Chapter 1

Linear operators on a Hilbert space

This chapter is mainly based on the first chapters of the book [Amr09]. All missing proofs can be found in this reference.

1.1 Hilbert space

Definition 1.1.1. A (complex) Hilbert space $\mathcal{H}$ *is a vector space on* $\mathbb{C}$ *with a strictly positive scalar product (or inner product), which is complete for the associated norm and which admits a countable basis. The scalar product is denoted by* $\langle\cdot,\cdot\rangle$ *and the corresponding norm by* $\|\cdot\|$.

In particular, note that for any $f, g, h \in \mathcal{H}$ and $\alpha \in \mathbb{C}$ the following properties hold:

(i) $\langle f, g\rangle = \overline{\langle g, f\rangle}$,

(ii) $\langle f + \alpha g, h\rangle = \langle f, h\rangle + \alpha\langle g, h\rangle$,

(iii) $\|f\|^2 = \langle f, f\rangle > 0$ if and only if $f \neq 0$.

From now on, the symbol $\mathcal{H}$ will always denote a Hilbert space.

Examples 1.1.2. *(i)* $\mathcal{H} = \mathbb{C}^d$ *with* $\langle\alpha, \beta\rangle = \sum_{j=1}^d \alpha_j \overline{\beta_j}$ *for any* $\alpha, \beta \in \mathbb{C}^d$,

(ii) $\mathcal{H} = l^2(\mathbb{Z})$ *with* $\langle a, b\rangle = \sum_{j\in\mathbb{Z}} a_j \overline{b_j}$ *for any* $a, b \in l^2(\mathbb{Z})$,

(iii) $\mathcal{H} = L^2(\mathbb{R}^d)$ *with* $\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,\mathrm{d}x$ *for any* $f, g \in L^2(\mathbb{R}^d)$.

Let us recall some useful inequalities: For any $f, g \in \mathcal{H}$ one has

(i) $|\langle f, g\rangle| \leq \|f\|\,\|g\|$ Schwartz inequality,

(ii) $\|f + g\| \leq \|f\| + \|g\|$,

(iii) $\|f + g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$,

(iv) $\big| \|f\| - \|g\| \big| \leq \|f - g\|$

the last 3 inequalities are called triangle inequalities. In addition, let us recall that $f, g \in \mathcal{H}$ are said *orthogonal* if $\langle f, g\rangle = 0$.

Definition 1.1.3. *A sequence* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *is* strongly convergent *to* $f_\infty \in \mathcal{H}$ *if* $\lim_{n\to\infty} \|f_n - f_\infty\| = 0$*, or is* weakly convergent *to* $f_\infty \in \mathcal{H}$ *if for any* $g \in \mathcal{H}$ *one has* $\lim_{n\to\infty}\langle f_n - f_\infty, g\rangle = 0$*.*

Clearly, a strongly convergent sequence is also weakly convergent. The converse is not true.

Definition 1.1.4. A subspace $\mathcal{M}$ *of a Hilbert space* $\mathcal{H}$ *is a linear subset of* $\mathcal{H}$*, or more precisely* $\forall f, g \in \mathcal{M}$ *and* $\alpha \in \mathbb{C}$ *one has* $f + \alpha g \in \mathcal{M}$*.*

Note that if $\mathcal{M}$ is closed, then $\mathcal{M}$ is a Hilbert space in itself, with the scalar product and norm inherited from $\mathcal{H}$.

Examples 1.1.5. *(i) If* $f_1, \ldots, f_n \in \mathcal{H}$*, then* $\mathsf{Vect}(f_1, \ldots, f_n)$ *is the closed vector space generated by the linear combinations of* $f_1, \ldots f_n$*.* $\mathsf{Vect}(f_1, \ldots, f_n)$ *is a closed subspace.*

(ii) If $\mathcal{M}$ *is a closed subspace of* $\mathcal{H}$*, then* $\mathcal{M}^\perp := \{f \in \mathcal{H} \mid \langle f, g\rangle = 0, \forall g \in \mathcal{M}\}$ *is a closed subspace of* $\mathcal{H}$*.*

Note that the closed subspace $\mathcal{M}^\perp$ is called *the orthocomplement of* $\mathcal{M}$ *in* $\mathcal{H}$. Indeed, one has:

Lemma 1.1.6 (Projection Theorem)**.** *Let* $\mathcal{M}$ *be a closed subspace of a Hilbert space* $\mathcal{H}$*. Then, for any* $f \in \mathcal{H}$ *there exist a unique* $f_1 \in \mathcal{M}$ *and a unique* $f_2 \in \mathcal{M}^\perp$ *such that* $f = f_1 + f_2$*.*

Let us recall that the dual $\mathcal{H}^*$ of the Hilbert space $\mathcal{H}$ consists in the set of all bounded linear functionals on $\mathcal{H}$, *i.e.* $\mathcal{H}^*$ consists in all mappings $\varphi : \mathcal{H} \to \mathbb{C}$ satisfying for any $f, g \in \mathcal{H}$ and $\alpha \in \mathbb{C}$

(i) $\varphi(f + \alpha g) = \varphi(f) + \alpha\varphi(g)$, (linearity)

(ii) $|\varphi(f)| \leq c\|f\|$, (boundedness)

where c is a constant independent of f. One sets

$$\|\varphi\|_{\mathcal{H}^*} := \sup_{0\neq f\in\mathcal{H}} \frac{|\varphi(f)|}{\|f\|}.$$

Note that if $g \in \mathcal{H}$, then g defines an element φ_g of $\mathcal{H}^*$ by setting $\varphi_g(f) := \langle f, g\rangle$.

Lemma 1.1.7 (Riesz Lemma)**.** *For any* $\varphi \in \mathcal{H}^*$*, there exists a unique* $g \in \mathcal{H}$ *such that for any* $f \in \mathcal{H}$

$$\varphi(f) = \langle f, g\rangle.$$

In addition, g *satisfies* $\|\varphi\|_{\mathcal{H}^*} = \|g\|$*.*

As a consequence, one often identifies $\mathcal{H}^*$ with $\mathcal{H}$ itself.

1.2 Bounded operators

First of all, let us recall that a linear map B between two complex vector spaces $\mathcal{M}$ and $\mathcal{N}$ satisfies $B(f+\alpha g) = Bf + \alpha Bg$ for all $f, g \in \mathcal{M}$ and $\alpha \in \mathbb{C}$.

