Course I

NAGOYA UNIVERSITY

Lesson 10
Inverse Functions

10A

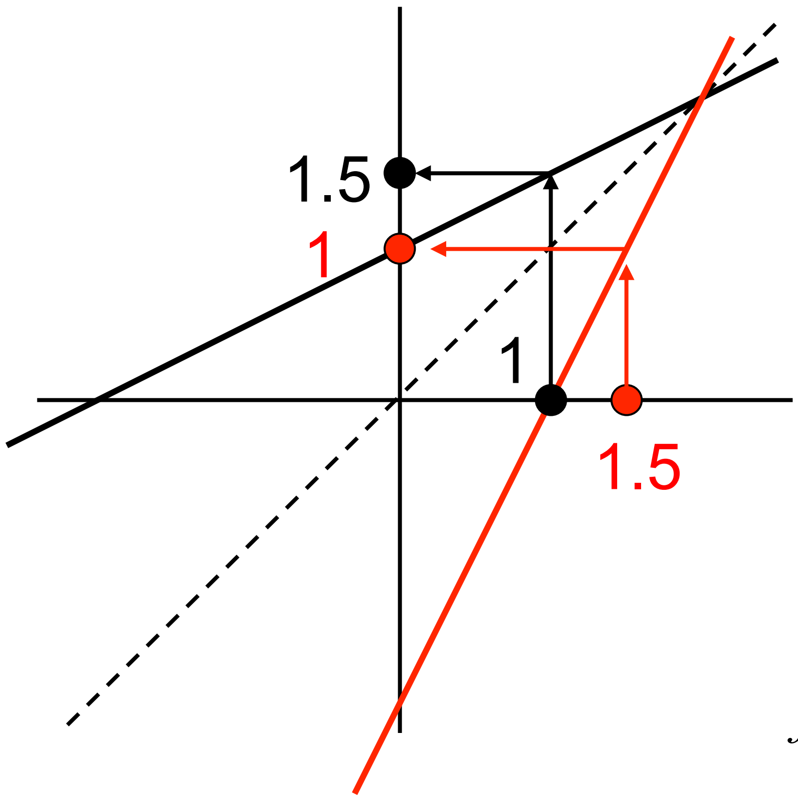
- Inverse functions
- Exponential Function and Logarithmic Function

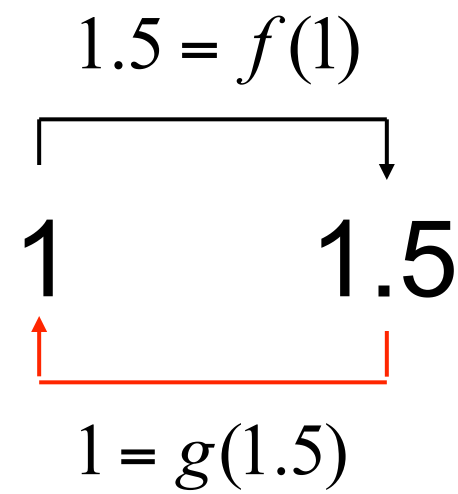
Inverse Function – Example 1

Example : Consider $y = f(x) = \frac{1}{2}x + 1$

The inverse function is given by exchanging x and y.

