})$, and moreover it is equivalent to the functor category* $\mathrm{mod}\,\mathcal{C}$.

This generalizes [BR, Theorem 3.4] which is for t-structure. One standard example of this theorem is the following: let A be a Noetherian ring with finite global dimension, then the standard t-structure of $D^b(\operatorname{mod} A)$ has a heart $\operatorname{mod} A$ with coheart $\operatorname{proj} A$, and we have an equivalence $\operatorname{mod} A \simeq \operatorname{mod}(\operatorname{proj} A)$ in this case.

For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on an exact category $\mathcal{B}$, We denote by ${}^{\perp_1}\mathcal{U}$ the subcategory such that $X \in {}^{\perp_1}\mathcal{U}$ if $\operatorname{Ext}^1_{\mathcal{B}}(X, \mathcal{U}) = 0$. We denote by

$$\mathcal{C} = \mathcal{U} \cap {}^{\perp_1}\mathcal{U}$$

the *coheart* of $(\mathcal{U}, \mathcal{V})$. We have the following theorem in exact category.

Theorem 1.12. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on an exact category $\mathcal{B}$ with enough projectives and injectives, if for any any object $U \in \mathcal{U}$, there exists an exact sequence $U' \rightarrowtail C \twoheadrightarrow U$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then the heart of $(\mathcal{U}, \mathcal{V})$ has enough projectives $H(\Omega\mathcal{C})$, and moreover it is equivalent to the functor category* $\operatorname{mod}(\mathcal{C}/\mathcal{P})$*, where $\mathcal{P}$ is the subcategory of projetive objects on $\mathcal{B}$.*

We also show that the condition $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$ on triangulated category is satisfied in many cases, for example, when $\mathcal{U}$ is covariantly finite in a Krull-Schmidt triangulated category. And for exact category case, see Examples 4.18 and 4.19.

2. Hearts of twin cotorsion pairs on exact categories

2.1. **Preliminaries.** First we briefly review the important properties of exact categories. For more details, we refer to [B]. Let $\mathcal{A}$ be an additive category, we call a pair of morphisms (i, d) a *weak short exact sequence* if i is the kernel of d and d is the cokernel of i. Let $\mathcal{E}$ be a class of weak short exact sequences of $\mathcal{A}$, stable under isomorphisms, direct sums and direct summands. If a weak short exact sequence (i, d) is in $\mathcal{E}$, we call it a *short exact sequence* and denote it by

$$X \overset{i}{\rightarrowtail} Y \overset{d}{\twoheadrightarrow} Z.$$

We call i an *inflation* and d a *deflation.* The pair $(\mathcal{A}, \mathcal{E})$ (or simply $\mathcal{A}$) is said to be an *exact category* if it satisfies the following properties:

(a) Identity morphisms are inflations and deflations.
(b) The composition of two inflations (resp. deflations) is an inflation (resp. deflation).
(c) If $X \overset{i}{\rightarrowtail} Y \overset{d}{\twoheadrightarrow} Z$ is a short exact sequence, for any morphisms $f : Z' \to Z$ and $g : X \to X'$, there are commutative diagrams

$$\begin{array}{ccc} Y' & \overset{d'}{\twoheadrightarrow} & Z' \\ {\scriptstyle f'}\downarrow & PB & \downarrow{\scriptstyle f} \\ Y & \underset{d}{\twoheadrightarrow} & Z \end{array} \qquad \begin{array}{ccc} X & \overset{i}{\rightarrowtail} & Y \\ {\scriptstyle g}\downarrow & PO & \downarrow{\scriptstyle g'} \\ X' & \underset{i'}{\rightarrowtail} & Y' \end{array}$$

where d' is a deflation and i' is an inflation, the left square being a pull-back and the right being a push-out.

We introduce the following properties of exact category, the proofs of which can be find in [B, §2]:

Proposition 2.1. *Consider a commutative square*

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \underset{i'}{\rightarrowtail} & B' \end{array}$$

in which i and i' are inflations. The following conditions are equivalent:

(a) *The square is a push-out.*

(b) *The sequence* $A \overset{\binom{i}{-f}}{\rightarrowtail} B \oplus A' \overset{(f'\ i')}{\twoheadrightarrow} B'$ *is short exact.*
(c) *The square is both a push-out and a pull-back.*
(d) *The square is a part of a commutative diagram*

$$\begin{array}{ccccc} A & \overset{i}{\rightarrowtail} & B & \twoheadrightarrow & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \| \\ A' & \underset{i'}{\rightarrowtail} & B' & \twoheadrightarrow & C \end{array}$$

with short exact rows.

Proposition 2.2. (a) *If* $X \overset{i}{\rightarrowtail} Y \overset{d}{\twoheadrightarrow} Z$ *and* $N \overset{g}{\rightarrowtail} M \overset{f}{\twoheadrightarrow} Y$ *are two short exact sequences, then there is a commutative diagram of short exact sequences*

$$\begin{array}{ccccc} N & = & N & & \\ \downarrow & & \downarrow{\scriptstyle g} & & \\ Q & \rightarrowtail & M & \twoheadrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle f} & & \| \\ X & \underset{i}{\rightarrowtail} & Y & \underset{d}{\twoheadrightarrow} & Z \end{array}$$

where the lower-left square is both a push-out and a pull-back.

(b) *If* $X \overset{i}{\rightarrowtail} Y \overset{d}{\twoheadrightarrow} Z$ *and* $Y \overset{g}{\rightarrowtail} K \overset{f}{\twoheadrightarrow} L$ *are two short exact sequences, then there is a commutative diagram of short exact sequences*

$$\begin{array}{ccccc} X & \overset{i}{\rightarrowtail} & Y & \overset{d}{\twoheadrightarrow} & Z \\ \| & & \downarrow{\scriptstyle g} & & \downarrow \\ X & \rightarrowtail & K & \twoheadrightarrow & R \\ & & \downarrow{\scriptstyle f} & & \downarrow \\ & & L & = & L \end{array}$$

where the upper-right square is both a push-out and a pull-back.

Let $\mathcal{A}$ be an exact category, an object P is called projective in $\mathcal{A}$ if for any deflation $f : X \to Y$ and any morphism $g : P \to Y$, there exists a morphism $h : P \to X$ such that $g = fh$. $\mathcal{A}$ is said to have enough projectives if for any object $X \in \mathcal{A}$, there is an object P which is projective in $\mathcal{A}$ and a deflation $p : P \to X$. Injective objects and having enough injectives are defined dually.

Throughout this paper, let $\mathcal{B}$ be a Krull-Schmidt exact category with enough projectives and injectives. Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the full subcategory of projectives (resp. injectives) of $\mathcal{B}$.

Definition 2.3. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{B}$ which are closed under direct summands. We call $(\mathcal{U}, \mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:
(a) $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U}, \mathcal{V}) = 0$.
(b) For any object $B \in \mathcal{B}$, there exits two short exact sequences

$$V_B \rightarrowtail U_B \twoheadrightarrow B, \quad B \rightarrowtail V^B \twoheadrightarrow U^B$$

satisfying $U_B, U^B \in \mathcal{U}$ and $V_B, V^B \in \mathcal{V}$.

Definition 2.4. A pair of cotorsion pairs $(\mathcal{S}, \mathcal{T})$, $(\mathcal{U}, \mathcal{V})$ on $\mathcal{B}$ is called a *twin cotorsion pair* if it satisfies:

$$\mathcal{S} \subseteq \mathcal{U}.$$

By definition and Lemma 2.6 this condition is equivalent to $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{S},\mathcal{V})=0$, and also to $\mathcal{V}\subseteq\mathcal{T}$.

Remark 2.5. (a) We also regard a cotorsion pair $(\mathcal{U},\mathcal{V})$ as a degenerated case of a twin cotorsion pair $(\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V})$.
(b) If $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ is a twin cotorsion pair on $\mathcal{B}$, then $(\mathcal{V}^{\mathrm{op}},\mathcal{U}^{\mathrm{op}}),(\mathcal{T}^{\mathrm{op}},\mathcal{S}^{\mathrm{op}})$ is a twin cotorsion pair on $\mathcal{B}^{\mathrm{op}}$.

By definition of a cotorsion pair, we can immediately conclude:

Lemma 2.6. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair of $\mathcal{B}$, then*
(a) *B belongs to $\mathcal{U}$ if and only if $\mathrm{Ext}^1_{\mathcal{B}}(B,\mathcal{V})=0$.*
(b) *B belongs to $\mathcal{V}$ if and only if $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U},B)=0$.*
(c) *$\mathcal{U}$ and $\mathcal{V}$ are closed under extension.*
(d) *$\mathcal{P}\subseteq\mathcal{U}$ and $\mathcal{I}\subseteq\mathcal{V}$.*

Definition 2.7. For any twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$, put
$$\mathcal{W}:=\mathcal{T}\cap\mathcal{U}.$$
(a) $\mathcal{B}^+$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence
$$V_B\rightarrowtail U_B\twoheadrightarrow B$$
where $U_B\in\mathcal{W}$ and $V_B\in\mathcal{V}$.
(b) $\mathcal{B}^-$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence
$$B\rightarrowtail T^B\twoheadrightarrow S^B$$
where $T^B\in\mathcal{W}$ and $S^B\in\mathcal{S}$.

By this definition we get $\mathcal{S}\subseteq\mathcal{U}\subseteq\mathcal{B}^-$ and $\mathcal{V}\subseteq\mathcal{T}\subseteq\mathcal{B}^+$.

Definition 2.8. Let $(\mathcal{S},\mathcal{T})$, $(\mathcal{U},\mathcal{V})$ be a twin cotorsion pair of $\mathcal{B}$, we denote the quotient of $\mathcal{B}$ by $\mathcal{W}$ as $\underline{\mathcal{B}}:=\mathcal{B}/\mathcal{W}$. For any morphism $f\in\mathrm{Hom}_{\mathcal{B}}(X,Y)$, we denote its image in $\mathrm{Hom}_{\underline{\mathcal{B}}}(X,Y)$ by $\underline{f}$. And for any subcategory $\mathcal{C}$ of $\mathcal{B}$, we denote by $\underline{\mathcal{C}}$ the subcategory of $\underline{\mathcal{B}}$ consisting of the same objects as $\mathcal{C}$. Put
$$\mathcal{H}:=\mathcal{B}^+\cap\mathcal{B}^-.$$
Since $\mathcal{H}\supseteq\mathcal{W}$, we have an additive full quotient subcategory
$$\underline{\mathcal{H}}:=\mathcal{H}/\mathcal{W}$$
which we call the *heart* of twin cotorsion pair $(\mathcal{S},\mathcal{T})$, $(\mathcal{U},\mathcal{V})$.
The heart of a cotorsion pair $(\mathcal{U},\mathcal{V})$ is defined to be the heart of twin cotorsion pair $(\mathcal{U},\mathcal{V})$, $(\mathcal{U},\mathcal{V})$.

We prove some useful lemmas for a twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ in the following:

Lemma 2.9. *Let $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ be a twin cotorsion pair on $\mathcal{B}$, then*
(a) *$\mathcal{B}^-$ is closed under direct summands. Moreover, if $X\in\mathcal{B}^-$ admits a short exact sequence*
$$X\rightarrowtail W\twoheadrightarrow U$$
where $W\in\mathcal{W}$ and $U\in\mathcal{U}$, then any direct summand X_1 of X admits a short exact sequence
$$X_1\rightarrowtail W\twoheadrightarrow Y$$
where $Y\in\mathcal{U}$.
(b) *$\mathcal{B}^+$ is closed under direct summands. Moreover, if $X\in\mathcal{B}^+$ admits a short exact sequence*
$$V\rightarrowtail W'\twoheadrightarrow X$$
where $W\in\mathcal{W}$ and $V\in\mathcal{V}$, then any direct summand X_2 of X admits a short exact sequence
$$Z\rightarrowtail W'\twoheadrightarrow X_2$$
where $Z\in\mathcal{V}$.

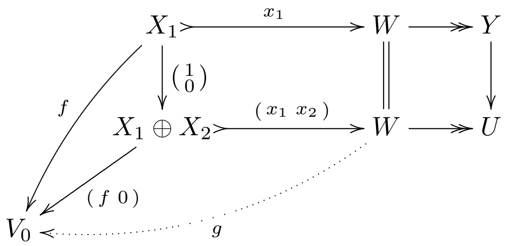
Proof. We only show (a), (b) is by dual.
Suppose $X_1 \oplus X_2$ admits a short exact sequence

$$X_1 \oplus X_2 \overset{(\, x_1 \; x_2 \,)}{\rightarrowtail} W \twoheadrightarrow U$$

where $U \in \mathcal{U}$ and $W \in \mathcal{W}$. Then $x_1 : X_1 \to W$ is also an inflation by the properties of exact category. Let x_1 admit a short exact sequence

$$X_1 \overset{x_1}{\rightarrowtail} W \twoheadrightarrow Y.$$

For any morphism $f : X_1 \to V_0$ where $V_0 \in \mathcal{V}$, consider a morphism $(\, f \; 0 \,) : X_1 \oplus X_2 \to V_0$. Since $\mathrm{Ext}^1_{\mathcal{B}}(U, V_0) = 0$, $(\, x_1 \; x_2 \,)$ is a left $\mathcal{V}$-approximation of W, there exists a morphism $g : W \to V_0$ such that $(\, f \; 0 \,) = (\, gx_1 \; gx_2 \,)$.



Hence $\mathrm{Hom}_{\mathcal{B}}(x_1, V_0) : \mathrm{Hom}_{\mathcal{B}}(W, V_0) \to \mathrm{Hom}_{\mathcal{B}}(X_1, V_0)$ is surjective. By the following exact sequence

$$\mathrm{Hom}_{\mathcal{B}}(W, V_0) \xrightarrow{\mathrm{Hom}_{\mathcal{B}}(x_1, V_0)} \mathrm{Hom}_{\mathcal{B}}(X_1, V_0) \xrightarrow{0} \mathrm{Ext}^1_{\mathcal{B}}(Y, V_0) \to \mathrm{Ext}^1_{\mathcal{B}}(W, V_0) = 0$$

we have $\mathrm{Ext}^1_{\mathcal{B}}(Y, V_0) = 0$, which implies $Y \in \mathcal{U}$. □

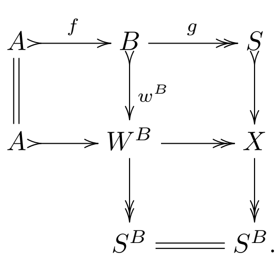
Lemma 2.10. (a) *If* $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} U$ *is a short exact sequence in* $\mathcal{B}$ *with* $U \in \mathcal{U}$*, then* $A \in \mathcal{B}^-$ *implies* $B \in \mathcal{B}^-$*.*

(b) *If* $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} S$ *is a short exact sequence in* $\mathcal{B}$ *with* $S \in \mathcal{S}$*, then* $B \in \mathcal{B}^-$ *implies* $A \in \mathcal{B}^-$*.*

Proof. (b) Since $B \in \mathcal{B}^-$, by definition, there exists a short exact sequence

$$B \overset{w^B}{\rightarrowtail} W^B \twoheadrightarrow S^B.$$

Take a push-out of g and w^B, by Proposition 2.2, we get a commutative diagram of short exact sequences



We thus get $X \in \mathcal{S}$ since $\mathcal{S}$ is closed under extension. This gives $A \in \mathcal{B}^-$.
(a) Since $A \in \mathcal{B}^-$, it admits a short exact sequence

$$A \overset{w^A}{\rightarrowtail} W^A \twoheadrightarrow S^A.$$

where $W^A \in \mathcal{W}$ and $S^A \in \mathcal{S}$. Since $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{S},\mathcal{T}) = 0$, w^A is a left $\mathcal{T}$-approximation of A. Thus there exists a commutative diagram of two short exact sequences

$$\begin{array}{ccccc} A & \rightarrowtail & W^A & \twoheadrightarrow & S^A \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ B & \underset{t^B}{\rightarrowtail} & T^B & \twoheadrightarrow & S^B. \end{array}$$

It suffices to show $T^B \in \mathcal{U}$.
Apply $\mathrm{Ext}^1_{\mathcal{B}}(-,\mathcal{V})$ to the following commutative diagram

$$\begin{array}{ccccc} A & \overset{f}{\rightarrowtail} & B & \twoheadrightarrow & U \\ {\scriptstyle w^A}\downarrow & & \downarrow{\scriptstyle t^B} & & \\ W^A & \longrightarrow & T^B & & \end{array}$$

since $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U},\mathcal{V}) = 0$, we obtain the following commutative diagram

$$\begin{array}{ccccc} & \boxed{\dashleftarrow} & \mathrm{Ext}^1_{\mathcal{B}}(T^B,\mathcal{V}) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{B}}(W^A,\mathcal{V}) = 0 \\ & & \downarrow{\scriptstyle \mathrm{Ext}^1_{\mathcal{B}}(t^B,\mathcal{V})} & & \downarrow \\ 0 = \mathrm{Ext}^1_{\mathcal{B}}(U,\mathcal{V}) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{B}}(B,\mathcal{V}) & \underset{\mathrm{Ext}^1_{\mathcal{B}}(f,\mathcal{V})}{\longrightarrow} & \mathrm{Ext}^1_{\mathcal{B}}(A,\mathcal{V}). \end{array}$$

It follows that $\mathrm{Ext}^1_{\mathcal{B}}(t^B,\mathcal{V}) = 0$. Then from the following exact sequence

$$0 = \mathrm{Ext}^1_{\mathcal{B}}(S^B,\mathcal{V}) \to \mathrm{Ext}^1_{\mathcal{B}}(T^B,\mathcal{V}) \xrightarrow{\mathrm{Ext}^1_{\mathcal{B}}(t^B,\mathcal{V})=0} \mathrm{Ext}^1_{\mathcal{B}}(B,\mathcal{V})$$

we get that $\mathrm{Ext}^1_{\mathcal{B}}(T^B,\mathcal{V}) = 0$, which implies that $T^B \in \mathcal{U}$. Thus $T^B \in \mathcal{W}$ and $B \in B^-$. □

Dually, the following holds.

Lemma 2.11. (a) *If* $T \rightarrowtail A \overset{f}{\twoheadrightarrow} B$ *is a short exact sequence in* $\mathcal{B}$ *with* $T \in \mathcal{T}$, *then* $B \in \mathcal{B}^+$ *implies* $A \in \mathcal{B}^+$.

(b) *If* $V \rightarrowtail A \overset{f}{\twoheadrightarrow} B$ *is a short exact sequence in* $\mathcal{B}$ *with* $V \in \mathcal{V}$, *then* $A \in \mathcal{B}^+$ *implies* $B \in \mathcal{B}^+$.

Now we give a proposition which is similar with [AR, Proposition 1.10] and useful in our article.

Proposition 2.12. *Let* $\mathcal{T}$ *be a subcategory of* $\mathcal{B}$ *satisfying*

(a) $\mathcal{P} \subseteq \mathcal{T}$.
(b) $\mathcal{T}$ *is contravariantly finite.*
(c) $\mathcal{T}$ *is closed under extension.*

Then we get a cotorsion pair $(\mathcal{T},\mathcal{V})$ *where*

$$\mathcal{V} = \{X \in \mathcal{B} \mid \mathrm{Ext}^1_{\mathcal{B}}(\mathcal{T},X) = 0\}.$$

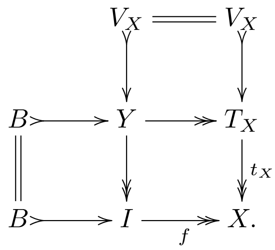
Proof. For any object $B \in \mathcal{B}$, it admits a short exact sequence

$$B \rightarrowtail I \overset{f}{\twoheadrightarrow} X$$

where $I \in \mathcal{I}$. By (a) and (b), we can take two short exact sequences

$$V_X \rightarrowtail T_X \overset{t_X}{\twoheadrightarrow} X, \qquad V_B \rightarrowtail T_B \overset{t_B}{\twoheadrightarrow} B$$

where t_X (resp. t_B) is a minimal right $\mathcal{T}$-approximation of X (resp. B). Since $\mathcal{T}$ is closed under extension, by Wakamatsu's Lemma, we obtain $V_X \in \mathcal{V}$ (resp. $V_B \in \mathcal{V}$). Take a pull-back of f and t_X, we get the following commutative diagram



Since $I, V \in \mathcal{V}$ and $\mathcal{V}$ is extension closed, we get $Y \in \mathcal{V}$. Thus B admits two short exact sequence

$$V_B \rightarrowtail T_B \twoheadrightarrow B, \quad B \rightarrowtail Y \twoheadrightarrow T_X$$

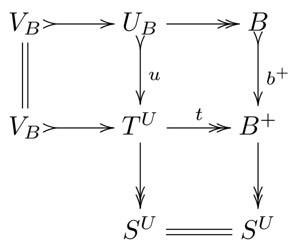
satisfying $V_B, Y \in \mathcal{V}$ and $T_B, T_X \in \mathcal{T}$. Hence by definition $(\mathcal{T}, \mathcal{V})$ is a cotorsion pair. □

2.2. $\underline{\mathcal{H}}$ **is preabelian.** In this section, we fix a twin cotorsion pair $(\mathcal{S}, \mathcal{T})$, $(\mathcal{U}, \mathcal{V})$, we will show that the heart $\underline{\mathcal{H}}$ of a twin cotorsion pair is preabelian.

Definition 2.13. For any $B \in \mathcal{B}$, define B^+ and $b^+ : B \to B^+$ as follows:
Take two short exact sequences:

$$V_B \rightarrowtail U_B \twoheadrightarrow B, \quad U_B \rightarrowtail T^U \twoheadrightarrow S^U$$

where $U_B \in \mathcal{U}$, $V_B \in \mathcal{V}$, $T^U \in \mathcal{T}$ and $S^U \in \mathcal{S}$. By Proposition 2.2, we get the following commutative diagram



(1)

where the upper-right square is both a push-out and a pull-back.

We can easily get the following Lemma.

Lemma 2.14. *By Definition 2.13, $B^+ \in \mathcal{B}^+$. Moreover, if $B \in \mathcal{B}^-$, then $B^+ \in \mathcal{H}$.*

Proof. Since $\mathcal{U}$ is closed under extension, we get $T^U \in \mathcal{U} \cap \mathcal{T} = \mathcal{W}$. Hence by definition $B^+ \in \mathcal{B}^+$. If $B \in \mathcal{B}^-$, by Lemma 2.10, B^+ also lies in $\mathcal{B}^-$. Thus $B^+ \in \mathcal{H}$. □

We give an important property of b^+ in the following proposition.

Proposition 2.15. *For any $B \in \mathcal{B}$ and $Y \in \mathcal{B}^+$,* $\mathrm{Hom}_{\mathcal{B}}(b^+, Y) : \mathrm{Hom}_{\mathcal{B}}(B^+, Y) \to \mathrm{Hom}_{\mathcal{B}}(B, Y)$ *is surjective and* $\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{b^+}, Y) : \mathrm{Hom}_{\underline{\mathcal{B}}}(B^+, Y) \to \mathrm{Hom}_{\underline{\mathcal{B}}}(B, Y)$ *is bijective.*

Proof. Let $y \in \mathrm{Hom}_{\mathcal{B}}(B, Y)$ be any morphism. By definition, there exists a short exact sequence

$$V_Y \rightarrowtail W_Y \xrightarrow{w_Y} Y.$$

Since $\mathrm{Ext}^1_{\mathcal{B}}(U_B, V_Y) = 0$, w_Y is a right $\mathcal{U}$-approximation of Y. Thus any $f \in \mathrm{Hom}_{\mathcal{B}}(U_B, Y)$ factors through W_Y.

$$\begin{array}{ccccc} & & & & U_B \\ & & & \swarrow^{g} & \downarrow{\scriptstyle f} \\ V_Y & \rightarrowtail & W_Y & \underset{w_Y}{\twoheadrightarrow} & Y \end{array}$$

As $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{S}, \mathcal{T}) = 0$, u is a left $\mathcal{T}$-approximation of U_B, we get the following commutative diagram:

$$\begin{array}{ccccc} U_B & \overset{u}{\rightarrowtail} & T^U & \twoheadrightarrow & S^U \\ {\scriptstyle g}\downarrow & \swarrow & & & \\ W_Y & & & & \\ {\scriptstyle w_Y}\downarrow & & & & \\ Y & & & & \end{array}$$

which implies that $\mathrm{Hom}_{\mathcal{B}}(u, Y) : \mathrm{Hom}_{\mathcal{B}}(T^U, Y) \to \mathrm{Hom}_{\mathcal{B}}(U_B, Y)$ is epimorphic. Hence when we apply $\mathrm{Hom}_{\mathcal{B}}(-, Y)$ to the diagram (1), we obtain the following exact sequence

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathcal{B}}(B^+, Y) & \xrightarrow{\mathrm{Hom}_{\mathcal{B}}(b^+, Y)} & \mathrm{Hom}_{\mathcal{B}}(B, Y) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{B}}(S, Y) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{B}}(B^+, Y) \\ & & \downarrow & & \| & & \downarrow \\ \mathrm{Hom}_{\mathcal{B}}(T^U, Y) & \xrightarrow{\mathrm{Hom}_{\mathcal{B}}(u, Y)} & \mathrm{Hom}_{\mathcal{B}}(U_B, Y) & \xrightarrow{0} & \mathrm{Ext}^1_{\mathcal{B}}(S^U, Y) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{B}}(T^U, Y) \end{array}$$

which implies that $\mathrm{Hom}_{\mathcal{B}}(b^+, Y)$ is an epimorphism. In particular, $\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{b^+}, Y)$ is an epimorphism.
It remains to show that $\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{b^+}, Y)$ is monomorphic. Suppose $q \in \mathrm{Hom}_{\mathcal{B}}(B^+, Y)$ satisfies $\underline{qb^+} = 0$, it follows that qb^+ factors through $\mathcal{W}$. Since w_Y is a right $\mathcal{U}$-approximation, there exists a morphism $a : B \to W_Y$ such that $w_Y a = qb^+$. Take a push-out of b^+ and a, we get the following commutative diagram of short exact sequences

$$\begin{array}{ccccc} B & \overset{b^+}{\rightarrowtail} & B^+ & \twoheadrightarrow & S^U \\ {\scriptstyle a}\downarrow & PO & \downarrow{\scriptstyle c'} & & \| \\ W_Y & \underset{c}{\rightarrowtail} & Q & \twoheadrightarrow & S^U. \end{array}$$

There exists a morphism $d : Q \to Y$ such that $dc = w_Y$ and $dc' = q$ by the definition of push-out. But $Q \in \mathcal{U}$ by Lemma 2.6, and w_Y is a right $\mathcal{U}$-approximation, we have that d factors through W_Y. Thus $q = dc'$ also factors through W_Y, and $\underline{q} = 0$. □

We give an equivalent condition for a special case when $B^+ = 0$ in $\underline{\mathcal{B}}$.

Lemma 2.16. *For any $B \in \mathcal{B}$, the following are equivalent.*

(a) $B^+ \in \mathcal{W}$.
(b) $B \in \mathcal{U}$.
(c) $\underline{b^+} = 0$ *in* $\underline{\mathcal{B}}$.

Proof. Consider the diagram (1) in Definition 2.13. We first prove that (b) implies (a).
Suppose (b) holds. Since $B \in \mathcal{U}$, we get $B^+ \in \mathcal{U}$. Thus $\mathrm{Ext}^1_{\mathcal{B}}(B^+, V_B) = 0$, and then t splits. Hence B^+ is a direct summand of $T^U \in \mathcal{W}$, which implies that $B^+ \in \mathcal{W}$.
Obviously (a) implies (c), now it suffices to show that (c) implies (b).
Since b^+ factors through $\mathcal{W}$, and t is a right $\mathcal{U}$-approximation of B^+, we get that b^+ factors through t. Hence by the definition of pull-back, the first row of diagram (1) splits, which implies that $B \in \mathcal{U}$. □

Now we give a dual construction.

