Chapter 5

Twisted crossed product C^*-algebras

This chapter is mainly dedicated to a brief introduction on twisted C^*-dynamical systems, twisted crossed products and on their representations. We mainly follow the survey article [MPR05] which is based on the standard references [BS70, Pac94, PR89, PR90]. To simplify, we undertake various hypotheses which are not needed for part of the arguments. Primarily, we assume that an *abelian* locally compact group acts upon an *abelian* C^*-algebra. This will allow us to use the Fourier transform and the Gelfand theory. Note that the general framework can easily be guessed from Section 3.1 on locally compact groups and from Section 3.4 on crossed product C^*-algebras. Note also that from now on, the additive notation will be used for the group, since in the applications we shall mainly consider the group $\mathbb{R}^d$.

To make the transition towards pseudodifferential operators and the magnetic case, we introduce at the end of the chapter a special type of twisted crossed products, in which the algebra is composed of continuous functions defined on the group. It is preceded and prepared by some considerations in group cohomology.

5.1 Twisted C^*-dynamical systems

Let us directly start with the definition of twisted dynamical systems. This definition corresponds to a generalization of Definition 3.3.1 in which no twist was introduced.

Definition 5.1.1. An (abelian) twisted C^*-dynamical system *consists in a quadruplet* $(\mathscr{C}, G, \theta, \omega)$*, where* $\mathscr{C}$ *is an abelian* C^**-algebra,* G *is a locally compact abelian group,* $\theta : G \to \mathsf{Aut}(\mathscr{C})$ *is a continuous homomorphism from* G *to the group of* $*$*-automorphisms of* $\mathscr{C}$ *(endowed with the pointwise convergence topology), and* ω *is a strictly continuous normalized* 2*-cocycle on* G *with values in the unitary group of the multiplier algebra of* $\mathscr{C}$*.*

Note that the pair (θ, ω) is often called *a twisted action of* G *on* $\mathscr{C}$. Very often, we shall use the shorter expression *twisted dynamical system* for the quadruplet $(\mathscr{C}, G, \theta, \omega)$.

Remark 5.1.2. *(i) Almost everything in this section would be true, with only some minor modifications, without assuming $\mathscr{C}$ and G to be abelian. However, our main interest lies in the connection between twisted dynamical systems and pseudodifferential theories. And for this purpose commutativity is extremely useful, almost essential. Therefore we do assume it from the very beginning.*

(ii) A strictly continuous 2-cocycle is a function $\omega : G \times G \to \mathscr{U}(\mathscr{C})$ (the unitary group in the multiplier algebra $\mathscr{M}(\mathscr{C})$ of $\mathscr{C}$), continuous with respect to the strict topology on $\mathscr{U}(\mathscr{C})$, and such that for all $x, y, z \in G$:

$$\omega(x+y,z)\,\omega(x,y) = \theta_x[\omega(y,z)]\,\omega(x,y+z). \tag{5.1.1}$$

We shall also assume it to be normalized:

$$\omega(x,0) = \omega(0,x) = 1, \qquad \text{for all } x \in G. \tag{5.1.2}$$

It is known that any automorphism of $\mathscr{C}$ extends uniquely to a $*$-automorphism of $\mathscr{M}(\mathscr{C})$ and, obviously, leaves $\mathscr{U}(\mathscr{C})$ invariant. By applying this fact to θ_x and by denoting the extension with the same symbol, one gives a sense to (5.1.1)*. Actually, by suitable particularizations in* (5.1.1)*, we get $\theta_{-x}[\omega(x,0)] = \omega(0,0) = \omega(0,x)$, $\forall x \in G$, hence for normalization it suffices to ask $\omega(0,0) = 1$. The required continuity (see Definition 2.5.14) can be rephrased in this abelian setting by saying that for any $\varphi \in \mathscr{C}$, the map*

$$G \times G \ni (x,y) \mapsto \varphi\omega(x,y) \in \mathscr{C}$$

is continuous. In fact Borel conditions could be imposed instead of continuity for most of the constructions and results; we do not pursue this here.

(iii) Since $\mathscr{C}$ is abelian, we know by Gelfand theory that there exists a locally compact space Ω such that $\mathscr{C}$ is isometrically $*$-isomorphic to $C_0(\Omega)$, i.e. *$\mathscr{C} \cong C_0(\Omega)$. If the C^*-algebra $C_0(\Omega)$ is not unital, then $C_b(\Omega)$, the C^*-algebra of all bounded and continuous complex functions on Ω, surely is. It contains $C_0(\Omega)$ as an essential ideal. In fact $C_b(\Omega)$ can be identified with the multiplier algebra $\mathscr{M}(\mathscr{C})$ of $\mathscr{C}$. Thus the unitary group of $\mathscr{C}$ is identified with $C(\Omega;\mathbb{T})$, the family of all continuous functions on Ω taking values in the group $\mathbb{T}$ of complex numbers of modulus* 1*. Moreover, the strict topology on $C(\Omega;\mathbb{T})$ coincides with the topology of uniform convergence on compact subsets of Ω.*

We can now go on with covariant representations, by slightly adapting Definition 3.3.4.

