Course I

Nagoya University

Lesson 6
Trigonometric Function **(III)**

6A

- Trigonometric Identities
- Exact Values of Some Trigonometric Function

Sum and Difference Identities for Sine and Cosine

An identity is an equality that is true for any value of the variable.
(*An equation* is an equality that is true only for certain values of the variable.)

In calculation, we may replace either member of the identity with the other. We use an identity to give an expression a more convenient form.

Sum and Difference Formulae

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Memorize !

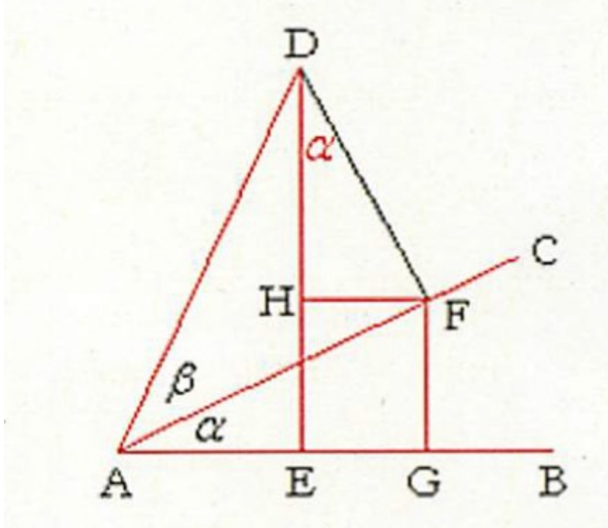
This formula is most important because all the identities in the following are based on this formula.

Proof of Sum Formula for Sine

We can prove these identities geometrically.

Example 1 Prove the formula $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

Ans.



From the figure

DE = AD $\sin(\alpha+\beta)$

=FG + DH

=FA $\sin\alpha$ +DF $\cos\alpha$

=AD $\cos\beta\sin\alpha$ + AD $\sin\beta\cos\alpha$

Therefore

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

Find Exact Values of Trigonometric Function

We can find values of some trigonometric functions from known cases.

Example 2 Find the values of (a) sin15° and (b) cos15°.

Ans.

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4},$$

$$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Sum and Difference Identities for Tangent

Example 3 Derive the sum and difference formulae of the tangent function.

We use the sine and cosine formulae.

$\div \cos\alpha\cos\beta$

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}=\frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}=\frac{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}{1+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

In summary

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}, \qquad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Exercise

Exercise 1 Find the values of (a) $\sin 75°$, (b) $\cos 75°$ and (c) $\tan 75°$.

Ans.

Pause the video and solve the problem.

Answer to the Exercise

Exercise 1 Find the values of (a) sin75°, (b) cos75° and (c) tan75°.

Ans.

(a) $$\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(b) $$\cos 75° = \cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(c) $$\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$= \frac{1 + 1/\sqrt{3}}{1 - 1 \times 1/\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

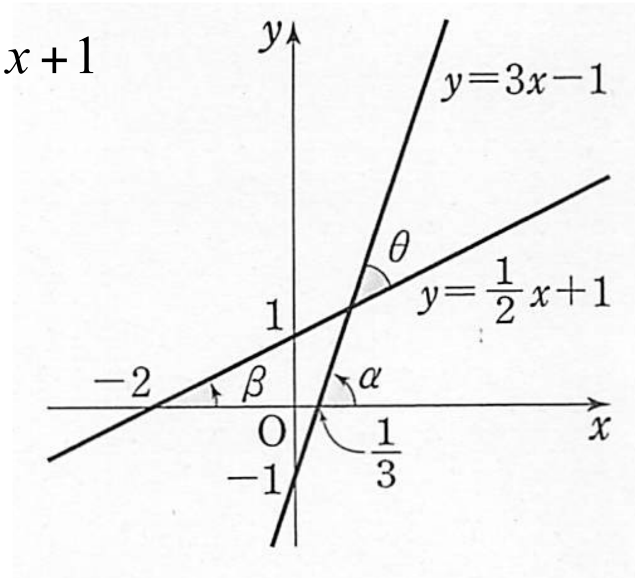
Exercise

Exercise 2 There are two lines $y = 3x - 1$ and $y = \frac{1}{2}x + 1$

Answer the following questions.