Definition 2.17. For any object $B \in \mathcal{B}$, we define $b^- : B^- \to B$ as follows Take the following two short exact sequences

$$B \rightarrowtail T^B \twoheadrightarrow S^B, \quad V_T \rightarrowtail U_T \twoheadrightarrow T^B$$

where $U_T \in \mathcal{U}$, $V_T \in \mathcal{V}$, $T^B \in \mathcal{T}$ and $S^B \in \mathcal{S}$. By Proposition 2.2, we get the following commutative diagram:

$$\begin{array}{ccccc} V_T & = & V_T & & \\ \downarrow & & \downarrow & & \\ B^- & \rightarrowtail & U_T & \twoheadrightarrow & S^B \\ {\scriptstyle b^-}\downarrow & & \downarrow & & \| \\ B & \rightarrowtail & T^B & \twoheadrightarrow & S^B. \end{array}$$

By duality, we get:

Proposition 2.18. *For any $B \in \mathcal{B}$, $B^- \in \mathcal{B}^-$ and $B \in \mathcal{B}^+$ implies $B^- \in \mathcal{H}$. For any $X \in \mathcal{B}^-$, $\operatorname{Hom}_{\mathcal{B}}(X, b^-) : \operatorname{Hom}_{\mathcal{B}}(X, B^-) \to \operatorname{Hom}_{\mathcal{B}}(X, B)$ is surjective and $\operatorname{Hom}_{\underline{\mathcal{B}}}(X, \underline{b^-}) : \operatorname{Hom}_{\underline{\mathcal{B}}}(X, B^-) \to \operatorname{Hom}_{\underline{\mathcal{B}}}(X, B)$ is bijective.*

Definition 2.19. For any morphism $f : A \to B$ with $A \in \mathcal{B}^-$, define C_f and $c_f : B \to C_f$ as follows:
By definition, there exists a short exact sequence

$$A \overset{w^A}{\rightarrowtail} W^A \twoheadrightarrow S^A.$$

Take a push-out of f and w^A, we get the following commutative diagram of short exact sequences

$$\begin{array}{ccccc} A & \overset{w^A}{\rightarrowtail} & W^A & \twoheadrightarrow & S^A \\ {\scriptstyle f}\downarrow & PO & \downarrow & & \| \\ B & \underset{c_f}{\rightarrowtail} & C_f & \underset{s}{\twoheadrightarrow} & S^A. \end{array} \tag{2}$$

By Lemma 2.10, $B \in \mathcal{B}^-$ implies $C_f \in \mathcal{B}^-$.
Dually, we have the following:

Definition 2.20. For any morphism $f : A \to B$ in $\mathcal{B}$ with $B \in \mathcal{B}^+$, define K_f and $k_f : K_f \to A$ as follows:
By definition, there exists a short exact sequence

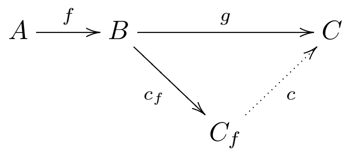
$$V_B \rightarrowtail W_B \overset{w_B}{\twoheadrightarrow} B.$$

Take a pull-back of f and w_B, we get the following commutative diagram of short exact sequences

$$\begin{array}{ccccc} V_B & \rightarrowtail & K_f & \overset{k_f}{\twoheadrightarrow} & A \\ \| & & \downarrow & PB & \downarrow{\scriptstyle f} \\ V_B & \rightarrowtail & W_B & \underset{w_B}{\twoheadrightarrow} & B \end{array} \tag{3}$$

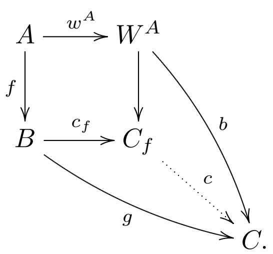
By Lemma 2.11, $A \in B^+$ implies $K_f \in B^+$.
The following lemma gives an important property of c_f:

Lemma 2.21. *Let $f : A \to B$ be any morphism in $\mathcal{B}$ with $A \in \mathcal{B}^-$, take the notation of Definition 2.19, then $c_f : B \to C_f$ satisfies the following properties:*
For any $C \in \mathcal{B}$ and any morphism $g \in \mathrm{Hom}_{\mathcal{B}}(B, C)$ satisfying $\underline{gf} = 0$, there exists a morphism $c : C_f \to C$ such that $cc_f = g$.



Moreover if $C \in \mathcal{B}^+$, then $\underline{c}$ is unique in $\underline{\mathcal{B}}$. The dual statement also holds for k_f in Definition 2.20.

Proof. Since $\underline{gf} = 0$, gf factors through $\mathcal{W}$. As $\mathrm{Ext}^1_{\mathcal{B}}(S_A, W^A) = 0$, w^A is a left $\mathcal{W}$-approximation of A. Hence there exists $b : W^A \to C$ such that $gf = bw^A$. Then by the definition of push-out, we get the following commutative diagram



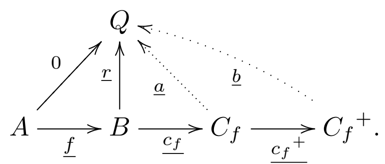
Now assume that $C \in \mathcal{B}^+$ and there exists $c' : C_f \to C$ such that $c'c_f = g$. Since $(c' - c)c_f = 0$, there exists a morphism $d : S^A \to C$ such that $c' - c = ds$. As C admits a short exact sequence

$$V_C \rightarrowtail W_C \xrightarrow{w_C} C$$

and w_C is a right $\mathcal{U}$-approximation of C, we obtain that there exists a morphism $e : S^A \to W_C$ such that $w_C e = d$. Hence $c' - c$ factors through W_C, and $\underline{c} = \underline{c'}$. □

Theorem 2.22. *For any twin cotorsion pair $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$, its heart $\underline{\mathcal{H}}$ is preabelian.*

Proof. We only show the construction of the cokernel. For any $A, B \in \mathcal{H}$ and any morphism $f : A \to B$, by Definition 2.19, since $A, B \in \mathcal{B}^-$, it follows $\underline{c_f f} = 0$ and $C_f \in \mathcal{B}^-$. By Proposition 2.15, there exists ${c_f}^+ : C_f \to {C_f}^+$ where ${C_f}^+ \in \mathcal{H}$ by Lemma 2.14. We claim that $\underline{{c_f}^+ c_f} : B \to {C_f}^+$ is the cokernel of $\underline{f}$. Let Q be any object in $\mathcal{H}$, and let $r : B \to Q$ be any morphism satisfying $\underline{rf} = 0$, then by Lemma 2.21 and Proposition 2.15, there exists a commutative diagram



The uniqueness of $\underline{b}$ follows from Lemma 2.21 and Proposition 2.15. □

Corollary 2.23. *Let $f : A \to B$ be a morphism in $\mathcal{H}$, the the followings are equivalent:*

(a) *$\underline{f}$ is epimorphic in $\underline{\mathcal{H}}$.*
(b) *${C_f}^+ \in \mathcal{W}$.*
(c) *$C_f \in \mathcal{U}$.*

Proof. The equivalence of (b) and (c) is given by Lemma 2.16.
By Theorem 2.22, $\underline{{c_f}^+ c_f}$ is the cokernel of $\underline{f}$ in $\underline{\mathcal{H}}$. The equivalence of (a) and (b) follows immediately by this argument. □

2.3. **Abelianess of the hearts of cotorsion pairs.** In this section we fix a cotorsion pair $(\mathcal{U}, \mathcal{V})$. We will prove that the heart $\underline{\mathcal{H}} = \mathcal{B}^+ \cap \mathcal{B}^- / \mathcal{U} \cap \mathcal{V}$ of a cotorsion pair is abelian.

Lemma 2.24. *Let $A, B \in \mathcal{H}$, and let*

$$C \stackrel{g}{\rightarrowtail} A \stackrel{f}{\twoheadrightarrow} B$$

be a short exact sequence in $\mathcal{B}$. If $\underline{f}$ is epimorphic in $\underline{\mathcal{H}}$, then C belongs to $\mathcal{B}^-$.

Proof. As $\underline{f}$ is epimorphic in $\underline{\mathcal{H}}$, we get $C_f \in \mathcal{U}$ by Corollary 2.23. By Definition 2.19, we get following commutative diagram

$$\begin{array}{ccccc} C & = & C & & \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} & & \\ A & \stackrel{w^A}{\rightarrowtail} & W^A & \twoheadrightarrow & U^A \\ {\scriptstyle f}\downarrow & PO & \downarrow & & \| \\ B & \underset{c_f}{\rightarrowtail} & C_f & \twoheadrightarrow & U^A. \end{array} \tag{4}$$

The middle column shows that $C \in \mathcal{B}^-$. □

We need the following lemma to prove our theorem.

Lemma 2.25. (a) *Let $f : A \to B$ be a morphism in $\mathcal{B}$ with $B \in \mathcal{B}^+$, then there exists a deflation $\alpha = (\, f \; -w_B \,) : A \oplus W_B \twoheadrightarrow B$ in $\mathcal{B}$ such that $\underline{\alpha} = \underline{f}$.*

(b) *Let $f : A \to B$ be a morphism in $\mathcal{B}$ with $A \in \mathcal{B}^-$, then there exists an inflation $\alpha = \begin{pmatrix} f \\ -w^A \end{pmatrix} : A \rightarrowtail B \oplus W^A$ in $\mathcal{B}$ such that $\underline{\alpha'} = \underline{f}$.*

Proof. We only show the first one, the second is dual.
As $B \in \mathcal{B}^+$, it admits a short exact sequence

$$V_B \rightarrowtail W_B \stackrel{w_B}{\twoheadrightarrow} B.$$

Take a pull-back of f and w_B, we get a commutative diagram

$$\begin{array}{ccccc} V_B & \rightarrowtail & C & \twoheadrightarrow & A \\ \| & & \downarrow & & \downarrow{\scriptstyle f} \\ V_B & \rightarrowtail & W_B & \underset{w_B}{\twoheadrightarrow} & B. \end{array}$$

By dual of Proposition 2.1, we get a short exact sequence

$$C \rightarrowtail A \oplus W_B \xrightarrow{\alpha = (\, f \; -w_B \,)} B$$

and consequently α is a deflation and $\underline{\alpha} = \underline{f}$. □

Theorem 2.26. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{B}$, its heart $\underline{\mathcal{H}}$ is an abelian category.*

Proof. Since $\underline{\mathcal{H}}$ is preabelian, it remains to show the following:

(a) If $\underline{f}$ is epimorphic in $\underline{\mathcal{H}}$, then $\underline{f}$ is a cokernel of some morphism in $\underline{\mathcal{H}}$.
(b) If $\underline{f}$ is monomorphic in $\underline{\mathcal{H}}$, then $\underline{f}$ is a kernel of some morphism in $\underline{\mathcal{H}}$.

We only show (a), since (b) is dual.
For any morphism $\underline{f} : A \to B$ which is epimorphic in $\underline{\mathcal{H}}$, by Lemma 2.25, it is enough to consider the case that f is a deflation.
Let f admit a short exact sequence:

$$C \overset{g}{\rightarrowtail} A \overset{f}{\twoheadrightarrow} B.$$

By Lemma 2.24, we have $C \in \mathcal{B}^-$. By Proposition 2.15, there exists

$$c^+ : C \to C^+$$

where C^+ lies in $\mathcal{H}$ by Lemma 2.14. As $A \in \mathcal{B}^+$, there exists $a : C^+ \to A$ such that $ac^+ = g$.

$$\begin{array}{ccc} C & \xrightarrow{\quad g \quad} & A \\ & {\scriptstyle c^+}\searrow \quad \nearrow{\scriptstyle a} & \\ & C^+. & \end{array}$$

Since $\underline{fac^+} = \underline{fg} = 0$, we have $\underline{fa} = 0$ by Proposition 2.15. We claim that $\underline{f}$ is the cokernel of $\underline{a}$.
Let Q be any object in $\mathcal{H}$ and $r : A \to Q$ be any morphism. By Proposition 2.15, $\underline{rg} = 0$ if and only if $\underline{ra} = 0$.
So it is enough to show that any $\underline{r}$ satisfying $\underline{rg} = 0$ factors through $\underline{f}$.
If $\underline{rg} = 0$, rg factors through $\mathcal{W}$. Consider the second column of diagram (4), since h is a left $\mathcal{V}$-approximation of C, there exists a morphism $c : W^A \to Q$ such that $rg = ch$. Since $h = w^A g$, we get that $(r - cw^A)g = 0$. Thus $r - cw^A$ factors through f, which implies that $\underline{r}$ factors through $\underline{f}$. ☐

2.4. $\underline{\mathcal{H}}$ **is semi-abelian.** In the following sections, we fix a twin cotorsion pair $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$.

Definition 2.27. A preabelian category $\mathcal{A}$ is called *left semi-abelian* if in any pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

in $\mathcal{A}$, α is an epimorphism whenever δ is a cokernel. *Right semi-abelian* is defined dually. $\mathcal{A}$ is called *semi-abelian* if it is both left and right semi-abelian. In this section we will prove that the heart $\underline{\mathcal{H}}$ of a twin cotorsion pair is semi-ableian.

Lemma 2.28. *If morphism $\beta \in \mathrm{Hom}_{\underline{\mathcal{H}}}(B, C)$ is a cokernel of a morphism $\underline{f} \in \mathrm{Hom}_{\underline{\mathcal{H}}}(A, B)$, then B admits a short exact sequence*

$$B \rightarrowtail C' \twoheadrightarrow S$$

where $C' \in \mathcal{H}$, $C \simeq C'$ in $\underline{\mathcal{H}}$ and $S \in \mathcal{S}$.

Proof. Let β be the cokernel of $\underline{f} : A \to B$. By Theorem 2.22, the cokernel of $\underline{f}$ is given by $\underline{{c_f}^+ c_f}$. Therefore ${C_f}^+ \simeq C$ in $\underline{\mathcal{H}}$. Consider diagram (4) and the diagram which induces $(C_f)^+$ by Definition 2.13:

$$\begin{array}{ccccc} V & \rightarrowtail & U & \twoheadrightarrow & C_f \\ \| & & \downarrow & & \downarrow{\scriptstyle {c_f}^+} \\ V & \rightarrowtail & T' & \twoheadrightarrow & {C_f}^+ \\ & & \downarrow & & \downarrow \\ & & S' & = & S' \end{array}$$

By Proposition 2.2, we obtain the following commutative diagram of short exact sequences

$$\begin{array}{ccccc}
B & \overset{c_f}{\rightarrowtail} & C_f & \twoheadrightarrow & S^A \\
\| & & {\scriptstyle {c_f}^+}\downarrow & & \downarrow \\
B & \rightarrowtail & {C_f}^+ & \twoheadrightarrow & Q \\
 & & \downarrow & & \downarrow \\
 & & S' & = & S'.
\end{array}$$

From the third column we get $Q \in \mathcal{S}$. Hence we get the required short exact sequence. ☐

Proposition 2.29. *Let* $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ *be a short exact sequence in* $\mathcal{B}$ *with* f *in* $\mathcal{H}$*. If* g *factors through* $\mathcal{U}$*, then* $\underline{f}$ *is epimorphic in* $\underline{\mathcal{H}}$*.*

Proof. By Corollary 2.23, it suffices to show that $C_f \in \mathcal{U}$.
By definition of $c_f : B \to C_f$, there is a commutative diagram of short exact sequences

$$\begin{array}{ccccc}
A & \overset{w^A}{\rightarrowtail} & W^A & \twoheadrightarrow & S^A \\
{\scriptstyle f}\downarrow & PO & \downarrow & & \| \\
B & \underset{c_f}{\rightarrowtail} & C_f & \twoheadrightarrow & S^A \\
{\scriptstyle g}\downarrow & & \downarrow & & \\
C & = & C. & &
\end{array}$$

Since $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{W}, \mathcal{V}) = 0$, we get the following commutative diagram of exact sequence

$$\begin{array}{ccccc}
\mathrm{Ext}^1_{\mathcal{B}}(C, \mathcal{V}) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{B}}(C_f, \mathcal{V}) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{B}}(W^A, \mathcal{V}) = 0 \\
\| & & \downarrow{\scriptstyle \mathrm{Ext}^1_{\mathcal{B}}(c_f, \mathcal{V})} & & \downarrow \\
\mathrm{Ext}^1_{\mathcal{B}}(C, \mathcal{V}) & \underset{\mathrm{Ext}^1_{\mathcal{B}}(g, \mathcal{V})}{\longrightarrow} & \mathrm{Ext}^1_{\mathcal{B}}(B, \mathcal{V}) & \underset{\mathrm{Ext}^1_{\mathcal{B}}(f, \mathcal{V})}{\longrightarrow} & \mathrm{Ext}^1_{\mathcal{B}}(A, \mathcal{V}).
\end{array}$$

Then $\mathrm{Ext}^1_{\mathcal{B}}(c_f, \mathcal{V})$ factors through $\mathrm{Ext}^1_{\mathcal{B}}(g, \mathcal{V})$. We have $\mathrm{Ext}^1_{\mathcal{B}}(g, \mathcal{V}) = 0$ since g factors through $\mathcal{U}$, thus we get $\mathrm{Ext}^1_{\mathcal{B}}(c_f, \mathcal{V}) = 0$. Then from the following exact sequence

$$0 = \mathrm{Ext}^1_{\mathcal{B}}(S^A, \mathcal{V}) \to \mathrm{Ext}^1_{\mathcal{B}}(C_f, \mathcal{V}) \xrightarrow{\mathrm{Ext}^1_{\mathcal{B}}(c_f, \mathcal{V}) = 0} \mathrm{Ext}^1_{\mathcal{B}}(B, \mathcal{V})$$

we obtain that $\mathrm{Ext}^1_{\mathcal{B}}(C_f, \mathcal{V}) = 0$, which implies $C_f \in \mathcal{U}$.

☐

Lemma 2.30. *Suppose* $X \in \mathcal{B}^-$ *admits a short exact sequence*

$$X \overset{x}{\rightarrowtail} B \twoheadrightarrow U$$

where $B \in \mathcal{H}$ *and* $U \in \mathcal{U}$*. Then the unique morphism* $\underline{b} \in \mathrm{Hom}_{\underline{\mathcal{H}}}(X^+, B)$ *given by Proposition 2.15 which satisfies* $\underline{bx^+} = \underline{x}$ *is epimorphic.*

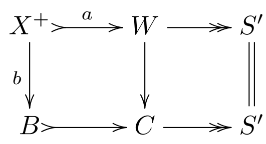
Proof. By Definition 2.13, there exists a short exact sequence

$$X \overset{x^+}{\rightarrowtail} X^+ \twoheadrightarrow S$$

where $S \in \mathcal{S}$. By Proposition 2.15, there exits $b : X^+ \to B$ such that $bx^+ = x$. Since $X \in \mathcal{B}^-$, we obtain $X^+ \in \mathcal{H}$ by Lemma 2.14. Hence X^+ admits a short exact sequence

$$X^+ \overset{a}{\rightarrowtail} W \twoheadrightarrow S'$$

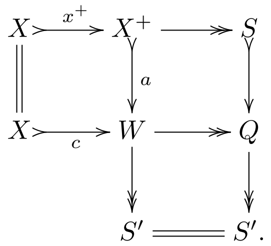
where $W \in \mathcal{W}$ and $S' \in \mathcal{S}$. Take a push-out of a and b, we get the following commutative diagram



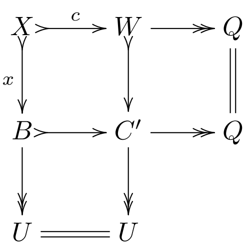
which induces a short exact sequence

$$X^+ \xrightarrow{\binom{b}{-a}} B \oplus W \twoheadrightarrow C$$

by Proposition 2.1. By Proposition 2.2, we obtain the following commutative diagram



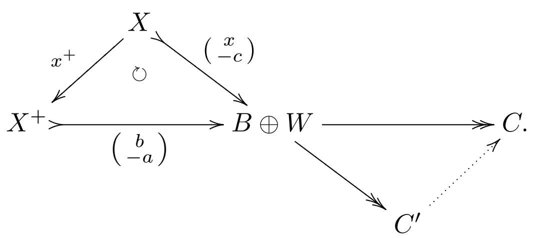
Take a push-out of x and c



from the second column we obtain that $C' \in \mathcal{U}$ and we get the following short exact sequence

$$X \xrightarrow{\binom{x}{-c}} B \oplus W \twoheadrightarrow C'$$

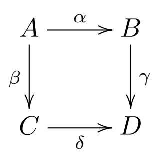
by Proposition 2.1. Thus we get the following commutative diagram



Hence by Proposition 2.29, $\underline{b}$ is epimorphic. □

We introduce the following lemma which is an analogue of [N1, Lemma 5.3].

Lemma 2.31. *Let*



be a pull-back diagram in $\underline{\mathcal{H}}$. *If there exists an object* $X \in \mathcal{B}^-$ *and morphisms* $x_B : X \to B$, $x_C : X \to C$ *which satisfy the following conditions, then* α *is epimorphic in* $\underline{\mathcal{H}}$.

(a) *The following diagram is commutative.*

$$\begin{array}{ccc} X & \xrightarrow{\underline{x_B}} & B \\ {\scriptstyle \underline{x_C}}\downarrow & & \downarrow{\scriptstyle \gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

(b) *There exists a short exact sequence* $X \rightarrowtail^{x_B} B \twoheadrightarrow U$ *with* $U \in \mathcal{U}$.

Proof. Take $x^+ : X \to X^+$ as in Definition 2.13. Then by Proposition 2.15, there exist $f_B : X^+ \to B$ and $f_C : X^+ \to C$ such that $\underline{f_B x^+} = \underline{x_B}$ and $\underline{f_C x^+} = \underline{x_C}$. By Lemma 2.30, $\underline{f_B}$ is epimorphic in $\underline{\mathcal{H}}$. As $\gamma\underline{x_B} = \delta\underline{x_C}$, we get $\gamma\underline{f_B x^+} = \delta\underline{f_C x^+}$, it follows by Proposition 2.15 that $\gamma\underline{f_B} = \delta\underline{f_C}$. By the definition of pull-back, there exists a morphism $\eta : X^+ \to A$ in $\underline{\mathcal{H}}$ which makes the following diagram commute.

$$\begin{array}{ccccc} X^+ & & & & \\ & \searrow^{\eta} & & \overset{\underline{f_B}}{\searrow} & \\ & {\scriptstyle \underline{f_C}}\searrow & A & \xrightarrow{\alpha} & B \\ & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ & & C & \xrightarrow[\delta]{} & D \end{array}$$

Since $\underline{f_B}$ is epimorphic, we obtain that α is also epimorphic. $\square$

Theorem 2.32. *For any twin cotorsion pair* $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$, *its heart* $\underline{\mathcal{H}}$ *is semi-abelian.*

Proof. By duality, we only show $\underline{\mathcal{H}}$ is left semi-abelian. Assume we are given a pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

in $\underline{\mathcal{H}}$ where δ is a cokernel. It suffices to show that α becomes epimorphic.
By Lemma 2.28, replacing D by an isomorphic one if necessary, we can assume that there exists an inflation $d : C \rightarrowtail D$ satisfying $\delta = \underline{d}$, which admits a short exact sequence

$$C \overset{d}{\rightarrowtail} D \twoheadrightarrow S$$

where $S \in \mathcal{S}$. As $D \in \mathcal{B}^+$, by Lemma 2.25 we can also assume that there exists an deflation $c : B \twoheadrightarrow D$ such that $\gamma = \underline{c}$. By Proposition 2.2, we get the following commutative diagram of short exact sequences

$$\begin{array}{ccccc} X & \overset{x_B}{\rightarrowtail} & B & \twoheadrightarrow & S \\ {\scriptstyle x_C}\downarrow & & \downarrow{\scriptstyle c} & & \| \\ C & \underset{d}{\rightarrowtail} & D & \twoheadrightarrow & S. \end{array}$$

it follows by Lemma 2.10 that $X \in \mathcal{B}^-$. Hence by Lemma 2.31 α is epimorphic in $\underline{\mathcal{H}}$. $\square$

2.5. **The case where $\underline{\mathcal{H}}$ becomes integral.** In this section we give a sufficient condition where the heart $\underline{\mathcal{H}}$ becomes integral.

Definition 2.33. A preabelian category $\mathcal{A}$ is called *left integral* if in any pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

in $\mathcal{A}$, α is an epimorphism whenever δ is an epimorphic. *Right integral* is defined dually. $\mathcal{A}$ is called *integral* if it is both left and right integral.

Let $\mathcal{C}$ be a subcategory of $\mathcal{B}$, denote by $\Omega\mathcal{C}$ (resp. $\Omega^-\mathcal{C}$) the subcateogy of $\mathcal{B}$ consisting of objects ΩC (resp. $\Omega^- C$) such that there exists a short exact sequence

$$\Omega C \rightarrowtail P_C \twoheadrightarrow C \ (P \in \mathcal{P}, C \in \mathcal{C})$$
$$(\text{resp. } C \rightarrowtail I^C \twoheadrightarrow \Omega^- C \ (I \in \mathcal{I}, C \in \mathcal{C})).$$

By definition we get $\mathcal{P} \subseteq \Omega\mathcal{C}$ and $\mathcal{I} \subseteq \Omega^-\mathcal{C}$. By Lemma 2.9 we get that for any cotorsion pair $(\mathcal{U},\mathcal{V})$ on $\mathcal{B}$, $\Omega\mathcal{U}$ and $\Omega^-\mathcal{V}$ are closed under direct summands.

Let $\mathcal{B}_1$ $\mathcal{B}_2$ be two subcategories of $\mathcal{B}$, recall that $\mathcal{B}_1 * \mathcal{B}_2$ is subcategory of $\mathcal{B}$ consisting of objects X such that there exists a short exact sequence

$$B_1 \rightarrowtail X \twoheadrightarrow \mathcal{B}_2$$

where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$.

Theorem 2.34. *If a twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ satisfies*

$$\mathcal{U} \subseteq \mathcal{S} * \mathcal{T},\ \mathcal{P} \subseteq \mathcal{W} \quad \textit{or} \quad \mathcal{T} \subseteq \mathcal{U} * \mathcal{V},\ \mathcal{I} \subseteq \mathcal{W}$$

then $\underline{\mathcal{H}}$ becomes integral.

Proof. According to [R, Proposition 6], a semi-abelian category is left integral if and only if it is right integral. By duality, it suffices to show that $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}, \mathcal{P} \subseteq \mathcal{W}$ implies that $\underline{\mathcal{H}}$ is left integral. Assume we are given a pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

in $\underline{\mathcal{H}}$ where δ is an epimorphism. It is sufficient to show that α is epimorphic.
Let $d : C \to D$ and $c : B \to D$ be morphisms satisfying $\delta = \underline{d}$ and $\gamma = \underline{c}$. Since δ is epimorphic, if we take $c_d : D \to C_d$ as in Definition 2.19

$$\begin{array}{ccccc} C & \overset{w^C}{\rightarrowtail} & W^C & \twoheadrightarrow & S^C \\ {\scriptstyle d}\downarrow & PO & \downarrow & & \| \\ D & \underset{c_d}{\rightarrowtail} & C_d & \underset{r}{\twoheadrightarrow} & S^C \end{array}$$

then $C_d \in \mathcal{U}$ by Corollary 2.23. By assumption $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$, C_d admits a short exact sequence

$$S_0 \overset{s_0}{\rightarrowtail} C_d \overset{t_0}{\twoheadrightarrow} T_0$$

with $S_0 \in \mathcal{S}$, $T_0 \in \mathcal{T}$. Since $B \in \mathcal{B}^-$ admits a short exact sequence

$$B \rightarrowtail W^B \twoheadrightarrow S^B$$

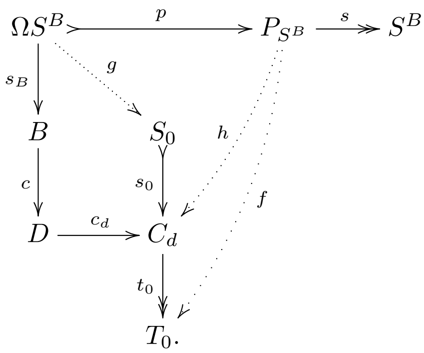
and S^B admits a short exact sequence

$$\Omega S^B \xrightarrow{p} P_{S^B} \overset{s}{\twoheadrightarrow} S^B$$

there exists a commutative diagram

$$\begin{array}{ccccc} \Omega S^B & \overset{p}{\rightarrowtail} & P_{S^B} & \overset{s}{\twoheadrightarrow} & S^B \\ {\scriptstyle s_B}\downarrow & & \downarrow & & \| \\ B & \rightarrowtail & W^B & \twoheadrightarrow & S^B. \end{array} \tag{5}$$

As $\mathrm{Ext}^1_{\mathcal{B}}(S_B, T_0) = 0$, p is a left $\mathcal{T}$-approximation of ΩS^B. Therefore there exists a morphism $f : P_{S^B} \to T_0$ such that $t_0 c_d c s_B = fp$. As $P_{S^B} \in \mathcal{P}$, there is a morphism $h : P_{S^B} \to C_d$ such that $f = t_0 h$. Since $t_0(c_d c s_B - hp) = 0$, there exists a morphism $g : \Omega S_B \to S_0$ such that $c_d c s_B - hp = s_0 g$. Then we get the following diagram



Take a push-out of p and g, we get the following commutative diagram

$$\begin{array}{ccccc} \Omega S^B & \overset{p}{\rightarrowtail} & P_{S^B} & \overset{s}{\twoheadrightarrow} & S^B \\ {\scriptstyle g}\downarrow & & \downarrow & & \| \\ S_0 & \rightarrowtail & Q & \twoheadrightarrow & S^B \end{array}$$

and a short exact sequence

$$\Omega S^B \overset{\left(\begin{smallmatrix} p \\ -g \end{smallmatrix}\right)}{\rightarrowtail} P_{S^B} \oplus S_0 \twoheadrightarrow Q$$

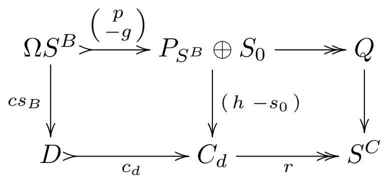
by Proposition 2.1 where $Q \in \mathcal{S}$. As Q admits a short exact sequence

$$\Omega Q \overset{k_Q}{\rightarrowtail} P_Q \overset{l_Q}{\twoheadrightarrow} Q$$

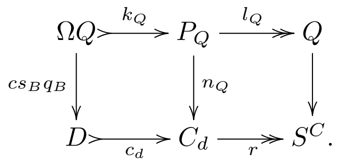
we get the following commutative diagram of short exact sequences

$$\begin{array}{ccccc} \Omega Q & \overset{k_Q}{\rightarrowtail} & P_Q & \overset{l_Q}{\twoheadrightarrow} & Q \\ {\scriptstyle q_B}\downarrow & & \downarrow & & \| \\ \Omega S^B & \underset{\left(\begin{smallmatrix} p \\ -g \end{smallmatrix}\right)}{\rightarrowtail} & P_{S^B} \oplus S_0 & \twoheadrightarrow & Q. \end{array} \tag{6}$$

Since $c_d c s_B = hp + s_0 g$, we obtain the following commutative diagram of short exact sequences.



