Course I

NAGOYA UNIVERSITY

Lesson 8
Exponential Functions

8A

- **Rational numbers and Irrational Numbers**

Numbers

Natural number

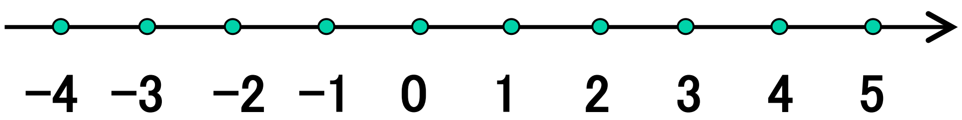
- Ordinary numbers used for counting and ordering.
- Zero is not included.

  [Ex.]　1,　2,　3,　4,　5, ⋯

Integer

- Numbers consisting of the natural numbers, their negative numbers and zero.

  [Ex.] ⋯, −3, −2, −1, 0, 1, 2, 3, ⋯
- Integers are expressed by the following points on the number line.



Rational numbers

Rational → $m : n$

- A rational number is any number that can be expressed as a fraction of two integers in the form $\dfrac{m}{n}$.
- The denominator n must not be equal to zero.

  [Ex.] $\dfrac{1}{2}\left(=\dfrac{2}{4}\right), \dfrac{3}{7}$
- Since n can be equal to 1, every integer is a rational number.

- The fraction is expressed by two types of decimal numbers

① Finite decimal expansion [Ex.] $\dfrac{1}{25} = 0.04$

② Infinite recurring decimal expansion [Ex.] $\dfrac{1}{27} = 0.037037037037\cdots$

Irrational numbers

- An irrational number is any real number that is not a rational , i.e., one that cannot be written as a fraction.
- Decimal expansion of irrational numbers never ends and never enters periodic pattern.

$$\frac{m}{n}$$

[Ex.]