(1) Let the angles made by these lines and the x-axis by **α** and **β**. Express the slopes of these lines by tangent function values.

(2) Find the angle **θ** $(0<\theta<\frac{\pi}{2})$ made by these two lines.



Ans.

Pause the video and solve the problem.

Uhh….

Exercise 2 There are two lines $y = 3x - 1$ and $y = \frac{1}{2}x + 1$

Answer the following questions.

(1) Let the angles made by these lines and the x-axis by **α** and **β**. Express the slopes of these lines by tangent function values.

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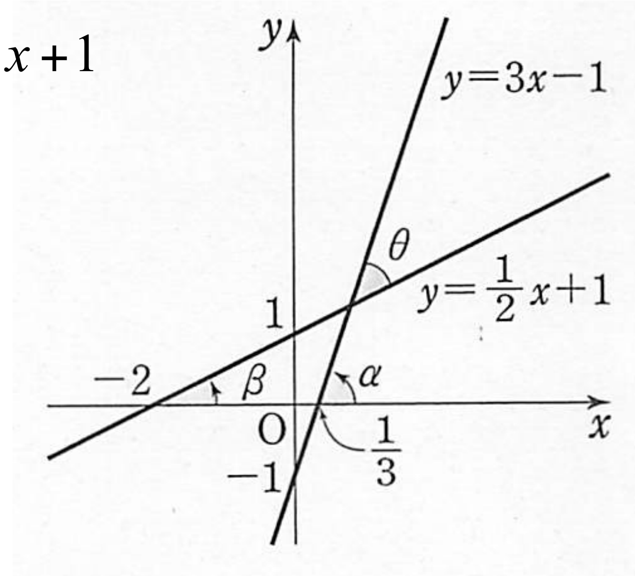

Ans.

(1) $\tan\alpha = 3, \quad \tan\beta = \frac{1}{2}$

(2) $\theta = \alpha - \beta$

$$\tan\theta = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{3 - \frac{1}{2}}{1 + 3\times\frac{1}{2}} = 1, \qquad \therefore \quad \theta = \frac{\pi}{4}$$

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Lesson 6
Trigonometric Function **(II)**

6B6B

- **Derivation of Trigonometric Identities**
- **Double-Angle Identities**
- **Half-Angle Identities**
- **Product-to-Some Identities**
- **Some-to-Product Identities**

Derivation of Trigonometric Identities

Three-Angle Identities

$$\sin 3\alpha = -4\sin^3\alpha + 3\sin\alpha$$

etc.

$\beta = 2\alpha$

Double-Angle Identities

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

etc.

$\alpha = \beta$

$\alpha \to \frac{\alpha}{2}$

Half-Angle Identities

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

etc.

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Solve inversely

Product-to-Sum Identities

$$\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\}$$

etc.

Put $\alpha+\beta = A,\ \alpha-\beta = B$

Sum-to-Product Identities

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

etc.

Double-Angle Identities

- $\sin 2\alpha = 2\sin\alpha\cos\alpha$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

  $= 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$
- $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

Example 4 Prove the double angle identity of tangent

Ans.

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$

: From definition

$$= \frac{\sin(\alpha+\alpha)}{\cos(\alpha+\alpha)} = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha}$$

: From sum identities

$$= \frac{2\tan\alpha}{1-\tan^2\alpha}$$

: Divide by $\cos^2\alpha$ and use $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

Half-Angle Identities

- $\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}$
- $\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}$
- $\tan^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{1+\cos\alpha}$

Example 5 Evaluate the values of $\sin 22.5°$, $\cos 22.5°$, $\tan 22.5°$

Ans. Since 22.5°is in quadrant I, all these values are positive.