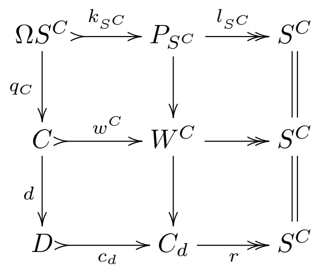
Thus we get the following commutative diagram



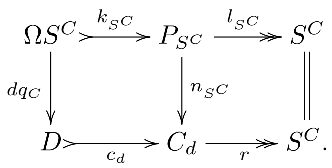
As $\mathcal{P} \subseteq \mathcal{W}$, we conclude that $\Omega Q \in \mathcal{B}^-$. Since S^C admits a short exact sequence

$$\Omega S^C \overset{k_{S^C}}{\rightarrowtail} P_{S^C} \overset{l_{S^C}}{\twoheadrightarrow} S^C$$

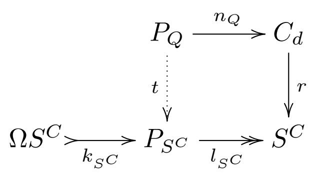
where $P_{S^C} \in \mathcal{P}$, hence we get the following commutative diagram of short exact sequence



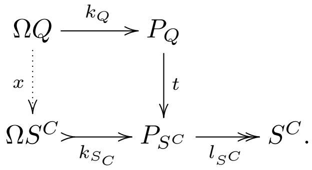
which induces the following diagram



As P_Q is projective, there exists a morphism $t : P_Q \to P_{S_C}$ such that $l_{S_C} t = r n_Q$.



Now it follows that $l_{S^C} t k_Q = r n_Q k_Q = r c_d c s_B q_B = 0$, thus there exists a morphism $x : \Omega Q \to \Omega S_C$ such that $k_{S^C} x = t k_Q$.



As $r n_{S^C} t = l_{S^C} t = r n_Q$, there exists a morphism $y : P_Q \to D$ such that $n_{S^C} t - n_Q = c_d y$. Therefore

$$c_d d q_C x = n_{S^C} k_{S^C} x = n_{S^C} t k_Q = (c_d y + n_Q) k_Q = c_d (y k_Q + c s_B q_B).$$

Then $dq_Cx = yk_Q + cs_Bq_B$, since c_d is monomorphic. Hence there exists a commutative diagram in $\underline{\mathcal{B}}$

$$\begin{array}{ccc} \Omega Q & \xrightarrow{\underline{s_Bq_B}} & B \\ {\scriptstyle \underline{q_Cx}}\downarrow & & \downarrow{\scriptstyle \underline{c}} \\ C & \xrightarrow[\underline{d}]{} & D. \end{array}$$

By Proposition 2.1, we get the following short exact sequences from (5) and (6):

$$\Omega Q \xrightarrow{\binom{q_B}{-k_Q}} \Omega S_B \oplus P_Q \twoheadrightarrow P_{S_B} \oplus S_0, \qquad \Omega S^B \xrightarrow{\binom{s_B}{-p}} B \oplus P_{S^B} \twoheadrightarrow W^B.$$

Then by Proposition 2.2, we get the following commutative diagram of short exact sequences

$$\begin{array}{ccccc} \Omega Q & \xrightarrow{\binom{q_B}{-k_Q}} & \Omega S^B \oplus P_Q & \twoheadrightarrow & P_{S^B} \oplus S_0 \\ \| & & \downarrow{\scriptstyle \begin{pmatrix} s_B & 0 \\ -p & 0 \\ 0 & 1 \end{pmatrix}} & & \downarrow \\ \Omega Q & \xrightarrow[\eta]{} & B \oplus P_{S^B} \oplus P_Q & \twoheadrightarrow & M \\ & & \downarrow & & \downarrow \\ & & W^B & = & W^B \end{array}$$

where $\eta = \underline{s_Bq_B}$. From the third column we get that $M \in \mathcal{U}$. By Lemma 2.31, we obtain that α is epimorphic. $\square$

2.6. **The case where $\underline{\mathcal{H}}$ becomes almost abelian.** In this section we give a sufficient condition when $\underline{\mathcal{H}}$ becomes almost abelian.

Definition 2.35. A preabelian category $\mathcal{A}$ is called *left almost abelian* if in any pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

in $\mathcal{A}$, α is a cokernel whenever δ is a cokernel. *Right almost abelian* is defined dually. $\mathcal{A}$ is called *almost abelian* if it is both left and right almost abelian.

We need the following proposition to show our result.

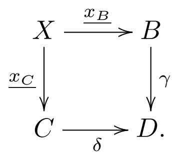
Proposition 2.36. [R, Proposition 2] *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms in a right (resp. left) semi-abelian category. If f and g are (co-)kernels, then gf is a (co-)kernel. If gf is a (co-)kernel, then f (resp. g) is a (co-)kernel.*

Use this proposition, we can prove the following lemma, which is an analogue of Lemma 2.31.

Lemma 2.37. *Let*

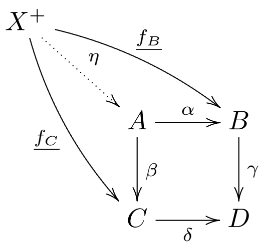
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \gamma} \\ C & \xrightarrow[\delta]{} & D \end{array}$$

be a pull-back diagram in $\underline{\mathcal{H}}$. Let $X \in \mathcal{B}^-$ and $x_B : X \to B$, $x_C : X \to C$ be morphisms which satisfy that $\underline{x_B}$ is a cokernel in the following commutative diagram



Then if $\mathcal{U} \subseteq \mathcal{T}$, we obtain α is a cokernel in $\underline{\mathcal{H}}$.

Proof. Since $\mathcal{U} \subseteq \mathcal{T}$, we get $\mathcal{H} = \mathcal{B}^-$. Take $x^+ : X \to X^+$ as in Definition 2.13. Then by Proposition 2.15, there exist $f_B : X^+ \to B$ and $f_C : X^+ \to C$ such that $\underline{f_B x^+} = \underline{x_B}$ and $\underline{f_C x^+} = \underline{x_C}$. Since $\underline{x_B}$ is a cokernel, by Proposition 2.36, $\underline{f_B}$ is also a cokernel in $\underline{\mathcal{H}}$. As $\gamma \underline{x_B} = \delta \underline{x_C}$, it follows by Proposition 2.15 that $\gamma \underline{f_B} = \delta \underline{f_C}$. By the definition of pull-back, there exists a morphism $\eta : X^+ \to A$ in $\underline{\mathcal{H}}$ which makes the following diagram commute.



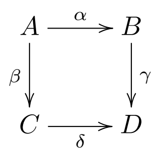
Since $\underline{f_B}$ is a cokernel, we obtain that α is also a cokernel by Proposition 2.36. $\square$

Theorem 2.38. *Let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be a twin cotorsion pair on $\mathcal{B}$ satisfying*

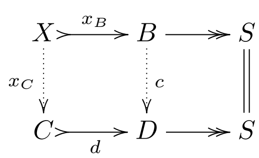
$$\mathcal{U} \subseteq \mathcal{T} \text{ or } \mathcal{T} \subseteq \mathcal{U}$$

then $\underline{\mathcal{H}}$ is almost abelian.

Proof. By [R, Proposition 3], a semi-abelian category is left almost abelian if and only if it is right almost abelian. By duality, it is enough to show that $\mathcal{U} \subseteq \mathcal{T}$ implies $\underline{\mathcal{H}}$ is left almost abelian.
Assume we are given a pull-back diagram

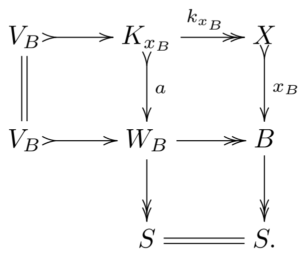


in $\underline{\mathcal{H}}$ where δ is a cokernel. It suffices to show that α becomes a cokernel.
Repeat the same argument as in Theorem 2.32, we get the following diagram

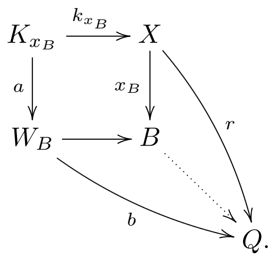


where $X \in \mathcal{B}^-$, $\underline{d} = \delta$ and $\underline{c} = \gamma$. According to Lemma 2.37, it suffices to show that $\underline{x_B}$ is a cokernel in $\underline{\mathcal{H}}$.

By Definition 2.20 and Proposition 2.2, we get the following commutative diagram



It follows that $K_{x_B} \in \mathcal{B}^- = \mathcal{H}$ and $\underline{k_{x_B} x_B} = 0$. Now let $r : X \to Q$ be any morphism in $\mathcal{H}$ such that $\underline{rk_{x_B}} = 0$, then rk_{x_B} factors through $\mathcal{W}$. Since $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{S}, \mathcal{T}) = 0$, a is a left $\mathcal{T}$-approximation of K_{x_B}, thus there exists a morphism $b : W_B \to Q$ such that $ab = rk_B$. By the definition of push-out, we get the following commutative diagram



Since $\underline{x_B}$ is epimorphic in $\underline{\mathcal{H}}$ by Proposition 2.29, the above diagram implies that $\underline{x_B}$ is the cokernel of $\underline{k_{x_B}}$. □

By Theorem 2.34, in the case of the above theorem, the heart $\underline{\mathcal{H}}$ also becomes integral. Then by [R, Theorem 2], $\underline{\mathcal{H}}$ is equivalent to a torsionfree class of a hereditary torsion theory in an abelian category induced by $\underline{\mathcal{H}}$. For more details, one can see [R, §4].

2.7. **Existence of enough projectives/injectives.** We call an object $P \in \underline{\mathcal{H}}$ (proper-)projective if for any epimorphism (resp. cokernel) $\alpha : X \to Y$ in $\underline{\mathcal{H}}$, there exists an exact sequence

$$\mathrm{Hom}_{\underline{\mathcal{H}}}(P, X) \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{H}}}(P,\alpha)} \mathrm{Hom}_{\underline{\mathcal{H}}}(P, Y) \to 0.$$

An (proper-)injective object is defined dually.
$\underline{\mathcal{H}}$ is said to have enough projectives if for any object $X \in \underline{\mathcal{H}}$, there is a cokernel $\delta : P \to X$ such that P is proper-projective. Having enough injectives is defined dually.

In this section we give sufficient conditions that the heart $\underline{\mathcal{H}}$ of a twin cotorsion pair has enough projectives and has enough injectives.

Lemma 2.39. *If a twin cotorsion pair $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ satisfies $\mathcal{U} \subseteq \mathcal{T}$, then we have $\Omega\mathcal{S} \subseteq \mathcal{H}$.*

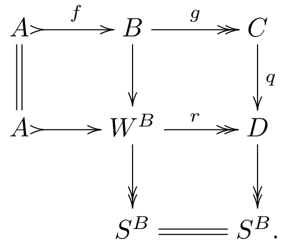
Proof. We first have $\mathcal{P} \subseteq \mathcal{U} = \mathcal{W}$, then by definition $\Omega\mathcal{S} \subseteq \mathcal{B}^-$. But we observe that $\mathcal{U} \subseteq \mathcal{T}$ implies $\mathcal{B}^+ = \mathcal{B}$, hence $\Omega\mathcal{S} \subseteq \mathcal{H}$. □

Proposition 2.40. *Let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be a twin cotorsion pair satisfying $\mathcal{U} \subseteq \mathcal{T}$, then any object in $\Omega\mathcal{S}$ is projective in $\underline{\mathcal{H}}$.*

Proof. Let B and C be any objects in $\mathcal{H}$ and let $p : \Omega S \to C$ be any morphism.
Let $\underline{g} : B \to C$ be a morphism which is epimorphic in $\underline{\mathcal{H}}$, by Lemma 2.25 we can assume that it admits a short exact sequence

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C.$$

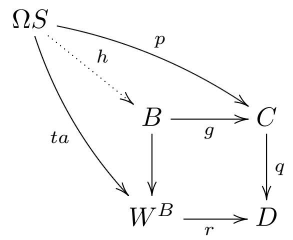
Since $B \in \mathcal{H}$ admits a short exact sequence $B \rightarrowtail W^B \twoheadrightarrow S^B$, then according to Proposition 2.2, there exists a commutative diagram



By Lemma 2.10, we obtain $D \in \mathcal{B}^- = \mathcal{H}$. Since $\underline{qg} = 0$ and $\underline{g}$ is epimorphic in $\underline{\mathcal{H}}$, we have $\underline{q} = 0$. By definition ΩS admits a short exact sequence

$$\Omega S \stackrel{a}{\rightarrowtail} P \twoheadrightarrow S \ \ (P \in \mathcal{P}, S \in \mathcal{S}).$$

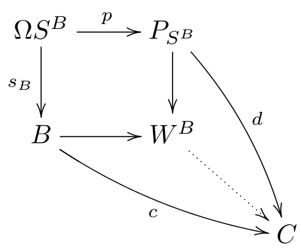
Since $\underline{qp} = 0$, qp factors through $\mathcal{W}$. As $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{S}, \mathcal{T}) = 0$, a is a left $\mathcal{T}$-approximation of ΩS. Thus there exists a morphism $s : P \to D$ such that $qp = sa$. Since P is projective, there exists a morphism $t : P \to W^B$ such that $s = rt$. Hence by the definition of pull-back, we get the following commutative diagram



which implies that ΩS is projective in $\underline{\mathcal{H}}$. □

Proposition 2.41. *Let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be a twin cotorsion pair satisfying $\mathcal{U} \subseteq \mathcal{T}$, then any object $\mathcal{B} \in \mathcal{H}$ admits an epimorphism $\alpha : \Omega S \to B$ in $\underline{\mathcal{H}}$.*

Proof. Let B be any object in $\mathcal{H}$, consider commutative diagram (5). By Proposition 2.1, the left square is a push-out. Now it suffices to show $\underline{s_B}$ is epimorphic in $\underline{\mathcal{H}}$.
Let $c : B \to C$ be any morphism in $\mathcal{H}$ such that $\underline{cs_B} = 0$, then cs_B factors through $\mathcal{W}$. Since p is a left $\mathcal{T}$-approximation of ΩS, there exits a morphism $d : P_{S^B} \to C$ such that $cs_B = dp$. Thus by the definition of push-out we have a commutative diagram



which implies $\underline{c} = 0$. Hence $\underline{s_B}$ is epimorphic in $\underline{\mathcal{H}}$. □

Moreover, we have

Proposition 2.42. *Let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be a twin cotorsion pair satisfying $\mathcal{U} \subseteq \mathcal{T}$, then an object B is projective in $\underline{\mathcal{H}}$ implies that $B \in \underline{\Omega \mathcal{S}}$.*

Proof. Suppose B is projective in $\underline{\mathcal{H}}$, consider the commutative diagram (5). By Proposition 2.41, $\underline{s_B}$ is epimorphic in $\underline{\mathcal{H}}$, thus B is a direct summand of ΩS^B in $\underline{\mathcal{H}}$. Hence by Lemma 2.9 B lies in $\underline{\Omega \mathcal{S}}$. □

From the following proposition we can get that in the case $\mathcal{U} \subseteq \mathcal{T}$ when the projectives in $\underline{\mathcal{H}}$ is enough.

Proposition 2.43. *Let $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$ be a twin cotorsion pair satisfying $\mathcal{U}\subseteq\mathcal{T}$, then $\underline{\mathcal{H}}$ has enough projectives if and only if any indecomposable object $B\in\mathcal{H}-\mathcal{U}$ admits a short exact sequence*

$$B \rightarrowtail S^1 \twoheadrightarrow S^2$$

where $S^1,S^2\in\mathcal{S}$.

Proof. We prove the "if" part first.
Since an object $B\in\mathcal{H}$ isomorphic to an object $B'\in\mathcal{H}$ in $\underline{\mathcal{H}}$ such that B' does not have any direct summand in $\mathcal{U}$, we can only consider the obejct $B\in\mathcal{H}$ not having any direct summand in $\mathcal{U}$. Thus by assumption, B admits a short exact sequence

$$B \rightarrowtail S^1 \twoheadrightarrow S^2$$

where $S^1,S^2\in\mathcal{S}$. As S^2 admits a short exact sequence

$$\Omega S^2 \overset{b}{\rightarrowtail} P_{S^2} \twoheadrightarrow S^2.$$

We have the following commutative diagram

$$\begin{array}{ccccc}
\Omega S^2 & \overset{b}{\rightarrowtail} & P_{S^2} & \twoheadrightarrow & S^2 \\
\downarrow{\scriptstyle a} & & \downarrow & & \| \\
B & \rightarrowtail & S^1 & \twoheadrightarrow & S^2.
\end{array}$$

Then we get a short exact sequence

$$\Omega S^2 \overset{\binom{a}{-b}}{\rightarrowtail} B\oplus P_{S^2} \twoheadrightarrow S^1$$

by Proposition 2.1. Since $B\oplus P_{S^2}$ admits a short exact sequence

$$V \rightarrowtail U \twoheadrightarrow B\oplus P_{S^2}$$

where $V\in\mathcal{V}$ and $U\in\mathcal{U}=\mathcal{W}$, we obtain the following commutative diagram by Proposition 2.2

$$\begin{array}{ccccc}
V & \rightarrowtail & Q & \overset{c}{\twoheadrightarrow} & \Omega S^2 \\
\| & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \binom{a}{-b}} \\
V & \rightarrowtail & U & \twoheadrightarrow & B\oplus P_{S^2} \\
 & & \downarrow & & \downarrow \\
 & & S^1 & = & S^1.
\end{array}$$

Thus $Q\in\mathcal{B}^-=\mathcal{H}$ and $\underline{ca}=0$. We claim that $\underline{a}$ is the cokernel of $\underline{c}$ in $\underline{\mathcal{H}}$.
If $r:\Omega S^2\to M$ is a morphism in $\mathcal{H}$ such that rc factors through $\mathcal{W}$, then there exists $e:U\to M$ such that $cr=ed$, since d is a left $\mathcal{T}$-approximation of Q. Hence by definition of push-out, we get the following commutative diagram

$$\begin{array}{ccccc}
Q & \overset{c}{\longrightarrow} & \Omega S^2 & & \\
\downarrow & & \downarrow{\scriptstyle \binom{a}{-b}} & \searrow{\scriptstyle r} & \\
U & \longrightarrow & B\oplus P_{S^2} & \dashrightarrow & M \\
 & \searrow{\scriptstyle e} & & &
\end{array}$$

which implies that $\underline{r}$ factors through $\underline{a}$. Since $\underline{a}$ is epimorphic in $\underline{\mathcal{H}}$ by Proposition 2.41, we get that $\underline{a}$ is the cokernel of $\underline{c}$.
Now we assume that $\underline{\mathcal{H}}$ has enough projectives.

By Proposition 2.42, all the projective objects in $\underline{\mathcal{H}}$ lie in $\underline{\Omega S}$. Let B be any indecomposable object in $\mathcal{H} - \mathcal{U}$ and $\beta : \Omega S \to B$ be a cokernel in $\underline{\mathcal{H}}$. Then by Lemma 2.28, we get a short exact sequence

$$\Omega S \overset{f}{\rightarrowtail} B' \twoheadrightarrow S'$$

where $B' \in \mathcal{H}$ and $B' \simeq B$ in $\underline{\mathcal{H}}$ and $S' \in \mathcal{S}$. Since ΩS admits a short exact sequence

$$\Omega S \overset{p}{\rightarrowtail} P_S \twoheadrightarrow S$$

we take a push-out of f and p, then we get the following commutative diagram

$$\begin{array}{ccccc}
\Omega S & \overset{f}{\rightarrowtail} & B' & \twoheadrightarrow & S' \\
{\scriptstyle p}\downarrow & & \downarrow & & \| \\
P_S & \rightarrowtail & Q' & \twoheadrightarrow & S' \\
\downarrow & & \downarrow & & \\
S & = & S. & &
\end{array}$$

From the second row we get $Q' \in \mathcal{S}$. Since B is indecomposable, it is a direct summand of B'. Hence by Lemma 2.9, B admits a short exact sequence

$$B \rightarrowtail Q' \twoheadrightarrow S''$$

where $S'' \in \mathcal{S}$. □

By duality, we have

Proposition 2.44. *Let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be a twin cotorsion pair satisfying $\mathcal{T} \subseteq \mathcal{U}$, then any object in $\underline{\mathcal{H}}$ is injective if and only if it lies in $\underline{\Omega^- \mathcal{V}}$.*

Proposition 2.45. *Let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be a twin cotorsion pair satisfying $\mathcal{T} \subseteq \mathcal{U}$, then any object $\mathcal{B} \in \mathcal{H}$ admits a monomorphism $\beta : B \to \Omega^- V$ in $\underline{\mathcal{H}}$ where $\Omega^- V \in \underline{\Omega^- \mathcal{V}}$.*

Proposition 2.46. *Let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be a twin cotorsion pair satisfying $\mathcal{T} \subseteq \mathcal{U}$, then the heart has enough injectives if and only if any object $B \in \mathcal{H} - \mathcal{T}$ admits a short exact sequence*

$$V_2 \rightarrowtail V_1 \twoheadrightarrow B$$

where $V_1, V_2 \in \mathcal{V}$.

2.8. **Localisation on the heart of a special twin cotorsion pair.** Let $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ be a twin cotorsion pair on $\mathcal{B}$ such that $\mathcal{T} = \mathcal{U}$, in this case we get $\mathcal{B}^+ = \mathcal{B}^- = \mathcal{B}$ and $\mathcal{W} = \mathcal{T}$, hence $\underline{\mathcal{H}} = \mathcal{B}/\mathcal{T}$. According to Theorem 2.34, $\mathcal{B}/\mathcal{T}$ is integral. Moreover, By Proposition 2.40 (resp. Proposition 2.44), we obtain that any object in $\Omega\mathcal{S}$ (resp. $\Omega^-\mathcal{V}$) is projective (resp. injective) in $\mathcal{B}/\mathcal{T}$.
Let R be the class of regular morphisms in $\mathcal{B}/\mathcal{T}$, then by Theorem [R, p173], the localisation $(\mathcal{B}/\mathcal{T})_R$ (if it exists) is abelian.

Till the end of this section we assume that $\mathcal{B}$ is skeletally small and k-linear over a field k and has a twin cotorsion pair $(\mathcal{S}, \mathcal{T}), (\mathcal{T}, \mathcal{V})$. We denote that by Proposition 2.12 it is equivalent to assume that $\mathcal{B}$ has a cotorsion pair $(\mathcal{S}, \mathcal{T})$ such that $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T}$ is contravariantly finite.

Let $\mathcal{D}$ be a category and R' is a class of morphisms on $\mathcal{D}$. If R' admits both a calculus of right fractions and a calculus of left fractions (for details, see [BM, §4]), then the Gabriel-Zisman localisation $\mathcal{D}_{R'}$ at R' (if it exists) has a very nice description. The objects in $\mathcal{D}_{R'}$ are the same as the objects in $\mathcal{D}$. The morphism from X to Y are of the form

$$X \overset{r}{\longleftarrow} A \overset{f}{\longrightarrow} Y$$

denoted by $[r, f]$ where r lies in R'.
The localisation functor from $\mathcal{D}$ to $\mathcal{D}_{R'}$ takes a morphism f to $[id, f]$. We denote this image by $[f]$. For $r \in R'$, $[r, id]$ is the inverse of $[r]$. We denote it x_r. Thus, every morphism has the form $[r, f] = [f]x_r$.

By [BM, Corollary 4.2], R admits both a calculus of right fractions and a calculus of left fractions.

For a subcategory $\mathcal{C} \subseteq \mathcal{B}$, we denote by $[\mathcal{C}]$ the full subcategory of $(\mathcal{B}/\mathcal{T})_R$ which has the same objects as $\mathcal{C}$.

Lemma 2.47. *We have* $\Omega\mathcal{S}/\mathcal{P} = \underline{\Omega\mathcal{S}} \simeq [\Omega\mathcal{S}]$.

Proof. We first show that a morphism $f : \Omega S \to B$ factors through $\mathcal{P}$ if and only if it factors through $\mathcal{T}$. Since $\mathcal{P} \subseteq \mathcal{U} = \mathcal{T}$, we only need to show f factors through $\mathcal{T}$ implies it factors through $\mathcal{P}$. Suppose f factors through $\mathcal{T}$. By definition ΩS admits the following short exact sequence

$$\Omega S \overset{q}{\rightarrowtail} P_S \twoheadrightarrow S$$

where $P_S \in \mathcal{P}$, $S \in \mathcal{S}$ and B admits the following short exact sequence

$$V_B \rightarrowtail W_B \overset{w_B}{\twoheadrightarrow} B.$$

As w_B is a right $\mathcal{U}$-approximation of B, there exists a morphism $a : \Omega S \to W_B$ such that $f = w_B a$. Since q is a left $\mathcal{T}$-approximation of ΩS, there exists a morphism $b : P \to W_B$ such that $bq = a$, hence $f = w_B bq$. Thus by definition we have $\Omega\mathcal{S}/\mathcal{P} = \underline{\Omega\mathcal{S}}$.
Let $L : \underline{\Omega\mathcal{S}} \to [\Omega\mathcal{S}]$ be the location of the localisation functor from $\mathcal{B}/\mathcal{T}$ to $(\mathcal{B}/\mathcal{T})_R$. We claim that it is an equivalence. Obviously it is dense, it is faithful by [BM, Lemma 4.4] and full by [BM, Lemma 5.4]. □

Denote by $\mathrm{Mod}\,\mathcal{C}$ the category of contravariant additive functors from a category $\mathcal{C}$ to $\mathrm{mod}\,k$ for any category $\mathcal{C}$. Let $\mathrm{mod}\,\mathcal{C}$ be the full subcategory of $\mathrm{Mod}\,\mathcal{C}$ consisting of objects A admitting an exact sequence:

$$\mathrm{Hom}_{\mathcal{C}}(-, C_1) \xrightarrow{\beta} \mathrm{Hom}_{\mathcal{C}}(-, C_0) \xrightarrow{\alpha} A \to 0$$

where $C_0, C_1 \in \mathcal{C}$.