Definition 5.1.3. A covariant representation *of an (abelian) twisted C^*-dynamical system $(\mathscr{C}, G, \theta, \omega)$ consists in a triple $(\mathcal{H}, \pi, U)$, where*

(i) $(\mathcal{H}, \pi)$ is a (non-degenerate) representation of $\mathscr{C}$,

(ii) $(\mathcal{H}, U)$ *is a strongly continuous map from* G *to* $\mathscr{U}(\mathcal{H})$ *which satisfies*

$$U_x U_y = \pi\big(w(x,y)\big) U_{x+y} \qquad \forall x, y \in G, \tag{5.1.3}$$

(iii) the following compatibility condition holds

$$\pi\big(\theta_x(\varphi)\big) = U_x \pi(\varphi) U_x^* \qquad x \in G, \varphi \in \mathscr{C}. \tag{5.1.4}$$

One observes that in this framework U is a sort of generalized projective representation of G. The usual notion of projective representation corresponds to the case in which for all $x, y \in G$, $\omega(x,y) \in \mathbb{T}$, *i.e.* $\omega(x,y)$ is a constant function on the spectrum Ω of $\mathscr{C}$.

For twisted C^*-dynamical systems, regular representations also exist, see Example 3.3.6 in the context of dynamical systems without twist. We present below the construction borrowed from Definition 3.10 of [PR89] (note that the conventions are slightly different from Example 3.3.6 since here the right action is used instead of the left action, but these modifications are not really relevant).

Example 5.1.4 (Regular representation)**.** *Let* $(\mathscr{C}, G, \theta, \omega)$ *be an (abelian) twisted* C^**-dynamical system, and let* $(\mathcal{H}, \pi)$ *be a faithful representation of* $\mathscr{C}$*. Consider the Hilbert space* $\tilde{\mathcal{H}} := L^2(G; \mathcal{H})$*, and define* $\tilde{\pi} : \mathscr{C} \to \mathscr{B}(\tilde{\mathcal{H}})$ *and* $\tilde{U} : G \to \mathscr{U}(\tilde{\mathcal{H}})$ *by*

$$[\tilde{\pi}(\varphi)h](x) := \pi\big(\theta_x(\varphi)\big)h(x) \qquad \text{and} \qquad [\tilde{U}_y h](x) := \pi\big(\omega(x,y)\big)h(x+y), \tag{5.1.5}$$

for any $\varphi \in \mathscr{C}$*,* $h \in \tilde{\mathcal{H}}$ *and* $x, y \in G$*. It is then checked straightforwardly that the triple* $\big(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{U}\big)$ *is a covariant representation of the (abelian) twisted* C^**-dynamical system.*

Exercise 5.1.5. *Check carefully the statements contained in the previous example.*

5.2 Twisted crossed product algebras

Let $(\mathscr{C}, G, \theta, \omega)$ be an (abelian) twisted dynamical system. As for the non-twisted case, we start by mixing together the algebra $\mathscr{C}$ and the space $C_c(G)$ in a way to form a *-algebra. We define $C_c(G; \mathscr{C})$, the set of compactly supported $\mathscr{C}$-valued functions, and endow it with the norm $\|f\|_1 := \int_G \|f(x)\| \mathrm{d}x$. Let us also fix an element τ of the set $\mathsf{End}(G)$ of continuous endomorphisms of G. Particular cases are $\mathbf{0}, \mathbf{1} \in \mathsf{End}(G)$, $\mathbf{0}(x) := 0$ and $\mathbf{1}(x) := x$, for all $x \in G$. Addition and subtraction of endomorphisms are well-defined. For elements f, g of $C_c(G; \mathscr{C})$ and for any point $x \in G$ we set

$$(f *_\tau^\omega g)(x) := \int_G \theta_{\tau(y-x)}\,[f(y)]\,\theta_{(\mathbf{1}-\tau)y}\,[g(x-y)]\,\theta_{-\tau x}\,[\omega(y, x-y)]\,\mathrm{d}y \tag{5.2.1}$$

and

$$f^{*_\tau^\omega}(x) := \theta_{-\tau x}[\omega(x,-x)^{-1}]\,\theta_{(\mathbf{1}-2\tau)x}\Big[\overline{f(-x)}\Big], \tag{5.2.2}$$

where $\overline{f(-x)}$ corresponds to the involution of $\mathscr{C}$ applied to $f(-x)$. Note that the expression (5.2.2) becomes much simpler if $\omega(x,-x) = 1$, which will be the case in most of the applications.

Exercise 5.2.1. *Check that the above product is associative, and that* $^{*^\omega_\tau}$ *is an involution.*

Remark 5.2.2. *In the corresponding Section 3.4, and more generally in the literature, only the special case* $\tau = \mathbf{0}$ *is considered. We introduced all these isomorphic structures because they help in understanding* τ*-quantizations in pseudodifferential theory.*

Lemma 5.2.3. *For two functions* f *and* g *in* $C_c(G;\mathscr{C})$ *and for* $\tau \in \mathsf{End}(G)$*, the function* $f *^\omega_\tau g$ *belongs to* $C_c(G;\mathscr{C})$*. With the composition law* $*^\omega_\tau$ *and the involution* $^{*^\omega_\tau}$*, the completion* $L^1(G;\mathscr{C})$ *of* $C_c(G;\mathscr{C})$ *with respect to the norm* $\|\cdot\|_1$ *is a* B^**-algebra. These* B^**-algebras are isomorphic for different* τ*'s.*

