NAGOYA UNIVERSITY

Lesson 2
Algebraic Equations and Functions

2A

- **Algebraic Equations**
- **First Order Equations**
- **Second Order Equations**

Algebraic Equation

Q. The weight of a box containing 8 apples is 1500g. The weight of the box is 300g. Assuming every apple has the same weight. What is the weight of one apple ?

A. Let the weight of one apple be x. Then we have

$8x + 300 = 1500$. Therefore $x = 150$ g.

First order equation

Equation

Statement of equality between two expressions consisting of varia and/or numbers is called an equation.

$$f(x) = g(x)$$

Algebraic Equation (Polynomial Equation)

$$a_0 x^n + a_1 x^{n-1} + a_{n-2} x^2 + \cdots + a_{n-1} x + a_n = 0$$

First Order Equations

Linear Equation

$$ax + b = 0 \quad (a \neq 0)$$

$$\therefore\ ax = -b \qquad \therefore\ x = -\frac{b}{a}$$: the root of this equation.

Simultaneous Linear Equations

Simultaneous equations are a set of equations containing multiple variables.

Example 1. Solve the following equations $\left.\begin{aligned} 2x + y = 8 \\ x + y = 6 \end{aligned}\right\}$

Ans. [*Substitution method*] From the second eq. $y = -x + 6$

Substituting this to the first eq.

$$2x + (-x + 6) = 8 \quad \therefore\ x = 2$$

[*Elimination method*] Subtracting the second eq. from the first eq. $x = 2$

Substituting this to the first eq. $4 + y = 8 \quad \therefore\ y = 4$

Quadratic Equation (Second Order Polynomial Equation)

Quadratic Equation

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

(1) Solve by factoring

If we can factor it in the following form $a(x-p)(x-q)=0$,

the roots are $x = p,\quad x = q$.

Example 2. Solve the following equations

(1) $x^2 + 8x + 12 = 0$ (2) $6x^2 - x - 15 = 0$

(1) $(x+2)(x+6) = 0 \quad \therefore x + 2 = 0$ or $x + 6 = 0$

Therefore, $x = -2$ or $x = -6$

The roots are $x = -2,\ -6$

(2) $(2x+3)(3x-5) = 0 \quad \therefore 2x + 3 = 0 \quad 3x - 5 = 0 \quad \therefore \quad x = -\frac{3}{2} \quad x = \frac{5}{3}$

The roots are $x = -\frac{3}{2}\ ,\ x = \frac{5}{3}$

(2) Solve by the Quadratic Formula

We cannot factor the following equation by observation.

Ex. $x^2 + 5x + 3 = 0$

Quadratic formula

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

If $b^2 - 4ac > 0$, there exist two distinct roots $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac = 0$, there exists one root (double root) $x = -\dfrac{b}{2a}$

If $b^2 - 4ac < 0$, there is no real root.

（*Refer to the next slide about the proof.*）

Derivation of the Quadratic Formula

The quadratic formula is derived as follows.

That was too easy!

$$ax^2 + bx + c = 0$$

$$\therefore \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

By completing the square, we have

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

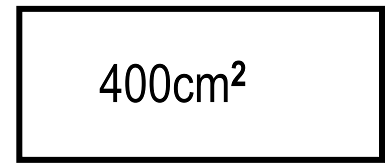
$$\therefore \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

: Quadratic formula

Exercise

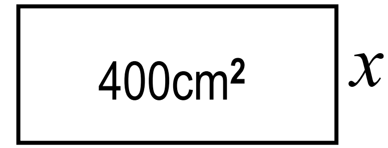
Exercise 1. There is a string of length 100cm. We want to make a rectangle with area $400cm^2$ by this string. What are the lengths of the sides of this rectangle ?



Pause the video and solve the problem.

Answer to the Exercise

Exercise 1. There is a string of length 100cm. We want to make a rectangle with area 400cm² by this strong. What are the lengths of two sides of this rectangle ?



Ans.

Let the length of one side be x . .