Since $\underline{\Omega\mathcal{S}} \simeq [\Omega\mathcal{S}]$, We have $\mathrm{mod}(\Omega\mathcal{S}/\mathcal{P}) \simeq \mathrm{mod}[\Omega\mathcal{S}]$.

We give the following proposition which is an analogue of [BM, Lemma 5.5] (for more details, see [BM, §5]).

Proposition 2.48. *If* $(\mathcal{S}, \mathcal{T}), (\mathcal{T}, \mathcal{V})$ *is a twin cotorsion pair on* $\mathcal{B}$ *which is skeletally small, and let* R *denote the class of morphisms which are both monomorphic and epimorphic in* $\mathcal{B}/\mathcal{T}$*, then*

(a) *The projectives in* $(\mathcal{B}/\mathcal{T})_R$ *are exactly the objects in* $\Omega\mathcal{S}$.
(b) *The category* $(\mathcal{B}/\mathcal{T})_R$ *has enough projectives.*

For convenience, for any objects $X, Y \in \mathcal{B}$, we denote $\mathrm{Hom}_{[\mathcal{B}]}(X, Y)$ by $[X, Y]$. For any morphism $f : X \to Y$, we denote $\mathrm{Hom}_{[\mathcal{B}]}(-, [\underline{f}])$ by $- \circ [\underline{f}]$ and $\mathrm{Hom}_{[\mathcal{B}]}([\underline{f}], -)$ by $[\underline{f}] \circ -$.

Now we can prove the following theorem.

Theorem 2.49. *Let* $\mathcal{B}$ *be a skeletally small, Krull-Schimdt,* k*-linear exact category with enough projecitves and injectives, containing a twin cotorsion pair*

$$(\mathcal{S}, \mathcal{T}), (\mathcal{T}, \mathcal{V}).$$

Let R *denote the class of morphisms which are both monomorphic and epimorphic in* $\mathcal{B}/\mathcal{T}$ *and* $(\mathcal{B}/\mathcal{T})_R$ *denote the localisation of* $\mathcal{B}/\mathcal{T}$ *at* R*, then*

$$(\mathcal{B}/\mathcal{T})_R \simeq \mathrm{mod}(\Omega\mathcal{S}/\mathcal{P}).$$

Proof. It suffices to show $(\mathcal{B}/\mathcal{T})_R \simeq \mathrm{mod}[\Omega\mathcal{S}]$.
From any object $B \in (\mathcal{B}/\mathcal{T})_R$, there is a projective presentation of B:

$$\Omega S_1 \xrightarrow{[\underline{d_1}]} \Omega S_0 \xrightarrow{[\underline{d_0}]} B \to 0$$

Let ΩS be any object in $[\Omega\mathcal{S}]$, we get the following exact sequence:

$$[\Omega S, \Omega S_1] \xrightarrow{\Omega S \circ [\underline{d_1}]} [\Omega S, \Omega S_0] \xrightarrow{\Omega S \circ [\underline{d_0}]} [\Omega S, B] \to 0$$

which induces a exact sequence in mod$[\Omega\mathcal{S}]$:

$$[-,\Omega S_1] \xrightarrow{-\circ[\underline{d_1}]} [-,\Omega S_0] \xrightarrow{-\circ[\underline{d_0}]} [-,B] \to 0$$

Now we can define a functor $\Phi : (\mathcal{B}/\mathcal{T})_R \to \text{mod}[\Omega\mathcal{S}]$ as follows:

$$B \mapsto [-,B],$$
$$[\underline{f}] \mapsto -\circ[\underline{f}].$$

• Let us prove that Φ is faithful.

For any morphism $[\underline{f}] : B \to B'$ we have the following commutative diagram

$$\begin{array}{ccccccc}
\Omega S_1 & \xrightarrow{[\underline{d_1}]} & \Omega S_0 & \xrightarrow{[\underline{d_0}]} & B & \longrightarrow & 0 \\
{\scriptstyle [\underline{f_1}]}\downarrow & & \downarrow{\scriptstyle [\underline{f_0}]} & & \downarrow{\scriptstyle [\underline{f}]} & & \\
\Omega S_1' & \xrightarrow[{[\underline{d_1'}]}]{} & \Omega S_0 & \xrightarrow[{[\underline{d_0'}]}]{} & B & \longrightarrow & 0
\end{array}$$

in $(\mathcal{B}/\mathcal{T})_R$ which induces a commutative diagram in mod$[\Omega\mathcal{S}]$

$$\begin{array}{ccccccc}
[-,\Omega S_1] & \xrightarrow{-\circ[\underline{d_1}]} & [-,\Omega S_0] & \xrightarrow{-\circ[\underline{d_0}]} & [-,B] & \longrightarrow & 0 \\
{\scriptstyle -\circ[\underline{f_1}]}\downarrow & & \downarrow{\scriptstyle -\circ[\underline{f_0}]} & & \downarrow{\scriptstyle -\circ[\underline{f}]} & & \\
[-,\Omega S_1'] & \xrightarrow[-\circ[\underline{d_1'}]]{} & [-,\Omega S_0'] & \xrightarrow[-\circ[\underline{d_0'}]]{} & [-,B'] & \longrightarrow & 0.
\end{array}$$

Hence if $-\circ[\underline{f}] = 0$, we obtain $-\circ[\underline{d_0' f_0}] = 0$, which implies $[\underline{d_0' f_0}] = 0$. Thus $[\underline{f}] = 0$.

• Let us prove that Φ is full.

For any morphism $\alpha : [-,B] \to [-,B']$, we have the following commutative diagram

$$\begin{array}{ccccccc}
[-,\Omega S_1] & \xrightarrow{-\circ[\underline{d_1}]} & [-,\Omega S_0] & \xrightarrow{-\circ[\underline{d_0}]} & [-,B] & \longrightarrow & 0 \\
{\scriptstyle \alpha_1}\downarrow & & {\scriptstyle \alpha_0}\downarrow & & \downarrow{\scriptstyle \alpha} & & \\
[-,\Omega S_1'] & \xrightarrow[-\circ[\underline{d_1'}]]{} & [-,\Omega S_0'] & \xrightarrow[-\circ[\underline{d_0'}]]{} & [-,B'] & \longrightarrow & 0.
\end{array}$$

in mod$[\Omega\mathcal{S}]$. By Yoneda's Lemma, there exists $[\underline{f_i}] : \Omega S_i \to \Omega S_i'$ such that $\alpha_i = -\circ[\underline{f_i}]$. Hence there is a commutative diagram

$$\begin{array}{ccccccc}
[-,\Omega S_1] & \xrightarrow{-\circ[\underline{d_1}]} & [-,\Omega S_0] & \xrightarrow{-\circ[\underline{d_0}]} & [-,B] & \longrightarrow & 0 \\
{\scriptstyle -\circ[\underline{f_1}]}\downarrow & & \downarrow{\scriptstyle -\circ[\underline{f_0}]} & & \downarrow{\scriptstyle -\circ[\underline{f}]} & & \\
[-,\Omega S_1'] & \xrightarrow[-\circ[\underline{d_1'}]]{} & [-,\Omega S_0'] & \xrightarrow[-\circ[\underline{d_0'}]]{} & [-,B'] & \longrightarrow & 0.
\end{array}$$

in $(\mathcal{B}/\mathcal{T})_R$, thus $\alpha = -\circ[\underline{f}]$.

• Let us prove that Φ is dense:

We first show that mod$[\Omega\mathcal{S}]$ is abelian. It is enough to show that $[\Omega\mathcal{S}]$ has pseudokernels. Let $\alpha : \Omega S_1 \to \Omega S_0$ be a morphism in $[\Omega\mathcal{S}]$, then since $(\mathcal{B}/\mathcal{T})_R$ is abelian, there exists a kernel $\beta : K \to \Omega S_1$ in $(\mathcal{B}/\mathcal{T})_R$. By Proposition 2.48, there exists a epimorphism $\gamma : \Omega S \to K$. We observe that $\beta\gamma$ is a pseudokernel of α.

Let $F \in \text{mod}[\Omega\mathcal{S}]$ which admits an exact sequence

$$[-,\Omega S_1] \xrightarrow{-\circ\gamma} [-,\Omega S_0] \to F \to 0$$

where $\gamma \in [\Omega S_1, \Omega S_0]$. Let $B = \text{Coker}\,\gamma$ then we get an exact sequence

$$\Omega S_1 \xrightarrow{\gamma} \Omega S_0 \to B \to 0$$

in $(\mathcal{B}/\mathcal{T})_R$. Hence $F \simeq [-, B]$. □

2.9. **Examples.** In this section we give several examples of twin cotorsion pair, and we also give some view of the relation between the heart of a cotorsion pair and the hearts of its two components.
First we introduce some notations. Let $\mathcal{C}$ be a subcategory of $\mathcal{B}$, we set

(a) $\mathcal{C}^{\perp_n} = \{X \in \mathcal{B} \mid \operatorname{Ext}^i_{\mathcal{B}}(\mathcal{C}, X) = 0,\ 0 < i \leq n\}$.
(b) ${}^{\perp_n}\mathcal{C} = \{X \in \mathcal{B} \mid \operatorname{Ext}^i_{\mathcal{B}}(X, \mathcal{C}) = 0,\ 0 < i \leq n\}$.
(c) $\mathcal{C}^{\perp} = \{X \in \mathcal{B} \mid \operatorname{Ext}^i_{\mathcal{B}}(\mathcal{C}, X) = 0,\ \forall i > 0\}$.
(d) ${}^{\perp}\mathcal{C} = \{X \in \mathcal{B} \mid \operatorname{Ext}^i_{\mathcal{B}}(X, \mathcal{C}) = 0,\ \forall i > 0\}$.

According to [HO, §7.2], we give the following definition.

Definition 2.50. A cotorsion pair $(\mathcal{U}, \mathcal{V})$ is called a hereditary cotorsion pair if $\operatorname{Ext}^i_{\mathcal{B}}(\mathcal{U}, \mathcal{V}) = 0, i > 0$.

The following proposition can be easily checked by definition.

Proposition 2.51. *For a cotorsion pair $(\mathcal{U}, \mathcal{V})$, the following conditions are equivalent.*

(a) $(\mathcal{U}, \mathcal{V})$ *is hereditary.*
(b) $\mathcal{V} = \mathcal{U}^{\perp}$.
(c) $\mathcal{U} = {}^{\perp}\mathcal{V}$.
(d) $\Omega\mathcal{U} \subseteq \mathcal{U}$.
(e) $\Omega^{-}\mathcal{V} \subseteq \mathcal{V}$.

Remark 2.52. We can call a pair of subcategories $(\mathcal{U}, \mathcal{V})$ a *co-t-structure* on $\mathcal{B}$ if it is a hereditary cotorsion pair, since by the proposition above the hereditary cotorsion pair on $\mathcal{B}$ is just an analogue of the co-*t*-structure on triangulated category.

Example 2.53. We introduce two trivial hereditary cotorsion pairs:

$$(\mathcal{P}, \mathcal{B}) \text{ and } (\mathcal{B}, \mathcal{I}).$$

We observe that in these two cases the hearts are 0. These two cotorsion pairs also form a twin cotorsion pair

$$(\mathcal{P}, \mathcal{B}), (\mathcal{B}, \mathcal{I}).$$

We observe that its heart is also 0.

Example 2.54. Let Λ be an artin algebra and T be a cotilting module of finite injective dimension, denote

$$\mathcal{X} := {}^{\perp}T \text{ and } \mathcal{Y} := ({}^{\perp}T)^{\perp}.$$

By [AR, Theorem 5.4, Corollary 5.10, Proposition 3.3.], $(\mathcal{X}, \mathcal{Y})$ is a hereditary cotorsion pair. By [AR, Proposition 3.3, (c, iii)], we get

$$\mathcal{W} \subseteq (\operatorname{mod}\Lambda)^{+} \subseteq \mathcal{Y}.$$

Dually, by [AR, Proposition 3.3, (d, iii)], we get

$$\mathcal{W} \subseteq (\operatorname{mod}\Lambda)^{-} \subseteq \mathcal{X}.$$

Then $\mathcal{H} = (\operatorname{mod}\Lambda)^{+} \cap (\operatorname{mod}\Lambda)^{-} \subseteq \mathcal{X} \cap \mathcal{Y} = \mathcal{W}$, hence $\underline{\mathcal{H}} = 0$.
By [AR, Proposition 1.8], $({}^{\perp_1}T, ({}^{\perp_1}T)^{\perp_1})$ is a cotorsion pair. According to [AR, §2], ${}^{\perp_1}T, ({}^{\perp_1}T)^{\perp_1}$ is also a a cotorsion pair. Hence by definition

$$({}^{\perp}T, ({}^{\perp}T)^{\perp}), ({}^{\perp_1}T, ({}^{\perp_1}T)^{\perp_1})$$

form a twin cotorsion pair. We can also observe that its heart is trivial.

In fact, we have

Proposition 2.55. *If one cotorsion pair in a twin cotorsion pair $(\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})$ is hereditary, then this twin cotorsion pair has a trivial heart, i.e. its heart is zero.*

Proof. We prove that if $(\mathcal{S},\mathcal{T})$ is hereditary, then $\mathcal{W}=\mathcal{V}\cap\mathcal{S}=\mathcal{B}^+\cap\mathcal{B}^-$, another part is by dual. For any object $B\in\mathcal{B}^-$, there is a short exact category

$$B\rightarrowtail W^B\twoheadrightarrow S^B.$$

Since we have the following exact sequence

$$0=\operatorname{Ext}^1_{\mathcal{B}}(W^B,\mathcal{T})\to\operatorname{Ext}^1_{\mathcal{B}}(B,\mathcal{T})\to\operatorname{Ext}^2_{\mathcal{B}}(S^B,\mathcal{T})=0$$

which implies $B\in\mathcal{S}$. Hence $\mathcal{B}^-=\mathcal{S}$. Dually, $\mathcal{B}^+=\mathcal{V}$. Hence $\mathcal{W}\subseteq\mathcal{B}^+\cap\mathcal{B}^-=\mathcal{V}\cap\mathcal{S}\subseteq\mathcal{W}$, this implies $\underline{\mathcal{H}}=0$. □

Recall that $\mathcal{M}$ is *n-cluster tilting* if it satisfies the following conditions

(a) $\mathcal{M}$ is contravariantly finite and covariantly finite in $\mathcal{B}$,
(b) $\mathcal{M}^{\perp_n}=\mathcal{M}$.
(c) ${}^{\perp_n}\mathcal{M}=\mathcal{M}$.

A 2-cluster tilting subcategory is usually called *cluster tilting* subcategory.
Let $\mathcal{M}$ be a cluster tilting subcategory of $\mathcal{B}$. Remark that $\mathcal{P}\subseteq\mathcal{M}$ and $\mathcal{I}\subseteq\mathcal{M}$. For each object $B\in\mathcal{B}$, we have two short exact sequences

$$B\overset{f}{\rightarrowtail}M\twoheadrightarrow N,$$

$$N'\rightarrowtail M'\overset{g}{\twoheadrightarrow}B$$

that f (resp. g) is a left (resp. right) $\mathcal{M}$-approximation of B. We observe $N\in{}^{\perp_1}\mathcal{M}=\mathcal{M}$ (resp. $N'\in\mathcal{M}^{\perp_1}=\mathcal{M}$), therefore $(\mathcal{M},\mathcal{M})$ is a cotorsion pair. In this case, $\mathcal{W}=\mathcal{M}$ and $\mathcal{B}^+=\mathcal{B}^-=\mathcal{B}$, thus $\underline{\mathcal{H}}=\underline{\mathcal{B}}=\mathcal{B}/\mathcal{M}$, which is abelian also by [DL].
Moreover, any object in $\Omega\mathcal{M}$ (resp. $\Omega^-\mathcal{M}$) is projective (resp. injective) in $\mathcal{B}/\mathcal{M}$, and by Proposition 2.43,2.46, $\mathcal{B}/\mathcal{M}$ has enough projectives and enough injectives.

Proposition 2.56. *A subcategory $\mathcal{M}$ in $\mathcal{B}$ is cluster tilting if and only if $(\mathcal{M},\mathcal{M})$ is a cotorsion pair on $\mathcal{B}$.*

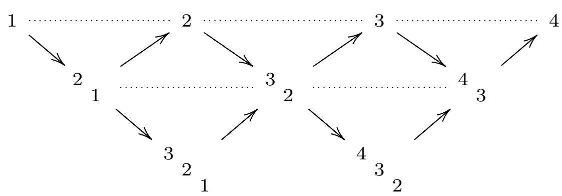
Proof. From the above discussion, we know that $(\mathcal{M},\mathcal{M})$ is a cotorsion pair if $\mathcal{M}$ is cluster tilting, so it remains to show the "only if" part. But it is just followed by the definition of cotorsion pair and Lemma 2.6. □

In the following examples, we denote by "$\circ$" in a quiver the objects belong to a subcategory and by "$\cdot$" the objects do not.

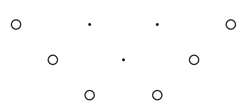
Example 2.57. Let Λ be the path algebra of the following quiver

$$1\longleftarrow 2\longleftarrow 3\longleftarrow 4$$

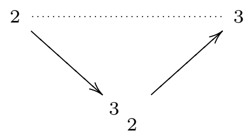
then we obtain the *AR*-quiver $\Gamma(\operatorname{mod}\Lambda)$ of $\operatorname{mod}\Lambda$.



Let $\mathcal{M}=\{X\in\operatorname{mod}\Lambda\mid\operatorname{Ext}^1_{\mathcal{B}}(X,\Lambda)=0\}$, then by [AR, Proposition 1.10, 1.9], $(\mathcal{M},\mathcal{M}^{\perp_1})$ is a cotorsion pair on $\operatorname{mod}\Lambda$. But

$\mathcal{M}=$ 

which consisting of all the direct sums of indecomposable projectives and indecomposable injectives. We observe that in fact $\mathcal{M} = \mathcal{M}^{\perp_1}$ and hence it is a cluster tilting subcategory. And the quiver of the quotient category $(\operatorname{mod}\Lambda)/\mathcal{M}$ is



which is equivalent to the AR-quiver of A_2.

Example 2.58. Take the notion of the former example, Let

$$\mathcal{M}' = \begin{array}{ccccccc} \circ & & \cdot & & \cdot & & \cdot \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

then by [AR, Proposition 1.10, 1.9], $(\mathcal{M}', \mathcal{M}'^{\perp_1})$ is a cotorsion pair and

$$\mathcal{M}'^{\perp_1} = \begin{array}{ccccccc} \circ & & \cdot & & \circ & & \circ \\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

hence it contains Λ. Obviously it is closed under extension and contravariantly finite, then by [AR, Proposition 1.10, 1.9], $(\mathcal{M}'^{\perp_1}, (\mathcal{M}'^{\perp_1})^{\perp_1})$ is also a cotorsion pair on $\operatorname{mod}\Lambda$ and

$$(\mathcal{M}'^{\perp_1})^{\perp_1} = \begin{array}{ccccccc} \circ & & \cdot & & \cdot & & \circ \\ & \cdot & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

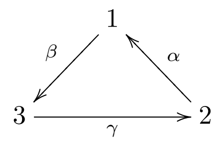
Thus we get a twin cotorsion pair

$$(\mathcal{M}', \mathcal{M}'^{\perp_1}), (\mathcal{M}'^{\perp_1}, (\mathcal{M}'^{\perp_1})^{\perp_1}).$$

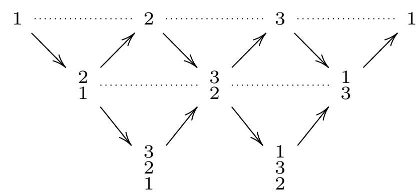
Then the quiver of $(\operatorname{mod}\Lambda)/\mathcal{M}'^{\perp_1}$ is $2 \to \begin{smallmatrix}3\\ \;2\end{smallmatrix}$. The quiver of quotient category $\Omega\mathcal{M}'/\mathcal{P}$ is just 2. Hence we get $((\operatorname{mod}\Lambda)/\mathcal{M}'^{\perp_1})_R \simeq \operatorname{mod}(\Omega\mathcal{M}'/\mathcal{P})$.

From Example 2.58, we see that there exist two cotorsion pairs which have non-trivial hearts form a twin cotorsion pair also having a non-trivial heart. From the following example, we see that even two components of a twin cotorsion pair have non-trivial hearts, the heart of the twin cotorsion pair itself can be zero.

Example 2.59. Let Λ be the k-algebra given by the quiver



and bound by $\alpha\beta = 0$ and $\beta\gamma\alpha = 0$. Then its AR-quiver $\Gamma(\operatorname{mod}\Lambda)$ is given by



Here, the first and the last columns are identified. Let

$$\mathcal{S}=\begin{array}{ccccccc}\cdot & & \circ & & \cdot & & \cdot\\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}\qquad \mathcal{T}=\begin{array}{ccccccc}\cdot & & \cdot & & \circ & & \cdot\\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

and

$$\mathcal{U}=\begin{array}{ccccccc}\circ & & \circ & & \cdot & & \circ\\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}\qquad \mathcal{V}=\begin{array}{ccccccc}\cdot & & \cdot & & \cdot & & \cdot\\ & \circ & & \cdot & & \circ & \\ & & \circ & & \circ & & \end{array}$$

The heart of cotorsion pair $(\mathcal{S},\mathcal{T})$ is add(1) and the heart of cotorsion pair $(\mathcal{U},\mathcal{V})$ is add(3). But when we consider the twin cotorsion pair $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$, we get $\mathcal{W}=\mathcal{V}$ and

$$(\operatorname{mod}\Lambda)^-/\mathcal{W}=\operatorname{add}(\,1\oplus 2\,)\text{ and }(\operatorname{mod}\Lambda)^+/\mathcal{W}=\operatorname{add}(\,3\,)$$

hence its heart is zero.

3. Half exact functors associated with general hearts on exact categories

To construct the associated half exact functor H, we first introduce two functors $\sigma^+:\underline{\mathcal{B}}\to\underline{\mathcal{B}}^+$ and $\sigma^-:\underline{\mathcal{B}}\to\underline{\mathcal{B}}^-$ in section 3.2, which are analogs of function functors associated with t-structures. In section 3.3, we show that these two functors commute. We prove the property of the half exact functor in section 3.4. The relationship between different hearts are studied in section 3.5. The last section contains several examples of our results.

3.1. **Preliminaries.** For briefly review of the important properties of exact categories, we refer to [L, §2]. For more details, we refer to [B]. We introduce the following properties used a lot in this paper, the proofs can be found in [B, §2].

We recall some in section 2, which also work for a single cotorsion pair.

Definition 3.1. For any $B\in\mathcal{B}$, we define B^+ and $\alpha_B:B\to B^+$ as follows:
Take two short exact sequences:

$$V_B\rightarrowtail U_B\xrightarrow{u_B}\twoheadrightarrow B\ ,\qquad U_B\overset{w'}{\rightarrowtail} W^0\twoheadrightarrow U^0$$

where $U_B,U^0\in\mathcal{U}$, W^0,$V_B\in\mathcal{V}$. In fact, $W^0\in\mathcal{W}$ since $\mathcal{U}$ is closed under extension. By Proposition 2.2, we get the following commutative diagram

$$\begin{array}{ccccc}V_B & \rightarrowtail & U_B & \xrightarrow{u_B} & B\\ \| & & \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle \alpha_B}\\ V_B & \rightarrowtail & W^0 & \xrightarrow[w]{} & B^+\\ & & \downarrow & & \downarrow\\ & & U^0 & = & U^0\end{array}\tag{7}$$

where the upper-right square is both a push-out and a pull-back.

By definition, $B^+\in\mathcal{B}^+$. We recall the following useful proposition.

Proposition 3.2. *For any $B\in\mathcal{B}$*

(a) *If $B\in\mathcal{B}^-$, then $B^+\in\mathcal{H}$.*
(b) *α_B is a left $\mathcal{B}^+$-approximation, and for an object $Y\in\mathcal{B}^+$, $\operatorname{Hom}_{\underline{\mathcal{B}}}(\underline{\alpha_B},Y):\operatorname{Hom}_{\underline{\mathcal{B}}}(B^+,Y)\to\operatorname{Hom}_{\underline{\mathcal{B}}}(B,Y)$ is bijective.*

By Proposition 3.2, we can define a functor σ^+ from $\underline{\mathcal{B}}$ to $\underline{\mathcal{B}}^+$ as follows:
For any object $B \in \mathcal{B}$, since all the $B^{+'}s$ are isomorphic to each other in $\underline{\mathcal{B}}$ by Proposition 3.2, we fix a B^+ for B. Let

$$\begin{aligned} \sigma^+ : \underline{\mathcal{B}} &\to \underline{\mathcal{B}}^+ \\ B &\mapsto B^+ \end{aligned}$$

and for any morphism $\underline{f} : B \to C$, we define $\sigma^+(\underline{f})$ as the unique morphism given by Proposition 3.2

$$\begin{array}{ccc} B & \xrightarrow{\underline{f}} & C \\ {\scriptstyle \underline{\alpha_B}}\downarrow & & \downarrow{\scriptstyle \underline{\alpha_C}} \\ B^+ & \underset{\sigma^+(\underline{f})}{\dashrightarrow} & C^+. \end{array}$$

Let $i^+ : \underline{\mathcal{B}}^+ \hookrightarrow \underline{\mathcal{B}}$ be the inclusion functor, then (σ^+, i^+) is an adjoint pair by Proposition 3.2.

Proposition 3.3. *The functor σ^+ has the following properties:*

(a) *σ^+ is an additive functor.*
(b) $\sigma^+|_{\underline{\mathcal{B}}^+} = \mathrm{id}_{\underline{\mathcal{B}}^+}$.
(c) *For any morphism $f : A \to B$, $\sigma^+(\underline{f}) = 0$ in $\underline{\mathcal{B}}$ if and only if f factors through $\mathcal{U}$. In particular, $\sigma^+(B) = 0$ if and only if $B \in \underline{\mathcal{U}}$.*

Proof. (a), (b) can be concluded easily by definition, we only prove (c).
The "if" part is followed by [L, Lemma 3.4].
Now suppose $\sigma^+(\underline{f}) = 0$ in $\underline{\mathcal{B}}$. By Proposition 3.2, we have the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xleftarrow{u_B} & U_B & \longleftarrow & V_B \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle w'} & & \| \\ A^+ & \xrightarrow[f^+]{} & B^+ & \xleftarrow[w]{} & W^0 & \longleftarrow & V_B \\ \downarrow & & \downarrow & & \downarrow & & \\ U_A^0 & \longrightarrow & U^0 & = & U^0 & & \end{array}$$

where $\underline{f}^+ = \sigma^+(\underline{f})$. Then f^+ factors through an object $W \in \mathcal{W}$.

$$\begin{array}{ccc} A^+ & \xrightarrow{f^+} & B^+ \\ & {\scriptstyle a}\searrow \quad \nearrow{\scriptstyle b} & \\ & W & \end{array}$$

Since w is a right $\mathcal{U}$-approximation of B^+, there exists a morphism $c : W \to W^0$ such that $b = wc$. Thus $\alpha_B f = f^+\alpha_A = ba\alpha_A = w(ca\alpha_A)$. By the definition of pull-back, there exists a morphism $d : A \to U_B$ such that $f = u_B d$. Thus f factors through $\mathcal{U}$. $\square$

Definition 3.4. For any object $B \in \mathcal{B}$, we define B^- and $\gamma_B : B^- \to B$ as follows:
Take the following two short exact sequences

$$B \overset{v^B}{\rightarrowtail} V^B \twoheadrightarrow U^B \ , \qquad V_0 \rightarrowtail W_0 \twoheadrightarrow V^B$$

where $V^B, V_0 \in \mathcal{V}$, and $W_0, U^B \in \mathcal{U}$. Then $W_0 \in \mathcal{W}$ holds since $\mathcal{V}$ is closed under extension. By Proposition 2.2, we get the following commutative diagram:

$$\begin{array}{ccccc} V_0 & = & V_0 & & \\ {\scriptstyle v}\downarrow & & \downarrow & & \\ B^- & \rightarrowtail & W_0 & \twoheadrightarrow & U^B \\ {\scriptstyle \gamma_B}\downarrow & & \downarrow & & \| \\ B & \underset{v^B}{\rightarrowtail} & V^B & \twoheadrightarrow & U^B. \end{array} \tag{8}$$

By definition $B^- \in \mathcal{B}^-$ and we have:

Proposition 3.5. [L, Proposition 3.6] *For any object $B \in \mathcal{B}$*

(a) *$B \in \mathcal{B}^+$ implies $B^- \in \mathcal{H}$.*
(b) *γ_B is a right $\mathcal{B}^-$-approximation. For any $X \in \mathcal{B}^-$, $\mathrm{Hom}_{\underline{\mathcal{B}}}(X, \underline{\gamma_B}) : \mathrm{Hom}_{\underline{\mathcal{B}}}(X, B^-) \to \mathrm{Hom}_{\underline{\mathcal{B}}}(X, B)$ is bijective.*

we define a functor σ^- from $\underline{\mathcal{B}}$ to $\underline{\mathcal{B}}^-$ as the dual of σ^+:

$$\begin{aligned} \sigma^- : \underline{\mathcal{B}} &\to \underline{\mathcal{B}}^- \\ B &\mapsto B^-. \end{aligned}$$

For any morphism $\underline{f} : B \to C$, we define $\sigma^-(\underline{f})$ as the unique morphism given by Proposition 3.5

$$\begin{array}{ccc} B^- & \overset{\sigma^-(\underline{f})}{\dashrightarrow} & C^- \\ {\scriptstyle \underline{\gamma_B}}\downarrow & & \downarrow{\scriptstyle \underline{\gamma_C}} \\ B & \underset{\underline{f}}{\longrightarrow} & C. \end{array}$$

Let $i^- : \underline{\mathcal{B}}^- \hookrightarrow \underline{\mathcal{B}}$ be the inclusion functor, then (i^-, σ^-) is an adjoint pair by Proposition 3.5.