Proof. The fact that $L^1(G;\mathscr{C})$ is stable under the product $*^\omega_\tau$ follows from the relations

$$\|\theta_{\tau(y-x)}\left[f(y)\right]\theta_{(\mathbf{1}-\tau)y}\left[g(x-y)\right]\theta_{-\tau x}\left[\omega(y,x-y)\right]\| \le \|f(y)\|\,\|g(x-y)\|,$$

and

$$\int_G \|(f *^\omega_\tau g)(x)\|\mathrm{d}x \le \int_G \Big[\int_G \|f(y)\|\,\|g(x-y)\|\mathrm{d}y\Big]\mathrm{d}x = \|f\|_1\|g\|_1.$$

The associativity of this composition law is easily deduced from the 2-cocycle property of ω. All the other requirements also follow by routine computations.

The isomorphisms are the mappings

$$m_{\tau,\tau'} : L^1(G;\mathscr{C}) \to L^1(G;\mathscr{C}), \quad (m_{\tau,\tau'}f)(x) := \theta_{(\tau'-\tau)x}[f(x)], \qquad x \in G.$$

On the first copy of $L^1(G;\mathscr{C})$ one considers the structure defined by τ' and on the second that defined by τ. Note the obvious relations $m_{\tau,\tau'}m_{\tau',\tau''} = m_{\tau,\tau''}$ and $[m_{\tau,\tau'}]^{-1} = m_{\tau',\tau}$ for all $\tau,\tau',\tau'' \in \mathsf{End}(G)$. □

We recall that *a* C^**-norm* on a *-algebra has to satisfy $\|A^*A\| = \|A\|^2$. Since C^*-norms have many technical advantages and since $\|\cdot\|_1$ has not this C^*-property, we shall make now some adjustments, valid in an abstract setting (see Definition 3.4.2 for a simplified version of the following construction). A B^*-algebra $\mathfrak{C}$ with norm $\|\cdot\|$ is called *an* A^**-algebra* when it admits a C^*-norm or, equivalently, when it has an injective representation in a Hilbert space [Tak02, Def. 9.19]. In this case we can consider *the standard* C^**-norm* on it, defined as the supremum of all the C^*-norms, that we shall denote by $|||\cdot|||$. A rather explicit formula for $|||\cdot|||$ is $|||A||| = \sup\{\|\pi(A)\|_{\mathscr{B}(\mathcal{H})} \mid (\mathcal{H},\pi)$ is a representation$\}$. One has by Lemma 2.4.14 that $|||A||| \le \|A\|$ for all $A \in \mathfrak{C}$. The completion with respect to this norm will be a C^*-algebra containing $\mathfrak{C}$ as a dense *-subalgebra. We call it *the enveloping* C^**-algebra of* $\mathfrak{C}$. It is known that $\big(L^1(G;\mathscr{C}), *^\omega_\tau, ^{*^\omega_\tau}, \|\cdot\|_1\big)$ is indeed an A^*-algebra[1].

[1]In the general setting of twisted crossed product C^*-algebra, this fact is not trivial. The argument uses the existence of an approximate unit, see [PR89, Rem. 2.6], [BS70, Thm. 3.3] and the Appendix of [PR90]. Fortunately, for our (abelian) twisted C^*-dynamical system, the regular representation induces the necessary injective representation of $L^1(G;\mathscr{C})$, as we shall see in the proof of Proposition 5.4.6.

Definition 5.2.4. The enveloping C^*-algebra of $\big(L^1(G;\mathscr{C}), *^{\omega}_{\tau}, {}^{*^{\omega}_{\tau}}, \|\cdot\|_1\big)$ is called *the twisted crossed product of $\mathscr{C}$ by G associated with the twisted action (θ,ω) and the endomorphism τ*. It will be denoted by $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$.

The C^*-algebra $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ has a rather abstract nature. But most of the time one uses efficiently the fact that $L^1(G;\mathscr{C})$ is a dense *-subalgebra, on which everything is very explicitly defined. Let us even observe that the algebraic tensor product $L^1(G)\odot\mathscr{C}$ may be identified with the dense *-subspace of $L^1(G;\mathscr{C})$ (hence of $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ also) formed of functions with finite-dimensional range. The isomorphism $m_{\tau,\tau'}$ extends nicely to an isomorphism from $\mathscr{C} \rtimes^{\omega}_{\theta,\tau'} G$ to $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$.

The next lemma shows clearly the importance of twisted crossed products as a way to bring together the information contained in a twisted dynamical system, see Theorem 3.4.1 for the untwisted version.