$x = \frac{1}{2}y + 1$ that is $y = g(x) = 2x - 2$





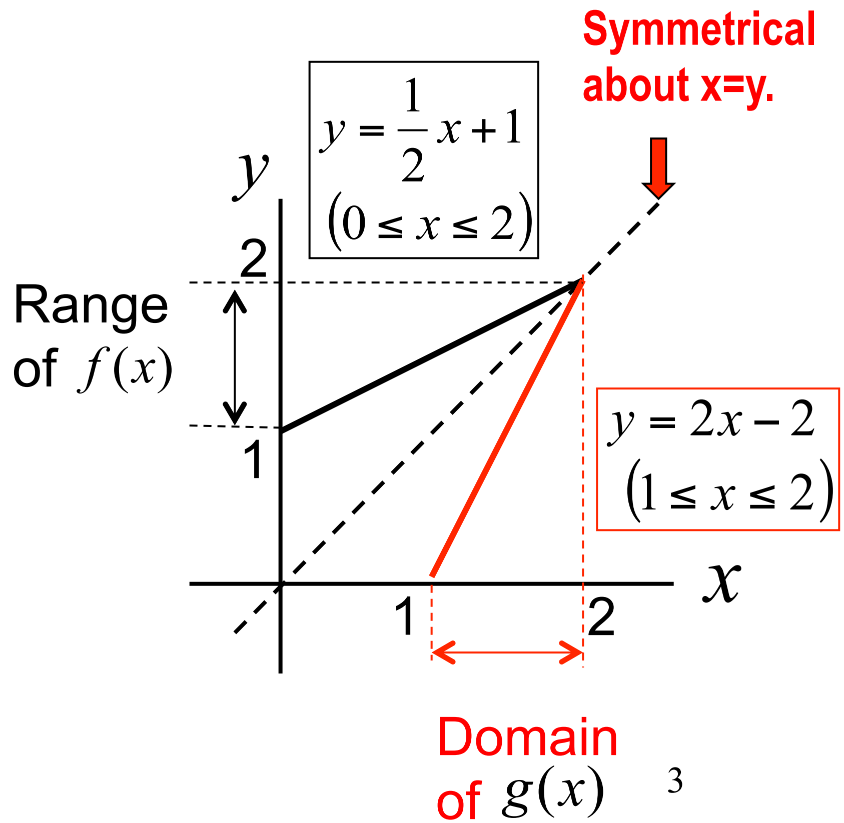
The inverse function $g(x)$ bring $f(1)$ back to 1.

Definition of Inverse Functions

Functions $f(x)$ and $g(x)$ are inverses of one another if: $f(g(x)) = x$ and $g(f(x)) = x$ for all values of x in their respective domains.

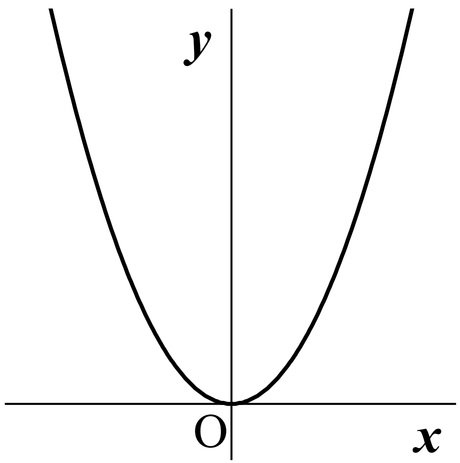
How to Find Inverse Function

1. Replace the variables.
   $y = f(x) \quad \rightarrow \quad x = f(y)$
2. Rearrange the expression.
   $x = f(y) \quad \rightarrow \quad y = g(x)$
3. Determine that the domain of $g(x)$ is the same as the range of $f(x)$.



Inverse Function – Example(2)