Proposition 3.6. *The functor σ^- has the following properties:*

(a) *σ^- is an additive functor.*
(b) *$\sigma^-|_{\underline{\mathcal{B}}^-} = \mathrm{id}_{\underline{\mathcal{B}}^-}$.*
(c) *For any morphism $f : A \to B$, $\sigma^-(\underline{f}) = 0$ in $\underline{\mathcal{B}}$ if and only if f factors through $\mathcal{V}$. In particular, $\sigma^-(B) = 0$ if and only if $B \in \underline{\mathcal{V}}$.*

3.2. **Reflection sequences and coreflection sequences.** In the following two sections we fix a cotorsion pair $(\mathcal{U}, \mathcal{V})$. The reflection (resp. coreflection) sequences [AN] are defined on triangulated categories, but the definitions of the similar concepts on exact categories are not simple.

Let $\mathcal{C}$ be a subcategory of $\mathcal{B}$, denote by $\Omega\mathcal{C}$ (resp. $\Omega^-\mathcal{C}$) the subcategory of $\mathcal{B}$ consisting of objects ΩC (resp. $\Omega^- C$) such that there exists a short exact sequence

$$\Omega C \rightarrowtail P_C \twoheadrightarrow C \ (P_C \in \mathcal{P}, C \in \mathcal{C})$$
$$(\text{resp. } C \rightarrowtail I^C \twoheadrightarrow \Omega^- C \ (I^C \in \mathcal{I}, C \in \mathcal{C})).$$

Lemma 3.7. *$\Omega\mathcal{U} \subseteq \mathcal{B}^-$ and $\Omega^-\mathcal{V} \subseteq \mathcal{B}^+$.*

Proof. We only prove the first one, the second is dual.
An object $\Omega U \in \Omega\mathcal{U}$ admits two short exact sequences

$$\Omega U \overset{q}{\rightarrowtail} P_U \twoheadrightarrow U, \qquad \Omega U \overset{v'}{\rightarrowtail} V^{\Omega U} \twoheadrightarrow U^{\Omega U}$$

where $U, U^{\Omega U} \in \mathcal{U}$, $V^{\Omega U} \in \mathcal{V}$ and $P_U \in \mathcal{P}$. It is enough to show that $V^{\Omega U} \in \mathcal{U}$. Since $\mathrm{Ext}^1_{\mathcal{B}}(U, V^{\Omega U}) = 0$, there exists a morphism $p : P_U \to V^{\Omega U}$ such that $pq = v'$.

$$\begin{array}{ccccc} \Omega U & \overset{q}{\rightarrowtail} & P_U & \twoheadrightarrow & U \\ \| & & \downarrow{\scriptstyle p} & & \downarrow \\ \Omega U & \underset{v'}{\rightarrowtail} & V^{\Omega U} & \twoheadrightarrow & U^{\Omega U} \end{array}$$

Now we get a short exact sequence $P_U \rightarrowtail V^{\Omega U} \oplus U \twoheadrightarrow U^{\Omega U}$. Since $\mathcal{U}$ is closed under extension and direct summands, $V^{\Omega U} \in \mathcal{U}$. Thus $\Omega U \in \mathcal{B}^-$. □

Definition 3.8. Let B be any object in $\mathcal{B}$.

(a) A *reflection sequence* for B is a short exact sequence

$$B \overset{z}{\rightarrowtail} Z \twoheadrightarrow U$$

where $U \in \mathcal{U}$, $Z \in \mathcal{B}^+$ and there exists a commutative diagram

$$\begin{array}{ccccc} \Omega U & \overset{q}{\rightarrowtail} & P_U & \twoheadrightarrow & U \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle p} & & \| \\ B & \underset{z}{\rightarrowtail} & Z & \twoheadrightarrow & U \end{array}$$

with $P_U \in \mathcal{P}$ and x factoring through $\mathcal{U}$.

(b) A *coreflection sequence* for B is a short exact sequence

$$V \rightarrowtail K \overset{k}{\twoheadrightarrow} B$$

where $V \in \mathcal{V}$, $K \in \mathcal{B}^-$ and there exists a commutative diagram

$$\begin{array}{ccccc} V & \rightarrowtail & K & \overset{k}{\twoheadrightarrow} & B \\ \| & & \downarrow & & \downarrow{\scriptstyle y} \\ V & \rightarrowtail & I^V & \twoheadrightarrow & \Omega^- V \end{array}$$

with $I^V \in \mathcal{I}$ and y factoring through $\mathcal{V}$.

Lemma 3.9. *Let B be an object in $\mathcal{B}$. Then*

(a) *The short exact seqeunce $B \overset{\alpha_B}{\rightarrowtail} B^+ \twoheadrightarrow U^0$ in (2.1) is a reflection sequence for B.*

(b) *The short exact seqeunce $V_0 \rightarrowtail B^- \overset{\gamma_B}{\twoheadrightarrow} B$ in (2.2) is a coreflection sequence for B.*

(c) *For any reflection sequence $B \overset{z}{\rightarrowtail} Z \twoheadrightarrow U$ for B, we have $Z \simeq B^+$ in $\underline{\mathcal{B}}$.*

(d) *For any coreflection sequence $V \rightarrowtail K \overset{k}{\twoheadrightarrow} B$ for B, we have $K \simeq B^-$ in $\underline{\mathcal{B}}$.*

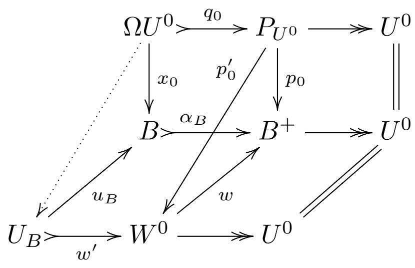
Proof. We only prove (a) and (c), the other two are dual.
(a) Since U^0 admits the following short exact sequence

$$\Omega U^0 \overset{q_0}{\rightarrowtail} P_{U^0} \twoheadrightarrow U^0$$

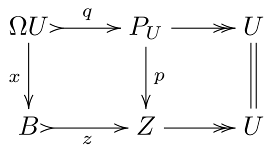
we get the following commutative diagram

$$\begin{array}{ccccc} \Omega U^0 & \overset{q_0}{\rightarrowtail} & P_{U^0} & \twoheadrightarrow & U^0 \\ {\scriptstyle x_0}\downarrow & & \downarrow{\scriptstyle p_0} & & \| \\ B & \underset{\alpha_B}{\rightarrowtail} & B^+ & \twoheadrightarrow & U^0. \end{array}$$

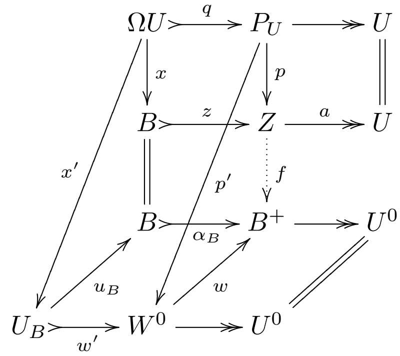
Since P_{U^0} is projective, there exists a morphism $p_0' : P_{U^0} \to W^0$ such that $wp_0' = p_0$, we get $\alpha_B x_0 = p_0 q_0 = wp_0' q_0$. Then x_0 factors through $U_B \in \mathcal{U}$ since (2.1) is a pull-back diagram.



Hence by definition $B \rightarrowtail^{\alpha_B} B^+ \twoheadrightarrow U^0$ is a reflection sequence for B.
(c) We first show that there exists a morphism $\underline{f} : Z \to B^+$ such that $\alpha_B = fz$.
The reflection sequence admits a commutative diagram



where the left square is a push-out by Proposition 2.1. Since x factors through $\mathcal{U}$, and u_B is a right $\mathcal{U}$-approximation of B, there exists a morphism $x' : \Omega U \to U_B$ such that $x = u_B x'$. Since $\mathrm{Ext}^1_{\mathcal{B}}(U, W^0) = 0$, there exists a morphism $p' : P_U \to W^0$ such that $w'x' = p'q$, thus $\alpha_B x = \alpha_B u_B x' = ww'x' = wp'q$. Then by the definition of push-out, there exists a morphism $f : Z \to B^+$ such that $\alpha_B = fz$.

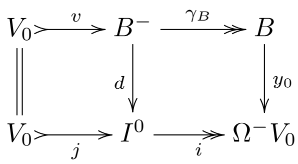


Since By Proposition 3.2, there is a morphism $g : B^+ \to Z$ such that $g\alpha_B = z$, we have a morphism $\underline{fg} : B^+ \to B^+$ such that $\underline{fg\alpha} = \underline{\alpha}$, which implies that $\underline{fg} = \underline{\mathrm{id}}_{B^+}$.
Now we prove that $\underline{gf} = \underline{\mathrm{id}}_Z$.
Since $(gf - \mathrm{id}_Z)z = 0$, we get a morphism $b : U \to B^+$ such that $gf - \mathrm{id}_Z = ba$. Since $\mathrm{Ext}^1_{\mathcal{B}}(U, V_B) = 0$, b factors through W^0, hence $\underline{gf} = \underline{\mathrm{id}}_Z$.
Thus $B^+ \simeq Z$ in $\underline{\mathcal{B}}$. $\square$

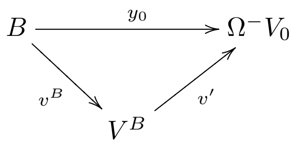
Proposition 3.10. *There exists an isomorphism of functors from $\underline{\mathcal{B}}$ to $\underline{\mathcal{H}}$*

$$\eta : \sigma^+ \circ \sigma^- \xrightarrow{\simeq} \sigma^- \circ \sigma^+.$$

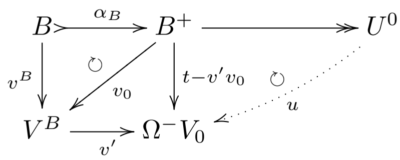
Proof. By Proposition 3.2 and 3.5 both $\sigma^+ \circ \sigma^-$ and $\sigma^- \circ \sigma^+$ are functors from $\underline{\mathcal{B}}$ to $\underline{\mathcal{H}}$.
By Lemma 3.9, We can take the following commutative diagram of short exact sequences



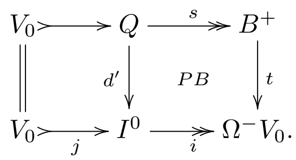
where y_0 factors through V^B since v^B is a left $\mathcal{V}$-approximation of $\mathcal{B}$.



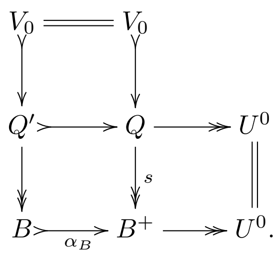
By Lemma 3.7 and Proposition 3.2, there exists a morphism $t : B^+ \to \Omega^- V_0$ such that $y_0 = t\alpha_B$. Since $\mathrm{Ext}^1_{\mathcal{B}}(U^0, V^B) = 0$, there exists a morphism $v_0 : B^+ \to V^B$ such that $v^B = v_0 \alpha_B$. Thus $t\alpha_B = v'v^B = v'v_0\alpha_B$, then we obtain that $t - v'v_0$ factors through U^0.



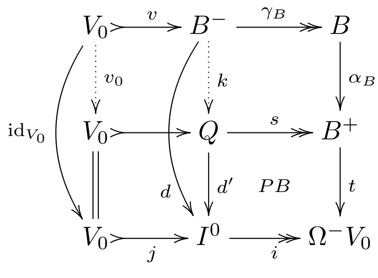
Since $\mathrm{Ext}^1_{\mathcal{B}}(U^0, V_0) = 0$, u factors through $I^0 \in \mathcal{V}$. Hence t factors through $\mathcal{V}$.
Take a pull-back of t and i, we get the following commutative diagram



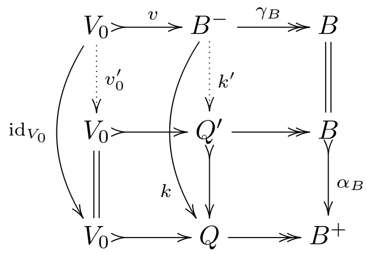
By [L, Lemma 2.11], we obtain $Q \in \mathcal{B}^+$. Now by Proposition 2.2, we get the following commutative diagram



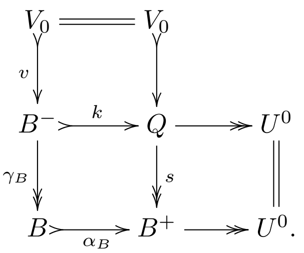
By the definition of pull-back, there exists a morphism $k : B \to Q$ such that $sk = \alpha_B\gamma_B$ and $d'k = d$. Hence we have the following diagram



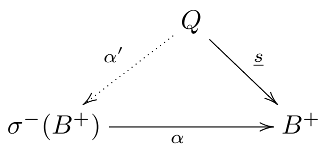
where the upper-left square commutes. Hence $jv_0 = d'kv = dv = j$, we can conclude that $v_0 = \mathrm{id}_{V_0}$ since j is monomorphic. By the same method we can get the following commutative diagram



where $v_0' = \mathrm{id}_{V_0}$. Therefore k' is isomorphic by [B, Corollary 3.2]. We obtain the following commutative diagram

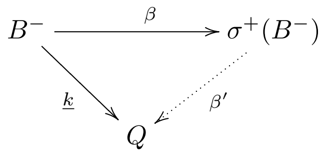


We get $Q \in \mathcal{B}^-$ by [L, Lemma 2.10], hence $Q \in \mathcal{H}$. Since t factors through $\mathcal{V}$, $V_0 \rightarrowtail Q \xrightarrow{s} B^+$ is a coreflection sequence for B^+. By Lemma 3.9, we have the following commutative diagram



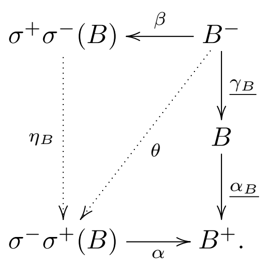
in $\underline{\mathcal{B}}$ where α' is isomorphic.

By duality we conclude that $B^- \overset{k}{\rightarrowtail} Q \twoheadrightarrow U^0$ is a reflection sequence for B^-. By Lemma 3.9, we have the following commutative diagram

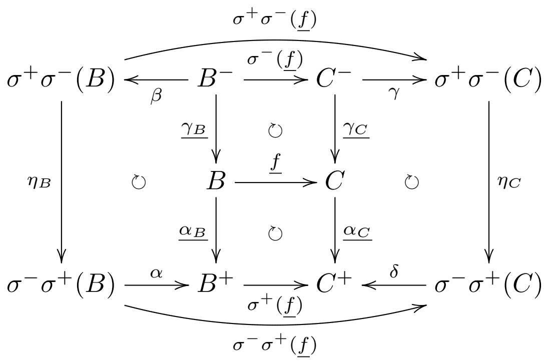


in $\underline{\mathcal{B}}$ where β' is isomorphic.

By Proposition 3.5, there exists a morphism $\theta : B^- \to \sigma^-\sigma^+(B)$ in $\underline{\mathcal{B}}$ such that $\alpha\theta = \underline{\alpha_B\gamma_B}$. Then by Proposition 3.2, there exists a unique morphism $\eta_B : \sigma^+\sigma^-(B) \to \sigma^-\sigma^+(B)$ such that $\eta_B\beta = \theta$. Hence we get the following commutative diagram



Then $\alpha\eta_B\beta = \underline{\alpha_B\gamma_B} = \underline{sk} = \alpha\alpha'\beta'\beta$, and we have $\eta_B = \alpha'\beta'$ by Proposition 3.2 and 3.5. Thus η_B is isomorphic. Let $\underline{f} : B \to C$ be a morphism in $\underline{\mathcal{B}}$, then we can get the following diagram by Proposition 3.2 and 3.5.



Since

$$\delta(\sigma^-\sigma^+(\underline{f}))\eta_B\beta = (\sigma^+(\underline{f}))\underline{\alpha_B\gamma_B} = \underline{\alpha_C\gamma_C}(\sigma^-(\underline{f})) = \delta\eta_C(\sigma^+\sigma^-(\underline{f}))\beta$$

we get $(\sigma^-\sigma^+(\underline{f}))\eta_B = \eta_C(\sigma^+\sigma^-(\underline{f}))$ by Proposition 3.2 and 3.5. Thus η is a natural isomorphism. □

3.3. **Half exact functor.** By Proposition 3.10, we have a natural isomorphism of functors from $\mathcal{B}$ to $\underline{\mathcal{H}}$

$$\sigma^+ \circ \sigma^- \circ \pi \simeq \sigma^- \circ \sigma^+ \circ \pi$$

where $\pi : \mathcal{B} \to \underline{\mathcal{B}}$ denotes the canonical functor. We denote $\sigma^- \circ \sigma^+ \circ \pi$ by

$$H : \mathcal{B} \to \underline{\mathcal{H}}.$$

The aim of this section is to show the following theorem.

Theorem 3.11. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathcal{B}$, the functor*

$$H : \mathcal{B} \to \underline{\mathcal{H}}$$

is half exact.

We call H the associated *half exact* functor to $(\mathcal{U}, \mathcal{V})$.

Proposition 3.12. *The functor H has the following properties:*

(a) *H is an additive functor.*
(b) *$H|_{\mathcal{H}} = \pi|_{\mathcal{H}}$.*
(c) *$H(\mathcal{U}) = 0$ and $H(\mathcal{V}) = 0$ hold. In particular, $H(\mathcal{P}) = 0$ and $H(\mathcal{I}) = 0$.*
(d) *For any reflection sequence $B \overset{z}{\rightarrowtail} Z \twoheadrightarrow U$ for B, $H(z)$ is an isomorphism in $\underline{\mathcal{H}}$.*
(e) *For any coreflection sequence $V \rightarrowtail K \overset{k}{\twoheadrightarrow} B$ for B, $H(k)$ is an isomorphism in $\underline{\mathcal{H}}$.*

Proof. (a) is followed by the definition of H and Proposition 3.3, 3.6 directly. Since $\mathcal{H} = \mathcal{B}^+ \cap \mathcal{B}^-$, by Proposition 3.3, 3.6, we get (b). By Proposition 3.3, $\sigma^+(\underline{\mathcal{B}}^+) = 0$, hence $H(\mathcal{U})$, $H(\mathcal{P}) = 0$ since $\mathcal{P} \subseteq \mathcal{U}$, dually we have $H(\mathcal{V}) = 0 = H(\mathcal{I})$. Hence (c) holds. For any reflection sequence, we have $H(z) = \sigma^- \circ \sigma^+(\underline{z}) = \sigma^-(\underline{g})$ where $g : B^+ \to Z$ is the morphism in the proof of Lemma 3.9. Since $\underline{g}$ is an isomorphism, we get $H(z)$ is an isomorphism in $\underline{\mathcal{H}}$. Thus (d) holds and by dual, (e) also holds. □

Lemma 3.13. *Let B be any object in $\mathcal{B}$,* $\mathrm{Hom}_{\underline{\mathcal{B}}}(\mathcal{U}, \mathcal{B}^+) = 0$ *and* $\mathrm{Hom}_{\underline{\mathcal{B}}}(B^-, \mathcal{V}) = 0$ *hold.*

Proof. We only show $\mathrm{Hom}_{\underline{\mathcal{B}}}(\mathcal{U}, \mathcal{B}^+) = 0$, the other one is dual.
Since $B \in \mathcal{B}^+$, it admits a short exact sequence $V_B \rightarrowtail W_B \twoheadrightarrow B$ where $W_B \in \mathcal{W}$. Then any morphism from an object in $\mathcal{U}$ to B factors through W_B, and the assertion follows. □

Lemma 3.14. *Let*

$$\begin{array}{ccccc} \Omega U & \overset{q}{\rightarrowtail} & P_U & \twoheadrightarrow & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} & & \| \\ A & \underset{g}{\rightarrowtail} & B & \underset{h}{\twoheadrightarrow} & U \end{array} \tag{9}$$

be a commutative diagram satisfying $U \in \mathcal{U}$ and $P_U \in \mathcal{P}$. Then the sequence

$$H(\Omega U) \xrightarrow{H(f)} H(A) \xrightarrow{H(g)} H(B) \to 0$$

is exact in $\underline{\mathcal{H}}$.

Proof. By Proposition 2.2, we get a commutative diagram by taking a pull-back of g and γ_B

$$\begin{array}{ccccc} V_0 & = & V_0 & & \\ \downarrow & & \downarrow & & \\ L & \overset{g'}{\rightarrowtail} & B^- & \twoheadrightarrow & U \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle \gamma_B} & & \| \\ A & \underset{g}{\rightarrowtail} & B & \underset{h}{\twoheadrightarrow} & U. \end{array}$$

By [L, Lemma 2.10], $L \in \mathcal{B}^-$. We can obtain a commutative diagram of short exact sequences

$$\begin{array}{ccccc} V_0 & \rightarrowtail & L & \overset{l}{\twoheadrightarrow} & A \\ \| & & \downarrow & & \downarrow{\scriptstyle g} \\ V_0 & \rightarrowtail & B^- & \overset{\gamma_B}{\twoheadrightarrow} & B \\ \| & & \downarrow & & \downarrow{\scriptstyle j} \\ V_0 & \rightarrowtail & I^0 & \twoheadrightarrow & \Omega^- V_0 \end{array}$$

where j factors through $\mathcal{V}$ by Lemma 3.9, hence

$$V_0 \rightarrowtail L \overset{l}{\twoheadrightarrow} A$$

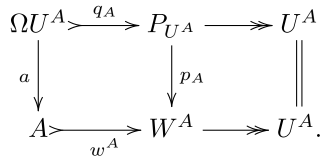
is a coreflection sequence for A. By Proposition 3.12, $H(l)$ and $H(\gamma_B)$ are isomorphic in $\underline{\mathcal{H}}$. Thus, replacing A by L and B by B^-, we may assume that $A, B \in \mathcal{B}^-$. Under this assumption, we show $H(g)$ is the cokernel of $H(f)$. We have $\Omega U \in \mathcal{B}^-$ by Lemma 3.7. For any $Q \in \mathcal{H}$, we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\underline{\mathcal{B}}}(H(B), Q) & \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(H(g),Q)} & \mathrm{Hom}_{\underline{\mathcal{B}}}(H(A), Q) & \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(H(f),Q)} & \mathrm{Hom}_{\underline{\mathcal{B}}}(H(\Omega U), Q) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_{\underline{\mathcal{B}}}(\sigma^+(B), Q) & \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(\sigma^+(\underline{g}),Q)} & \mathrm{Hom}_{\underline{\mathcal{B}}}(\sigma^+(A), Q) & \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(\sigma^+(\underline{f}),Q)} & \mathrm{Hom}_{\underline{\mathcal{B}}}(\sigma^+(\Omega U), Q) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_{\underline{\mathcal{B}}}(B, Q) & \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{g},Q)} & \mathrm{Hom}_{\underline{\mathcal{B}}}(A, Q) & \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{f},Q)} & \mathrm{Hom}_{\underline{\mathcal{B}}}(\Omega U, Q). \end{array}$$

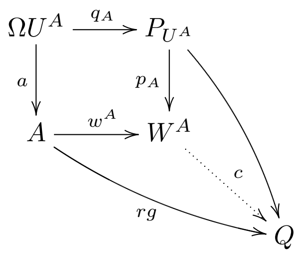
So it suffices to show the following sequence

$$0 \to \mathrm{Hom}_{\underline{\mathcal{B}}}(B, Q) \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{g},Q)} \mathrm{Hom}_{\underline{\mathcal{B}}}(A, Q) \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{f},Q)} \mathrm{Hom}_{\underline{\mathcal{B}}}(\Omega U, Q)$$

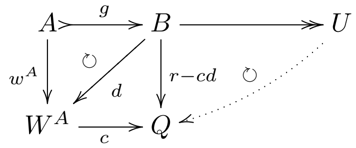
is exact.
We first show that $\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{g},Q)$ is injective. Let $r : B \to Q$ be any morphism such that $\underline{rg} = 0$. Take a commutative diagram of short exact sequences



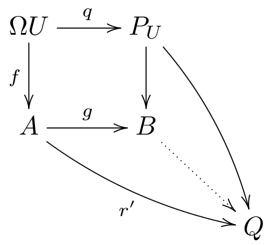
Since rga factors through $\mathcal{W}$ and $\mathrm{Ext}^1_{\mathcal{B}}(U^A, \mathcal{W}) = 0$, it factors through q_A. Thus there exists $c : W^A \to Q$ such that $cw^A = rg$.



As $\mathrm{Ext}^1_{\mathcal{B}}(U, W^A) = 0$, there exists $d : B \to W^A$ such that $w^A = dg$. Hence $rg = cw^A = cdg$, then $r - cd$ factors through U.



Since $\mathrm{Hom}_{\underline{\mathcal{B}}}(U,Q) = 0$ by Lemma 3.13, we get that $\underline{r} = 0$.
Assume $r' : A \to Q$ satisfies $\underline{r'f} = 0$, since $\mathrm{Ext}^1_{\mathcal{B}}(U, \mathcal{W}) = 0$, $r'f$ factors through q. As the left square of (3) is a push-out, we get the following commutative diagram.



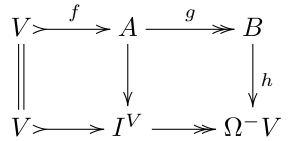
Hence r' factors through g. This shows the exactness of

$$\mathrm{Hom}_{\underline{\mathcal{B}}}(B,Q) \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{g},Q)} \mathrm{Hom}_{\underline{\mathcal{B}}}(A,Q) \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{B}}}(\underline{f},Q)} \mathrm{Hom}_{\underline{\mathcal{B}}}(\Omega U,Q).$$

□

Dually, we have the following:

Lemma 3.15. *Let*



be a commutative diagram satisfying $V \in \mathcal{V}$ and $I^V \in \mathcal{I}$. Then the sequence

$$0 \to H(A) \xrightarrow{H(g)} H(B) \xrightarrow{H(h)} H(\Omega^- V)$$

is exact in $\underline{\mathcal{H}}$.

Now we are ready to prove Theorem 3.11.