Lemma 5.2.5. *Let $(\mathcal{H},\pi,U)$ be a covariant representation of the (abelian) twisted C^*-dynamical system $(\mathscr{C},G,\theta,\omega)$, and let $\tau\in\mathsf{End}(G)$. Then $\pi\rtimes_\tau U$ defined on $L^1(G;\mathscr{C})$ by*

$$(\pi\rtimes_\tau U)f := \int_G \pi\left[\theta_{\tau y}\big(f(y)\big)\right] U_y \,\mathrm{d}y$$

extends to a representation of $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$, called the integrated form of (π,U). *One has $\pi\rtimes_{\tau'} U = (\pi\rtimes_\tau U)\circ m_{\tau,\tau'}$ if $\tau,\tau'\in\mathsf{End}(G)$.*

Proof. Some easy computations show that $\pi\rtimes_\tau U$ is a representation of the B^*-algebra $\big(L^1(G;\mathscr{C}), *^{\omega}_{\tau}, {}^{*^{\omega}_{\tau}}\big)$. Then, by taking into account that $\|(\pi\rtimes_\tau U)f\| \le \|f\|_1,\ \forall f\in L^1(G;\mathscr{C})$, one gets that $\pi\rtimes_\tau U$ extends to $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ by density and, by approximation, this extension has all the required algebraic properties.

The relation $\pi\rtimes_{\tau'} U = (\pi\rtimes_\tau U)\circ m_{\tau,\tau'}$ is checked readily on $L^1(G;\mathscr{C})$ and obviously extends to the full twisted crossed product. □

Let us mention that an analogue of Theorem 3.4.8 also holds in this more general setting. Indeed, one can recover the covariant representation from $\pi\rtimes_\tau U$. Actually, there is a bijective correspondence between covariant representations of a twisted dynamical system and non-degenerate representations of the twisted crossed product. This correspondence preserves equivalence, irreducibility and direct sums. We do not give explicit formulae, since we do not use them.

5.3 Group cohomology

We recall some definitions in group cohomology. They will be used in the next sections to show that standard matters as gauge invariance and τ-quantizations have a cohomological flavour. Now they will serve to isolate twisted dynamical systems for which a generalization of the Schrödinger representation exists.

Let G be an abelian, locally compact group and $\mathscr{U}$ a topological abelian group. Note that in our applications $\mathscr{U}$ will usually not be locally compact, being the unitary group of the multiplier algebra of an abelian C^*-algebra, as in Section 5.1. We also assume that there exists a continuous action θ of G by automorphisms of $\mathscr{U}$. We shall use for G and $\mathscr{U}$ additive and multiplicative notations, respectively.

The class of all continuous functions : $G^n \to \mathscr{U}$ is denoted by $C^n(G;\mathscr{U})$; it is obviously an abelian group (we use once again multiplicative notations). Elements of $C^n(G;\mathscr{U})$ are called *(continuous) n-cochains.* For any $n \in \mathbb{N}$, we define *the coboundary map* $\delta^n : C^n(G;\mathscr{U}) \ni \rho \mapsto \delta^n(\rho) \in C^{n+1}(G;\mathscr{U})$ by

$$\begin{aligned}&[\delta^n(\rho)]\,(x_1,\dots,x_n,x_{n+1})\\ &:= \theta_{x_1}\left[\rho(x_2,\dots,x_{n+1})\right]\prod_{j=1}^{n}\rho(x_1,\dots,x_j+x_{j+1},\dots,x_{n+1})^{(-1)^j}\rho(x_1,\dots,x_n)^{(-1)^{n+1}}.\end{aligned}$$

It is easily shown that δ^n is a group morphism and that $\delta^{n+1}\big(\delta^n(\rho)\big) = 1$ for any $n \in \mathbb{N}$. It follows that $\mathsf{Ran}(\delta^n) \subset \mathsf{Ker}(\delta^{n+1})$.

Definition 5.3.1. *(i)* $Z^n(G;\mathscr{U}) := \mathsf{Ker}(\delta^n)$ *is called* the set of n-cocycles (on G, with coefficients in $\mathscr{U}$).

(ii) $B^n(G;\mathscr{U}) := \mathsf{Ran}(\delta^{n-1})$ *is called* the set of n-coboundaries.

Let us note that $Z^n(G;\mathscr{U})$ and $B^n(G;\mathscr{U})$ are subgroups of $C^n(G;\mathscr{U})$, and that $B^n(G;\mathscr{U}) \subset Z^n(G;\mathscr{U})$.

Definition 5.3.2. *The quotient* $H^n(G;\mathscr{U}) := Z^n(G;\mathscr{U})/B^n(G;\mathscr{U})$ *is called* the n'th group of cohomology (of G with coefficients in $\mathscr{U}$). *Its elements are called* classes of cohomology.

In the sequel, we shall need only the cases $n = 0, 1, 2$, which we outline now for convenience. For $n = 0$, parts of the definitions are simple conventions. We set $C^0(G;\mathscr{U}) := \mathscr{U}$. One has $[\delta^0(\varphi)]\,(x) = \theta_x(\varphi)\varphi^{-1}$, for any $\varphi \in \mathscr{U}$, $x \in G$. This implies that $Z^0(G;\mathscr{U}) = \{\varphi \in \mathscr{U} \mid \varphi \text{ is a fixed point}\}$. By convention, $B^0(G;\mathscr{U}) = \{1\}$.

The mapping $\delta^1 : C^1(G;\mathscr{U}) \to C^2(G;\mathscr{U})$ is given by

$$\left[\delta^1(\lambda)\right](x,y) = \lambda(x)\,\theta_x[\lambda(y)]\,\lambda(x+y)^{-1}.$$

Thus a 1-cochain λ is in $Z^1(G;\mathscr{U})$ if it is *a crossed morphism*, *i.e.* if it satisfies $\lambda(x)\theta_x[\lambda(y)] = \lambda(x+y)$ for any $x, y \in G$. Particular cases are the elements of $B^1(G;\mathscr{U})$ (called *principal morphisms*), those of the form $\lambda(x) = \theta_x(\varphi)\varphi^{-1}$ for some $\varphi \in \mathscr{U}$.