Definition 1.2.1. *A map $B : \mathcal{H} \to \mathcal{H}$ is* a bounded linear operator *if $B : \mathcal{H} \to \mathcal{H}$ is a linear map, and if there exists $c \in \mathbb{R}$ such that $\|Bf\| \leq c\|f\|$ for all $f \in \mathcal{H}$. The set of* all bounded linear operators on $\mathcal{H}$ *is denoted by $\mathscr{B}(\mathcal{H})$.*

For any $B \in \mathscr{B}(\mathcal{H})$, one sets

$$\|B\| := \sup_{0 \neq f \in \mathcal{H}} \frac{\|Bf\|}{\|f\|}. \tag{1.2.1}$$

and call it *the norm of* B. Note that the same notation is used for the norm of an element of $\mathcal{H}$ and for the norm of an element of $\mathscr{B}(\mathcal{H})$, but this does not lead to any confusion.

Lemma 1.2.2. *If $B \in \mathscr{B}(\mathcal{H})$, then $\|B\| = \sup_{f,g \in \mathcal{H} \text{ with } \|f\|=\|g\|=1} |\langle Bf, g\rangle|$.*

Definition 1.2.3. *A sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ is* uniformly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if $\lim_{n\to\infty} \|B_n - B_\infty\| = 0$, is* strongly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f \in \mathcal{H}$ one has $\lim_{n\to\infty} \|B_n f - B_\infty f\| = 0$, or is* weakly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f, g \in \mathcal{H}$ one has $\lim_{n\to\infty}\langle B_n f - B_\infty f, g\rangle = 0$. In these cases, one writes respectively $u-\lim_{n\to\infty} B_n = B_\infty$, $s-\lim_{n\to\infty} B_n = B_\infty$ and $w-\lim_{n\to\infty} B_n = B_\infty$.*

Clearly, uniform convergence implies strong convergence, and strong convergence implies weak convergence. The reverse statements are not true.

Lemma 1.2.4. *For any $B \in \mathscr{B}(\mathcal{H})$, there exists a unique $B^* \in \mathscr{B}(\mathcal{H})$ such that for any $f, g \in \mathcal{H}$*

$$\langle Bf, g\rangle = \langle f, B^* g\rangle.$$

The operator B^* is called *the adjoint of* B, and the proof of this statement involves the Riesz Lemma.

Proposition 1.2.5. *The following properties hold:*

(i) $\mathscr{B}(\mathcal{H})$ is an algebra,

(ii) The map $\mathscr{B}(\mathcal{H}) \ni B \mapsto B^* \in \mathscr{B}(\mathcal{H})$ is an involution,

(iii) $\mathscr{B}(\mathcal{H})$ is complete with the norm $\|\cdot\|$,

(iv) One has $\|B^*\| = \|B\|$ and $\|B^*B\| = \|B\|^2$.

As a consequence of these properties, $\mathscr{B}(\mathcal{H})$ is a C^*-algebra, as we shall see later on.

Definition 1.2.6. *For any $B \in \mathscr{B}(\mathcal{H})$ one sets*

$$\mathsf{Ran}(B) := B\mathcal{H} = \{f \in \mathcal{H} \mid f = Bg \text{ for some } g \in \mathcal{H}\},$$

and call this set the range of B.

Definition 1.2.7. *An operator $B \in \mathscr{B}(\mathcal{H})$ is* invertible *if the equation $Bf = 0$ only admits the solution $f = 0$. In such a case, there exists a linear map $B^{-1} : \mathsf{Ran}(B) \to \mathcal{H}$ which satisfies $B^{-1}Bf = f$ for any $f \in \mathcal{H}$, and $BB^{-1}g = g$ for any $g \in \mathsf{Ran}(B)$. If B is invertible and $\mathsf{Ran}(B) = \mathcal{H}$, then $B^{-1} \in \mathscr{B}(\mathcal{H})$ and B is said* boundedly invertible *or* invertible in $\mathscr{B}(\mathcal{H})$.

Note that the two conditions B invertible and $\mathsf{Ran}(B) = \mathcal{H}$ imply $B^{-1} \in \mathscr{B}(\mathcal{H})$ is a consequence of the Closed graph Theorem.

Remark 1.2.8. *In the sequel, we shall use the notation $\mathbf{1} \in \mathscr{B}(\mathcal{H})$ for the operator defined on any $f \in \mathcal{H}$ by $\mathbf{1}f = f$, and $\mathbf{0} \in \mathscr{B}(\mathcal{H})$ for the operator defined by $\mathbf{0}f = 0$.*

Lemma 1.2.9 (Neumann series)**.** *If $B \in \mathscr{B}(\mathcal{H})$ and $\|B\| < 1$, then the operator $(\mathbf{1}-B)$ is invertible in $\mathscr{B}(\mathcal{H})$, with*

$$(\mathbf{1} - B)^{-1} = \sum_{n=0}^{\infty} B^n,$$

and with $\big\|(\mathbf{1} - B)^{-1}\big\| \le (1 - \|B\|)^{-1}$.

Note that we have used the identity $B^0 = \mathbf{1}$.

1.3 Special classes of operators

Definition 1.3.1. *An element $U \in \mathscr{B}(\mathcal{H})$ is* a unitary operator *if $UU^* = \mathbf{1}$ and if $U^*U = \mathbf{1}$.*

Note that in this case, U is boundedly invertible with $U^{-1} = U^*$. Indeed, observe first that $Uf = 0$ implies $f = U^*(Uf) = U^*0 = 0$. Secondly, for any $g \in \mathcal{H}$, one has $g = U(U^*g)$, and thus $\mathsf{Ran}(U) = \mathcal{H}$. Finally, the equality $U^{-1} = U^*$ follows from the unicity of the inverse.

Definition 1.3.2. *An element $P \in \mathscr{B}(\mathcal{H})$ is* an orthogonal projection *if $P = P^2 = P^*$.*

In this case, $P\mathcal{H}$ is a closed subspace of $\mathcal{H}$. Alternatively, for each closed subspace $\mathcal{M}$ of $\mathcal{H}$, there exists an orthogonal projection P such that $P\mathcal{H} = \mathcal{M}$.