Proof. Let

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

be any short exact sequence in $\mathcal{B}$. By Proposition 2.1, we can get the following commutative diagram:

$$\begin{array}{ccccc}
\Omega U^A & \overset{b}{\rightarrowtail} & P_{U^A} & \twoheadrightarrow & U^A \\
{\scriptstyle a}\downarrow & & \downarrow & & \| \\
A & \overset{v^A}{\rightarrowtail} & V^A & \twoheadrightarrow & U^A \\
{\scriptstyle f}\downarrow & PO & \downarrow{\scriptstyle e} & & \| \\
B & \underset{c}{\rightarrowtail} & D & \twoheadrightarrow & U^A \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle d} & & \\
C & = & C. & &
\end{array}$$

From the first and second row from the top, we get an exact sequence $H(\Omega U^A) \xrightarrow{H(a)} H(A) \to 0$ by Lemma 3.14. From the first and the third row from the top, we get an exact sequence $H(\Omega U^A) \xrightarrow{H(fa)} H(B) \xrightarrow{H(c)} H(D) \to 0$ by Lemma 3.14. From the middle column, we get an exact sequence $0 \to H(D) \xrightarrow{H(d)} H(C)$ by Lemma 3.15. Now we can obtain an exact sequence $H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C)$. □

Now we prove the following general observation on half exact functors.

Corollary 3.16. *Let* $\mathcal{A}$ *be an abelian category and* $F : \mathcal{B} \to \mathcal{A}$ *be a half exact functor satisfying* $F(\mathcal{P}) = 0$ *and* $F(\mathcal{I}) = 0$. *Then for any short exact sequence*

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

in $\mathcal{B}$, *there exist morphisms* $h : C \to \Omega^- A$ *and* $h' : \Omega C \to A$ *such that the following sequence*

$$\cdots \xrightarrow{F(\Omega h')} F(\Omega A) \xrightarrow{F(\Omega f)} F(\Omega B) \xrightarrow{F(\Omega g)} F(\Omega C) \xrightarrow{F(h')} F(A) \xrightarrow{F(f)} F(B)$$
$$\xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(\Omega^- A) \xrightarrow{F(\Omega^- f)} F(\Omega^- B) \xrightarrow{F(\Omega^- g)} F(\Omega^- C) \xrightarrow{F(\Omega^- h)} \cdots$$

is exact in $\mathcal{A}$.

Proof. Since $F(\mathcal{P}) = 0$ (resp. $F(\mathcal{I}) = 0$), the functor F can be regarded as a functor from $\mathcal{B}/\mathcal{P}$ (resp. $\mathcal{B}/\mathcal{I}$) to $\mathcal{A}$.
For convenience, we fix the following commutative diagram:

$$\begin{array}{ccccc}
\Omega A & \overset{q_A}{\rightarrowtail} & P_A & \overset{p_A}{\twoheadrightarrow} & A \\
{\scriptstyle \Omega f}\downarrow & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle f} \\
\Omega B & \overset{q_B}{\rightarrowtail} & P_B & \overset{p_B}{\twoheadrightarrow} & B \\
{\scriptstyle \Omega g}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle g} \\
\Omega C & \underset{q_C}{\rightarrowtail} & P_C & \underset{p_C}{\twoheadrightarrow} & C.
\end{array}$$

Since

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

admits two commutative diagrams

$$\begin{array}{ccccc} \Omega C & \overset{q_C}{\rightarrowtail} & P_C & \overset{p_C}{\twoheadrightarrow} & C \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle l} & & \| \\ A & \underset{f}{\rightarrowtail} & B & \underset{g}{\twoheadrightarrow} & C, \end{array} \qquad \begin{array}{ccccc} A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \\ \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle h} \\ A & \rightarrowtail & I^A & \underset{j}{\twoheadrightarrow} & \Omega^- A \end{array}$$

we get two short exact sequences by Proposition 2.1:

$$\Omega C \overset{\binom{-q_C}{h'}}{\rightarrowtail} P_C \oplus A \overset{(l\ f)}{\twoheadrightarrow} B, \qquad B \overset{\binom{i}{g}}{\rightarrowtail} I^A \oplus C \overset{(-j\ h)}{\twoheadrightarrow} \Omega^- A.$$

They induce two exact sequences

$$F(\Omega C) \xrightarrow{F(h')} F(A) \xrightarrow{F(f)} F(B), \qquad F(B) \xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(\Omega^- A).$$

by Theorem 3.11. Now it is enough to show that

(a) $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ induces an exact sequence

$$F(\Omega A) \xrightarrow{F(\Omega f)} F(\Omega B) \xrightarrow{F(\Omega g)} F(\Omega C) \xrightarrow{F(h')} F(A).$$

(b) $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ induces an exact sequence

$$F(C) \xrightarrow{F(h)} F(\Omega^- A) \xrightarrow{F(\Omega^- f)} F(\Omega^- B) \xrightarrow{F(\Omega^- g)} F(\Omega^- C).$$

We only show the first one, the second is by dual.

The short exact sequence $\Omega C \overset{\binom{-q_C}{h'}}{\rightarrowtail} P_C \oplus A \overset{(l\ f)}{\twoheadrightarrow} B$ admits the following commutative diagram

$$\begin{array}{ccccc} \Omega B & \overset{q_B}{\rightarrowtail} & P_B & \overset{p_B}{\twoheadrightarrow} & B \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \binom{k'}{m}} & & \| \\ \Omega C & \underset{\binom{-q_C}{h'}}{\rightarrowtail} & P_C \oplus A & \underset{(l\ f)}{\twoheadrightarrow} & B \end{array}$$

which induces the following exact sequence

$$\Omega B \overset{\binom{-q_B}{x}}{\rightarrowtail} P_A \oplus \Omega C \xrightarrow{\begin{pmatrix} k' & -q_C \\ m & h' \end{pmatrix}} \!\!\!\twoheadrightarrow P_C \oplus A.$$

We prove that $x + \Omega g$ factors through $\mathcal{P}$.

Since $fm + lk' = p_B \Rightarrow gfm + glk' = gp_B \Rightarrow p_C k' = p_C k$, there exists a morphism $n : P_B \to \Omega C$ such that $k - k' = q_C n$. Thus we have $q_C n q_B = k q_B - k' q_B = q_C \Omega g + q_C x$, which implies that $x + \Omega g = n q_B$.

Hence we obtain an exact sequence $F(\Omega B) \xrightarrow{F(\Omega g)} F(\Omega C) \xrightarrow{F(h')} F(A)$.

Since we have the following commutative diagram

$$\begin{array}{ccccc} \Omega A & \overset{q_A}{\rightarrowtail} & P_A & \overset{p_A}{\twoheadrightarrow} & A \\ {\scriptstyle x'}\downarrow & & \downarrow{\scriptstyle \binom{s}{t}} & & \downarrow{\scriptstyle \binom{0}{1}} \\ \Omega B & \underset{\binom{-q_B}{x}}{\rightarrowtail} & P_B \oplus \Omega C & \underset{\begin{pmatrix} k' & -q_C \\ m & h' \end{pmatrix}}{\twoheadrightarrow} & P_C \oplus A \end{array}$$

we can show that $x' + \Omega f$ factors through $\mathcal{P}$ using the same method.

Hence we get the following exact sequence

$$F(\Omega A) \xrightarrow{F(\Omega f)} F(\Omega B) \xrightarrow{F(\Omega g)} F(\Omega C) \xrightarrow{F(h')} F(A).$$

Now we obtain a long exact sequence

$$\cdots \xrightarrow{F(\Omega h')} F(\Omega A) \xrightarrow{F(\Omega f)} F(\Omega B) \xrightarrow{F(\Omega g)} F(\Omega C) \xrightarrow{F(h')} F(A) \xrightarrow{F(f)} F(B)$$
$$\xrightarrow{F(g)} F(C) \xrightarrow{F(h)} F(\Omega^- A) \xrightarrow{F(\Omega^- f)} F(\Omega^- B) \xrightarrow{F(\Omega^- g)} F(\Omega^- C) \xrightarrow{F(\Omega^- h)} \cdots$$

in $\underline{\mathcal{H}}$. □

Since $H(\mathcal{P}) = H(\mathcal{I}) = 0$, we can see from this proposition that H has the property we claimed in the introduction.

For two subcategories $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B}$, we denote $\operatorname{add}(\mathcal{B}_1 * \mathcal{B}_2)$ by the subcategory which consists by the objects X which admits a short exact sequence

$$B_1 \rightarrowtail X \oplus Y \twoheadrightarrow B_2$$

where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$.

Proposition 3.17. *For any cotorsion pair $(\mathcal{U}, \mathcal{V})$ on $\mathcal{B}$ and any object $B \in \mathcal{B}$, the following are equivalent.*

(a) $H(B) = 0$.
(b) $B \in \operatorname{add}(\mathcal{U} * \mathcal{V})$.

Proof. We first prove that (a) implies (b).
By Proposition 3.6, since $H(B) = \sigma^- \circ \sigma^+(B) = 0$, we get that $B^+ \in \mathcal{V}$, hence from the following commutative diagram

$$\begin{array}{ccccc}
V_B & \rightarrowtail & U_B & \overset{u_B}{\twoheadrightarrow} & B \\
\| & & \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle \alpha_B} \\
V_B & \rightarrowtail & W^0 & \underset{w}{\twoheadrightarrow} & B^+ \\
 & & \downarrow & & \downarrow \\
 & & U^0 & = & U^0
\end{array}$$

we get a short exact sequence $U_B \rightarrowtail B \oplus W^0 \twoheadrightarrow B^+$, which implies that $B \in \operatorname{add}(\mathcal{U} * \mathcal{V})$.
We show that (b) implies (a).
This is followed by Theorem 3.11 and Proposition 3.12. □

We denote $\operatorname{add}(\mathcal{U} * \mathcal{V})$ by $\mathcal{K}$.

The kernel of H becomes simple in the following cases.

Corollary 3.18. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair on $\mathcal{B}$, then*

(a) *If $\mathcal{U} \subseteq \mathcal{V}$, then $H(B) = 0$ if and only if $B \in \mathcal{V}$.*
(b) *If $\mathcal{V} \subseteq \mathcal{U}$, then $H(B) = 0$ if and only if $B \in \mathcal{U}$.*

Proof. This is followed by Proposition 3.17 directly. □

3.4. **Relationship between different hearts.** The half exact functor constructed in the previous section gives a useful to study the relationship between the hearts of different cotorsion pairs on $\mathcal{B}$. First, we start with fixing some notations

Let $i \in \{1, 2\}$. Let $(\mathcal{U}_i, \mathcal{V}_i)$ be a cotorsion pair on $\mathcal{B}$ and $\mathcal{W}_i = \mathcal{U}_i \cap \mathcal{V}_i$. Let $\mathcal{B}_i^+$ and $\mathcal{B}_i^-$ be the subcategories of B defined in (1.1) and (1.2).

Let $\mathcal{H}_i := \mathcal{B}_i^+ \cap \mathcal{B}_i^-$, then $\mathcal{H}_i/\mathcal{W}_i$ is the heart of $(\mathcal{U}_i, \mathcal{V}_i)$. Let $\pi_i : \mathcal{B} \to \mathcal{B}/\mathcal{W}_i$ be the canonical functor and $\iota_i : \mathcal{H}_i/\mathcal{W}_i \hookrightarrow \mathcal{B}/\mathcal{W}_i$ be the inclusion functor.

If $H_2(\mathcal{W}_1) = 0$, which means $\mathcal{W}_1 \subseteq \mathcal{K}_2$ by Proposition 3.17, then there exists a functor $h_{12} : \mathcal{B}/\mathcal{W}_1 \to \mathcal{H}_2/\mathcal{W}_2$ such that $H_2 = h_{12}\pi_1$.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\pi_1} & \mathcal{B}/\mathcal{W}_1 \\ & {\scriptstyle H_2}\searrow \quad \swarrow{\scriptstyle h_{12}} & \\ & \mathcal{H}_2/\mathcal{W}_2 & \end{array}$$

Hence we get a functor $\beta_{12} := h_{12}\iota_1 : \mathcal{H}_1/\mathcal{W}_1 \to \mathcal{H}_2/\mathcal{W}_2$.

Lemma 3.19. *The following conditions are equivalent to each other.*

(a) $H_1(\mathcal{U}_2) = H_1(\mathcal{V}_2) = 0$.
(b) $\mathcal{K}_2 \subseteq \mathcal{K}_1$.

Proof. By Proposition 3.12 and Theorem 3.11, (b) implies (a). Now we prove that (a) implies (b). By Proposition 3.17, we get $\mathcal{U}_2 \subseteq \mathcal{K}_1$ and $\mathcal{V}_2 \subseteq \mathcal{K}_1$. Let $X \in \mathcal{K}_2$, then by definition, it admits a short exact sequence

$$U_2 \rightarrowtail X \oplus Y \twoheadrightarrow V_2$$

where $U_2 \in \mathcal{U}_2$ and $V_2 \in \mathcal{V}_2$. Since $U_2, V_2 \in \mathcal{K}_1$, by definition, there exist two objects A and B such that $U_2 \oplus A, V_2 \oplus B \in \mathcal{U}_1 * \mathcal{V}_1$. Thus we get a short exact sequence

$$U_2 \oplus A \rightarrowtail X \oplus Y \oplus A \oplus B \twoheadrightarrow V_2 \oplus B.$$

Hence $X \in \operatorname{add}((\mathcal{U}_1 * \mathcal{V}_1) * (\mathcal{U}_1 * \mathcal{V}_1)) = \operatorname{add}(\mathcal{U}_1 * (\mathcal{V}_1 * \mathcal{U}_1) * \mathcal{V}_1) = \operatorname{add}(\mathcal{U}_1 * \mathcal{U}_1 * \mathcal{V}_1 * \mathcal{V}_1) = \operatorname{add}(\mathcal{U}_1 * \mathcal{V}_1)$, which implies that $\mathcal{K}_2 \subseteq \mathcal{K}_1$. □

Proposition 3.20. *The functor β_{12} is half exact. Moreover, if $\mathcal{K}_1 \subseteq \mathcal{K}_2$, then β_{12} is exact and $(\mathcal{H}_1 \cap \mathcal{K}_2)/\mathcal{W}_1$ is a Serre subcategory of $\mathcal{H}_1/\mathcal{W}_1$.*

Proof. Let $0 \to A \xrightarrow{\rho} B \xrightarrow{\mu} C \to 0$ be a short exact sequence in $\mathcal{H}_1/\mathcal{W}_1$, then μ admits a morphism $g : B \twoheadrightarrow C$ such that $\pi_1(g) = \beta$. We get the following commutative diagram

$$\begin{array}{ccccc} V_C & \rightarrowtail & K_g & \overset{k_g}{\twoheadrightarrow} & B \\ \| & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle g} \\ V_C & \rightarrowtail & W_C & \underset{w_C}{\twoheadrightarrow} & C \end{array}$$

where $V_C \in \mathcal{V}_1$ and $W_C \in \mathcal{W}_1$. Then we obtain a short exact sequence

$$K_g \overset{\binom{-a}{k_g}}{\rightarrowtail} B \oplus W_C \overset{(g\ w_C)}{\twoheadrightarrow} C.$$

By [L, Lemma 4.1], $K_g \in \mathcal{B}_j^-$. By [L, Definition 3.8], $K_g \in \mathcal{B}_1^+$. Hence $K_g \in \mathcal{H}_1$. By [L, Theorem 4.3], μ is the cokernel of $\pi_1(k_g)$. By dual of [L, Theorem 3.10], $\pi_1(k_g)$ is the kernel of μ. Hence $K_g \simeq A$ in $\mathcal{H}_1/W_1$. By Theorem 3.11, We get the an exact sequence

$$H_2(K_g) \xrightarrow{H_2(k_g)} H_2(B) \xrightarrow{H_2(g)} H_2(C)$$

which implies the following following exact sequence

$$\beta_{12}(A) \xrightarrow{\beta_{12}(\rho)} \beta_{12}(B) \xrightarrow{\beta_{12}(\mu)} \beta_{12}(C).$$

Hence β_{12} is half exact. Now we prove that if $\mathcal{K}_1 \subseteq \mathcal{K}_2$, which means $H_2(\mathcal{U}_1) = 0 = H_2(\mathcal{V}_1)$, then β_{12} is exact.
In this case, we only need to show that $\beta_{12}(\rho)$ is a monomorphism and $\beta_{12}(\mu)$ is an epimorphism. We

show that $\beta_{12}(\mu)$ is an epimorphism, the other part is by dual.
Since we have the following commutative diagram

$$\begin{array}{ccccc}
B & \overset{w^B}{\rightarrowtail} & W^B & \twoheadrightarrow & U^B \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle b} & & \Vert \\
C & \underset{c_g}{\rightarrowtail} & C_g & \underset{s}{\twoheadrightarrow} & U^B
\end{array}$$

where $W_B \in \mathcal{W}_1$ and $U^B \in \mathcal{U}_1$. Since μ is epimorphism, by [L, Corollary 3.11], $C_g \in \mathcal{U}_1$. Since we have the following short exact sequence

$$B \overset{\binom{g}{-h}}{\rightarrowtail} C \oplus W^B \overset{(c_g\ b)}{\twoheadrightarrow} C_g.$$

By Theorem 3.11, We have an exact sequence $H_2(B) \xrightarrow{H_2(g)} H_2(C) \to 0$, which induces the following exact sequence

$$\beta_{12}(B) \xrightarrow{\beta_{12}(\mu)} \beta_{12}(C) \to 0.$$

Now we prove that $(\mathcal{H}_1 \cap \mathcal{K}_2)/\mathcal{W}_1$ is a Serre subcategory of $\mathcal{H}_1/\mathcal{W}_1$.
Let $0 \to A \xrightarrow{\rho} B \xrightarrow{\mu} C \to 0$ be a short exact sequence in $\mathcal{H}_1/\mathcal{W}_1$.
If $B \in (\mathcal{H}_1 \cap \mathcal{K}_2)/\mathcal{W}_1$, since β_{12} is exact and $\beta_{12}(B) = 0$ by Proposition 3.17, we have $\beta_{12}(A) = 0 = \beta_{12}(C)$, which implies that $A, C \in (\mathcal{H}_1 \cap \mathcal{K}_2)/\mathcal{W}_1$.
If $A, C \in (\mathcal{H}_1 \cap \mathcal{K}_2)/\mathcal{W}_1$, since we have the following short exact sequence

$$K_g \overset{\binom{-a}{k_g}}{\rightarrowtail} B \oplus W_C \overset{(g\ w_C)}{\twoheadrightarrow} C.$$

in $\mathcal{B}$ such that $K_g \simeq A$ in $\mathcal{H}_1/W_1$, we get that $\mathcal{B} \in \operatorname{add}((\mathcal{U}_1 * \mathcal{V}_1) * (\mathcal{U}_1 * \mathcal{V}_1)) = \operatorname{add}(\mathcal{U}_1 * \mathcal{V}_1)$. Hence $B \in (\mathcal{H}_1 \cap \mathcal{K}_2)/\mathcal{W}_1$. □

We prove the following proposition, and we recall that a similar property has been proved for triangulated case in [ZZ, Lemma 6.3].

Proposition 3.21. *Let $(\mathcal{U}_1, \mathcal{V}_1)$, $(\mathcal{U}_2, \mathcal{V}_2)$ be cotorsion pairs on $\mathcal{B}$. If $\mathcal{W}_1 \subseteq \mathcal{K}_2 \subseteq \mathcal{K}_1$, then we have a natural isomorphism $\beta_{21}\beta_{12} \simeq \mathrm{id}_{\mathcal{H}_1/\mathcal{W}_1}$ of functors.*

Proof. Let $B \in \mathcal{H}_1$. By Definition 3.1 and 3.4, we get the following commutative diagrams

$$\begin{array}{ccccc}
V_B & \rightarrowtail & U_B & \twoheadrightarrow & B \\
\Vert & & \downarrow & & \downarrow{\scriptstyle s_B} \\
V_B & \rightarrowtail & W^0 & \twoheadrightarrow & B_2^+ \\
 & & \downarrow & & \downarrow \\
 & & U^0 & = & U^0,
\end{array}
\qquad
\begin{array}{ccccc}
V_0 & = & V_0 & & \\
\downarrow & & \downarrow & & \\
(B_2^+)_2^- & \rightarrowtail & W_0 & \twoheadrightarrow & U^{B_2} \\
{\scriptstyle t_B}\downarrow & & \downarrow & & \Vert \\
B_2^+ & \rightarrowtail & V^{B_2} & \twoheadrightarrow & U^{B_2}
\end{array}$$

where $U_B, U^0, U^{B_2} \in \mathcal{U}_2$, $V_B, V^{B_2}, V_0 \in \mathcal{V}_2$, $W_0, W^0 \in \mathcal{W}_2$ and $(B_2^+)_2^- = H_2(B)$ in $\mathcal{B}/\mathcal{W}_2$. By Lemma 3.19, we get $H_1(\mathcal{U}_2) = H_1(\mathcal{V}_2) = 0$, by Lemma 3.9 and Theorem 3.11, we get two isomorphisms $B \xrightarrow{H_1(s_B)} H_1(B_2^+)$ and $H_1((B_2^+)_2^-) \xrightarrow{H_1(t_B)} H_1(B_2^+)$ in $\mathcal{H}_1/\mathcal{W}_1$. Since $H_1((B_2^+)_2^-) = \beta_{21}\beta_{12}(B)$, we get a isomorphism $\rho_B := H_1(t_B)^{-1}H_1(s_B) : B \to \beta_{21}\beta_{12}(B)$ on $\mathcal{H}_1/\mathcal{W}_1$. Let $f : B \to C$ be a morphism in $\mathcal{H}_1$, we also denote it image in $\mathcal{H}_1/\mathcal{W}_1$ by f. By the definition of H_2, we get the following commutative diagrams in $\mathcal{B}$

$$\begin{array}{ccc} B & \xrightarrow{s_B} & B_2^+ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^+} \\ C & \xrightarrow[s_C]{} & C_2^+, \end{array} \qquad \begin{array}{ccc} (B_2^+)_2^- & \xrightarrow{t_B} & B_2^+ \\ {\scriptstyle (f^+)^-}\downarrow & & \downarrow{\scriptstyle f^+} \\ (C_2^+)_2^- & \xrightarrow[t_C]{} & C_2^+ \end{array}$$

where $\pi_2((f^+)^-) = H_2(f)$. Hence we obtain the following commutative diagram in $\mathcal{H}_1/\mathcal{W}_1$

$$\begin{array}{ccc} B & \xrightarrow{\rho_B} & \beta_{21}\beta_{12}(B) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta_{21}\beta_{12}(f)} \\ C & \xrightarrow[\rho_C]{} & \beta_{21}\beta_{12}(C) \end{array}$$

which implies that $\beta_{21}\beta_{12} \simeq \mathrm{id}_{\mathcal{H}_1/\mathcal{W}_1}$. □

According to Proposition 3.21, we obtain the following corollary immediately.

Corollary 3.22. *If $\mathcal{K}_1 = \mathcal{K}_2$, then we have an equivalence $\mathcal{H}_1/\mathcal{W}_1 \simeq \mathcal{H}_2/\mathcal{W}_2$ between two hearts.*

Let $S = \{\alpha \in \mathrm{Mor}(\mathcal{H}_2/\mathcal{W}_2) \mid \mathrm{Ker}(\alpha), \mathrm{Coker}(\alpha) \in (\mathcal{H}_2 \cap \mathcal{K}_1/\mathcal{W}_2\}$ and let $\overline{\mathcal{H}_2}$ be localization of $\mathcal{H}_2/\mathcal{W}_2$ respect to $(\mathcal{H}_2 \cap \mathcal{K}_1/\mathcal{W}_2$, then $\overline{\mathcal{H}_2}$ is abelian. Since β_{21} is exact and $\mathrm{Ker}(\beta_{21}) = (\mathcal{H}_2 \cap \mathcal{K}_1/\mathcal{W}_2$, we get the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}_2/\mathcal{W}_2 & \xrightarrow{\beta_{21}} & \mathcal{H}_1/\mathcal{W}_1 \\ & {\scriptstyle L}\searrow \quad \nearrow{\scriptstyle \overline{\beta_{21}}} & \\ & \overline{\mathcal{H}_2} & \end{array}$$

where L is the localization functor which is exact and $\overline{\beta_{21}}$ is a faithful exact functor. Since $\overline{\beta_{21}}L\beta_{12} \simeq \mathrm{id}_{\mathcal{H}_1/\mathcal{W}_1}$, we get that $L\beta_{12}$ is fully-faithful. Now we prove that $L\beta_{12}$ is dense under the assumption of Proposition 3.21.

Let $B \in \mathcal{H}_2$, by Definition 3.1 and 3.4, we get the following commutative diagrams

$$\begin{array}{ccccc} V_B & \rightarrowtail & U_B & \twoheadrightarrow & B \\ \| & & \downarrow & & \downarrow{\scriptstyle s_B} \\ V_B & \rightarrowtail & W^0 & \twoheadrightarrow & B_1^+ \\ & & \downarrow & & \downarrow \\ & & U^0 & = & U^0, \end{array} \qquad \begin{array}{ccccc} V_0 & & = & & V_0 \\ \downarrow & & & & \downarrow \\ (B_1^+)_1^- & \rightarrowtail & W_0 & \twoheadrightarrow & U^{B_1} \\ {\scriptstyle t_B}\downarrow & & \downarrow & & \| \\ B_1^+ & \rightarrowtail & V^{B_1} & \twoheadrightarrow & U^{B_1} \end{array}$$

where $U_B, U^0, U^{B_1} \in \mathcal{U}_1$, $V_B, V^{B_1}, V_0 \in \mathcal{V}_1$, $W_0, W^0 \in \mathcal{W}_1$ and $(B_1^+)_1^- = H_1(B)$ in $\mathcal{B}/\mathcal{W}_1$. Since $H_2(W_1) = 0$, we get the following exact sequences by Theorem 3.11

$$H_2(U_B) \to B \to H_2(B_1^+) \to H_2(U^0),$$

$$H_2(V_0) \to H_2((B_1^+)_1^-) \to H_2(B_1^+) \to H_2(V^{B_1}).$$

One can check that $H_2(\mathcal{U}_1), H_2(\mathcal{V}_1) \subseteq (\mathcal{H}_2 \cap \mathcal{K}_1/\mathcal{W}_2$ by definition. Since $\overline{\mathcal{H}_2}$ is abelian and L is exact, we get $B \simeq H_2(B_1^+) \simeq H_2((B_1^+)_1^-) = L\beta_{12}\beta_{21}(B)$ in $\overline{\mathcal{H}_2}$, which implies that $L\beta_{12}$ is dense.

Now we get the following theorem.

Theorem 3.23. *Let $(\mathcal{U}_1, \mathcal{V}_1)$, $(\mathcal{U}_2, \mathcal{V}_2)$ be cotorsion pairs on $\mathcal{B}$. If $\mathcal{W}_1 \subseteq \mathcal{K}_2 \subseteq \mathcal{K}_1$, then we have an equivalence $L\beta_{12} : \mathcal{H}_1/\mathcal{W}_1 \to \overline{\mathcal{H}_2}$.*

In the rest of this section, we discuss about the relationship between the heart of a twin cotorsion pair and the hearts of its two components.

First we recall the definition of the twin cotorsion pair. A pair of cotorsion pairs $(\mathcal{U}_1, \mathcal{V}_1)$, $(\mathcal{U}_2, \mathcal{V}_2)$ is called a twin cotorsion pair if $\mathcal{U}_1 \subseteq \mathcal{U}_2$. This condition is equivalent to $\mathcal{V}_2 \subseteq \mathcal{V}_1$ and also equivalent to $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{U}_1, \mathcal{V}_2) = 0$. We introduce some notations.

Let $\mathcal{W}_t := \mathcal{V}_1 \cap \mathcal{U}_2$.

(a) $\mathcal{B}_t^+$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence
$$V_B \rightarrowtail U_B \twoheadrightarrow B$$
where $U_B \in \mathcal{W}_t$ and $V_B \in \mathcal{V}_2$.

(b) $\mathcal{B}_t^-$ is defined to be the full subcategory of $\mathcal{B}$, consisting of objects B which admits a short exact sequence
$$B \rightarrowtail V^B \twoheadrightarrow U^B$$
where $V^B \in \mathcal{W}_t$ and $U^B \in \mathcal{U}_1$.

Denote
$$\mathcal{H}_t := \mathcal{B}_t^+ \cap \mathcal{B}_t^-.$$
Then $\mathcal{H}_t/\mathcal{W}_t$ is called the *heart* of $(\mathcal{U}_1, \mathcal{V}_1), (\mathcal{U}_2, \mathcal{V}_2)$.

