Nagoya University
G30 Preparatory Lecture

Mathematics

Course I : Functions and Equations
Course II : Calculus
Course III: Linear Algebra

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Course I : Functions and Equations

Flow of This Lecture Video

1. **Lecture A** (About 20 min.)
    - Listen to a lecture.
2. Exercise A
    - Practice solving related problems.
    - Pause the video and solve the problem by yourself.
    - If you cannot, *see* the preceding lecture repeatedly.
3. Explanation of the answers (About 5 min.)

4. **Lecture B** (About 20 min.)
5. Exercise B
6. Explanation of the answers

Contents

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Lesson 1
Polynomials and Factoring

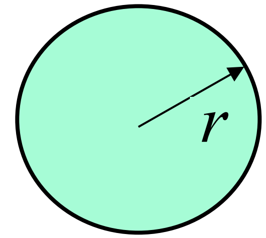
1A

- Polynomials
- Addition, Subtraction, and Multiplication

Polynomials

Q. What is the area of a circle with radius r cm ?

A. πr^2 (cm^2) : Monomial

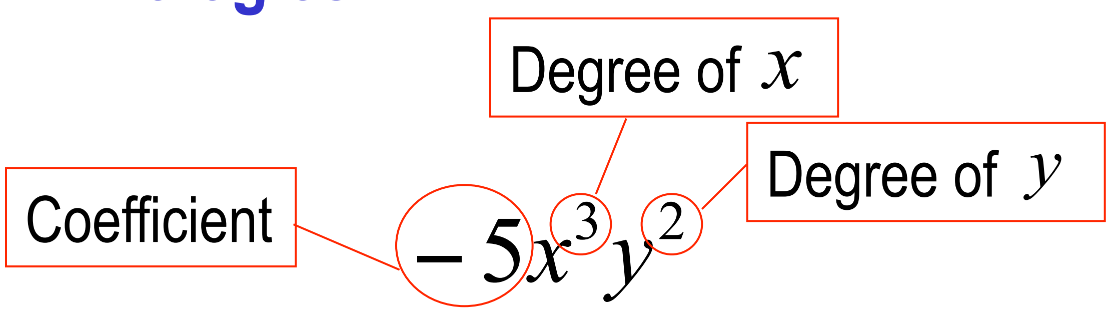


Q. What is the price for 5 shortcakes of x yen and 2 cream puffs of y yen?

A. $5x + 2y$ (yen) : Polynomial

***Polynomial** is an expression of finite length constructed from variables and constant, using only the operation of addition, subtraction and multiplication.*

Some terminologies



Some Notes

Note 1 : Degree of a polynomial is the largest degree of any one term.
(Ex: Degree of $2x^3 + 2x + 1$ is 3.)

Note 2 : The following terms are not polynomials.

$$3x^{\frac{1}{2}} + 1 \ (= 3\sqrt{x} + 1) \qquad 2x^{-2} (= 1/x^2)$$

Note 3 : Simplify by combining like terms.

$$5x^3 + 2x + 1 - 2x^2 + x \quad \rightarrow \quad 3x^3 - 2x^2 + 3x + 1$$

Note 4 : Write in the descending order of the degree

$$2x^3 + 2x + 1 - x^2 \quad \rightarrow \quad 2x^3 - x^2 + 2x + 1$$

Basic Process of Multiplication

(two-term)×(two-term)

$$(ax+b)(cx+d) = \overset{①}{acx^2} + \overset{②}{adx} + \overset{③}{bcx} + \overset{④}{bd}$$

$$= acx^2 + (ad+bc)x + bd$$

Steps:
① First terms
② Outer terms
③ Inner terms
④ Last terms

In general

Use the distribution property and combine.

Example 1. Find the product $(2x^2+x-3)(x+2)$

Ans.

<Method 1>

$$(2x^2+x-3)(x+2)$$
$$= (2x^2+x-3)x + (2x^2+x-3)\cdot 2$$
$$= 2x^3 + x^2 - 3x + 4x^2 + 2x - 6$$
$$= 2x^3 + 5x^2 - x - 6$$

<Method 2>

$$\begin{array}{rrrr} & 2x^2 & +x & -3 \\ \times) & & x & +2 \\ \hline 2x^3 & +x^2 & -3x & \\ & 4x^2 & +2x & -6 \\ \hline 2x^3 & +5x^2 & -x & -6 \end{array}$$

Some Formulas

Memorize the following formulas for quick calculation.

1. $(a+b)^2 = a^2 + 2ab + b^2$

   $(a-b)^2 = a^2 - 2ab + b^2$

   $b \to (-b)$

2. $(a+b)(a-b) = a^2 - b^2$

3. $(x+a)(x+b) = x^2 + (a+b)x + ab$

4. $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$

5. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

   $(a-b)^2 = a^3 - 3a^2b + 3ab^2 - b^3$

   $b \to (-b)$

Exercise

Exercise 1. Perform the indicated operation for the following polynomials.

$$A = 2x^3 - x^2 - 1, \qquad B = -x^2 + 2x + 1$$

(1) Add A and B.
(2) Subtract B from A.
(3) Multiply A and B.

Pause the video and solve the problem by yourself.

Answer to the Exercise

Exercise 1. Perform the indicated operation for the following polynomials.

$$A = 2x^3 - x^2 - 1, \qquad B = -x^2 + 2x + 1$$

(1) Add A and B. (2) Subtract B from A. (3) Multiply A and B.

Ans.