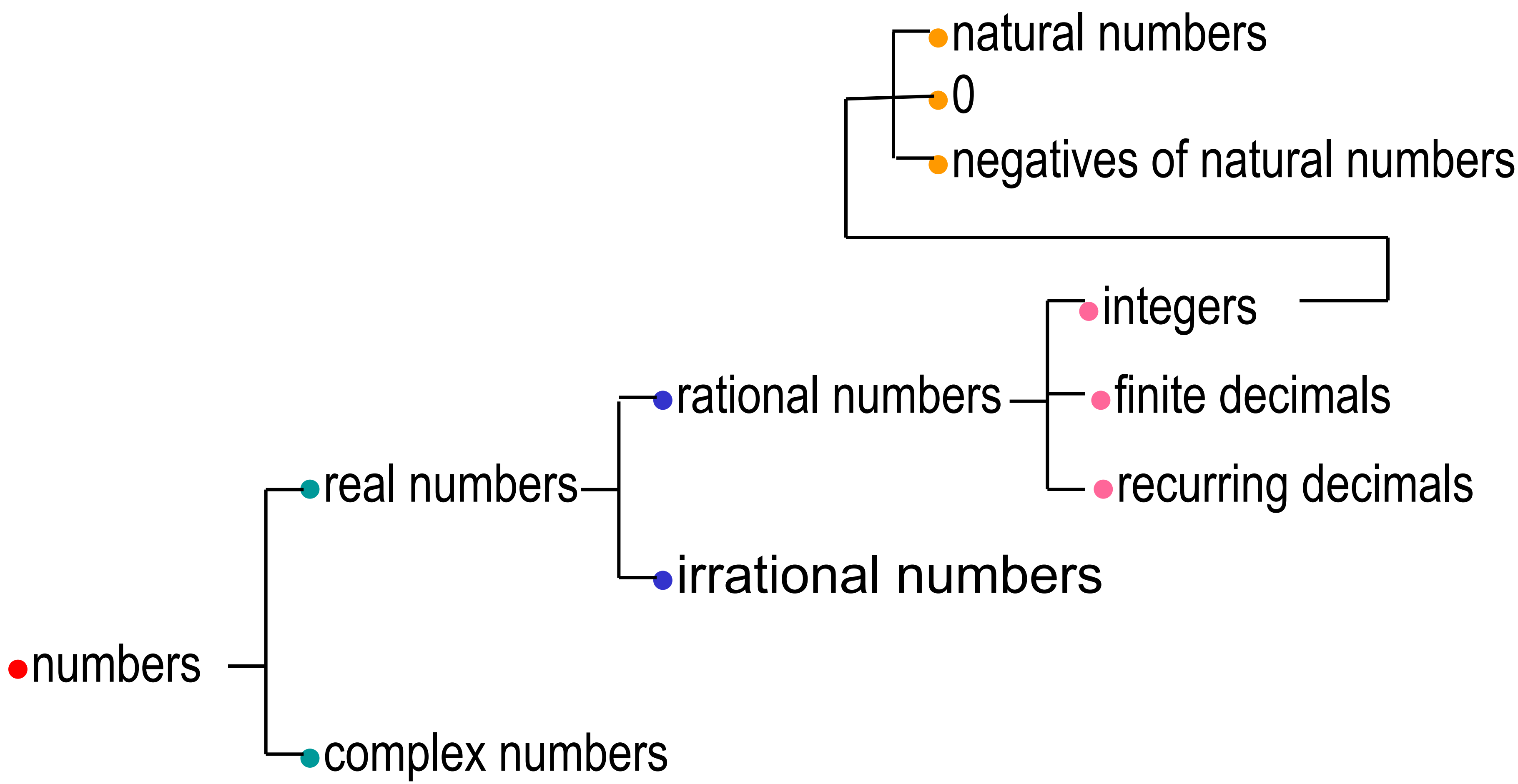
$$\sqrt{2} = 1.41421356\cdots$$

$$\pi = 3.14159265\cdots$$

Real number

- Real numbers include all the rational numbers and all the irrational numbers.
- Real numbers are expressed by a number line.

Number System



Example 1. Prove that $\sqrt{2}$ is an irrational number.

Ans.

If $\sqrt{2}$ is a rational number, we can express it as a fraction $\frac{m}{n}$ in lowest term. Then

$$\frac{m}{n} = \sqrt{2} \qquad \therefore \frac{m}{n} \cdot \frac{m}{n} = \frac{mm}{nn} = 2$$

Because m and n has no common divisor except 1, mm and nn also has no common divisor. Therefore, it is impossible to be

$$\frac{mm}{nn} = 2$$

Therefore, $\sqrt{2}$ is not a rational number, that is, it is an irrational number.

Example 2. Express the following recurring decimal as a fraction.

0.037037037037······

Ans.

Since the number repeat at every three digits, we consider the difference between x and $1000x$.

$$
\begin{array}{rrl}
 & 1000x = & 37.037037037037\cdots \\
-) & x = & 0.037037037037\cdots \\
\hline
 & 999x = & 37
\end{array}
$$

Therefore

$$x = \frac{37}{999} = \frac{1}{27}$$

I got it !

Rationalization of the Denominator

In order to make the expression simple and calculation easier, the irrational number in the denominator are eliminated in the following way.

Example 3. Rationalize the denominator of the following numbers.

(1) $\dfrac{6}{\sqrt{2}}$ (2) $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$

Ans.

(1) $\dfrac{6}{\sqrt{2}} = \dfrac{6\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \dfrac{6\times\sqrt{2}}{2} = 3\sqrt{2}$

(2) We use the relationship $(a+b)(a-b) = a^2 - b^2$

Then

$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}} = \frac{\left(\sqrt{2}+\sqrt{3}\right)}{\left(\sqrt{2}-\sqrt{3}\right)} = \frac{\left(\sqrt{2}+\sqrt{3}\right)^2}{\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)}$$

$$= \frac{2+2\sqrt{2}\sqrt{3}+3}{2-3} = -5-2\sqrt{6}$$

Exercise

Exercise 1. Calculate the following

$$\frac{\sqrt{5}}{\sqrt{3}-1}-\frac{\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

Ans.

Pause the video and solve the problem.

Answer to the Exercise

Exercise 2. Calculate the following

$$\frac{\sqrt{5}}{\sqrt{3}-1}-\frac{\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

Ans.

$$\frac{\sqrt{5}}{\sqrt{3}-1}-\frac{\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{5}\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}-\frac{\sqrt{3}\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)}$$

$$=\frac{\sqrt{5}\sqrt{3}+\sqrt{5}}{2}-\frac{\sqrt{3}\sqrt{5}-3}{2}=\frac{3+\sqrt{5}}{2}$$