$$\sin^2 22.5^\circ=\frac{1-\cos 45^\circ}{2}=\left(1-\frac{1}{\sqrt{2}}\right)/2=\frac{\sqrt{2}-1}{2\sqrt{2}}=\frac{2-\sqrt{2}}{4}\qquad\therefore\quad \sin 22.5^\circ=\frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\cos^2 22.5^\circ=\frac{1+\cos 45^\circ}{2}=\frac{2+\sqrt{2}}{4}\qquad\therefore\quad \cos 22.5^\circ=\frac{\sqrt{2+\sqrt{2}}}{2}$$

Therefore

$$\tan 22.5^\circ=\frac{\sin 22.5^\circ}{\cos 22.5^\circ}=\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}=\sqrt{\frac{(2-\sqrt{2})^2}{4-2}}=\frac{2-\sqrt{2}}{\sqrt{2}}=\sqrt{2}-1$$

Product-to-Sum Identities

$$\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}$$

$$\cos\alpha\sin\beta = \frac{1}{2}\{\sin(\alpha+\beta)-\sin(\alpha-\beta)\}$$

$$\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\{\cos(\alpha+\beta)-\cos(\alpha-\beta)\}$$

Example 6 Evaluate the values of $\sin 15^\circ \sin 75^\circ$

Ans.

$$\sin 15^\circ \sin 75^\circ = -\frac{1}{2}\{\cos(15^\circ+75^\circ)-\cos(15^\circ-75^\circ)\}$$

$$= -\frac{1}{2}\{\cos 90^\circ - \cos(-60^\circ)\} = \frac{1}{2}\cos 60^\circ = \frac{1}{4}$$

Sum-to-Product Identities

P-to-S

$\alpha + \beta = A$

$\alpha - \beta = B$

S-to-P

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Example 7 Evaluate the values of $\cos 10^\circ + \cos 110^\circ + \cos 230^\circ$

Ans.

$$\begin{aligned}\cos 10^\circ + \cos 110^\circ + \cos 230^\circ &= (\cos 10^\circ + \cos 230^\circ) + \cos 110^\circ \\ &= 2\cos 120^\circ \cos(-110^\circ) + \cos 110^\circ \\ &= -\cos 110^\circ + \cos 110^\circ = 0\end{aligned}$$

Exercise

Exercise 3 Let $s = \sin\theta$ and $c = \cos\theta$. Express the following functions by s and c.

(1) $\sin 4\theta$ (2) $\cos 4\theta$

Ans.

Pause the video and solve the problem.

Answer to the Exercise

Exercise 3 Let $s = \sin\theta$ and $c = \cos\theta$. Express the following functions by s and c.

(1) $\sin 4\theta$ (2) $\cos 4\theta$

Ans.

(1) $\sin 4\theta = \sin(2\times 2\theta) = 2\sin 2\theta\cos 2\theta = 4sc(1-2s^2)$

(2) $\cos 4\theta = \cos(2\times 2\theta) = 2\cos^2 2\theta - 1 = 2(2\cos^2\theta - 1)^2 - 1$

$= 2(2c^2-1)^2 - 1$

$= 2(4c^4 - 4c^2 + 1) - 1 = 8c^4 - 8c^2 + 1$

Exercise

Exercise 4 When $\tan x = -2$, evaluate $\sin 2x + \cos 2x$

[Hint] Divide the expression by $\sin^2 x + \cos^2 x = 1$.

Ans.

Pause the video and solve the problem by yourself.

Exercise 4 When $\tan x = -2$, evaluate $\sin 2x + \cos 2x$

[Hint] Divide the expression by $\sin^2 x + \cos^2 x = 1$.

Ans.

$$\sin 2x + \cos 2x = \frac{\sin 2x + \cos 2x}{1} = \frac{2\sin x \cos x + \cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x}$$

$$= \frac{2\tan x + 1 - \tan^2 x}{\tan^2 x + 1} = \frac{2(-2) + 1 - (-2)^2}{(-2)^2 + 1} = -\frac{7}{5}$$