(1)

$$\begin{array}{rrrrr} & 2x^3 & -x^2 & & -1 \\ +) & & -x^2 & +2x & +1 \\ \hline & 2x^3 & -2x^2 & +2x & \end{array}$$

(2)

$$\begin{array}{rrrrr} & 2x^3 & -x^2 & & -1 \\ -) & & -x^2 & +2x & +1 \\ \hline & 2x^3 & & -2x & -2 \end{array}$$

(3)

$$\begin{array}{rrrrrrr} & & 2x^3 & -x^2 & & -1 & \\ \times) & & & -x^2 & +2x & +1 & \\ \hline -2x^5 & +x^4 & & +x^2 & & & \\ & +4x^4 & -2x^3 & & -2x & & \\ & & +2x^3 & -x^2 & & -1 & \\ \hline -2x^5 & +5x^4 & & & -2x & -1 & \end{array}$$

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Lesson 1
Polynomials and Factoring

1B

- Factoring Polynomials
- How to factor polynomials
- Factor theorem

What is Factoring ?

Factoring a polynomial is the opposite process of multiplying polynomials.

Expansion

$$(x+2)(x+3) = x^2 + 5x + 6$$

Factoring

Factoring a number

Prime numbers

$$12 = 2 \times 6 \qquad 12 = \frac{1}{2} \times 24 \qquad 12 = \underline{2 \times 2 \times 3}$$

Completely factored

Factoring a polynomial

$$x^4 - 16 = (x^2+4)(x^2-4) = \underline{(x^2+4)(x+2)(x-2)}$$

Completely factored

How to Factor Polynomials

There is no perfect method to succeed factoring. Sometimes **insight** will help. However, there are several useful rules as follows.

1. **Factor out the largest common factor.**

[Ex.] $2x^3 + 10x^2 + 12x = 2x(x^2 + 5x + 6)$ ←Rule 1

2. **Use a formula of factoring.**

$(a + b)$ ab

[Ex.] $2x^3 + 10x^2 + 12x = 2x(x^2 + 5x + 6)$

$= 2x(x + 2)(x + 3)$ ←Rule 2

a b

3. **Rearrange by the variable with the lowest degree.**

[Ex.] $x^3 + 3x^2 + 2xy + 6y = (2x + 6)y + (x^3 + 3x^2)$ ←Rule 3

$= 2y(x + 3) + x^2(x + 3)$ ←Rule 1

$= (x^2 + 2y)(x + 3)$ ←Rule 1

4. **Use "Factor Theorem"**

To be discussed later.

Several Advanced Examples

Example 2. Factor the following polynomials.

(1) $2x^2+5xy+3y^2-3x-5y-2$ (2) x^4-10x^2+9 .

Ans.

(1) Every valuable has the same degree. In such a case, rearrange by any one of them.

$$
\begin{aligned}
&2x^2+5xy+3y^2-3x-5y-2\\
&=2x^2+(5y-3)x+(3y^2-5y-2)\\
&=2x^2+(5y-3)x+(y-2)(3y+1)\\
&=\{x+(y-2)\}\{2x+(3y+1)\}\\
&=(x+y-2)(2x+3y+1)
\end{aligned}
$$

$$
\begin{array}{ccccc}
1 & \times & y-2 & \rightarrow & 2y-4\\
2 & & 3y+1 & \rightarrow & 3y+1\\
\hline
 & & & & 5y-3
\end{array}
$$

(2) The polynomial has 4th degree. In such a case, observe the form carefully and use the similarity to the polynomial with a lower degree.

Put $X=x^2$

Then $x^4-10x^2+9=X^2-10X+9=(X-1)(X-9)$

$=(x^2-1)(x^2-9)=(x+1)(x-1)(x+3)(x-3)$

Factor Theorem

Factor Theorem

A polynomial $f(x)$ has a factor $(x-k)$ if and only if $f(k)=0$.

This theorem is commonly applied to the problems of factoring a polynomial and finding the roots of a polynomial equation (this will be explained later.)

Steps of application:

1. Guess a number k and confirm $f(k)=0$.
2. Divide $f(x)$ by $(x-k)$ and obtain $g(x)=f(x)/(x-k)$.
3. Then, $f(x)$ is factored to $f(x)=(x-k)g(x)$.
4. It is easier to find factors of $g(x)$ than that of $f(x)$.

Example 3. Factor the polynomial $f(x)=x^3+4x^2+x-6$.

Ans. $f(1)=0$ Therefore

$$x^3+4x^2+x-6=(x-1)(x^2+5x+6)$$
$$=(x-1)(x-2)(x-3)$$

$$\begin{array}{r|l} & x^2+5x+6 \\ \hline x-1 & x^3+4x^2+x-6 \\ & \underline{x^3-x^2} \\ & 5x^2+x-6 \\ & \underline{5x^2-5x} \\ & 6x-6 \\ & \underline{6x-6} \\ & 0 \end{array}$$

Exercise

Exercise 2. Factor the following polynomials.

(1) $4x^2 + 8x - 21$ (2) $x^3 + 7x^2 + 8x + 2$

Pause the video and solve the problem by yourself.

Answer to the Exercise

Exercise 2. Factor the following polynomials.

(1) $4x^2 + 8x - 21$ (2) $x^3 + 7x^2 + 8x + 2$

Ans.

(1) Remember the formula $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$

By trials, we find that $a = 2,\ b = -3,\ c = 2,\ d = 7$ satisfy the condition.

(Refer to the right-side calculation)

Therefore $4x^2 + 8x - 21 = (2x-3)(2x+7)$

2	-3	→	-6
2	7	→	14
4	-21		8

(2) We use Factor Theorem.

By trials, we find $x = -1$ satisfy $f(-1) = 0$.

By dividing $x^3 + 7x^2 + 8x + 2$ by $x + 1$, we have

$$\frac{x^3 + 7x^2 + 8x + 2}{x+1} = x^2 + 6x + 2$$

Therefore

$$x^3 + 7x^2 + 8x + 2 = (x^2 + 6x + 2)(x+1)$$