Lesson 8
Exponential Functions

8B

- **Exponential Functions**

Positive Integer Exponent

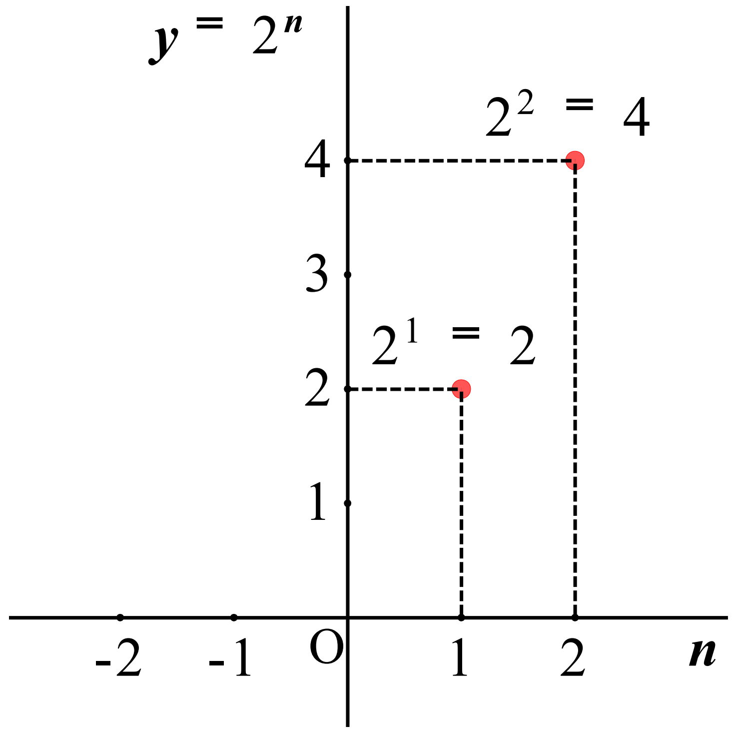
Power

The product whose factors are equal.

[Ex.] $2^1 = 2,$

$2^2 = 2 \times 2 = 4,$

$2^3 = 2 \times 2 \times 2 = 8, \ \cdots$



exponent

$$a^n = \underbrace{a \times a \times \cdots\cdots \times a}_{n}$$

base

n : integer

Theorems on Power

Multiplication

$$a^n a^m = a^{n+m}, \quad (a^n)^m = a^{nm}, \quad (ab)^n = a^n b^n$$

Division

$$\frac{a^n}{a^m} = \begin{cases} a^{n-m} & \text{if } n > m \\ \dfrac{1}{a^{m-n}} & \text{If } m > n \\ 1 & \text{if } m = n \end{cases}$$

$\Rightarrow$

$$\frac{a^n}{a^m} = a^{n-m}$$

[Ex.] $\frac{a^8}{a^3} = a^5, \quad \frac{a^3}{a^8} = \frac{1}{a^5}, \quad \frac{a^6}{a^6} = 1$

Extension to Negative Integer Exponent

$$\frac{a^n}{a^m} = a^{n-m}$$

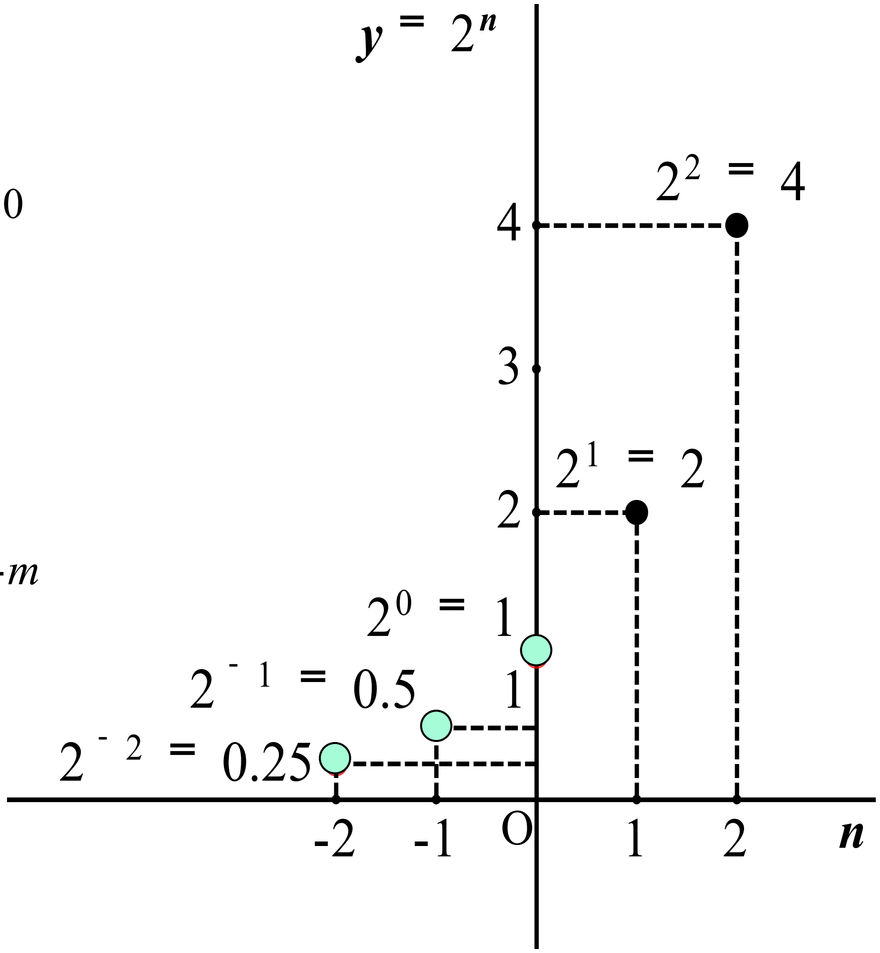
If we put $m = n$, we have $\frac{a^n}{a^n} = a^0$