Proposition 3.24. *Let $(\mathcal{U}_1, \mathcal{V}_1), (\mathcal{U}_2, \mathcal{V}_2)$ be a twin cotorsion pair on $\mathcal{B}$ and $f : A \to B$ be a morphism in $\mathcal{H}_t$, then $H_k(f) = 0$ ($k = 1$ or 2) if and only if f factors through $\mathcal{W}_t$.*

Proof. We only prove the case $k = 2$, the other case is by dual.
The "if" is followed directly by Proposition 3.3. Now we prove the "only if" part.
Since $H_k(f) = 0$, by Proposition 3.6 and 3.10, we get in the following commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \overset{u_B}{\twoheadleftarrow} & U_B & \leftarrowtail & V_B \\
\downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle w'} & & \| \\
A^+ & \xrightarrow[f^+]{} & B^+ & \underset{w}{\twoheadleftarrow} & W^0 & \leftarrowtail & V_B \\
\downarrow & & \downarrow & & \downarrow & & \\
U^0_A & \longrightarrow & U^0 & = & U^0 & &
\end{array}$$

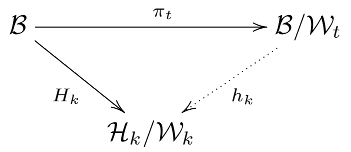
which is similar as in Proposition 3.3, where $U^0_A, U^0 \in \mathcal{U}_2$, $V_B \in \mathcal{V}_2$, $U_B \in \mathcal{W}_t$ and $W^0 \in \mathcal{W}_2$, f^+ factors through an object $V \in \mathcal{V}_2$. Since $A, B \in \mathcal{H}_t$, by [L, Lemma 2.10], $A^+, B^+ \in \mathcal{B}_t^-$. Hence there exits a diagram

$$\begin{array}{ccccc}
A^+ & \overset{w^A}{\rightarrowtail} & W^A & \twoheadrightarrow & U^A \\
{\scriptstyle f^+}\downarrow & \searrow^{a} & & & \\
 & & V & & \\
 & \swarrow_{b} & & & \\
B^+ & \rightarrowtail & W^B & \twoheadrightarrow & U^B
\end{array}$$

where $W^A, W^B \in \mathcal{W}_t$ and $U^A, U^B \in \mathcal{U}_1$. Since $\mathrm{Ext}^1_{\mathcal{B}}(U^A, V) = 0$, there exists a morphism $c : W^A \to V$ such that $f^+ = bcw^A$. Now using the same argument as in Proposition 3.3, we get that f factors through $U_B \in \mathcal{W}_t$. $\square$

Let $\pi_t : \mathcal{B} \to \mathcal{B}/\mathcal{W}_t$ be the canonical functor and $\iota_t : \mathcal{H}_t/\mathcal{W}_t \hookrightarrow \mathcal{B}/\mathcal{W}_t$ be the inclusion functor.

Let $k \in \{1,2\}$, since $H_k(W_t) = 0$ by Proposition 3.12, there exists a functor $h_k : \mathcal{B}/\mathcal{W}_t \to \mathcal{H}_k/\mathcal{W}_k$ such that $H_k = h_k \pi_t$.



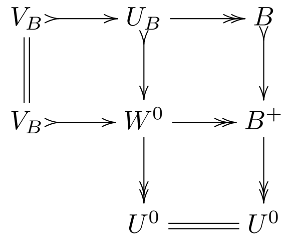
Hence we get a functor $\beta_k := h_k \iota_t : \mathcal{H}_t/\mathcal{W}_t \to \mathcal{H}_k/\mathcal{W}_k$ and the following corollary.

Corollary 3.25. *Let $(\mathcal{U}_1,\mathcal{V}_1),(\mathcal{U}_2,\mathcal{V}_2)$ be a twin cotorsion pair on $\mathcal{B}$, then $\beta_k : \mathcal{H}_t/\mathcal{W}_t \to \mathcal{H}_k/\mathcal{W}_k$ ($k \in \{1,2\}$) is faithful.*

This corollary also implies that if $\mathcal{H}_1/\mathcal{W}_1 = 0$ or $\mathcal{H}_2/\mathcal{W}_2 = 0$, $\mathcal{H}_t/\mathcal{W}_t$ is also zero.
Moreover, we have the following proposition.

Proposition 3.26. *Let $(\mathcal{U}_1,\mathcal{V}_1),(\mathcal{U}_2,\mathcal{V}_2)$ be a twin cotorsion pair on $\mathcal{B}$. If $\mathcal{H}_t/\mathcal{W}_t = 0$, then $\mathcal{H}_1 \subseteq \mathcal{U}_2$ and $\mathcal{H}_2 \subseteq \mathcal{V}_1$.*

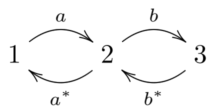
Proof. We only prove that $\mathcal{H}_t/\mathcal{W}_t = 0$ implies $\mathcal{H}_1 \subseteq \mathcal{U}_2$, the other one is by dual.
Let $B \in \mathcal{H}_1$, since $B_1^- \subseteq B_t^-$ by definition, in the following diagram



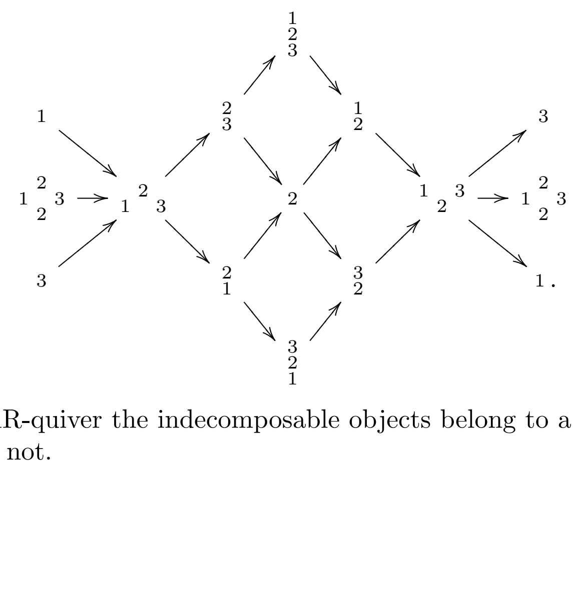
where $U_B \in \mathcal{U}_2$, $V_B \in \mathcal{V}_2$, $U^0 \in \mathcal{U}_1$ and $W^0 \in \mathcal{W}_t$, we get $B^+ \in \mathcal{H}_t$ by [L, Lemma 2.10]. If $\mathcal{H}_t/\mathcal{W}_t = 0$, then $B^+ \in \mathcal{W}_t$. By [L, Lemma 3.4], $B \in \mathcal{U}_2$. □

3.5. **Examples.**

Example 3.27. Let Λ be the k-algebra given by the quiver

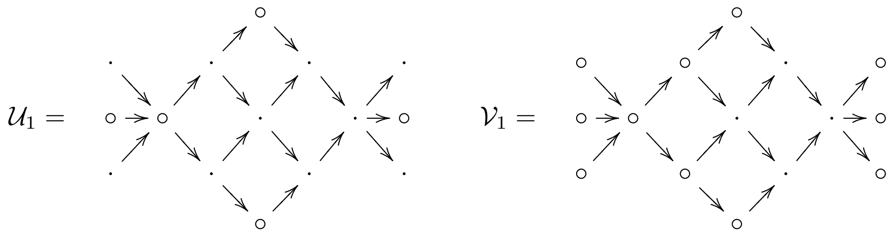


and bounded by the relations $a^*a = 0 = bb^*$, $aa^* = b^*b$. The AR-quiver of $\mathcal{B} = \operatorname{mod}\Lambda$ is given by

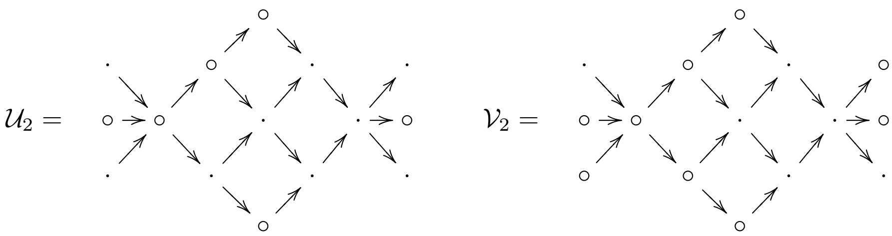


We denote by "∘" in the AR-quiver the indecomposable objects belong to a subcategory and by "·" the indecomposable objects do not.

Let $\mathcal{U}_1$ and $\mathcal{V}_1$ be the full subcategories of $\operatorname{mod}\Lambda$ given by the following diagram.



The heart $\mathcal{H}_1/\mathcal{W}_1 = \operatorname{add}(2)$ and $\mathcal{H}_1 \simeq \operatorname{mod}(\mathcal{U}_1/\mathcal{P})$ by [DL, Theorem 3.2]. Now let $\mathcal{U}_2$ and $\mathcal{V}_2$ be the full subcategories of $\operatorname{mod}\Lambda$ given by the following diagram.



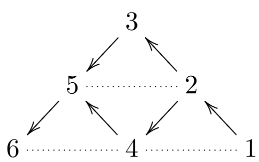
The heart $\mathcal{H}_2/\mathcal{W}_2 = \operatorname{add}(1\,,\,2)$. Since $\mathcal{W}_1 = \mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \mathcal{V}_2 \subseteq \mathcal{V}_1$, by Theorem 3.23, $\overline{\mathcal{H}}_2 \simeq \mathcal{H}_1/\mathcal{W}_1$. Moreover, $\mathcal{V}_1/\mathcal{U}_1$ has a triangulated category structure, and $(\mathcal{U}_2/\mathcal{U}_1, \mathcal{V}_2/\mathcal{U}_1)$ is a cotorsion pair on it. The Serre subcategory $(\mathcal{H}_2 \cap \mathcal{K}_1)/\mathcal{W}_2 = \operatorname{add}(1)$ is the heart of $(\mathcal{U}_2/\mathcal{U}_1, \mathcal{V}_2/\mathcal{U}_1)$.

Recall that a subcategory $\mathcal{M}$ of $\mathcal{B}$ is called rigid if $\operatorname{Ext}^1_{\mathcal{B}}(\mathcal{M}, \mathcal{M}) = 0$, $\mathcal{M}$ is cluster tilting if it satisfies

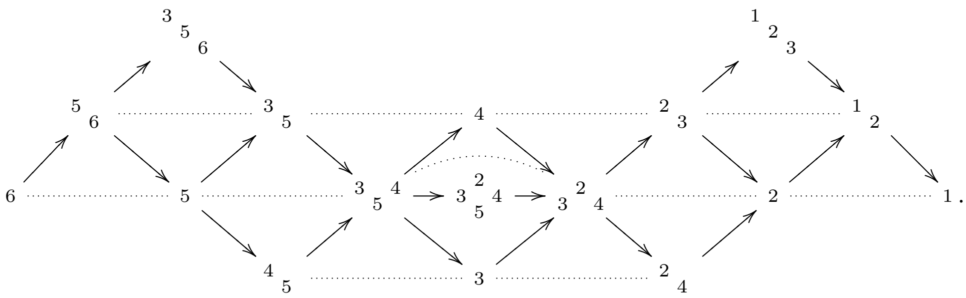
(a) $\mathcal{M}$ is contravariantly finite and covariantly finite in $\mathcal{B}$.
(b) $X \in \mathcal{M}$ if and only if $\operatorname{Ext}^1_{\mathcal{B}}(X, \mathcal{M}) = 0$.
(c) $X \in \mathcal{M}$ if and only if $\operatorname{Ext}^1_{\mathcal{B}}(\mathcal{M}, X) = 0$.

If $\mathcal{M}$ is a cluster tilting subcategory of $\mathcal{B}$, then $(\mathcal{M}, \mathcal{M})$ is a cotorsion pair on $\mathcal{B}$ (see [L, Proposition 10.5]).. In this case we have $\mathcal{H} = \mathcal{B}^- = \mathcal{B}^+ = \mathcal{B}$, $\sigma^- = \sigma^+ = \mathrm{id}$ and $H = \pi$.

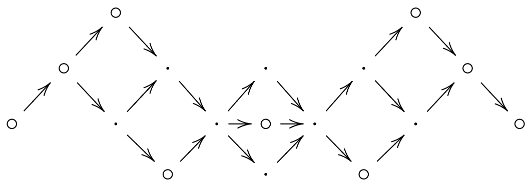
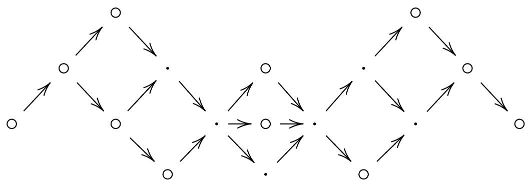
Example 3.28. Let Λ be the k-algebra given by the quiver



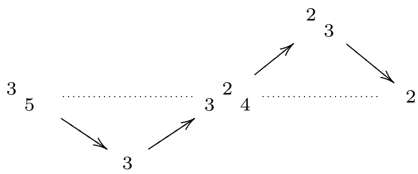
with mesh relations. The AR-quiver of $\mathcal{B} := \operatorname{mod}\Lambda$ is given by



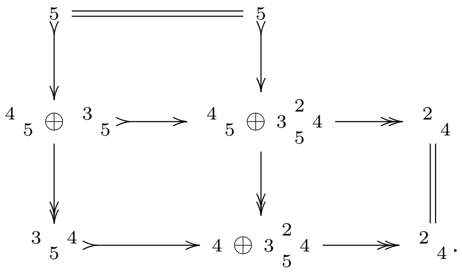
Let $\mathcal{U}_1$ and $\mathcal{V}_1$ be the full subcategories of $\operatorname{mod}\Lambda$ given by the following diagram.

$\mathcal{U}_1 =$  $\mathcal{V}_1 =$ 

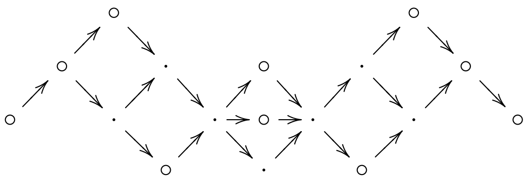
Then $(\mathcal{U}_1, \mathcal{V}_1)$ is a cotorsion pair on $\operatorname{mod}\Lambda$. The heart $\mathcal{H}_1/\mathcal{W}_1$ is the following.



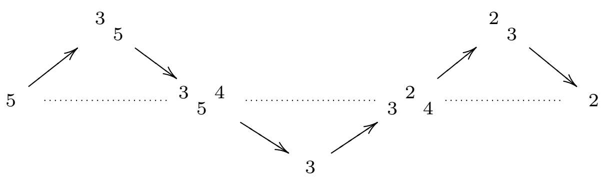
The only indecomposable object which does not lie in $\mathcal{H}_1$ or $\mathcal{U}_1$,$\mathcal{V}_1$ is ${}^{3}{}_{5}{}^{4}$, since we have the following commutative diagram



We get $H_1({}^{3}{}_{5}{}^{4}) = {}^{3}{}_{5}$ since ${}^{4}{}_{5} \in \mathcal{P}$. Let

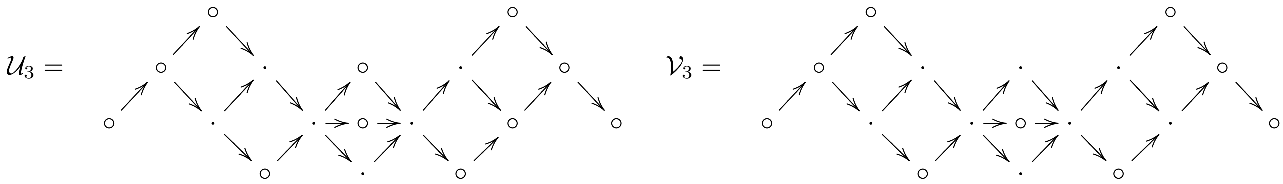
$\mathcal{M} =$ 

Since $\mathcal{M}$ is a cluster tilting subcategory of $\mathcal{B}$, $(\mathcal{U}_2, \mathcal{V}_2) = (\mathcal{M}, \mathcal{M})$ is a cotorsion pair. The heart $\mathcal{H}_2/\mathcal{W}_2 = \operatorname{mod}\Lambda/\mathcal{M}$ is the following.

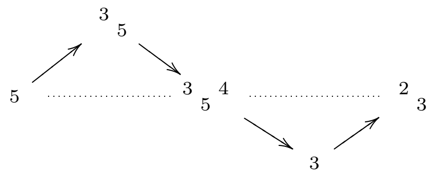


Since $\mathcal{U}_1 \subseteq \mathcal{M} \subseteq \mathcal{V}_1$, we have $\mathcal{W}_1 \subseteq \mathcal{K}_2 \subseteq \mathcal{K}_1$. Since We get $H_1({}^{4}{}_{5}{}^{3}) = {}^{3}{}_{5}$, we get that β_{21} is exact. But β_{12} is not exact, since ${}^{3}{}_{5} \rightarrowtail 3 \twoheadrightarrow {}_{3}{}^{2}{}_{4}$ is a short exact sequence in $\mathcal{H}_1/\mathcal{W}_1$ but not a short exact sequence in $\mathcal{H}_2/\mathcal{W}_2$. In this case, $(\mathcal{H}_2 \cap \mathcal{K}_1)/\mathcal{W}_2$ is add(5), we can see that $\overline{\mathcal{H}}_2 \simeq \mathcal{H}_1/\mathcal{W}_1$.

Let $\mathcal{U}_3$ and $\mathcal{V}_3$ be the full subcategories of $\operatorname{mod}\Lambda$ given by the following diagram.



and the heart $\mathcal{H}_3/\mathcal{W}_3$ is the following.



Hence we get $\mathcal{H}_1/\mathcal{W}_1 \simeq \mathcal{H}_3/\mathcal{W}_3$. But we find that $\mathcal{U}_3 \subsetneqq \mathcal{K}_1$ and $\mathcal{V}_1 \subsetneqq \mathcal{K}_3$, which implies that the condition Corollary 3.22 is not necessary for the equivalence of two hearts.

By Theorem 3.11 and Proposition 3.16, we get:

Proposition 3.29. *Let $\mathcal{M}$ be a cluster tilting subcategory of $\mathcal{B}$. Then the canonical functor*

$$\pi : \mathcal{B} \to \mathcal{B}/\mathcal{M}$$

is half exact. Moreover, every short exact sequence

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

in B induces a long exact sequence

$$\cdots \xrightarrow{\Omega h'} \Omega A \xrightarrow{\Omega f} \Omega B \xrightarrow{\Omega g} \Omega C \xrightarrow{h'} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Omega^- A \xrightarrow{\Omega^- f} \Omega^- B \xrightarrow{\Omega^- g} \Omega^- C \xrightarrow{\Omega^- h} \cdots$$

in the abelian category $\mathcal{B}/\mathcal{M}$.

Example 3.30. Let $\mathcal{M}$ be a cluster tilting subcategory of $\mathcal{B}$ (for instance, see [DL, Example 4.2]). Then we have a half exact functor

$$\begin{aligned} G:\ & \mathcal{B} \to \operatorname{mod}\mathcal{M}/\mathcal{P} \\ & X \mapsto \operatorname{Ext}^1_{\mathcal{B}}(-, X)|_{\mathcal{M}}. \end{aligned}$$

This is a composition of the half exact functor $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{M}$ given by Proposition 3.29 and an equivalence

$$\begin{aligned} \mathcal{B}/\mathcal{M} & \xrightarrow{\simeq} \operatorname{mod}\mathcal{M}/\mathcal{P} \\ X & \mapsto \operatorname{Ext}^1_{\mathcal{B}}(-, X)|_{\mathcal{M}}. \end{aligned}$$

given by [DL, Theorem 3.2]. By Proposition 3.18, $G(X) = 0$ if and only if $X \in \mathcal{M}$.

A more general case is given as follows. If $\mathcal{M}$ is a rigid subcategory of $\mathcal{B}$ which is contravariantly finite and contains $\mathcal{P}$, then by [L, Proposition 2.12], $(\mathcal{M}, \mathcal{M}^{\perp_1})$ is a cotorsion pair where $\mathcal{M}^{\perp_1} = \{X \in \mathcal{B} \mid \operatorname{Ext}^1_{\mathcal{B}}(\mathcal{M}, X) = 0\}$. Since $\mathcal{M}$ is rigid, we have $\mathcal{M} \subseteq \mathcal{M}^{\perp_1}$. In this case we have $\mathcal{B}^+ = \mathcal{B}$, $\mathcal{B}^- = \mathcal{H}$, $\sigma^+ = \mathrm{id}$ and $H = \sigma^- \circ \pi$. By [DL, Theorem 3.2], there exists an equivalence between $\underline{\mathcal{H}}$ and $\operatorname{mod}(\mathcal{M}/\mathcal{P})$. Hence by Theorem 3.11, we get the following example:

Example 3.31. Let $\mathcal{M}$ be a rigid subcategory of $\mathcal{B}$ which is contravariantly finite and contains $\mathcal{P}$ (for instance, see [DL, Example 4.3]). Then there exists a half exact functor

$$\begin{aligned} G:\ & \mathcal{B} \to \operatorname{mod}\mathcal{M}/\mathcal{P} \\ & X \mapsto \operatorname{Ext}^1_{\mathcal{B}}(-, \sigma^-(X))|_{\mathcal{M}} \end{aligned}$$

which is a composition of H and the equivalence

$$\begin{aligned} \underline{\mathcal{H}} &\xrightarrow{\simeq} \operatorname{mod} \mathcal{M}/\mathcal{P} \\ Y &\mapsto \operatorname{Ext}^1_{\mathcal{B}}(-, Y)|_{\mathcal{M}} \end{aligned}$$

given by [DL, Theorem 3.2]. By Proposition 3.18, $G(X) = 0$ if and only if $X \in \mathcal{M}^{\perp_1}$.

4. Hearts of cotorsion pairs are functor categories over cohearts

In this section, we give an equivalence between hearts and the functor categories over cohearts.

4.1. Hearts on triangulated categories.

Let $\mathcal{T}$ be a triangulated category.

Definition 4.1. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{T}$ which are closed under direct summands. We call $(\mathcal{U}, \mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

- (a) $\operatorname{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}[1]) = 0$.
- (b) For any object $T \in \mathcal{T}$, there exists a triangle $T[-1] \to V_T \to U_T \to T$ satisfying $U_T \in \mathcal{U}$ and $V_T \in \mathcal{V}$.

For a cotorsion pairs $(\mathcal{U}, \mathcal{V})$, let $\mathcal{W} := \mathcal{U} \cap \mathcal{V}$. We denote the quotient of $\mathcal{T}$ by $\mathcal{W}$ as $\underline{\mathcal{T}} := \mathcal{T}/\mathcal{W}$. For any morphism $f \in \operatorname{Hom}_{\mathcal{T}}(X, Y)$, we denote its image in $\operatorname{Hom}_{\underline{\mathcal{T}}}(X, Y)$ by $\underline{f}$. For any subcategory $\mathcal{D} \supseteq \mathcal{W}$ of $\mathcal{T}$, we denote by $\underline{\mathcal{D}}$ the full subcategory of $\underline{\mathcal{T}}$ consisting of the same objects as $\mathcal{D}$. Let

$$\mathcal{T}^+ := \{T \in \mathcal{T} \mid U_T \in \mathcal{W}\}, \quad \mathcal{T}^- := \{T \in \mathcal{T} \mid V^T \in \mathcal{W}\}.$$

Let

$$\mathcal{H} := \mathcal{T}^+ \cap \mathcal{T}^-$$

we call the additive subcategory $\underline{\mathcal{H}}$ the *heart* of cotorsion pair $(\mathcal{U}, \mathcal{V})$. Under these settings, Abe, Nakaoka [AN] introduced the homological functor $H : \mathcal{T} \to \underline{\mathcal{H}}$ associated with $(\mathcal{U}, \mathcal{V})$. We often use the following property of H: $H(\mathcal{U}) = 0 = H(\mathcal{V})$.

For the coheart $\mathcal{C} := \mathcal{U}[-1] \cap {}^{\perp}\mathcal{U}$, since $\mathcal{C} \subseteq \mathcal{T}^-$, for any object $C \in \mathcal{C}$, by definition of H we get the following commutative diagram the following commutative diagram

$$\begin{array}{ccccccc}
V_C & \longrightarrow & U_C & \longrightarrow & C & \longrightarrow & V_C[1] \\
\| & & \downarrow & & \downarrow{\scriptstyle i} & & \| \\
V_C & \longrightarrow & W_C & \longrightarrow & H(C) & \longrightarrow & V_C[1] \\
 & & \downarrow & & \downarrow & & \\
 & & U'_C & = & U'_C & & \\
 & & \downarrow & & \downarrow & & \\
 & & U_C[1] & \longrightarrow & C[1] & &
\end{array} \tag{10}$$

where $U_C, U'_C \in \mathcal{U}$, $V_C \in \mathcal{V}$ and $W_C \in \mathcal{W}$. Moreover, $H(i)$ is an isomorphism in $\underline{\mathcal{H}}$ by [AN, Proposition 3.8, Theorem 5.7].

For the coheart, we have the following proposition which implies that $\operatorname{mod} \mathcal{C}$ is an abelian category.

Proposition 4.2. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair that $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, then the coheart $\mathcal{C}$ has pseudo-kernels.*

Proof. Let $f : C_1 \to C_2$ be a morphism in $\mathcal{C}$, we can extend it to a triangle $T \xrightarrow{g} C_1 \xrightarrow{f} C_2 \to T[1]$. Since we have a cotorsion pair $(\mathcal{U}, \mathcal{V})$, the pair $(\mathcal{U}[-1], \mathcal{V}[-1])$ is also a cotorsion pair on $\mathcal{T}$. Hence T admits a triangle $V[-1] \to U[-1] \xrightarrow{h} T \xrightarrow{j} V$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Since $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, $U[-1]$ admits a triangle $C \xrightarrow{a} U[-1] \xrightarrow{b} U' \to C[1]$. We obtain that $f(gha) = 0$ and we claim that $gha : C \to C_1$ is a pseudo-kernel of f.
Let $g' : C' \to C$ be morphism in $\mathcal{C}$ such that fg', then there exists a morphism $x : C' \to T$ such that $g' = gx$. Since $\mathrm{Hom}_{\mathcal{T}}(C', V) = 0$, we have $jx = 0$, hence there exists a morphism $y : C' \to U[-1]$ such that $x = hy$. Since $\mathrm{Hom}_{\mathcal{T}}(C', U) = 0$, we have $by = 0$, hence there exists a morphism $z : C' \to C$ such that $y = az$. Thus $g' = (gha)z$, which means that $gha : C \to C_1$ is a pseudo-kernel of f.coheart □

We will prove the following theorem.

Theorem 4.3. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair that $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, then $\underline{\mathcal{H}}$ has enough projectives $H(\mathcal{C})$ and is equivalent to* $\mathrm{mod}\,\mathcal{C}$.

Let's start with an important property for H.

Proposition 4.4. *The functor $H|_{\mathcal{C}} : \mathcal{C} \to H(\mathcal{C})$ is an equivalence.*

Proof. By definition we get that H is dense on $\mathcal{C}$. We only have to check that $H|_{\mathcal{C}}$ is fully-faithful.
Let $C_1, C_2 \in \mathcal{C}$, since C_i, $i = 1, 2$ admits a triangle

$$C_i \to H(C_i) \to U_i \to C_i[1]$$

where $U_i \in \mathcal{U}$, let $f \in \mathrm{Hom}_{\mathcal{T}}(C_1, C_2)$, by [N, Proposition 4.3], we get a commutative diagram

$$\begin{array}{ccccccc} C_1 & \longrightarrow & H(C_1) & \longrightarrow & U_1 & \longrightarrow & C_1[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^+} & & \downarrow & & \downarrow \\ C_2 & \longrightarrow & H(C_2) & \longrightarrow & U_2 & \longrightarrow & C_2[1]. \end{array}$$

where $\underline{f}^+ = H(f)$. If $H(f) = 0$, f factors through $\mathcal{U}$ by [L2, Proposition 2.5]. Since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, \mathcal{U}) = 0$, we get $f = 0$ which means H is faithful on $\mathcal{C}$.
Let $g \in \mathrm{Hom}_{\mathcal{T}}(H(C_1), H(C_2))$, since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, \mathcal{U}) = 0$, we can still get the following commutative diagram

$$\begin{array}{ccccccc} C_1 & \longrightarrow & H(C_1) & \longrightarrow & U_1 & \longrightarrow & C_1[1] \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow \\ C_2 & \longrightarrow & H(C_2) & \longrightarrow & U_2 & \longrightarrow & C_2[1]. \end{array}$$

Then we have $\underline{g} = H(f')$. Thus H is full on $\mathcal{C}$. □

Now we prove the following theorem.