For $n = 2$ one encounters a situation which was already taken into account in the definition of twisted dynamical systems. The formula for the coboundary map is

$$\left[\delta^2(\omega)\right](x,y,z) = \theta_x[\omega(y,z)]\,\omega(x+y,z)^{-1}\,\omega(x,y+z)\,\omega(x,y)^{-1}.$$

Thus a 2-cocycle is just a function satisfying the relation (5.1.1). $B^2(G;\mathscr{U})$ is composed of 2-cocycles of the form $\omega(x,y) = \lambda(x)\,\theta_x[\lambda(y)]\,\lambda(x+y)^{-1}$ for some 1-cochain λ.

In the applications, we shall consider for $\mathscr{U}$ the unitary group of an algebra of functions defined on the group G itself. An example of special importance will be the group $\mathscr{U} = C(G;\mathbb{T})$, endowed with the strict topology, which correspond to the unitary group of the multiplier algebra of $C_0(G)$. In this case, the groups of cohomology are particularly simple.

Lemma 5.3.3. *For any locally compact abelian group G and for any $n \geq 1$, one has $H^n\big(G; C(G;\mathbb{T})\big) = \{1\}$.*

Proof. Let $\rho^n \in Z^n\big(G; C(G;\mathbb{T})\big)$, i.e. ρ^n is a continuous n-cochain satisfying for any $y_1, \ldots, y_{n+1} \in G$

$$\theta_{y_1}\left[\rho^n(y_2,\ldots,y_{n+1})\right] \prod_{j=1}^{n} \rho^n(y_1,\ldots,y_j+y_{j+1},\ldots,y_{n+1})^{(-1)^j} \rho^n(y_1,\ldots,y_n)^{(-1)^{n+1}} = 1.$$

We set in this relation $y_1 = q$, $y_j = x_{j-1}$ for $j \geq 2$ and rephrase it as

$$\begin{aligned}&\theta_q\left[\rho^n(x_1,\ldots,x_n)\right]\\ =&\rho^n(q+x_1,x_2,\ldots,x_n)\prod_{j=1}^{n-1}\rho^n(q,x_1,\ldots,x_j+x_{j+1},\ldots,x_n)^{(-1)^j}\rho^n(q,x_1,\ldots,x_{n-1})^{(-1)^n},\end{aligned}$$

which is an identity in $C(G;\mathbb{T})$. One calculates both sides at the point $x = 0$ and obtain

$$\begin{aligned}\left[\rho^n(x_1,\ldots,x_n)\right](q) &= \left[\rho^n(q+x_1,x_2,\ldots,x_n)\right](0)\\ &\cdot\prod_{j=1}^{n-1}\left[\rho^n(q,x_1,\ldots,x_j+x_{j+1},\ldots,x_n)^{(-1)^j}\right](0)\left[\rho^n(q,x_1,\ldots,x_{n-1})^{(-1)^n}\right](0).\end{aligned}$$

This means exactly $\rho^n = \delta^{n-1}(\rho^{n-1})$ for

$$\left[\rho^{n-1}(z_1,\ldots,z_{n-1})\right](q) := \left[\rho^n(q,z_1,\ldots,z_{n-1})\right](0) \tag{5.3.1}$$

and thus any n-cocyle is at least formally a n-coboundary.

We show now that ρ^{n-1} has the right continuity properties. Let us recall that if $C(G;\mathbb{T})$ is endowed with the topology of uniform convergence on compact sets of G and if Y is a locally compact space, then $C\big(Y; C(G;\mathbb{T})\big)$ can naturally be identified with $C(G\times Y;\mathbb{T})$ (the proof of this statement is an easy exercise). So ρ^n can be interpreted as an element of $C(G\times G^n;\mathbb{T})$. Being obtained from ρ^n by a restriction ρ^{n-1} belongs to $C(G^n;\mathbb{T})$, and thus can be interpreted as an element of $C\big(G^{n-1}; C(G;\mathbb{T})\big) \equiv C^{n-1}\big(G; C(G;\mathbb{T})\big)$, which finishes the proof. □

Let us add one more definition which will play a crucial role in the sequel.

Definition 5.3.4. *Let $\mathscr{U}$ be a topological abelian group endowed with a continuous action θ of G by automorphisms of $\mathscr{U}$, and let $\omega \in Z^2(G;\mathscr{U})$. We say that ω is* pseudo-trivial *if there exists another topological abelian group $\mathscr{U}'$ with a similar action θ' of G such that $\mathscr{U}$ is a subgroup of $\mathscr{U}'$, for each $x \in G$ one has $\theta_x = \theta'_x|_{\mathscr{U}}$, and such that $\omega \in B^2(G;\mathscr{U}')$.*

Thus, to produce pseudo-trivial 2-cocycles, one has to find some $\omega \in B^2(G;\mathscr{U}')$ such that $\omega(x,y) \in \mathscr{U} \subset \mathscr{U}'$ for any $x,y \in G$ and such that $(x,y) \mapsto \omega(x,y) \in \mathscr{U}$ is continuous with respect to the topology of $\mathscr{U}$. This is possible in principle because the product $\lambda(x)\theta_x[\lambda(y)][\lambda(x+y)]^{-1}$ can be better-behaved than any of its factors. The particular choice $[\lambda(z)](q) = [\omega(q,z)](0)$ we made in (5.3.1) will lead later on to the familiar transversal gauge for magnetic systems.