Therefore, we define $a^0 = 1$

If we put $n = 0$, we have $\frac{a^0}{a^m} = a^{-m}$

Therefore, we define

$$a^{-n} = \frac{1}{a^n}$$



[Ex.] $2^0 = 1$, $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

Extension to Rational Exponent

Rational number

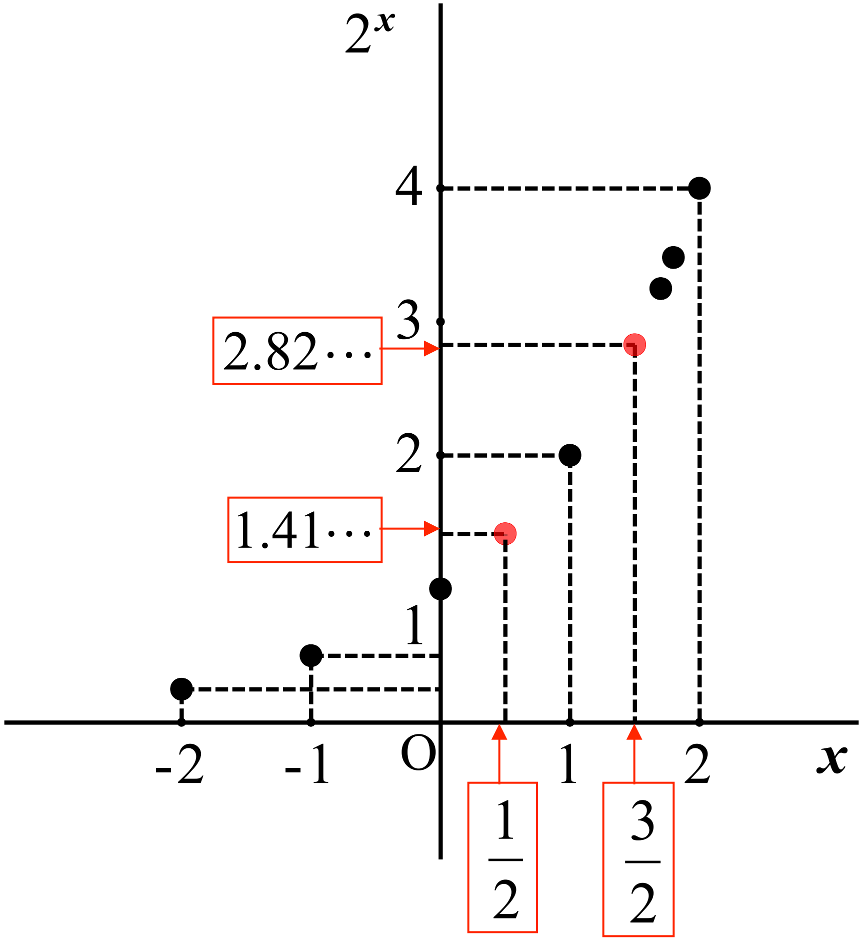
a^{x} $\qquad a^{\frac{m}{n}}$

From the rule $(a^n)^m = a^{nm}$, we have

$$\left(a^{\frac{m}{n}}\right)^n = a^{\frac{m}{n}\times n} = a^m$$

Therefore

$$a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$$



[Ex.]