Theorem 4.5. *If $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, then $\underline{\mathcal{H}}$ has enough projectives $H(\mathcal{C})$.*

Proof. We first prove that $H(\mathcal{C})$ is projective in $\underline{\mathcal{H}}$.
Let $\underline{f} : A \to B$ be an epimorphism in $\underline{\mathcal{H}}$, since $A \in \mathcal{T}^-$, we get the following commutative diagram in $\mathcal{T}$

$$\begin{array}{ccccccc} U^A[-1] & \longrightarrow & A & \longrightarrow & W^A & \longrightarrow & U^A \\ \| & & \downarrow{\scriptstyle f} & & \downarrow & & \| \\ U^A[-1] & \longrightarrow & B & \xrightarrow[g]{} & D & \longrightarrow & U^A. \end{array} \tag{11}$$

First we show that $D \in \mathrm{add}(\mathcal{U} * \mathcal{V})$. We can get the following triangle $A \to B \oplus W^A \to D \to A[1]$. Apply H to this triangle, we have an sequence $A \xrightarrow{\underline{f}} B \xrightarrow{H(g)} H(D)$, since $\underline{f}$ is epimorphic, we obtain $H(g) = 0$. Apply H to the second row of the above diagram, we get an exact sequence $B \xrightarrow{H(g)} H(D) \to 0$, which implies $H(D) = 0$. By [L2, Proposition 4.7], this means that $D \in \mathrm{add}(\mathcal{U} * \mathcal{V})$.
Denote $B \oplus W^A$ by B', from the second square (8) we get a triangle $A \xrightarrow{f'} B' \xrightarrow{g'} D \to A[1]$ where $\underline{f'} = \underline{f}$. Since $D \in \mathcal{K}$, it admits a triangle $U_D \to D \oplus D' \to V_D \to U_D[1]$. Now let $\underline{h} : H(C) \to B'$ be a morphism in $\underline{\mathcal{H}}$ where $C \in \mathcal{C}$. Since $\mathrm{Hom}_{\mathcal{T}}(C, V_C) = 0 = \mathrm{Hom}_{\mathcal{T}}(C, U_C)$, by (10), $g'hi = 0$, we have the following commutative diagram

$$\begin{array}{ccccccc}
C & \xrightarrow{i} & H(C) & \longrightarrow & U & \longrightarrow & C[1] \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow \\
A & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & D & \longrightarrow & A[1].
\end{array}$$

Apply H to this diagram, since $H(i)$ is an isomorphism in $\underline{\mathcal{H}}$, we have the following commutative diagram

$$\begin{array}{ccccc}
 & & H(C) & & \\
 & \swarrow^{H(j)H(i)^{-1}} & \downarrow{\scriptstyle \underline{h}} & & \\
A & \xrightarrow[\underline{f}]{} & B & \longrightarrow & 0.
\end{array}$$

This implies that $H(\mathcal{C})$ is projective in $\underline{\mathcal{H}}$.
Since $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, and $\mathcal{H} \subseteq \mathcal{U}[-1] * \mathcal{U} = \mathcal{C} * \mathcal{U}$, any object $A \in \mathcal{H}$ admits a triangle $C_A \to A \to U' \to C_A[1]$, apply H to this triangle, we get an exact sequence $H(C_A) \to A \to 0$ in $\underline{\mathcal{H}}$. $\square$

Now we show the main result of this section.

Theorem 4.6. *If* $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, *then* $\underline{\mathcal{H}} \simeq \mathrm{mod}\,\mathcal{C}$.

Proof. It is enough to show that $\underline{\mathcal{H}} \simeq \mathrm{mod}\, H(\mathcal{C})$ since $\mathcal{C} \simeq H(\mathcal{C})$.
Define

$$\begin{aligned} F : \underline{\mathcal{H}} &\to \mathrm{mod}\, H(\mathcal{C}) \\ A &\mapsto \mathrm{Hom}_{\underline{\mathcal{T}}}(-, A)|_{H(\mathcal{C})}. \end{aligned}$$

Now we show that F is dense.
Let $N \in \mathrm{mod}\, H(\mathcal{C})$, we have an exact sequence

$$\mathrm{Hom}_{H(\mathcal{C})}(-, P_1) \xrightarrow{\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{f})} \mathrm{Hom}_{H(\mathcal{C})}(-, P_0) \to N \to 0$$

where $P_1, P_0 \in H(\mathcal{C})$. Since $\underline{\mathcal{H}}$ is abelian, we have a exact sequence $P_1 \xrightarrow{\underline{f}} P_0 \to Y \to 0$ Now apply $\mathrm{Hom}_{\underline{\mathcal{T}}}(H(\mathcal{C}), -)$ to this exact sequence, we have

$$\mathrm{Hom}_{H(\mathcal{C})}(-, P_1) \xrightarrow{\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{f})} \mathrm{Hom}_{H(\mathcal{C})}(-, P_0) \to \mathrm{Hom}_{\underline{\mathcal{T}}}(-, H(Y))|_{H(\mathcal{C})} \to 0$$

Hence $N \simeq \mathrm{Hom}_{\underline{\mathcal{T}}}(-, H(Y))|_{H(\mathcal{C})}$.
We prove that F is faithful.
Let $\underline{f} : A \to B$ be a morphism in $\underline{\mathcal{H}}$ such that $F(\underline{f}) = 0$. Since $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, A admits a triangle $C_A \xrightarrow{i} A \to U \to C_A[1]$, and C_A admits a triangle $C_A \xrightarrow{g} H(C) \xrightarrow{h} U' \to C[1]$. Since there exists a morphism $j : H(C) \to B$ such that $i = jg$, we have $\underline{fj} = 0$, hence fi factors through $\mathcal{W}$, then $\underline{f}H(i) = 0$. Since $H(i)$ is epimorphic, we get $\underline{f} = 0$
We prove that F is full.
Let $\alpha : \mathrm{Hom}_{\underline{\mathcal{T}}}(-, A_1)|_{H(\mathcal{C})} \to \mathrm{Hom}_{\underline{\mathcal{T}}}(-, A_2)|_{H(\mathcal{C})}$ be a morphism in $\mathrm{mod}\, H(\mathcal{C})$. By Theorem 4.5, A_i

admits an exact sequence $P'_{A_i} \xrightarrow{g_i} P_{A_i} \xrightarrow{f_i} A_i \to 0$ such that $P'_{A_i}, P_{A_i} \in H(\mathcal{C})$, we get the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{H(\mathcal{C})}(-,P'_{A_1}) & \xrightarrow{\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{g_1})} & \mathrm{Hom}_{H(\mathcal{C})}(-,P_{A_1}) & \xrightarrow{\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{f_1})} & \mathrm{Hom}_{\underline{\mathcal{T}}}(-,A_1)|_{H(\mathcal{C})} \to 0 \\
\downarrow{\scriptstyle \mathrm{Hom}_{H(\mathcal{C})}(-,\underline{a})} & & \downarrow{\scriptstyle \mathrm{Hom}_{H(\mathcal{C})}(-,\underline{b})} & & \downarrow{\scriptstyle \alpha} \\
\mathrm{Hom}_{H(\mathcal{C})}(-,P'_{A_2}) & \xrightarrow[\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{g_2})]{} & \mathrm{Hom}_{H(\mathcal{C})}(-,P_{A_2}) & \xrightarrow[\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{f_2})]{} & \mathrm{Hom}_{\underline{\mathcal{T}}}(-,A_2)|_{H(\mathcal{C})} \to 0
\end{array}$$

by Yoneda's Lemma. Hence we get the following commutative commutative diagram

$$\begin{array}{ccccc}
P'_{A_1} & \xrightarrow{\underline{g_1}} & P_{A_1} & \xrightarrow{\underline{f_1}} & A_1 \\
\downarrow{\scriptstyle \underline{a}} & & \downarrow{\scriptstyle \underline{b}} & & \vdots{\scriptstyle \underline{c}} \\
P'_{A_2} & \xrightarrow{\underline{g_2}} & P_{A_2} & \xrightarrow{\underline{f_2}} & A_2
\end{array}$$

Hence $\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{c}) = \alpha$. $\square$

Note that the condition $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$ is satisfied in many cases. The following proposition is given as an example.

Proposition 4.7. *If $\mathcal{U}$ is covariantly finite and $\mathcal{T}$ is Krull-Schimdt, then $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$.*

Proof. If $\mathcal{U}$ is covariantly finite and $\mathcal{T}$ is Krull-Schimdt, then $({}^{\perp_1}\mathcal{U},\mathcal{U})$ is a cotorsion pair. Hence any object $U \in \mathcal{U}$ admits a triangle $U' \to C[1] \to U \to U'[1]$, which implies that $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$. $\square$

4.2. **Hearts on exact categories.** Let $\mathcal{B}$ be a exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$.

Since $\Omega\mathcal{C} \subseteq \mathcal{B}^-$ by [L2, Lemma 3.2], for any object $\Omega C \in \Omega\mathcal{C}$, by definition of H we get from the following commutative diagram

$$\begin{array}{ccccc}
V_{\Omega C} & \rightarrowtail & U_{\Omega C} & \twoheadrightarrow & \Omega C \\
\| & & \downarrow & & \downarrow{\scriptstyle a} \\
V_{\Omega C} & \rightarrowtail & W_{\Omega C} & \twoheadrightarrow & H(\Omega C) \\
 & & \downarrow & & \downarrow \\
 & & U & = & U
\end{array}$$

where $H(a)$ is an isomorphism by [L2, Theorem 4.1, Proposition 4.2].

For the coheart, we have the following proposition which implies that $\mathrm{mod}(\mathcal{C}/\mathcal{P})$ is an abelian category.

Proposition 4.8. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair that for any any object $U \in \mathcal{U}$, there exists an exact sequence $U' \rightarrowtail C \twoheadrightarrow U$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$,, then the quotient category $\mathcal{C}/\mathcal{P}$ has pseudo-kernels.*

Proof. This is an analog of Proposition 4.2. $\square$

We will prove the following theorem.

Theorem 4.9. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair. Let $\mathcal{C} := \mathcal{U} \cap {}^{\perp_1}\mathcal{U}$ and $\Omega\mathcal{C} = \{X \in \mathcal{B} \mid X$ admits $X \rightarrowtail P \twoheadrightarrow C$ where $P \in \mathcal{P}$ and $C \in \mathcal{C}\}$. If for any any object $U \in \mathcal{U}$, there exists an exact sequence $U' \rightarrowtail C \twoheadrightarrow U$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then the heart of $(\mathcal{U},\mathcal{V})$ has enough projectives $H(\Omega\mathcal{C})$ and is equivalent to* $\mathrm{mod}(\mathcal{C}/\mathcal{P})$.

We prove the theorem in several steps. We denote the quotient of $\mathcal{B}$ by $\mathcal{P}$ as $\overline{\mathcal{B}} := \mathcal{B}/\mathcal{P}$. For any morphism $f \in \mathrm{Hom}_{\mathcal{B}}(X,Y)$, we denote its image in $\mathrm{Hom}_{\overline{\mathcal{B}}}(X,Y)$ by $\overline{f}$.

Lemma 4.10. *We have an equivalence $\overline{\mathcal{C}} \simeq \overline{\Omega\mathcal{C}}$.*

Proof. For any morphism $f : C \to C'$ in $\mathcal{C}$, we have the following commutative diagram

$$\begin{array}{ccccc} \Omega C & \rightarrowtail & P & \twoheadrightarrow & C \\ \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle f} \\ \Omega C' & \rightarrowtail & P' & \twoheadrightarrow & C' \end{array}$$

We can define a functor $G : \overline{\mathcal{C}} \to \overline{\Omega\mathcal{C}}$ such that $G(C) = \Omega C$ and $G(\overline{f}) = \overline{g}$. G is well defined since if f factors through $P'' \in \mathcal{P}$, then it factors through P', which implies g factors through P, hence $\overline{g} = 0$. We prove that G is an equivalence.
(i) We first prove that G is faithful.
If $\overline{g} = 0$, it factors through an projective object P_0. By the definition of $\mathcal{C}$, we get $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{C}, \mathcal{P}) = 0$, hence we have the following

$$\begin{array}{ccccc} \Omega C & \rightarrowtail & P & \twoheadrightarrow & C \\ & \searrow & \swarrow{\scriptstyle \cdots} & & \\ \downarrow{\scriptstyle g} & P_0 & \downarrow & & \downarrow{\scriptstyle f} \\ & \swarrow & & & \\ \Omega C' & \rightarrowtail & P' & \twoheadrightarrow & C'. \end{array}$$

This implies that f factors through P', hence $\overline{f} = 0$.
(ii) We prove that G is full.
For the following diagram

$$\begin{array}{ccccc} \Omega C & \rightarrowtail & P & \twoheadrightarrow & C \\ \downarrow{\scriptstyle g} & & & & \\ \Omega C' & \rightarrowtail & P' & \twoheadrightarrow & C' \end{array}$$

since $\mathrm{Ext}^1_{\mathcal{B}}(\mathcal{C}, \mathcal{P}) = 0$, we can get a commutative diagram

$$\begin{array}{ccccc} \Omega C & \rightarrowtail & P & \twoheadrightarrow & C \\ \downarrow{\scriptstyle g} & & \vdots & & \vdots{\scriptstyle f} \\ \Omega C' & \rightarrowtail & P' & \twoheadrightarrow & C' \end{array}$$

hence $G(\overline{f}) = \overline{g}$.
By the definition of $\Omega\mathcal{C}$, G is dense.
Hence G is an equivalence. □

Since $H(\mathcal{P}) = 0$, we have the following commutative diagram

$$\begin{array}{ccc} \Omega\mathcal{C} & \xrightarrow{H} & H(\Omega\mathcal{C}) \\ & {\scriptstyle \pi}\searrow \quad \nearrow{\scriptstyle \overline{H}} & \\ & \overline{\Omega\mathcal{C}} & \end{array}$$

where π is the quotient functor.

Proposition 4.11. $\overline{H} : \overline{\Omega\mathcal{C}} \to H(\Omega\mathcal{C})$ *is an equivalence.*

Proof. By definition we get that $\overline{H}$ is dense. Now we only have to check that $\overline{H}$ is fully-faithful.
Let $\Omega C_1, \Omega C_2 \in \Omega\mathcal{C}$, since ΩC_i, $i = 1, 2$ admits a short exact sequence

$$\Omega C_i \rightarrowtail H(\Omega C_i) \twoheadrightarrow U_i$$

where $U_i \in \mathcal{U}$. Let $\overline{f} \in \mathrm{Hom}_{\overline{\mathcal{B}}}(\Omega C_1, \Omega C_2)$, by [L, Proposition 3.3], we get the commutative diagram

$$\begin{array}{ccccc} \Omega C_1 & \rightarrowtail & H(\Omega C_1) & \twoheadrightarrow & U_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^+} & & \downarrow \\ \Omega C_2 & \rightarrowtail & H(\Omega C_2) & \twoheadrightarrow & U_2. \end{array}$$

where $\underline{f^+} = H(f)$. If $H(f) = 0$, f factors through $\mathcal{U}$ by [L2, Proposition 2.5]. Since $\mathrm{Hom}_{\overline{\mathcal{B}}}(\Omega\mathcal{C}, \mathcal{U}) = 0$, we get $\overline{f} = 0$ which means J is faithful on $\overline{\Omega\mathcal{C}}$.

Let $\underline{g} \in \mathrm{Hom}_{\underline{\mathcal{B}}}(H(\Omega C_1), H(\Omega C_2))$, since $\mathrm{Hom}_{\overline{\mathcal{B}}}(\Omega\mathcal{C}, \mathcal{U}) = 0$, we get that in the following diagram

$$\begin{array}{ccccc} \Omega C_1 & \overset{a_1}{\rightarrowtail} & H(\Omega C_1) & \overset{b_1}{\twoheadrightarrow} & U_1 \\ & & \downarrow{\scriptstyle g} & & \\ \Omega C_2 & \underset{a_2}{\rightarrowtail} & H(\Omega C_2) & \underset{b_2}{\twoheadrightarrow} & U_2. \end{array}$$

b_2ga_1 factors through an object $P \in \mathcal{P}$. Hence we have two morphisms $c : \Omega C_1 \to P$ and $d : P \to U_2$ such that $dc = b_2ga_1$. Since P is projective, there exists a morphism $p : P \to H(\Omega C_2)$ such that $d = b_2p$. Hence $b_2(ga_1 - pc) = 0$. Then there is a morphism $f' : \Omega C_1 \to \Omega C_2$ such that $f'a_2 = ga_1 - pc$. now we get a commutative diagram

$$\begin{array}{ccccc} \Omega C_1 & \overset{a_1}{\rightarrowtail} & H(\Omega C_1) & \overset{b_1}{\twoheadrightarrow} & U_1 \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} & & \downarrow \\ \Omega C_2 & \underset{a_2}{\rightarrowtail} & H(\Omega C_2) & \underset{b_2}{\twoheadrightarrow} & U_2. \end{array}$$

where $H(f') = \underline{g'}$. Since $H(\mathcal{P}) = 0$, we get $\underline{g'}H(a_1) = H(f')H(a_2) = \underline{g}H(a_1)$. Since $H(a_1) = 0$, we have $\underline{g'} = \underline{g}$. Hence $\overline{\overline{H}}$ is full. $\square$

Lemma 4.12. *If for any any object $U \in \mathcal{U}$, there exists an exact sequence $U' \rightarrowtail C \twoheadrightarrow U$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then any object $X \in \mathcal{H}$ admits a short exact sequence $X \rightarrowtail U \twoheadrightarrow C$ where $U \in \mathcal{U}$.*

Proof. Since $X \in \mathcal{H}$, it admits a short exact sequence $X \rightarrowtail W \overset{a}{\twoheadrightarrow} U$ where $W \in \mathcal{W}$ and $U \in \mathcal{U}$. Since $\mathcal{U} \subseteq \{Y \in \mathcal{B} \mid \text{Y admits } U' \rightarrowtail C \twoheadrightarrow Y\}$, U also admits a short exact sequence $U' \rightarrowtail C \overset{b}{\twoheadrightarrow} U$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$. Take a push-out of a and b, we get the following commutative diagram

$$\begin{array}{ccccc} & & U' & = & U' \\ & & \downarrow & & \downarrow \\ X & \rightarrowtail & U'' & \twoheadrightarrow & C \\ \| & & \downarrow & & \downarrow{\scriptstyle b} \\ X & \rightarrowtail & W & \overset{a}{\twoheadrightarrow} & U \end{array}$$

Since $\mathcal{U}$ is closed under extension, we have $U'' \in \mathcal{U}$. $\square$

Now we are ready to prove the main theorem of this section.

Theorem 4.13. *If for any any object $U \in \mathcal{U}$, there exists an exact sequence $U' \rightarrowtail C \twoheadrightarrow U$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then $\underline{\mathcal{H}}$ has enough projectives $H(\Omega\mathcal{C})$.*

Proof. We first prove that $H(\Omega\mathcal{C})$ is projective in $\underline{\mathcal{H}}$.
Let $\underline{f}: A \to B$ be an epimorphism in $\underline{\mathcal{H}}$, it admits the following commutative diagram in $\mathcal{B}$

$$\begin{array}{ccccc} A & \rightarrowtail & W^A & \twoheadrightarrow & U^A \\ \downarrow{\scriptstyle f} & & \downarrow & & \| \\ B & \overset{}{\underset{g}{\rightarrowtail}} & D & \twoheadrightarrow & U^A. \end{array}$$

We can get the following short exact sequence $A \rightarrowtail B \oplus W^A \twoheadrightarrow D$. Apply H to this triangle, we have the following exact sequence $A \xrightarrow{\underline{f}} B \xrightarrow{H(g)} H(D)$ which implies that $H(g) = 0$. Apply H to the second row of the above diagram, we get the following exact sequence $B \xrightarrow{H(g)} H(D) \to 0$, which implies $H(D) = 0$. This means that $D \in \mathcal{K}$ (see [L2, Proposition 4.7]).

Now we can assume that f admits a short exact sequence: $A \overset{f'}{\rightarrowtail} B' \overset{g'}{\twoheadrightarrow} D$ such that D admits a short exact sequence $U_D \rightarrowtail D \oplus D' \twoheadrightarrow V_D$. Now let $C \in \mathcal{C}$. Since $\mathrm{Hom}_{\overline{\mathcal{B}}}(\Omega C, V_C) = 0 = \mathrm{Hom}_{\overline{\mathcal{B}}}(\Omega C, U_C)$, $g'hi$ factors through $\mathcal{P}$. Hence as in the proof of Proposition 4.11, there is a morphism $j : C \to A$ such that $fj - ha$ factors through $\mathcal{P}$. Since $H(a)$ is an isomorphism in $\underline{\mathcal{H}}$, we have the following commutative diagram

$$\begin{array}{ccccc} & & H(C) & & \\ & \swarrow{\scriptstyle H(j)H(a)^{-1}} & \downarrow{\scriptstyle \underline{h}} & & \\ A & \underset{\underline{f}}{\longrightarrow} & B & \longrightarrow & 0. \end{array}$$

This implies that $H(\mathcal{C})$ is projective in $\underline{\mathcal{H}}$.
Since $\mathcal{U} \subseteq \{Y \in \mathcal{B} \mid \text{Y admits } U' \rightarrowtail C \twoheadrightarrow Y \text{ where } U' \in \mathcal{U} \text{ and } C \in \mathcal{C}\}$, by Lemma 4.12, any object $X \in \mathcal{H}$ admits a short exact sequence $X \rightarrowtail U \twoheadrightarrow C$ where $U \in \mathcal{U}$ and $C \in \mathcal{C}$. Hence we get the following commutative diagram

$$\begin{array}{ccccc} \Omega\mathcal{C} & \rightarrowtail & P & \twoheadrightarrow & C \\ \downarrow{\scriptstyle x} & & \downarrow & & \| \\ X & \rightarrowtail & U & \twoheadrightarrow & C \end{array}$$

which implies that $H(x)$ is an epimorphism. ☐

Theorem 4.14. *If for any object $U \in \mathcal{U}$, there exists an exact sequence $U' \rightarrowtail C \twoheadrightarrow U$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then $\underline{\mathcal{H}} \simeq \mathrm{mod}\,\overline{\mathcal{C}}$.*

Proof. This is an analog of Theorem 4.6. ☐

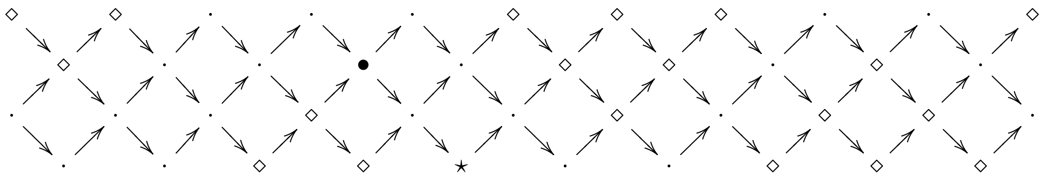
Proposition 4.15. *If $\mathcal{U}$ is covariantly finite and contains $\mathcal{I}$, $\mathcal{B}$ is Krull-Schmidt , then for any object $U \in \mathcal{U}$, there exists an exact sequence $U' \rightarrowtail C \twoheadrightarrow U$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$.*

Proof. This is an analog of Proposition 4.7. ☐

4.3. **Examples.** In this section we give several examples of our main theorem.
The first example comes from [KZ, Corollary 4.4].

Example 4.16. Let $\mathcal{M}$ be a cluster tilting subcategory of $\mathcal{T}$, then $(\mathcal{M}, \mathcal{M})$ is a cotorsion pair with coheart $\mathcal{M}[-1]$. This cotorsion pair satisfies the condition in Theorem 4.6, we get an equivalence $\mathcal{T}/\mathcal{M} \simeq \mathrm{mod}(\mathcal{M}[-1])$ where $\mathcal{T}/\mathcal{M}$ is the heart of $(\mathcal{M}, \mathcal{M})$.

Example 4.17. Let k be a field.



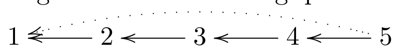
The above diagram is a part of $D^b(\mathrm{mod}\, k\mathbf{A}_4)$ which continues infinitely in both sides. Let $\mathcal{U}$ be the objects in $\diamond$, then $(\mathcal{U},\mathcal{U}^{\perp_1})$ is a cotorsion pair. The coheart $\mathcal{C}$ of it is in $\bullet$, and the heart $\underline{\mathcal{H}}$ of $(\mathcal{U},\mathcal{U}^{\perp_1})$ is in $\star$. By Proposition 4.7, we have $\underline{\mathcal{H}} \simeq \mathrm{mod}\, \mathcal{C}$.

For exact category case, we have the following example in which the cluster category case is included.

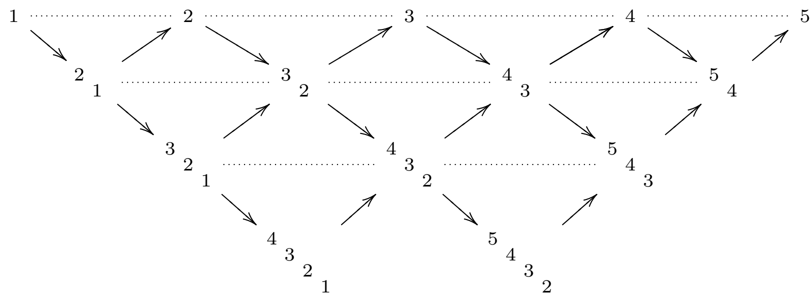
Example 4.18. Let $\mathcal{M}$ be a contravariantly finite rigid subcategory of $\mathcal{B}$ which contains $\mathcal{P}$, let $\mathcal{M}_L = \{X \in \mathcal{B} \mid \text{X admits } X \rightarrowtail M_1 \twoheadrightarrow M_2\}$, then by [DL, Theorem 3.2], we have $\mathcal{M}_L/\mathcal{M} \simeq \mathrm{mod}(\mathcal{M}/\mathcal{P})$. This is a special case of our theorem since $\mathcal{M}_L/\mathcal{M}$ is the heart of cotorsion pair $(\mathcal{M},\mathcal{M}^{\perp_1})$ and $\mathcal{M} = \mathcal{M} \cap {}^{\perp_1}\mathcal{M}$.

In the following example, we denote by "$\circ$" in a quiver the objects belong to a subcategory and by "$\cdot$" the objects do not. The following example is one of the smallest ones with not so small hearts.

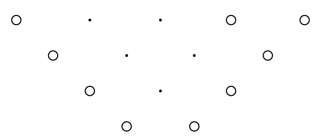
Example 4.19. Let Λ be the path algebra of the following quiver



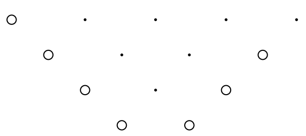
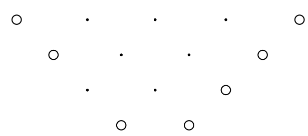
then we obtain the $\mathcal{B} = \mathrm{mod}\,\Lambda$.



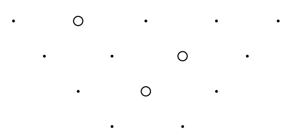
Let

$\mathcal{M} =$ 

Then $(\mathcal{M},\mathcal{M}^{\perp_1})$ is a cotorsion pair on $\mathcal{B}$ and the coheart $\mathcal{C} = {}^{\perp_1}\mathcal{M} \cap \mathcal{M} = {}^{\perp_1}\mathcal{M}$

${}^{\perp_1}\mathcal{M} =$  $\mathcal{M}^{\perp_1} =$ 

We get $\mathcal{C}/\mathcal{P} = \mathrm{add}(\begin{smallmatrix}5&&\\&4&\\&&3\end{smallmatrix} \oplus \begin{smallmatrix}5&\\&4\end{smallmatrix})$. And the heart is the following.

$\underline{\mathcal{H}} =$ 

We can see that $\mathrm{mod}(\mathcal{C}/\mathcal{P}) \simeq \underline{\mathcal{H}}$.

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