Then, we have $x(50-x)=400$

$$\therefore \quad x^2-50x+400=0$$

$$\therefore \quad x=\frac{50\pm\sqrt{50^2-4\times1\times400}}{2\times1}=40,\ 10$$

The side lengths are 40 cm and 10 cm

[Note] When your got the answer, confirm that the solutions satisfy the physical meanings. In this problem, x must be between 0cm and 50cm.

Lesson 2
Algebraic Equation and Functions

2B

- **What is a Function ?**
- **Linear Functions.**
- **Quadratic Functions.**

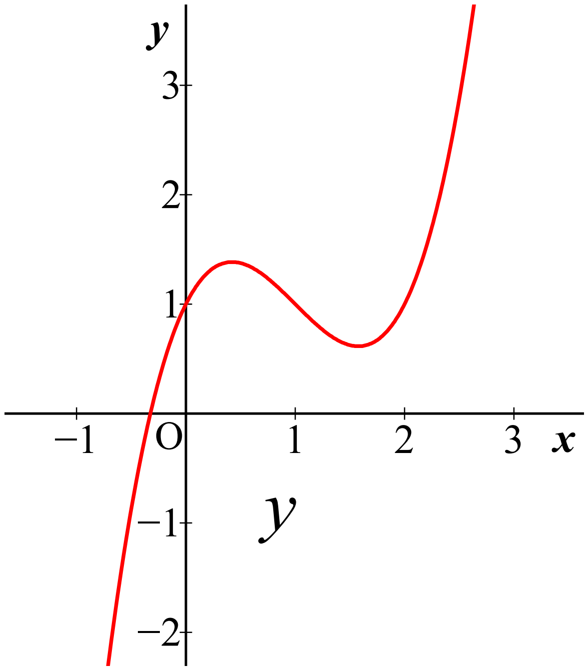
Function

- When a quantity y depends on or is determined by another quantity x or quantities $x_1,\ x_2,\ \cdots,\ x_n$, we say that y is a function of x or $x_1,\ x_2,\ \cdots,\ x_n$.

- This relationship is expressed by

  $y = f(x)$ or $y = f(x_1,\ x_2,\ \cdots,\ x_n)$

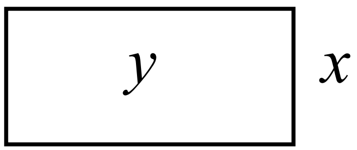
- Here, y is called the dependent variable and x is the independent variable.



$y = x^3 - 3x^2 + 2x + 1$

Example

Example 3. Suppose that a rectangle is made by a string with length 100cm. Represent the area y by the length x of one side.

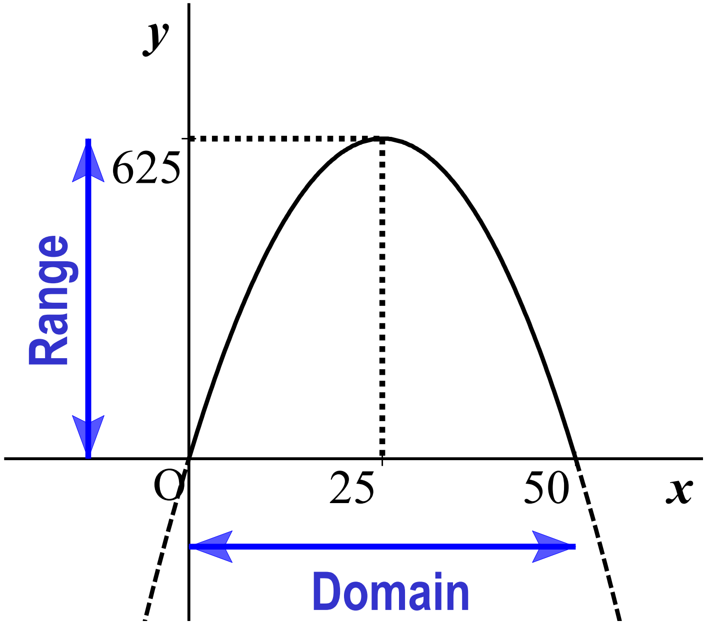


Ans. $y = (50 - x)x = -x^2 + 50x \quad (50 > x > 0)$

In this example, the area y is a function of the side length x.