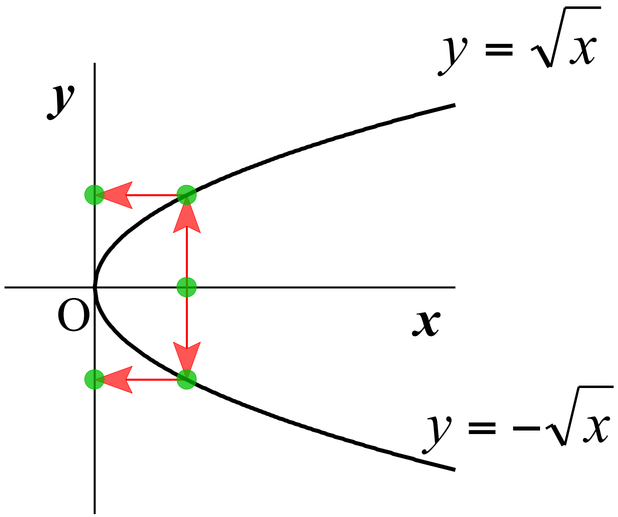
Parabola



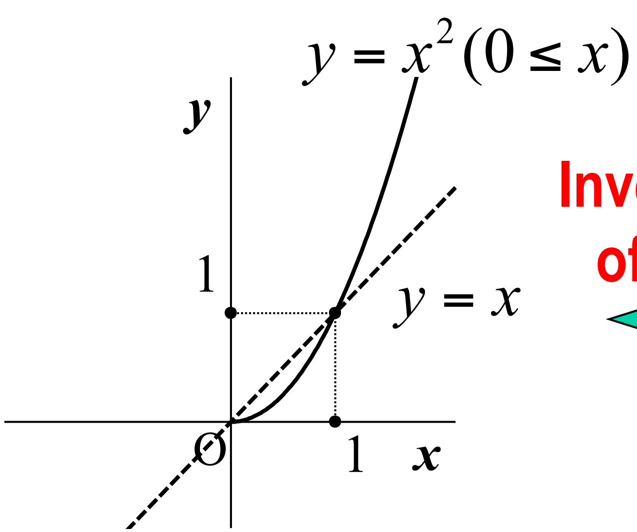
(a)

Not inverse

Because (b) is not a function

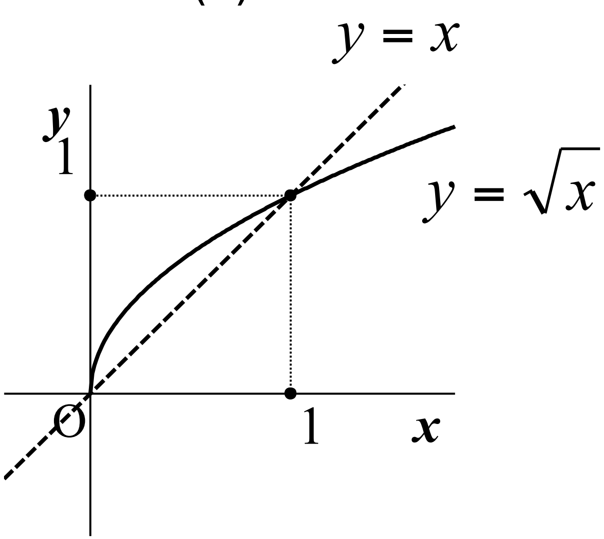


(b)



(c)