Now, for any family $\{g_j, h_j\}_{j=1}^n \subset \mathcal{H}$ and for any $f \in \mathcal{H}$ one sets

$$A_n f := \sum_{j=1}^{n} \langle f, g_j\rangle\, h_j. \tag{1.3.1}$$

Then $A_n \in \mathscr{B}(\mathcal{H})$, and $\mathsf{Ran}(A_n) \subset \mathsf{Vect}(h_1, \dots, h_n)$. Such an operator A_n is called *a finite rank operator*. In fact, any operator $B \in \mathscr{B}(\mathcal{H})$ with $\dim\big(\mathsf{Ran}(B)\big) < \infty$ is a finite rank operator.

Exercise 1.3.3. *For the operator A_n defined in* (1.3.1)*, give an upper estimate for $\|A_n\|$ and compute A_n^*.*

Definition 1.3.4. *An element $B \in \mathscr{B}(\mathcal{H})$ is* a compact operator *if there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of finite rank operators such that $\lim_{n\to\infty} \|A_n - B\| = 0$. The set of all compact operators is denoted by $\mathscr{K}(\mathcal{H})$.*

Proposition 1.3.5. *The following properties hold:*

(i) $B \in \mathscr{K}(\mathcal{H}) \Longleftrightarrow B^* \in \mathscr{K}(\mathcal{H})$,

(ii) $\mathscr{K}(\mathcal{H})$ *is a $*$-algebra, complete for the norm $\|\cdot\|$,*

(iii) If $B \in \mathscr{K}(\mathcal{H})$ and $A \in \mathscr{B}(\mathcal{H})$, then AB and BA belong to $\mathscr{K}(\mathcal{H})$.

As a consequence, $\mathscr{K}(\mathcal{H})$ is a C^*-algebra and an ideal of $\mathscr{B}(\mathcal{H})$.

Extension 1.3.6. *There are various subalgebras of $\mathscr{K}(\mathcal{H})$, for example the algebra of Hilbert-Schmidt operators, the algebra of trace class operators, and more generally the Schatten classes. Note that these algebras are not closed with respect to the norm $\|\cdot\|$ but with respect to some stronger norms $|||\cdot|||$. These algebras are ideals in $\mathscr{B}(\mathcal{H})$.*

1.4 Operator valued maps

Let I be an open interval on $\mathbb{R}$, and let us consider a map $F : I \to \mathscr{B}(\mathcal{H})$.

Definition 1.4.1. *The map F is* continuous in norm *on I if for all $x \in I$*

$$\lim_{\varepsilon\to 0} \big\|F(x+\varepsilon) - F(x)\big\| = 0.$$

The map F is strongly continuous *on I if for any $f \in \mathcal{H}$ and all $x \in I$*

$$\lim_{\varepsilon\to 0} \big\|F(x+\varepsilon)f - F(x)f\big\| = 0.$$

The map F is weakly continuous *on I if for any $f, g \in \mathcal{H}$ and all $x \in I$*

$$\lim_{\varepsilon\to 0} \big\langle \big(F(x+\varepsilon) - F(x)\big)f, g\big\rangle = 0.$$

One writes respectively $u - \lim_{\varepsilon\to 0} F(x+\varepsilon) = F(x)$, $s - \lim_{\varepsilon\to 0} F(x+\varepsilon) = F(x)$ and $w - \lim_{\varepsilon\to 0} F(x+\varepsilon) = F(x)$.

Definition 1.4.2. *The map F is* differentiable in norm *on I if there exists a map $F' : I \to \mathscr{B}(\mathcal{H})$ such that*

$$\lim_{\varepsilon\to 0} \Big\|\frac{1}{\varepsilon}\big(F(x+\varepsilon) - F(x)\big) - F'(x)\Big\| = 0.$$

The definitions for strongly differentiable *and* weakly differentiable *are similar.*

If I is an open interval of $\mathbb{R}$ and if $F : I \to \mathscr{B}(\mathcal{H})$, one defines $\int_I F(x)\,\mathrm{d}x$ as a Riemann integral (limit of finite sums over a partition of I) if this limiting procedure exists and is independent of the partitions of I. Note that these integrals can be defined in the weak topology, in the strong topology or in the norm topology (and in other topologies). For example, if $F : I \to \mathscr{B}(\mathcal{H})$ is strongly continuous and if $\int_I \|F(x)\|\mathrm{d}x < \infty$, then the integral $\int_I F(x)\,\mathrm{d}x$ exists in the strong topology.

Proposition 1.4.3. *Let I is an open interval of $\mathbb{R}$ and $F : I \to \mathscr{B}(\mathcal{H})$ such that $\int_I F(x)\,\mathrm{d}x$ exists (in an appropriate topology). Then,*

(i) For any $B \in \mathscr{B}(\mathcal{H})$ one has

$$B \int_I F(x)\,\mathrm{d}x = \int_I BF(x)\,\mathrm{d}x \quad \text{and} \quad \Big(\int_I F(x)\,\mathrm{d}x\Big)B = \int_I F(x)B\,\mathrm{d}x,$$

(ii) one also has $\left\| \int_I F(x)\,\mathrm{d}x \right\| \leq \int_I \|F(x)\|\,\mathrm{d}x$,

(iii) If $\mathscr{C} \subset \mathscr{B}(\mathcal{H})$ is a subalgebra of $\mathscr{B}(\mathcal{H})$, closed with respect to a norm $|||\cdot|||$, and if the map $F : I \to \mathscr{C}$ is continuous with respect to this norm and satisfies $\int_I |||F(x)|||\,\mathrm{d}x < \infty$, then $\int_I F(x)\,\mathrm{d}x$ exists, belongs to $\mathscr{C}$ and satisfies

$$\left|\left|\left| \int_I F(x)\,\mathrm{d}x \right|\right|\right| \leq \int_I |||F(x)|||\,\mathrm{d}x.$$

Note that the last statement is very useful, for example when $\mathscr{C} = \mathscr{K}(\mathcal{H})$ or any Schatten class.

1.5 Unbounded operators

In this section, we define an extension of the notion of bounded linear operators. Obviously, the following definitions and results are also valid for bounded linear operators.