Domain : The complete set of *possible values* of the independent variable.

Range : The complete set of all possible *resulting values* of the dependent variable.



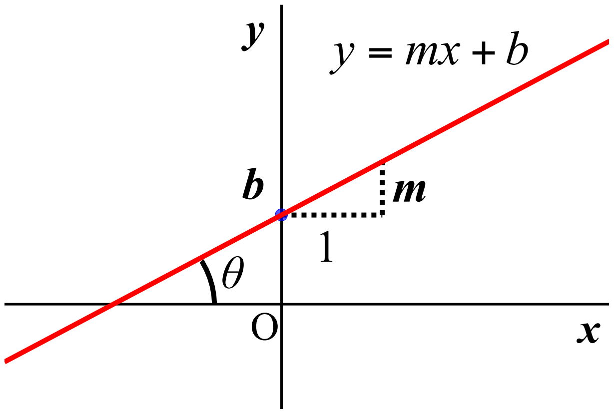
Linear Function (First-Degree Polynomial Function of One Variable)

1. Slope-intercept form

$$y = mx + b \qquad (m \neq 0)$$

m : slope

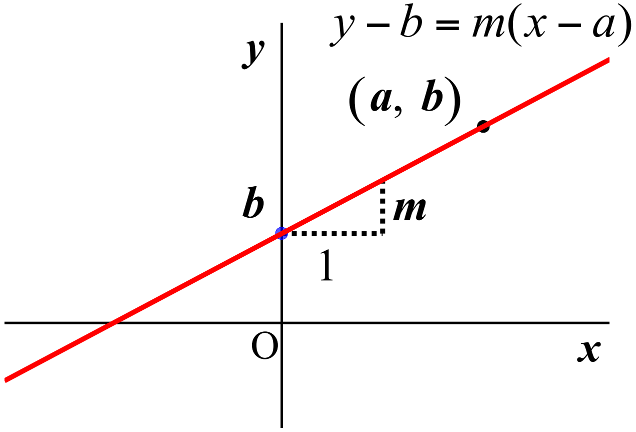
b : y-intercept point



2. Point-slope form

$$y - b = m(x - a) \qquad (m \neq 0)$$

The line passes the point (a, b)



3. General form

$$ax + by + c = 0$$

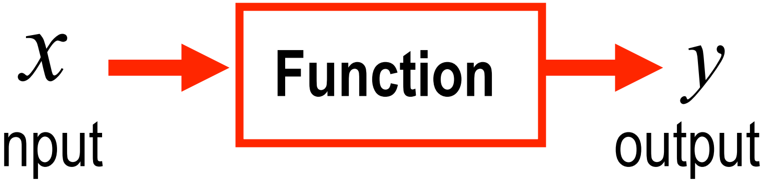
Note

Strictly speaking .....

The third form $ax + by + c = 0$ is a linear equation but not a linear fucntion.

Definition of a function:

- Function defined by $y = f(x)$ has a meaning of projection (mapping) from x to y .



- Functions are mathematical ideas that take one or more variables and produce a variable.