Let us emphasize that most of the time pseudo-triviality cannot be improved to a bona fide triviality. Very often, all the functions λ for which one has $\omega = \delta^1(\lambda)$ do not take all their values in $\mathscr{U}$ or miss the right continuity. We shall outline such a situation in the next section.

5.4 Standard twisted crossed products

When trying to transform the formalism of twisted crossed products into a pseudodifferential theory, one has to face the possible absence of an analogue of the Schrödinger representation and this would lead us too far from the initial motivation. The existence of a generalized Schrödinger representation is assured by the pseudo-triviality of the 2-cocycle, and thus we restrict ourselves to a specific class of twisted dynamical systems. In the same time we also restrict to algebras $\mathscr{C}$ of complex continuous functions on G. This also is not quite compulsory for a pseudodifferential theory, but it leads to a simple implementation of pseudo-triviality (by Lemma 5.3.3) and covers easily the important magnetic case.

We first extend of framework introduced in Assumption 4.3.1.

Definition 5.4.1. *Let G be an locally compact abelian group. We call G-*algebra *a C^*-subalgebra $\mathscr{C}$ of $BC_u(G)$ which is G-invariant,* i.e. *$\theta_x(\varphi) := \varphi(\cdot + x) \in \mathscr{C}$ for any $\varphi \in \mathscr{C}$ and $x \in G$, and which contains $C_0(G)$.*

The C^*-algebra $BC_u(G)$ is the largest one on which the action θ of translations with elements of G is norm-continuous. But we shall denote by θ_x even the x-translation on $C(G)$, the *-algebra of all continuous complex functions on G (which is not a normed algebra if G is not compact). The restriction of θ_x on $BC(G)$ is only strictly continuous.

Note that in the previous definition, the assumption $C_0(G) \subset \mathscr{C}$ implies that G can be identified with a dense subset of the Gelfand spectrum Ω of $\mathscr{C}$. If $\mathscr{C}$ is unital, then Ω is a compactification of G, see the beginning of Section 4.3 for the special case $G = \mathbb{R}^d$.

Now, if $\mathscr{C}$ is a G-algebra, then $(\mathscr{C}, G, \theta)$ is a C^*-dynamical system. If we twist it, we get:

Definition 5.4.2. A standard twisted dynamical system *is an (abelian) twisted C^*-dynamical system $(\mathscr{C}, G, \theta, \omega)$ for which $\mathscr{C}$ is a G-algebra. The C^*-algebra $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ is called* a standard twisted crossed product.

Proposition 5.4.3. *If $(\mathscr{C}, G, \theta, \omega)$ is a standard twisted dynamical system, then ω is pseudo-trivial.*

Note that a slightly more general statement and proof is provided in [MPR05, Prop.2.14]. In our context, if is sufficient to observe that $\mathscr{U}(\mathscr{C})$ can naturally be identified with a subgroup of $C(G;\mathbb{T})$, and that the strict topology on $\mathscr{U}(\mathscr{C})$ is finer than the strict topology of $C(G;\mathbb{T})$. The 2-cocycle ω can hence be considered as an element of $Z^2\big(G;C(G;\mathbb{T})\big)$, which coincides with $B^2\big(G;C(G;\mathbb{T})\big)$ by Lemma 5.3.3, and this proves the statement.

Remark 5.4.4. *If ω,ω' are two cohomologous elements of $Z^2\big(G;\mathscr{U}(\mathscr{C})\big)$,* i.e. *$\omega = \delta^1(\lambda)\omega'$ for some $\lambda \in C^1\big(G;\mathscr{U}(\mathscr{C})\big)$, then the C^*-algebras $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ and $\mathscr{C} \rtimes^{\omega'}_{\theta,\tau} G$ are naturally isomorphic: on $L^1(G;\mathscr{C})$ the isomorphism is given by $\left[i^{\lambda}_{\tau}(f)\right](x) := \theta_{-\tau x}[\lambda(x)]f(x)$. Thus $C_0(G) \rtimes^{\omega}_{\theta,\tau} G$ does not depend on ω but on its class of cohomology; this will be strengthened in Proposition 5.4.6. However this does not work if λ only belongs to $C^1\big(G;C(G;\mathbb{T})\big)$ and $\mathscr{C}$ is not $C_0(G)$; in general $\theta_{-\tau x}[\lambda(x)]f(x)$ gets out of $\mathscr{C}$ and i^{λ}_{τ} is no longer well-defined. For ω and ω' defining different classes of cohomology, $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ and $\mathscr{C} \rtimes^{\omega'}_{\theta,\tau} G$ are in general different C^*-algebras.*