Definition 1.5.1. *A* linear operator on *$\mathcal{H}$ is a pair $\big(A, \mathsf{D}(A)\big)$, where $\mathsf{D}(A)$ is a subspace of $\mathcal{H}$ and A is a linear map from $\mathsf{D}(A)$ to $\mathcal{H}$. $\mathsf{D}(A)$ is called* the domain of A. *One says that the operator $\big(A, \mathsf{D}(A)\big)$ is* densely defined *if $\mathsf{D}(A)$ is dense in $\mathcal{H}$.*

Note that one often just says *the linear operator* A, but that its domain $\mathsf{D}(A)$ is implicitly taken into account. For such an operator, its range $\mathsf{Ran}(A)$ is defined by

$$\mathsf{Ran}(A) := A\,\mathsf{D}(A) = \{f \in \mathcal{H} \mid f = Ag \text{ for some } g \in \mathsf{D}(A)\}.$$

In addition, one defines the kernel $\mathsf{Ker}(A)$ of A by

$$\mathsf{Ker}(A) := \{f \in \mathsf{D}(A) \mid Af = 0\}.$$

Example 1.5.2. *Let* $\mathcal{H} := L^2(\mathbb{R})$ *and consider the operator* X *defined by* $[Xf](x) = xf(x)$ *for any* $x \in \mathbb{R}$. *Clearly,* $\mathsf{D}(X) = \{f \in \mathcal{H} \mid \int_{\mathbb{R}} |xf(x)|^2 \,\mathrm{d}x < \infty\} \subsetneq \mathcal{H}$. *In addition, by considering the family of functions* $\{f_y\}_{y\in\mathbb{R}} \subset \mathsf{D}(X)$ *with* $f_y(x) := e^{|x-y|^2}$, *one easily observes that* $\sup_{0\neq f\in\mathsf{D}(X)} \frac{\|Xf\|}{\|f\|} = \infty$, *which can be compared with* (1.2.1).

Definition 1.5.3. *For any pair of linear operators* $\big(A, \mathsf{D}(A)\big)$ *and* $\big(B, \mathsf{D}(B)\big)$ *satisfying* $\mathsf{D}(A) \subset \mathsf{D}(B)$ *and* $Af = Bf$ *for all* $f \in \mathsf{D}(A)$, *one says that* $\big(B, \mathsf{D}(B)\big)$ *is* an extension of $\big(A, \mathsf{D}(A)\big)$ *to* $\mathsf{D}(B)$, *or that* $\big(A, \mathsf{D}(A)\big)$ *is the restriction of* $\big(B, \mathsf{D}(B)\big)$ *to* $\mathsf{D}(A)$.

Let us note that if $\big(A, \mathsf{D}(A)\big)$ is densely defined and if there exists $c \in \mathbb{R}$ such that $\|Af\| \leq c\|f\|$ for all $f \in \mathsf{D}(A)$, then there exists a natural continuous extension $\overline{A}$ of A with $\mathsf{D}(\overline{A}) = \mathcal{H}$. This extension satisfies $\overline{A} \in \mathscr{B}(\mathcal{H})$ with $\|\overline{A}\| \leq c$, and is called *the closure of the operator* A.

Exercise 1.5.4. *Construct this natural extension and show that* $\|\overline{A}\| \leq c$.

Let us stress that the sum $A + B$ for two linear operators is *a priori* only defined on the subspace $\mathsf{D}(A) \cap \mathsf{D}(B)$, and that the product AB is *a priori* defined only on the subspace $\{f \in \mathsf{D}(B) \mid Bf \in \mathsf{D}(A)\}$. These two sets can be very small.

Definition 1.5.5. *Let* $\big(A, \mathsf{D}(A)\big)$ *be a densely defined linear operator on* $\mathcal{H}$. The adjoint A^* of A *is the operator defined by*

$$\mathsf{D}(A^*) := \big\{f \in \mathcal{H} \mid \exists f^* \in \mathcal{H} \text{ with } \langle f^*, g\rangle = \langle f, Ag\rangle \text{ for all } g \in \mathsf{D}(A)\big\}$$

and $A^*f := f^*$ *for all* $f \in \mathsf{D}(A^*)$.

Let us note that the density of $\mathsf{D}(A)$ is necessary to ensure that A^* is well defined. Indeed, if f_1^*, f_2^* satisfy for all $g \in \mathsf{D}(A)$

$$\langle f_1^*, g\rangle = \langle f, Ag\rangle = \langle f_2^*, g\rangle,$$

then $\langle f_1^* - f_2^*, g\rangle = 0$ for all $g \in \mathsf{D}(A)$, and this equality implies $f_1^* = f_2^*$ only if $\mathsf{D}(A)$ is dense in $\mathcal{H}$. Note also that once $\big(A^*, \mathsf{D}(A^*)\big)$ is defined, one has

$$\langle A^*f, g\rangle = \langle f, Ag\rangle \qquad \forall f \in \mathsf{D}(A^*) \text{ and } \forall g \in \mathsf{D}(A).$$

Lemma 1.5.6. *Let* $\big(A, \mathsf{D}(A)\big)$ *be a densely defined linear operator on* $\mathcal{H}$. *Then*

$$\mathsf{Ker}(A^*) = \mathsf{Ran}(A)^{\perp}.$$

Proof. Let $f \in \mathsf{Ker}(A^*)$, *i.e.* $f \in \mathsf{D}(A^*)$ and $A^*f = 0$. Then, for all $g \in \mathsf{D}(A)$, one has

$$0 = \langle A^*f, g\rangle = \langle f, Ag\rangle$$

meaning that $f \in \mathsf{Ran}(A)^{\perp}$. Conversely, if $f \in \mathsf{Ran}(A)^{\perp}$, then for all $g \in \mathsf{D}(A)$ one has

$$\langle f, Ag\rangle = 0 = \langle 0, g\rangle$$

meaning that $f \in \mathsf{D}(A^*)$ and $A^*f = 0$, by the definition of the adjoint of A. □

Definition 1.5.7. *A densely defined linear operator* $\big(A, \mathsf{D}(A)\big)$ *is* self-adjoint *if* $\mathsf{D}(A^*) = \mathsf{D}(A)$ *and* $A^*f = Af$ *for all* $f \in \mathsf{D}(A)$.

Note that whenever the operator A is self-adjoint one has

$$\langle Af, g\rangle = \langle f, Ag\rangle \qquad \forall f, g \in \mathsf{D}(A).$$

Let us stress that self-adjoint operators are very important in relation with quantum mechanics: any physical system is described with such an operator. Self-adjoint operators are the natural generalisation of Hermitian matrices.