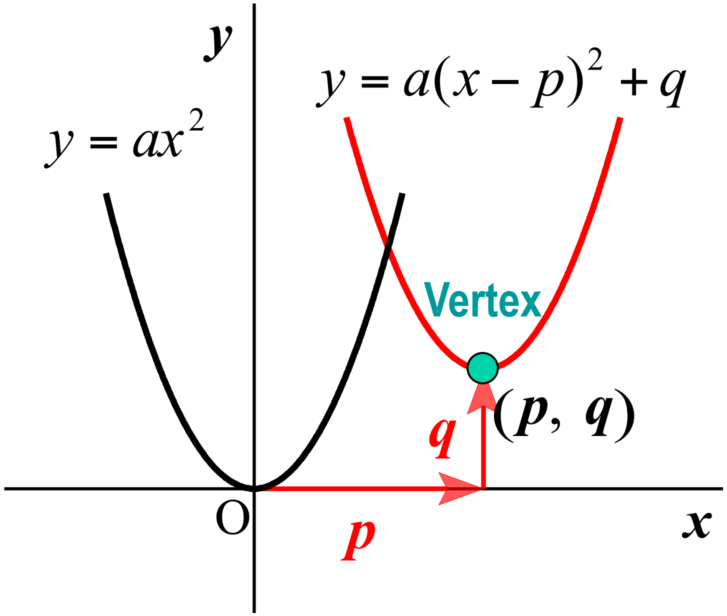
Quadratic Function (Second-Degree Polynomial Function)

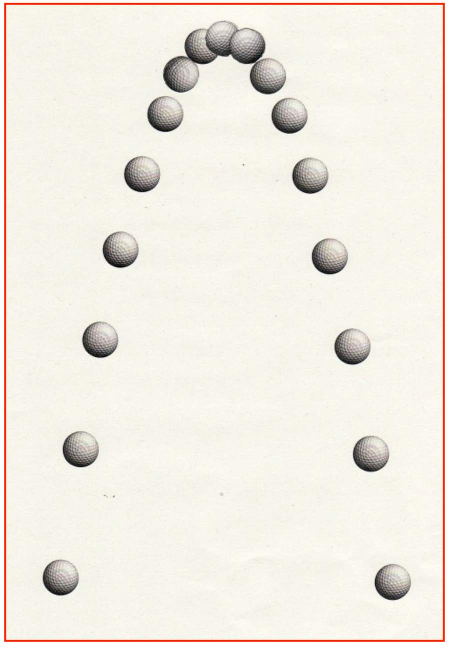
1. General form

$$y = ax^2 + bx + c \qquad (a \neq 0)$$

2. Standard Form

$$y = a(x - p)^2 + q \qquad (a \neq 0)$$





Trajectory of a thrown ball is a parabola represented by a quadratic function

- p = horizontal shift
- q = vertical shift

Derivation of the Standard Form

Example 4. Derive the standard form of a quadratic equation from its general form.

Ans.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left\{x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right\} + c$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right\} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}$$

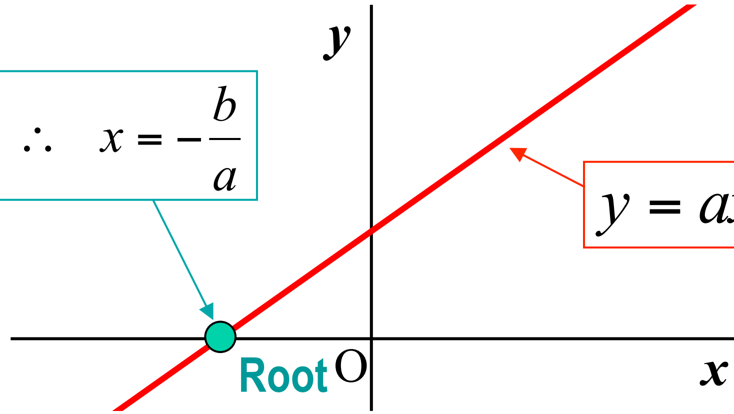
$$\therefore \quad p = -\frac{b}{2a}, \qquad q = -\frac{b^2 - 4ac}{4a^2}$$