In the sequel we fix a standard twisted dynamical system $(\mathscr{C},G,\theta,\omega)$. One observes that the untwisted system $(\mathscr{C},G,\theta)$ always has an obvious covariant representation $(\mathcal{H},\pi,U)$, with $\mathcal{H} := L^2(G)$ (with the Haar measure), $\pi(\varphi) \equiv \varphi(X) =$ the multiplication operator with φ, and $[U_y u](x) := u(x+y)$. Note that the right action is again considered, as in Example 5.1.4. Let us now choose $\lambda \in C^1\big(G;C(G;\mathbb{T})\big)$ such that $\delta^1(\lambda) = \omega$ (this identity taking place in $Z^2\big(G;C(G;\mathbb{T})\big)$). We set $U^{\lambda}_y := \pi\big(\lambda(y)\big)U_y$. Explicitly, for any $x \in G$ and $u \in \mathcal{H}$, $\left[U^{\lambda}_y u\right](x) = [\lambda(y)](x)u(x+y) \equiv \lambda(x;y)u(x+y)$. Let us already mention that the point (ii) in the next proposition is at the root of gauge invariance for magnetic pseudodifferential operators.

Proposition 5.4.5. *(i) $(\mathcal{H},\pi,U^{\lambda})$ is a covariant representation of $(\mathscr{C},G,\theta,\omega)$,*

(ii) If μ is another element of $C^1\big(G;C(G;\mathbb{T})\big)$ such that $\delta^1(\mu) = \omega$, then there exists $\varphi \in C(G;\mathbb{T})$ such that $\mu(x) = \theta_x(\varphi)\varphi^{-1}\lambda(x)$, $\forall x \in G$. Moreover, $U^{\mu}_x = \pi(\varphi^{-1})U^{\lambda}_x\pi(\varphi)$ for all $x \in G$.

Proof. The proof of the first statement consists in trivial verifications. For the second statement, one first notes that $\mu\lambda^{-1}$ belongs to $\mathsf{Ker}(\delta^1) = Z^1\big(G;C(G;\mathbb{T})\big)$. Since this set is equal to $B^1\big(G;C(G;\mathbb{T})\big)$ by Lemma 5.3.3, there exists $\varphi \in C^0\big(G;C(G;\mathbb{T})\big) \equiv C(G;\mathbb{T})$ satisfying $\mu(x) = \theta_x(\varphi)\varphi^{-1}\lambda(x)$, $\forall x \in G$. The last claim of the proposition follows from $\pi[\theta_x(\varphi)]U_x = U_x\pi(\varphi)$. $\square$

We call $(\mathcal{H},\pi,U^{\lambda})$ *the Schrödinger covariant representation associated with the* 1-*cochain* λ. Let us now recall the detailed form of the composition laws on $L^1(G;\mathscr{C})$. For simplicity we shall use notations as $f(x;y)$ for $[f(y)](x)$ and $\omega(x;y,z)$ for $[\omega(y,z)](x)$. With these notations and for any $f,g \in L^1(G;\mathscr{C})$, the relations (5.2.1) and (5.2.2) read respectively

$$(f *^{\omega}_{\tau} g)(q;x) = \int_G f\big(q+\tau(y-x);y\big)\, g\big(q+(\mathbf{1}-\tau)y;x-y\big)\, \omega\big(q-\tau x;y,x-y\big)\,\mathrm{d}y$$

and

$$(f^{*^\omega_\tau})(q;x) = \omega\big(q-\tau x; x, -x\big)^{-1}\overline{f\big(q+(\mathbf{1}-2\tau)x;-x\big)},$$

where x, y, q are elements of G.

Let us also denote for convenience by $\mathfrak{Rep}^\lambda_\tau$ the integrated representation $\pi \rtimes_\tau U^\lambda$ in $L^2(G)$ of the twisted crossed product $\mathscr{C} \rtimes^\omega_{\theta,\tau} G$, see also Lemma 5.2.5. Its explicit action on $f \in L^1(G;\mathscr{C})$ and $u \in L^2(G)$ is given by

$$\begin{aligned}\left[\big(\mathfrak{Rep}^\lambda_\tau(f)\big)\,u\right](x) &= \int_G f(x+\tau y; y)\,\lambda(x;y)\,u(x+y)\,\mathrm{d}y \\ &= \int_G f\big((\mathbf{1}-\tau)x+\tau y; y-x\big)\,\lambda(x;y-x)\,u(y)\,\mathrm{d}y.\end{aligned}$$

We gather some important properties of $\mathfrak{Rep}^\lambda_\tau$ in:

Proposition 5.4.6. *(i)* $\mathfrak{Rep}^\lambda_\tau[C_0(G)\rtimes^\omega_{\theta,\tau} G] = \mathscr{K}\big(L^2(G)\big)$*, the* C^**-algebra of all compact operators in* $L^2(G)$*.*

(ii) $\mathfrak{Rep}^\lambda_\tau$ *is a irreducible and faithful representation of* $\mathscr{C} \rtimes^\omega_{\theta,\tau} G$ *in* $L^2(G)$*, for any* G*-algebra* $\mathscr{C}$*,*

(iii) In the setting of Proposition 5.4.5.(ii), one has $\mathfrak{Rep}^\mu_\tau(f) = \pi(\varphi^{-1})\mathfrak{Rep}^\lambda_\tau(f)\pi(\varphi)$*.*