Extension 1.5.8. *Self-adjoint operators are a special class of closed and symmetric linear operators. These notions, as well as the graph or the essential self-adjointness of an operator are important topics for the study of unbounded linear operators.*

1.6 Resolvent and spectrum

Definition 1.6.1. *For a closed*[1] *linear operator* A*, a value* $z \in \mathbb{C}$ *is* an eigenvalue of A *if there exists* $f \in \mathsf{D}(A)$*,* $f \neq 0$*, such that* $Af = zf$*. In such a case, the element* f *is called* an eigenfunction of A associated with the eigenvalue z*. The set of all eigenvalues of* A *is denoted by* $\sigma_p(A)$.

Lemma 1.6.2. *Let* A *be a self-adjoint operator on* $\mathcal{H}$*. Then,*

(i) All eigenvalues of A *are real,*

(ii) Two eigenfunctions of A *associated with two different eigenvalues of* A *are orthogonal.*

Proof. (i) Assume that $Af = zf$ for some $z \in \mathbb{C}$ and $f \in \mathsf{D}(A)$ with $f \neq 0$. Then, one has

$$z\|f\|^2 = \langle zf, f\rangle = \langle Af, f\rangle = \langle f, Af\rangle = \langle f, zf\rangle = \overline{z}\|f\|^2,$$

which implies that $z \in \mathbb{R}$.

(ii) Assume that $Af = \lambda f$ and that $Ag = \mu g$ with $\lambda, \mu \in \mathbb{R}$ and $\lambda \neq \mu$, and $f, g \in \mathsf{D}(A)$, with $f \neq 0$ and $g \neq 0$. Then

$$\lambda\langle f, g\rangle = \langle Af, g\rangle = \langle f, Ag\rangle = \mu\langle f, g\rangle,$$

which implies that $\langle f, g\rangle = 0$, or in other words that f and g are orthogonal. □

[1]An operator A is closed if the three conditions (i) $f_n \in \mathsf{D}(A)$, (ii) $s-\lim_{n\to\infty} f_n = f$, (iii) $\{Af_n\}$ is strongly Cauchy, imply that $f \in \mathsf{D}(A)$ and $s-\lim_{n\to\infty} Af_n = Af$. Note that any self-adjoint operator as well as any bounded operator is closed.

By analogy to the bounded case, we say that A is *invertible* if $\mathsf{Ker}(A) = \{0\}$. In this case, the inverse A^{-1} gives a bijection from $\mathsf{Ran}(A)$ onto $\mathsf{D}(A)$. Note now that if z is an eigenvalue of a linear operator A, then $(A - z)$ is not invertible since $(A - z)f = 0$ for some $f \in \mathsf{D}(A)$ with $f \neq 0$. Then, *the spectrum of the operator* A is a generalization of the notion of eigenvalues which is based on the previous observation.

Definition 1.6.3. The resolvent set $\rho(A)$ *of a closed linear operator* A *is defined by*

$$\begin{aligned}\rho(A) :=& \big\{z \in \mathbb{C} \mid (A - z) \text{ is invertible in } \mathscr{B}(\mathcal{H})\big\}\\ =& \big\{z \in \mathbb{C} \mid \mathsf{Ker}(A - z) = \{0\} \text{ and } \mathsf{Ran}(A - z) = \mathcal{H}\big\}.\end{aligned}$$

The spectrum $\sigma(A)$ of A *is the complement of* $\rho(A)$ *in* $\mathbb{C}$*,* i.e. $\sigma(A) := \mathbb{C} \setminus \rho(A)$*.*

Definition 1.6.4. *For any closed linear operator* A *and for any* $z \in \rho(A)$*, the operator* $(A - z)^{-1} \in \mathscr{B}(\mathcal{H})$ *is called* the resolvent of A at the point z*.*

Exercise 1.6.5. *For any closed linear operator* A *and any* $z_1, z_2 \in \rho(A)$*, show the* first resolvent equation*, namely*

$$(A - z_1)^{-1} - (A - z_2)^{-1} = (z_1 - z_2)(A - z_1)^{-1}(A - z_2)^{-1}. \tag{1.6.1}$$

Lemma 1.6.6. *The spectrum of a self-adjoint operator* A *is real,* i.e. $\sigma(A) \subset \mathbb{R}$*.*

Proof of Lemma 1.6.6. Let us consider $z = \lambda + i\varepsilon$ with $\varepsilon \neq 0$, and show that $z \in \rho(A)$. Indeed, for any $f \in \mathsf{D}(A)$ one has

$$\begin{aligned}\|(A - z)f\|^2 &= \|(A - \lambda)f - i\varepsilon f\|^2\\ &= \big\langle (A - \lambda)f - i\varepsilon f, (A - \lambda)f - i\varepsilon f\big\rangle\\ &= \|(A - \lambda)f\|^2 + \varepsilon^2\|f\|^2.\end{aligned}$$

It follows that $\|(A - z)f\| \geq |\varepsilon|\|f\|$, and thus $A - z$ is invertible.

Now, for any for any $g \in \mathsf{Ran}(A - z)$ let us observe that

$$\|g\| = \big\|(A - z)(A - z)^{-1}g\big\| \geq |\varepsilon|\big\|(A - z)^{-1}g\big\|.$$

Equivalently, it means for all $g \in \mathsf{Ran}(A - z)$, one has

$$\big\|(A - z)^{-1}g\big\| \leq \frac{1}{|\varepsilon|}\|g\|. \tag{1.6.2}$$

Let us finally observe that $\mathsf{Ran}(A - z)$ is dense in $\mathcal{H}$. Indeed, by Lemma 1.5.6 one has

$$\mathsf{Ran}(A - z)^{\perp} = \mathsf{Ker}\,\big((A - z)^*\big) = \mathsf{Ker}(A^* - \overline{z}) = \mathsf{Ker}(A - \overline{z}) = \{0\}$$

since all eigenvalues of A are real. Thus, the operator $(A - z)^{-1}$ is defined on the dense domain $\mathsf{Ran}(A - z)$ and satisfies the estimate (1.6.2). As explained just before the Exercise 1.5.4, it means that $(A - z)^{-1}$ continuously extends to an element of $\mathscr{B}(\mathcal{H})$, and therefore $z \in \rho(A)$. □

1.7 Spectral theory for self-adjoint operators

1.7.1 Stieltjes measures

Let us consider a function $F : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

(i) F is monotone non-decreasing, *i.e.* $\lambda \geq \mu \Longrightarrow F(\lambda) \geq F(\mu)$,

(ii) F is right continuous, *i.e.* $F(\lambda) = F(\lambda + 0) := \lim_{\varepsilon \searrow 0} F(\lambda + \varepsilon)$ for all $\lambda \in \mathbb{R}$,

(iii) $F(-\infty) := \lim_{\lambda \to -\infty} F(\lambda) = 0$ and $\rho := F(+\infty) := \lim_{\lambda \to \infty} F(\lambda) < \infty$.