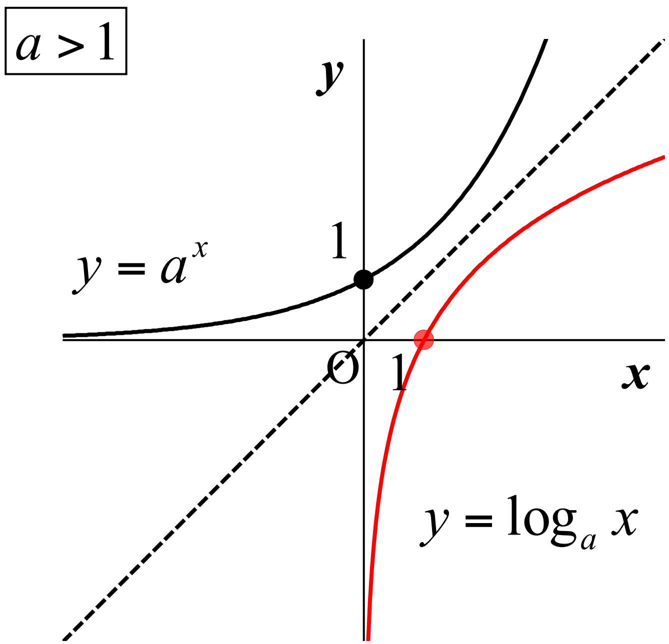
Inverse function of each other

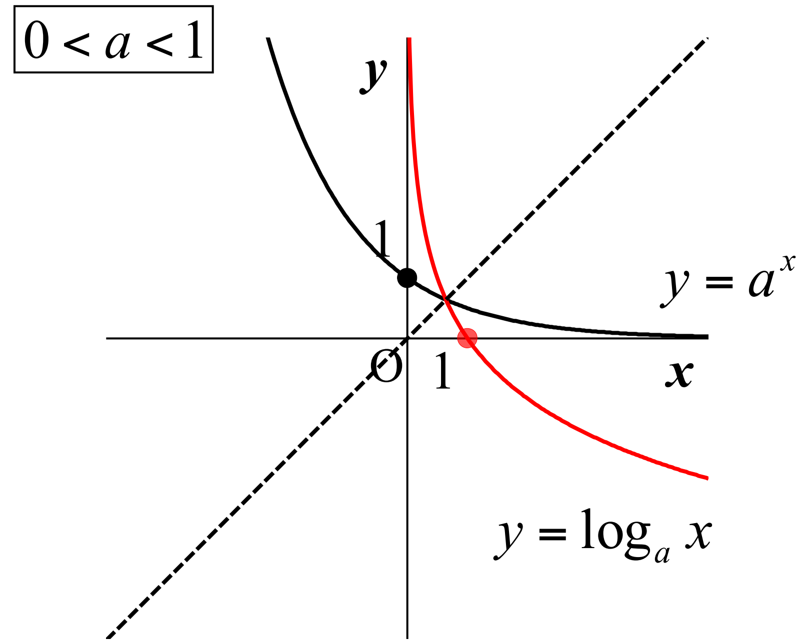


(d)

Exponential Function & Logarithmic Function

Exponential function $\quad y = a^x \ \left(a > 0,\ a \neq 0\right)$





The inverse function of is called a logarithmic function and written as

$$y = \log_a x$$

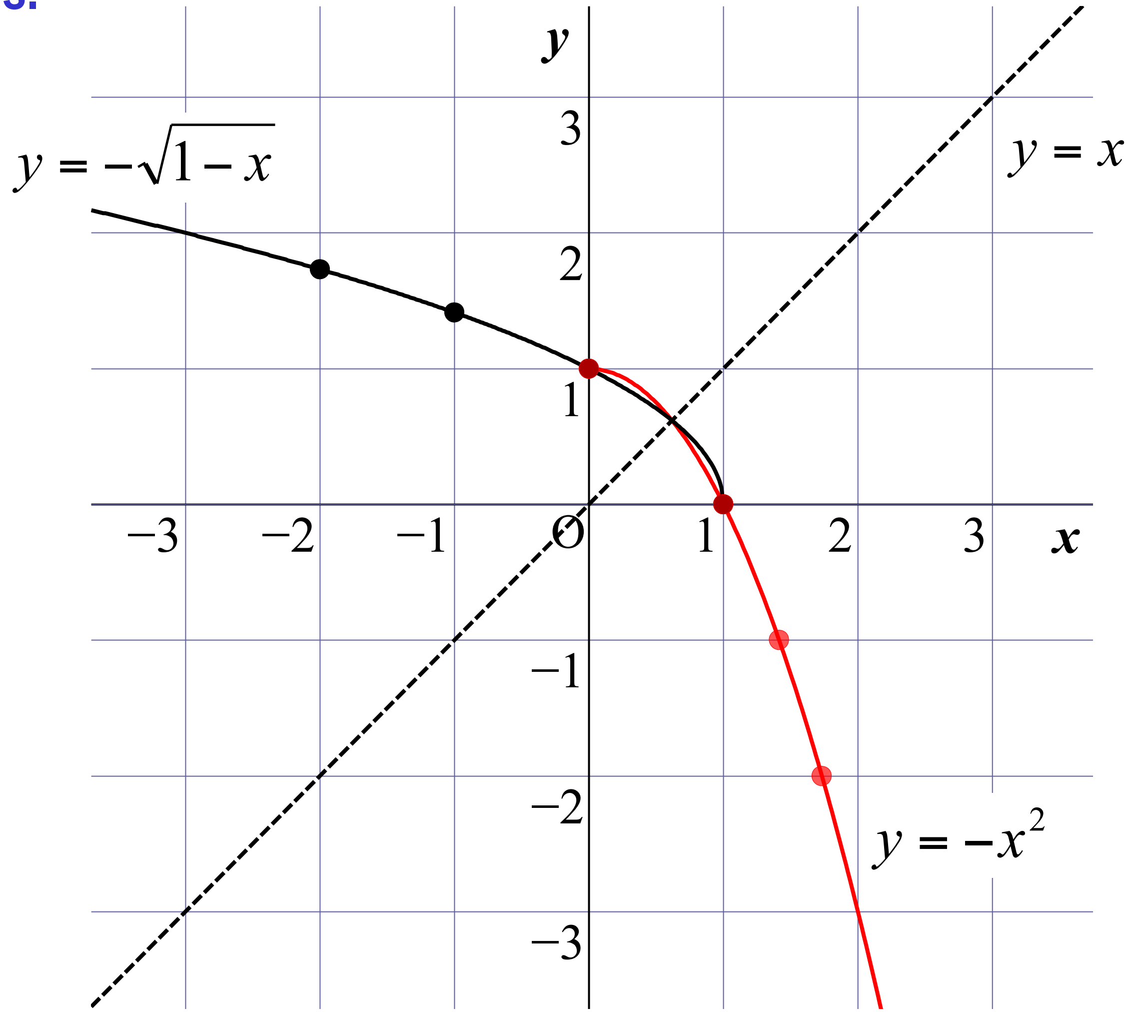
- The domain is the positive real number.
- The range is the whole real number.
- The graph always passes the point (1, 0)