$$2^{\frac{1}{2}} = \sqrt{2} = 1.414\cdots$$

$$2^{\frac{3}{2}} = \sqrt{2^3} = 2\sqrt{2} = 2.828\cdots$$

Extension to Irrational Exponent

Irrational number

a^{x}

The value of a power with an irrational exponent is defined as the limit of a power with a rational exponent

[Ex.] $a^{\sqrt{2}} = ?$

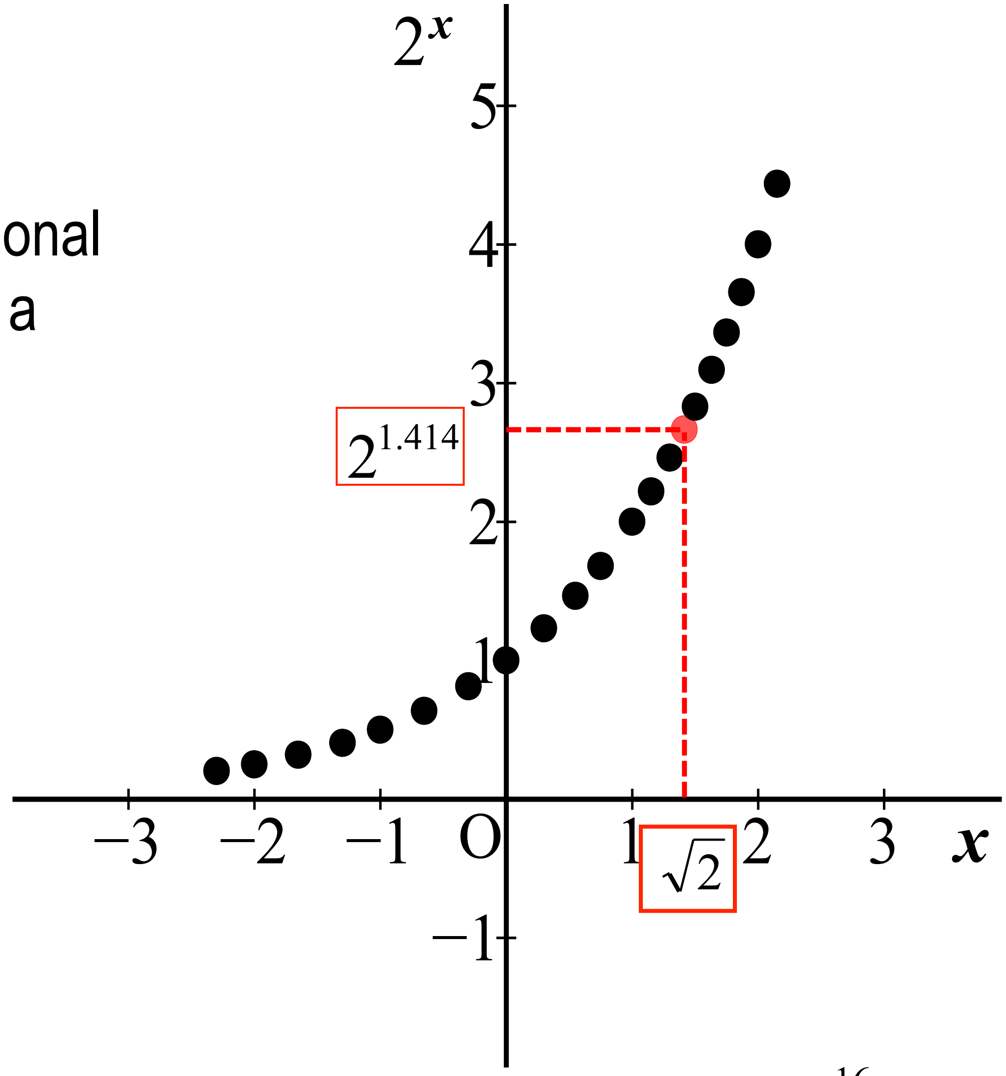
$2^{1} = 2.00000$

$2^{1.4} = 2.63902$

$2^{1.41} = 2.65737$

$2^{1.414} = 2.66475$

$\cdots$



Exponential Function

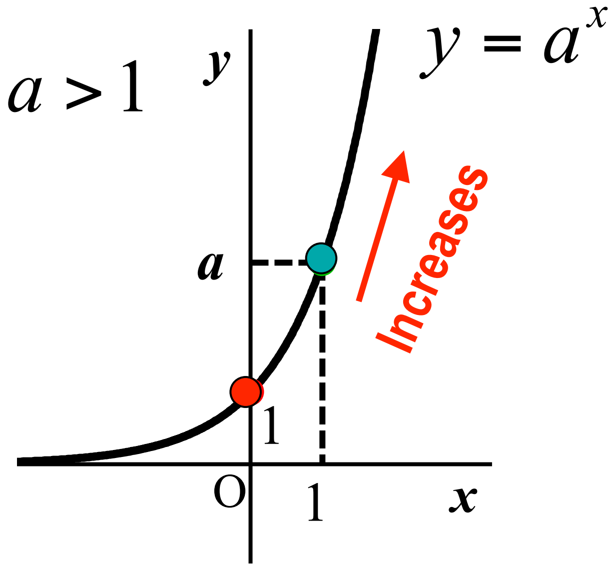
Exponential function