Note that $F(\lambda + 0) := \lim_{\varepsilon \searrow 0} F(\lambda + \varepsilon)$ and $F(\lambda - 0) := \lim_{\varepsilon \searrow 0} F(\lambda - \varepsilon)$ exist since F is a monotone and bounded function.

With a function F having these properties, one can associate a bounded Borel measure m_F on $\mathbb{R}$, called *Stieltjes measure*, starting with

$$m_F\big((a, b]\big) := F(b) - F(a), \qquad a, b \in \mathbb{R}$$

and extending then this definition to all Borel sets of $\mathbb{R}$. With this definition, note that $m_F(\mathbb{R}) = \rho$ and that

$$m_F\big((a, b)\big) = F(b - 0) - F(a), \qquad m_F\big([a, b]\big) = F(b) - F(a - 0)$$

and therefore $m_F\big(\{a\}\big) = F(a) - F(a - 0)$ is different from 0 if F is not continuous at the point a.

Note that starting with a bounded Borel measure m on $\mathbb{R}$ and setting $F(\lambda) := m\big((-\infty, \lambda]\big)$, then F satisfies the conditions (i)-(iii) and the associated Stieltjes measure m_F verifies $m_F = m$.

Theorem 1.7.1. *Any Stieltjes measure m admits a unique decomposition*

$$m = m_p + m_{ac} + m_{sc}$$

where m_p is a pure point measure, *m_{ac} is* an absolutely continuous measure *with respect to the Lebesgue measure on $\mathbb{R}$, and m_{sc} is* a singular continuous measure *with respect to the Lebesgue measure $\mathbb{R}$.*

This result is based on *Lebesgue Decomposition Theorem*. Let us simply stress that m_{sc} is singular with respect to the Lebesgue measure but $m_{sc}(\{\lambda\}) = 0$ for any $\lambda \in \mathbb{R}$. On the other hand, for any Borel set V, $m_p(V) = \sum_{\lambda \in V} m(\{\lambda\})$, where this sum contains at most a countable number of contributions.

1.7.2 Spectral measures

We shall now define a spectral measure, by analogy with the Stieltjes measure defined in the previous section.

Definition 1.7.2. A spectral family, *or* a resolution of the identity, *is a family* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *of orthogonal projections in* $\mathcal{H}$ *satisfying:*

(i) The family is non-decreasing, i.e. $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}$,

(ii) The family is strongly right continuous, i.e. $E_\lambda = E_{\lambda+0} = s-\lim_{\varepsilon\searrow 0} E_{\lambda+\varepsilon}$,

(iii) $s-\lim_{\lambda\to-\infty} E_\lambda = \mathbf{0}$ *and* $s-\lim_{\lambda\to\infty} E_\lambda = \mathbf{1}$,

It is important to observe that the condition (i) implies that the elements of the families are *commuting, i.e.* $E_\lambda E_\mu = E_\mu E_\lambda$. We also define *the support of the spectral family* as the following subset of $\mathbb{R}$:

$$\operatorname{supp}\{E_\lambda\} = \{\mu\in\mathbb{R} \mid E_{\mu+\varepsilon} - E_{\mu-\varepsilon} \neq \mathbf{0},\ \forall\varepsilon > 0\}.$$

With such a spectral family one first defines

$$E\big((a,b]\big) := E_b - E_a, \qquad a,b\in\mathbb{R}, \tag{1.7.1}$$

and extends this definition to all Borel sets on $\mathbb{R}$ (we denote by $\mathcal{A}_B$ the set of all Borel sets on $\mathbb{R}$). One ends up with a projection-valued map $E:\mathcal{A}_B\to\mathbb{R}$ which satisfies $E(\emptyset)=\mathbf{0}$, $E(\mathbb{R})=\mathbf{1}$, $E(V_1)E(V_2)=E(V_1\cap V_2)$ for any Borel sets V_1,V_2. In addition,

$$E\big((a,b)\big) = E_{b-0} - E_a, \qquad E\big([a,b]\big) = E_b - E_{a-0}$$

and therefore $E\big(\{a\}\big) = E_a - E_{a-0}$.

Definition 1.7.3. *The map* $E:\mathcal{A}_B\to\mathbb{R}$ *defined by* (1.7.1) *is called* the spectral measure *associated with the family* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$. *This spectral measure is* bounded from below *if there exists* $\lambda_-\in\mathbb{R}$ *such that* $E_\lambda=\mathbf{0}$ *for all* $\lambda<\lambda_-$.

Let us note that for any spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ and any $f\in\mathcal{H}$ one can set

$$F_f(\lambda) := \|E_\lambda f\|^2 = \langle E_\lambda f, f\rangle.$$

Then, one easily checks that the function F_f satisfies the conditions (i)-(iii) of the beginning of Section 1.7.1. Thus, one can associate with each element $f\in\mathcal{H}$ a finite Stieltjes measure m_f on $\mathbb{R}$ which satisfies $m_f(V) = \|E(V)f\|^2 = \langle E(V)f,f\rangle$ for any $V\in\mathcal{A}_B$.

Our next aim is to define integrals of the form

$$\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda) \tag{1.7.2}$$

for a continuous function $\varphi : [a, b] \to \mathbb{C}$ and for any spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$. Such integrals can be defined in the sense of Riemann-Stieltjes by first considering a partition $a = x_0 < x_1 < ... < x_n = b$ of $[a, b]$ and a collection $\{y_j\}$ with $y_j \in (x_{j-1}, x_j)$ and by defining the operator

$$\sum_{j=1}^{n} \varphi(y_j) E\big((x_{j-1}, x_j]\big). \tag{1.7.3}$$

It turns out that by considering finer and finer partitions of $[a, b]$, the corresponding expression (1.7.3) strongly converges to an element of $\mathscr{B}(\mathcal{H})$ which is independent of the successive choice of partitions. The resulting operator is denoted by (1.7.2).