Equations and Graphs of Functions

Linear Equation and Linear Function

$$ax + b = 0 \quad \therefore \quad x = -\frac{b}{a}$$

$$y = ax + b$$



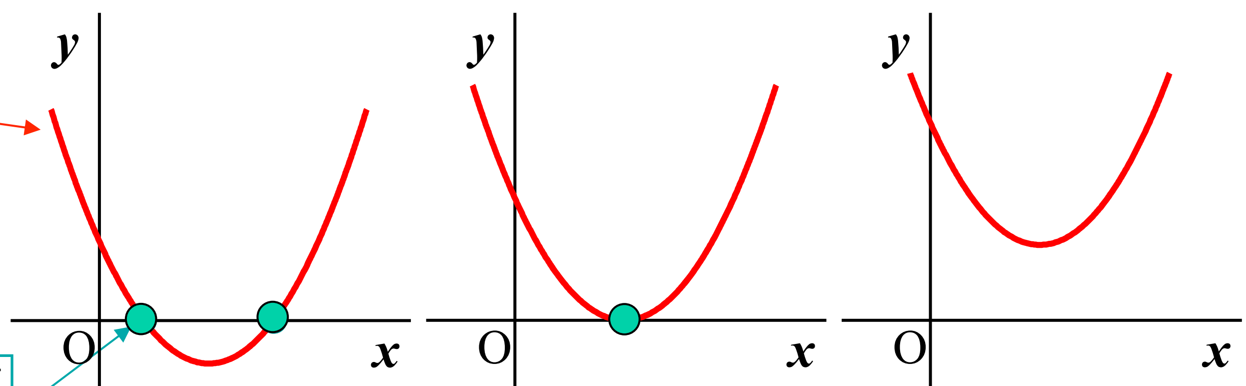
Quadratic Equation and Quadratic Function

Case of $a > 0$

$$y = a(x-p)^2 + q$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

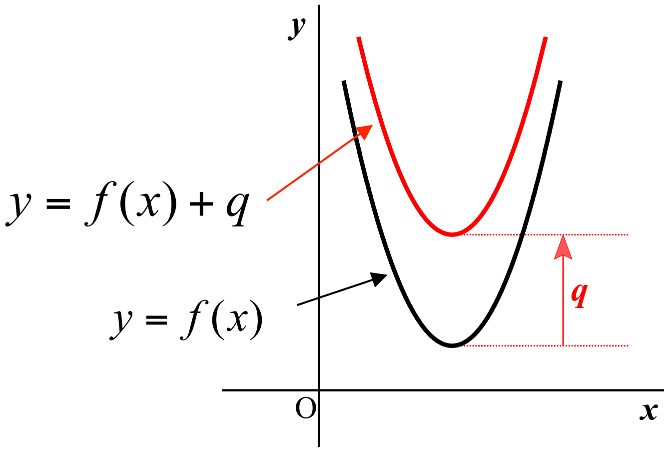
$D = b^2 - 4ac > 0$	$D = b^2 - 4ac = 0$	$D = b^2 - 4ac < 0$



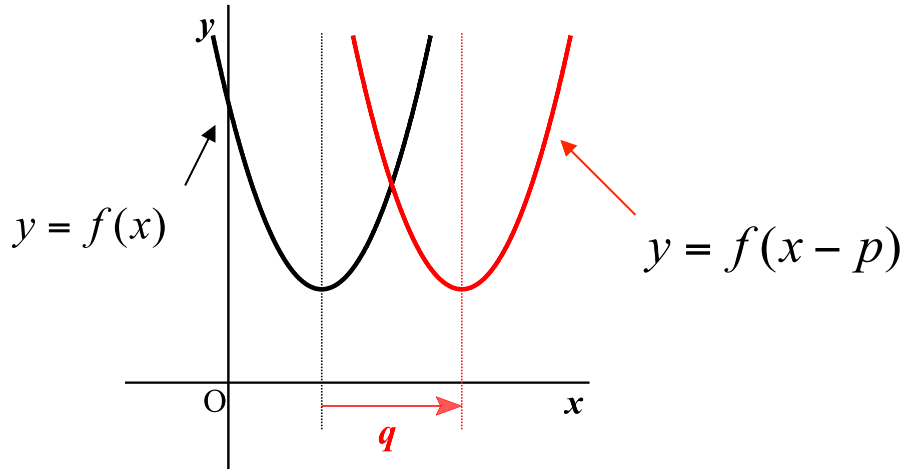
Two real roots | Double root | No real root