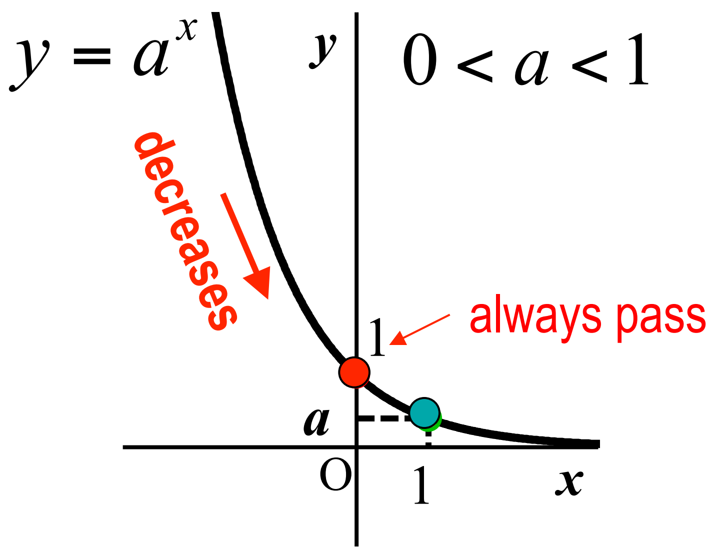
Positive real number (Range)

All real number (Domain)

$$y = a^{x} \qquad (a > 0,\ a \neq 1)$$

Characteristics

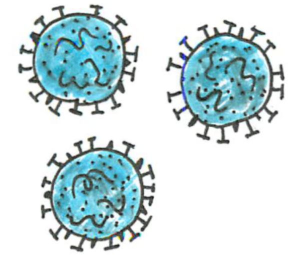




Phenomena with Exponential Growth (1)

SARS Case (Virus)

2002-11 First patient was found in China
2003-03 Patients 1622 (58 died)
2003-05 Patients 8860 (792 died)



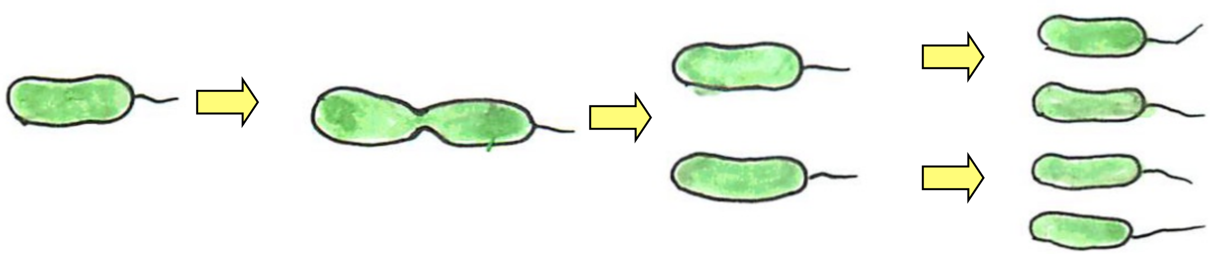
Growth by Doubling

$$y = 2^x$$

Assumption : Each bacteria splits into two cells in every one hour.