Proposition 1.7.4 (Spectral integrals)**.** *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be a spectral family, let $-\infty < a < b < \infty$ and let $\varphi : [a, b] \to \mathbb{C}$ be continuous. Then one has*

(i) $\Big\|\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda)\Big\| = \sup_{\mu\in[a,b]\cap \mathrm{supp}\{E_\lambda\}} |\varphi(\mu)|$,

(ii) $\Big(\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda)\Big)^* = \int_a^b \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda)$,

(iii) For any $f \in \mathcal{H}$, $\Big\|\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda) f\Big\|^2 = \int_a^b |\varphi(\lambda)|^2\, m_f(\mathrm{d}\lambda)$,

(iv) If $\psi : [a, b] \to \mathbb{C}$ is continuous, then

$$\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda) \ \cdot \int_a^b \psi(\lambda)\, E(\mathrm{d}\lambda) = \int_a^b \varphi(\lambda)\, \psi(\lambda)\, E(\mathrm{d}\lambda).$$

Let us now observe that if the support $\mathrm{supp}\{E_\lambda\}$ is bounded, then one can consider

$$\int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda) = s - \lim_{M\to\infty} \int_{-M}^{M} \varphi(\lambda)\, E(\mathrm{d}\lambda). \tag{1.7.4}$$

Similarly, by taking property (iii) of the previous proposition into account, one observes that this limit can also be taken if $\varphi \in L^\infty(\mathbb{R}, \mathbb{C})$. On the other hand, if φ is not bounded on $\mathbb{R}$, the r.h.s. of (1.7.4) is not necessarily well defined. In fact, if φ is not bounded on $\mathbb{R}$ and if $\mathrm{supp}\{E_\lambda\}$ is not bounded either, then the r.h.s. of (1.7.4) is an unbounded operator and can only be defined on a dense domain of $\mathcal{H}$.

Lemma 1.7.5. *Let $\varphi : \mathbb{R} \to \mathbb{C}$ be continuous, and let us set*

$$\mathsf{D}_\varphi := \Big\{ f \in \mathcal{H} \mid \int_{-\infty}^{\infty} |\varphi(\lambda)|^2\, m_f(\mathrm{d}\lambda) < \infty \Big\}.$$

Then the pair $\Big(\int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda), \mathsf{D}_\varphi\Big)$ defines a densely defined linear operator on $\mathcal{H}$. This operator is self-adjoint if and only if φ is a real function.

A function φ of special interest is the function defined by the identity function id, namely $\mathrm{id}(\lambda) = \lambda$.

Definition 1.7.6. *For any spectral family* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$*, the operator* $\left(\int_{-\infty}^{\infty} \lambda\, E(\mathrm{d}\lambda), \mathsf{D}_{\mathrm{id}}\right)$ *with*

$$\mathsf{D}_{\mathrm{id}} := \left\{ f \in \mathcal{H} \mid \int_{-\infty}^{\infty} \lambda^2\, m_f(\mathrm{d}\lambda) < \infty \right\}$$

is called the self-adjoint operator associated with $\{E_\lambda\}$*.*

By this procedure, any spectral family defines a self-adjoint operator on $\mathcal{H}$. *The spectral Theorem* corresponds to the converse statement:

Theorem 1.7.7 (Spectral Theorem)**.** *With any self-adjoint operator* $(A, \mathsf{D}(A))$ *on a Hilbert space* $\mathcal{H}$ *one can associate a unique spectral family* $\{E_\lambda\}$*, called* the spectral family of A*, such that* $\mathsf{D}(A) = \mathsf{D}_{\mathrm{id}}$ *and* $A = \int_{-\infty}^{\infty} \lambda\, E(\mathrm{d}\lambda)$*.*

In summary, there is a bijective correspondence between self-adjoint operators and spectral families. This theorem extends the fact that any $n \times n$ hermitian matrix is diagonalizable. The proof of this theorem is not trivial and is rather lengthy. In the sequel, we shall assume it, and state various consequences of this theorem.

Extension 1.7.8. *Study the proof the Spectral Theorem, starting with the version for bounded self-adjoint operators.*

1.7.3 Bounded functional calculus

Let A be a self-adjoint operator in $\mathcal{H}$ and $\{E_\lambda\}$ be the corresponding spectral family.

Definition 1.7.9. *For any bounded and continuous function* $\varphi : \mathbb{R} \to \mathbb{C}$ *one sets* $\varphi(A) \in \mathscr{B}(\mathcal{H})$ *for the operator defined by*

$$\varphi(A) := \int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda).$$

Exercise 1.7.10. *Show the following equality:* $\mathrm{supp}\{E_\lambda\} = \sigma(A)$*. Note that part of the proof consists in showing that if* $\varphi_z(\lambda) = (\lambda - z)^{-1}$ *for some* $z \in \rho(A)$*, then* $\varphi_z(A) = (A - z)^{-1}$*, where the r.h.s. has been defined in Section 1.6.*

For the next statement, we set $C_b(\mathbb{R})$ for the set of all continuous and bounded complex functions on $\mathbb{R}$.

Proposition 1.7.11. *a) For any* $\varphi \in C_b(\mathbb{R})$ *one has*

(i) $\varphi(A) \in \mathscr{B}(\mathcal{H})$ *and* $\|\varphi(A)\| = \sup_{\lambda\in\sigma(A)} |\varphi(\lambda)|$*,*

(ii) $\varphi(A)^* = \overline{\varphi}(A)$*, and* $\varphi(A)$ *is self-adjoint if and only if* φ *is real,*

(iii) $\varphi(A)$ is unitary if and only if $|\varphi(\lambda)| = 1$.

b) The map $C_b(\mathbb{R}) \ni \varphi \mapsto \varphi(A) \in \mathscr{B}(\mathcal{H})$ is a $*$-homomorphism.