Proof. (i) Since $\delta^1(\lambda) = \omega$ in $Z^2\big(G; C(G;\mathbb{T})\big)$, we can then consider the following isomorphism

$$\begin{aligned}&i^\lambda_\tau : \Big(L^1\big(G;C_0(G)\big), *^1_{\mathbf{0}}, {}^{*^1_{\mathbf{0}}}\Big) \to \Big(L^1\big(G;C_0(G)\big), *^\omega_\tau, {}^{*^\omega_\tau}\Big)\,, \\ &\left[i^\lambda_\tau(f)\right](x) = \theta_{-\tau x}\left[\lambda^{-1}(x)\,f(x)\right],\end{aligned} \tag{5.4.1}$$

that extends to an isomorphism between the non-twisted crossed product $C_0(G) \rtimes^1_{\theta,\mathbf{0}} G$ and the twisted crossed product $C_0(G) \rtimes^\omega_{\theta,\tau} G$ (this is consistent with Remark 5.4.4). One easily checks that $\mathfrak{Rep}^\lambda_\tau\big[i^\lambda_\tau(f)\big] = \int_G \pi[f(x)]U_x\,\mathrm{d}x$ for all f in $\Big(L^1(G;\mathscr{C}), *^1_{\mathbf{0}}, {}^{*^1_{\mathbf{0}}}\Big)$. But it is known that the image of $C_0(G)\rtimes^1_{\theta,\mathbf{0}} G$ through the representation $\pi \rtimes U \equiv \mathfrak{Rep}^1_{\mathbf{0}}$ is equal to the algebra $\mathscr{K}\big(L^2(G)\big)$ of compact operators in $L^2(G)$, *cf.* for example [GI02, Cor. 4.1].

(ii) Since $C_0(G) \subset \mathscr{C}$, then $C_0(G) \rtimes^\omega_{\theta,\tau} G$ can be identified to a C^*-subalgebra of $\mathscr{C} \rtimes^\omega_{\theta,\tau} G$ and the irreducibility of $\mathfrak{Rep}^\lambda_\tau\big(\mathscr{C} \rtimes^\omega_{\theta,\tau} G\big)$ follows from the irreducibility of $\mathscr{K}\big(L^2(G)\big)$, by (i).

Let us now recall that the regular representation of the twisted dynamical system $(\mathscr{C}, G, \theta, \omega)$ has been introduced in Example 5.1.4. In particular, we can choose in this representation the Hilbert space $L^2(G)$ and the representation π of $\mathscr{C}$ by operators of multiplication. One thus obtained the representation $\big(L^2\big(G;L^2(G)\big), \tilde\pi, \tilde U\big)$, with the maps $\tilde\pi$ and $\tilde U$ defined in (5.1.5). Since $L^2\big(G;L^2(G)\big)$ is canonically isomorphic to $L^2(G\times G)$, let us set $\xi(\cdot;x) := \xi(x)$ and introduce the unitary operator

$W^\lambda : L^2(G \times G) \to L^2(G \times G)$, $[W^\lambda \xi](x; y) := \lambda(x; y)\, \xi(x; x + y)$. Its adjoint is given by $[(W^\lambda)^* \xi](x; y) = \lambda^{-1}(x; y - x)\, \xi(x; y - x)$. Some easy computations show then that $\big[(W^\lambda)^* \tilde{\pi}(\varphi)\, W^\lambda \xi\big]\, (x; y) = \varphi(y)\, \xi(x; y)$. Moreover, one has

$$\begin{aligned}\Big[(W^\lambda)^* \tilde{U}_z\, W^\lambda\, \xi\Big]\, (x; y) &= \lambda^{-1}(x; y - x)\, \omega(x; y - x, z)\, \lambda(x; y - x + z)\, \xi(x; y + z) \\ &= \lambda(y; z)\, \xi(x; y + z),\end{aligned}$$

where we have used that $\omega = \delta^1(\lambda)$. Equivalently, one has $(W^\lambda)^*\, \tilde{\pi}(\varphi)\, W^\lambda = \mathbf{1} \otimes \varphi(X)$ and $(W^\lambda)^*\, U_z\, W^\lambda = \mathbf{1} \otimes \lambda(X; z) U_z \equiv \mathbf{1} \otimes U^\lambda_z$ in $L^2(G) \otimes L^2(G)$. Thus the regular representation is unitarily equivalent to the representation $(L^2(G) \otimes L^2(G), \mathbf{1} \otimes \pi, \mathbf{1} \otimes U^\lambda)$. Since the regular representation induces a faithful representation $\tilde{\pi} \rtimes U$ of $\mathscr{C} \rtimes^{\omega}_{\theta,\mathbf{0}} G$ in $L^2\big(G; L^2(G)\big)$, *cf.* Theorem 3.11 of [PR89], the Schrödinger representation induces a faithful representation of $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ in $L^2(G)$ for any $\tau \in \mathsf{End}(G)$.

(iii) The proof of this statement consists in a simple verification. □

Exercise 5.4.7. *Check that the map i^λ_τ introduced in the previous proof defines an isomorphism between the B^*-algebras $\Big(L^1\big(G; C_0(G)\big), *^1_{\mathbf{0}}, {}^{*^1_{\mathbf{0}}}\Big)$ and $\big(L^1\big(G; C_0(G)\big), *^\omega_\tau, {}^{*^\omega_\tau}\big)$.*