Shifting of a Graph

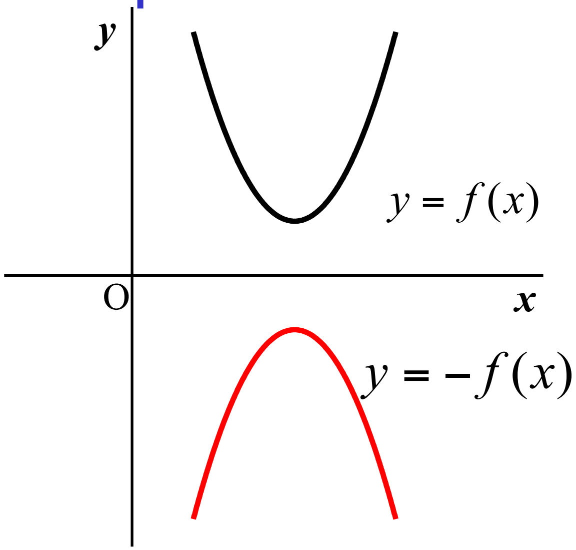
Parallel shift (upward)



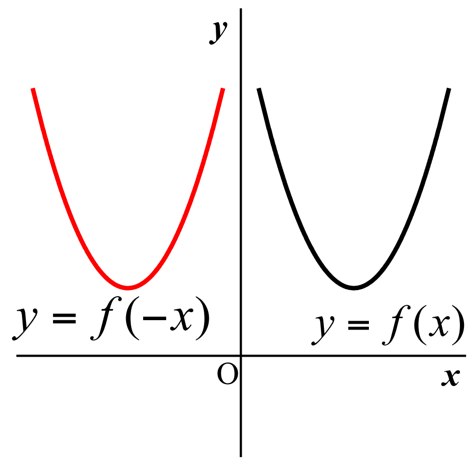
Parallel shift (rightward)



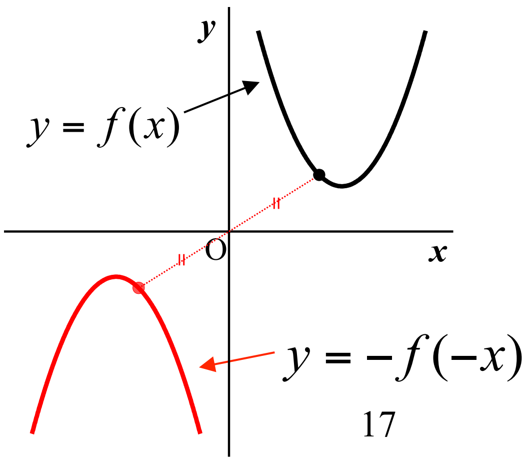
Symmetrical shift with respect to the x-axis



Symmetrical shift with respect to the y-axis



Symmetrical shift with respect to the origin



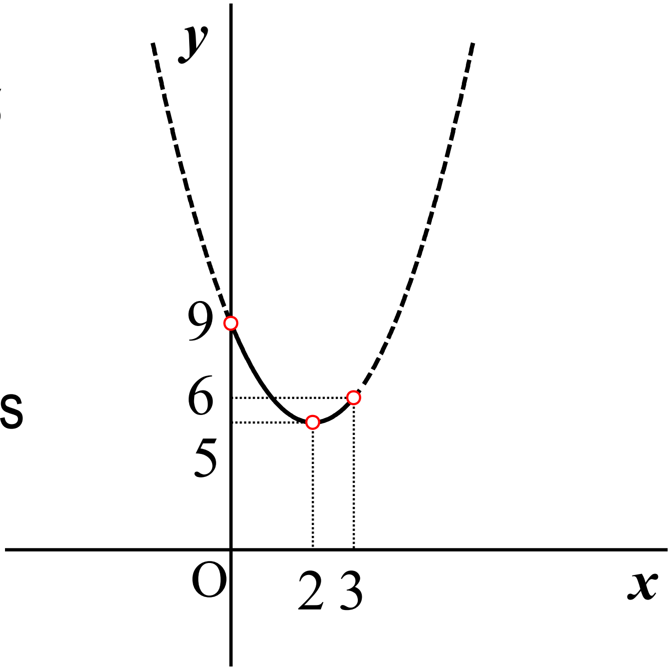
Maximum and Minimum Values of a Function