In the point (iii) above, one can consider the function $\varphi_t \in C_b(\mathbb{R})$ defined by $\varphi_t(\lambda) := e^{-it\lambda}$ for any fixed $t \in \mathbb{R}$. Then, if one sets $U_t := \varphi_t(A)$ one can observe that $U_t U_s = U_{t+s}$ and that the map $\mathbb{R} \ni t \mapsto U_t \in \mathscr{B}(\mathcal{H})$ is strongly continuous. Such a family $\{U_t\}_{t\in\mathbb{R}}$ is called *a strongly continuous unitary group.*

Theorem 1.7.12 (Stone Theorem)**.** *There exists a bijective correspondence between self-adjoint operators on $\mathcal{H}$ and strongly continuous unitary groups on $\mathcal{H}$. More precisely, if A is a self-adjoint operator on $\mathcal{H}$, then $\{e^{-itA}\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group, while if $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group, one sets*

$$\mathsf{D}(A) := \Big\{ f \in \mathcal{H} \mid \exists\, s - \lim_{t\to 0} \frac{1}{t}[U_t - 1]f \Big\}$$

and for $f \in \mathsf{D}(A)$ one sets $Af = s - \lim_{t\to 0} \frac{i}{t}[U_t - 1]f$.

Remark 1.7.13. *If the inverse Fourier transform $\check{\varphi}$ of φ belongs to $L^1(\mathbb{R})$, then the following equality holds*

$$\varphi(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{\varphi}(t)\, e^{-itA}\, \mathrm{d}t.$$

1.7.4 Spectral parts of a self-adjoint operator

In this section, we consider a fixed self-adjoint operator A (and its associated spectral family $\{E_\lambda\}$), and show that there exists a natural decomposition of the Hilbert space $\mathcal{H}$ with respect to this operator. First of all, recall from Lemma 1.6.6 that the spectrum of any self-adjoint operator is real. In addition, let us recall that for any $\mu \in \mathbb{R}$, one has

$$\mathsf{Ran}\big(E(\{\mu\})\big) = \{f \in \mathcal{H} \mid E(\{\mu\})f = f\}.$$

Then, one observes that the following equivalence holds:

$$f \in \mathsf{Ran}\big(E(\{\mu\})\big) \iff f \in \mathsf{D}(A) \ \text{ with } Af = \mu f.$$

Indeed, this can be inferred from the equality

$$\|Af - \mu f\|^2 = \int_{-\infty}^{\infty} |\lambda - \mu|^2\, m_f(\mathrm{d}\lambda)$$

which itself can be deduced from the point (iii) of Proposition 1.7.4. Indeed, since the integrand is strictly positive for each $\lambda \neq \mu$, one can have $\|Af - \mu f\| = 0$ if and only if $m_f(V) = 0$ for any Borel set V on $\mathbb{R}$ with $\mu \notin V$. In other words, the measure m_f is supported only on $\{\mu\}$.

Definition 1.7.14. *The set of all $\mu \in \mathbb{R}$ such that* $\mathsf{Ran}\,\big(E(\{\mu\})\big) \neq 0$ *is called* the point spectrum of A *or* the set of eigenvalues of A. *One then sets*

$$\mathcal{H}_p(A) := \bigoplus \mathsf{Ran}\,\big(E(\{\mu\})\big)$$

where the sum extends over all eigenvalues of A.

In accordance with what has been presented in Theorem 1.7.1, we define two additional subspaces of $\mathcal{H}$.

Definition 1.7.15.

$$\begin{aligned}\mathcal{H}_{ac}(A) &:= \big\{f \in \mathcal{H} \mid m_f \text{ is an absolutely continuous measure}\big\}\\ &= \big\{f \in \mathcal{H} \mid \text{ the function } \lambda \mapsto \|E_\lambda f\|^2 \text{ is absolutely continuous}\big\},\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{sc}(A) &:= \big\{f \in \mathcal{H} \mid m_f \text{ is a singular continuous measure}\big\}\\ &= \big\{f \in \mathcal{H} \mid \text{ the function } \lambda \mapsto \|E_\lambda f\|^2 \text{ is singular continuous}\big\},\end{aligned}$$

for which the comparison measure is always the Lebesgue measure on $\mathbb{R}$.

Theorem 1.7.16. *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$.*

a) This Hilbert space can be decomposed as follows

$$\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A),$$

and the restriction of the operator A to one of these subspaces defines a self-adjoint operator denoted respectively by A_p, A_{ac} and A_{sc}.

b) For any $\varphi \in C_b(\mathbb{R})$, one has the decomposition

$$\varphi(A) = \varphi(A_p) \oplus \varphi(A_{ac}) \oplus \varphi(A_{sc}).$$

Moreover, the following equality holds

$$\sigma(A) = \sigma(A_p) \cup \sigma(A_{ac}) \cup \sigma(A_{sc}).$$

Note that one often writes $E_p(A)$, $E_{ac}(A)$ and $E_{sc}(A)$ for the orthogonal projection on $\mathcal{H}_p(A)$, $\mathcal{H}_{ac}(A)$ and $\mathcal{H}_{sc}(A)$, respectively, and with these notations one has $A_p = AE_p(A)$, $A_{ac} = AE_{ac}(A)$ and $A_{sc} = AE_{sc}(A)$. In addition, note that the relation between the set of eigenvalues $\sigma_p(A)$ introduced in Definition 1.6.1 and the set $\sigma(A_p)$ is

$$\sigma(A_p) = \overline{\sigma_p(A)}.$$

Two additional sets are often introduced in relation with the spectrum of A, namely $\sigma_d(A)$ and $\sigma_{ess}(A)$.

Definition 1.7.17. *An eigenvalue* λ *belongs to* the discrete spectrum $\sigma_d(A)$ of A *if and only if* $\mathsf{Ran}\left(E(\{\lambda\})\right)$ *is of finite dimension, and* λ *is isolated from the rest of the spectrum of* A*.* The essential spectrum $\sigma_{ess}(A)$ of A *is the complementary set of* $\sigma_d(A)$ *in* $\sigma(A)$*, or more precisely*

$$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A).$$

We end this section with an other characterization of the spectrum of the operator A.

Proposition 1.7.18 (Weyl's criterion)**.** *Let* A *be a self-adjoint operator in a Hilbert space* $\mathcal{H}$*.*

a) A real number λ *belongs to* $\sigma(A)$ *if and only if there exists a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset D(A)$ *such that* $\|f_n\| = 1$ *and* $s - \lim_{n\to\infty}(A - \lambda)f_n = 0$*.*

b) A real number λ *belongs to* $\sigma_{ess}(A)$ *if and only if there exists a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset D(A)$ *such that* $\|f_n\| = 1$*,* $w - \lim_{n\to\infty} f_n = 0$ *and* $s - \lim_{n\to\infty}(A - \lambda)f_n = 0$*.*