End of Hour	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...	24
Bacteria - starting with one	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	...	16777216

24 hr

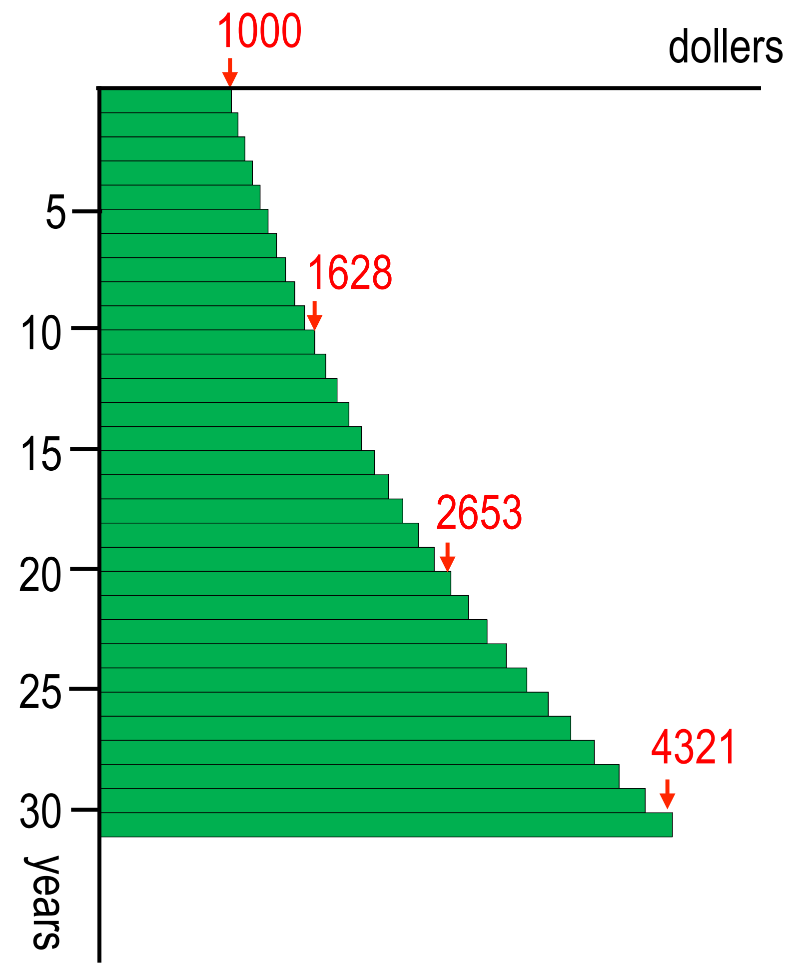


Phenomena with Exponential Growth (2)

Compound interest calculation

$$y = y_0 (1 + r)^x$$

Year (x), Principal (y_0), Interest (r)



[Ex.]