Example 5. Find the maximum and minimum values of the following function.

$$y = x^2 - 4x + 9 \qquad (0 \le x \le 3)$$

Ans.

$$y = x^2 - 4x + 9 = (x^2 - 4x + 4) + 5 = (x-2)^2 + 5$$



We can illustrate the graph of this function as shown in the right side
Considering the domain, we have the coordinates of the following three points.

The left boundary : (0, 9)
The vertex: (2, 5)
The right boundary: (3, 6)

Therefore, the maximum value is 9 and the minimum value is 5.

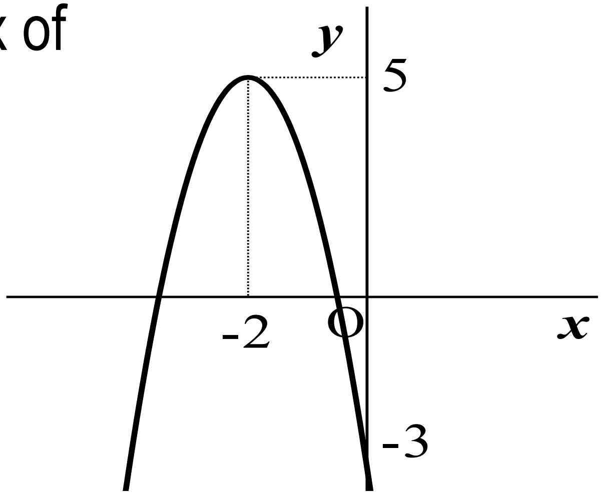
Exercise 2. (1) Find the coordinates of the vertex of

$$y = 2x^2 - 4x + 5$$

(2) Find the coefficients of the quadratic function

$$y = ax^2 + bx + c$$

which represents the graph in the right side.



Pause the video and solve the problem.

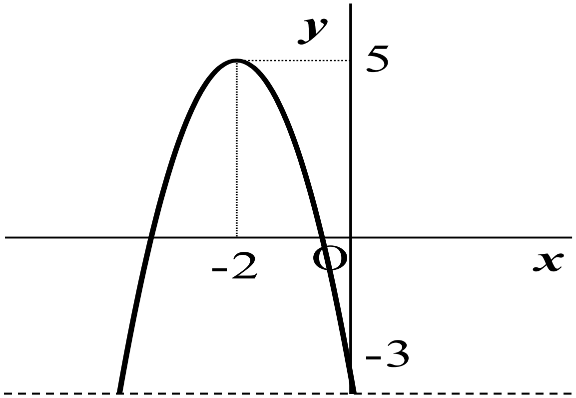
Exercise

Exercise 2. (1) Find the coordinates of the vertex of the function $y = 2x^2 - 12x + 17$

(2) Find the coefficients of the function

$$y = ax^2 + bx + c$$

which represents the graph.



(1) $2x^2 - 12x + 17 = 2(x^2 - 6x) + 17 = 2\{(x-3)^2 - 3^2\} + 17 = 2(x-3)^2 - 1$

Therefore, the coordinate of the vertex is (3, -1)

(2) From the coordinate of the vertex, the function is represented by

$$y = a(x+2)^2 + 5$$

Substituting the coordinate of the y -intercept (0, -3) into this function, we have

$$-3 = a(0+2)^2 + 5 \quad \therefore a = -2$$

Therefore,

$$y = -2(x+2)^2 + 5 \qquad \therefore \quad y = -2x^2 - 8x - 3$$