Principal $1000

Annual compound interest rate 5 %

1 Year later $1050.00

10 Years later $1628.89

20 Years later $ 2653.30

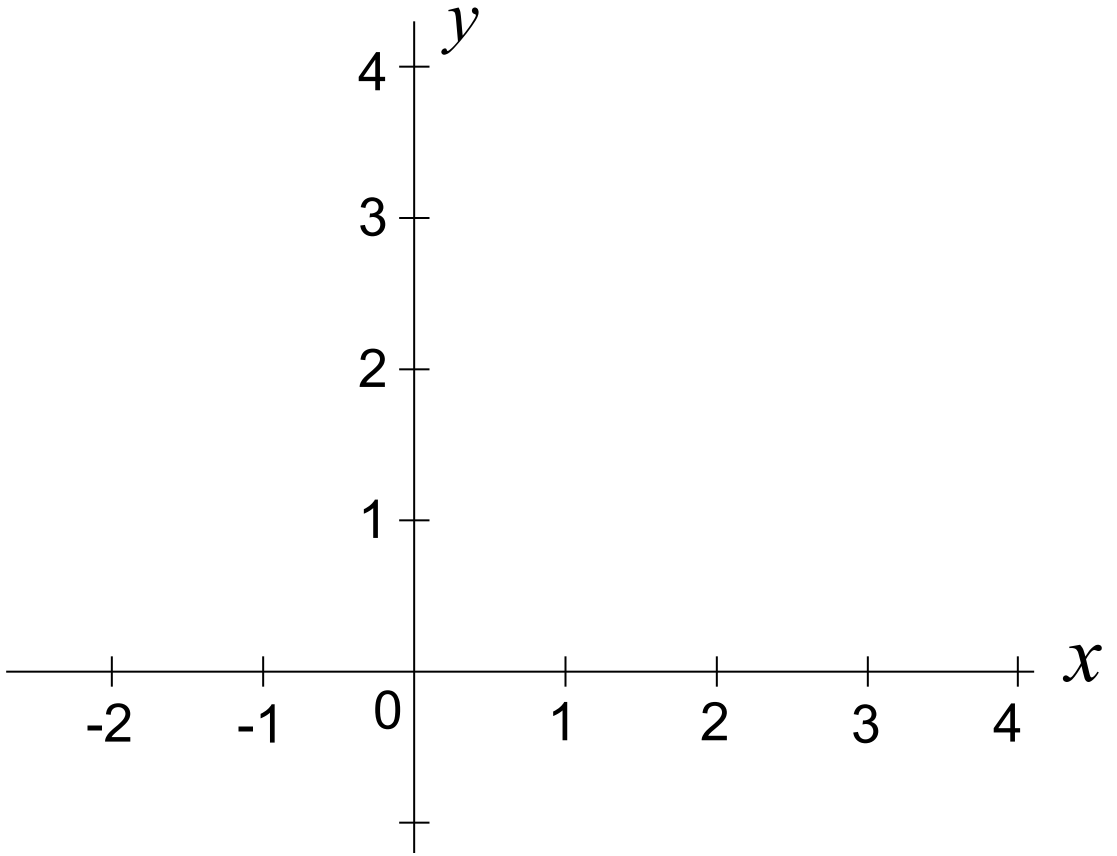
30 Years later $ 4321.94

Exercise

Exercise 2 Illustrate the graphs of the following functions on the given coordinates and discuss their relationship.

(1) $y = 2^x$ (2) $y = \frac{2^x}{4}$

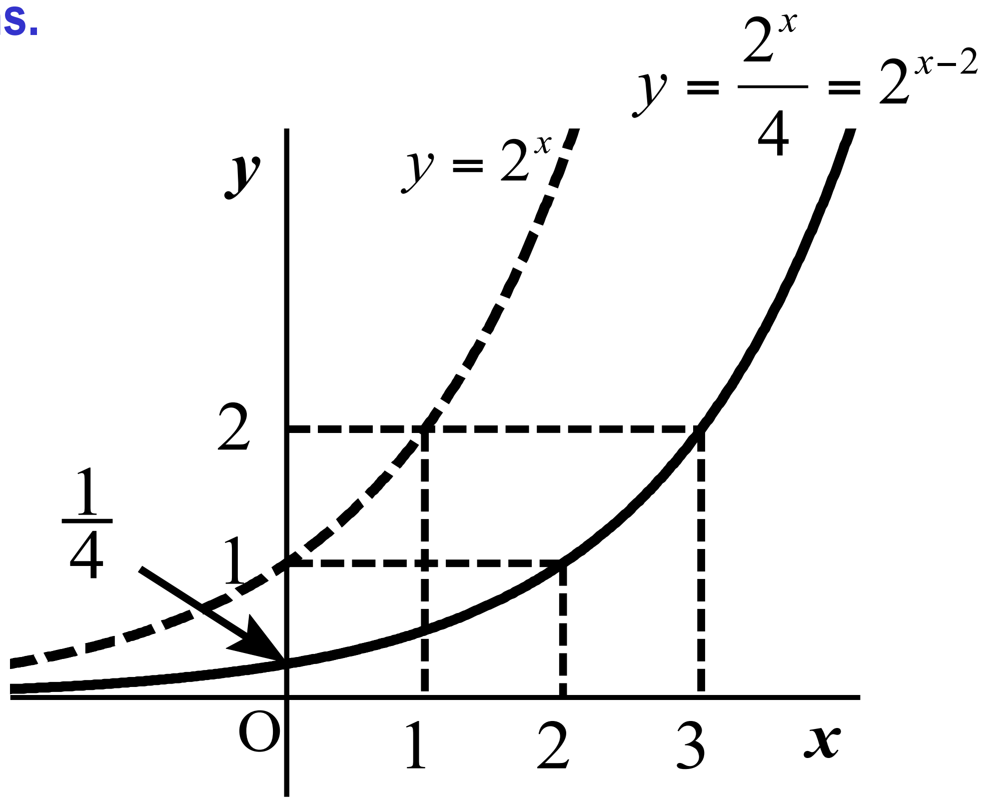
Ans.



Example 4 Illustrate the graphs of the following functions on the given coordinates and discuss their relationship.

(1) $y = 2^x$ (2) $y = \dfrac{2^x}{4}$

Ans.



Since $y = \dfrac{2^x}{4} = 2^{x-2}$, it is obtained by shifting the graph $y = 2^x$ rightward by 2.