Uhh.... What?

Example

Example 1. Illustrate the function $y=-\sqrt{1-x}$ and its inverse function on the given coordinate plane.

Ans.



$y=-\sqrt{1-x}$

x	1	0	-1	-2	-3
y	0	1	$\sqrt{2}$	$\sqrt{3}$	2

⇩

- Take symmetrical points about x=y,
  that is,
- replace their values

x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	1	0	-1	-2	-3

Exercise

Exercise 1. (1) Illustrate the function $y = \dfrac{x}{x+2} \quad (x \geq 0)$

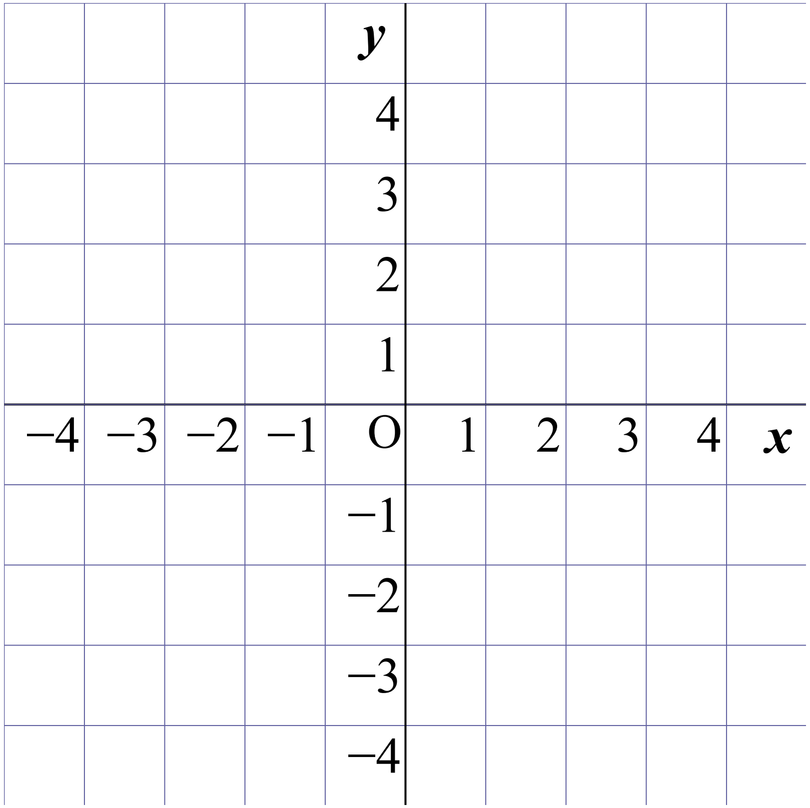
(2) Illustrate the inverse function by mapping symmetrically about $y = x$

(3) Find the domain of the inverse function.

(4) Find the expression of the inverse function by replacing x and y.

Ans.

Pause the video and solve the problem.



Answer to the Exercise

Exercise 1. (1) Illustrate the function $y = \dfrac{x}{x+2} \quad (x \geq 0)$

(2) Illustrate the inverse function by mapping symmetrically about $y = x$

(3) Find the domain and the range of the inverse function.

(4) Find the expression of the inverse function by replacing x and y.

Ans. (1) This expression becomes

$$y = 1 - \frac{2}{x+2}$$

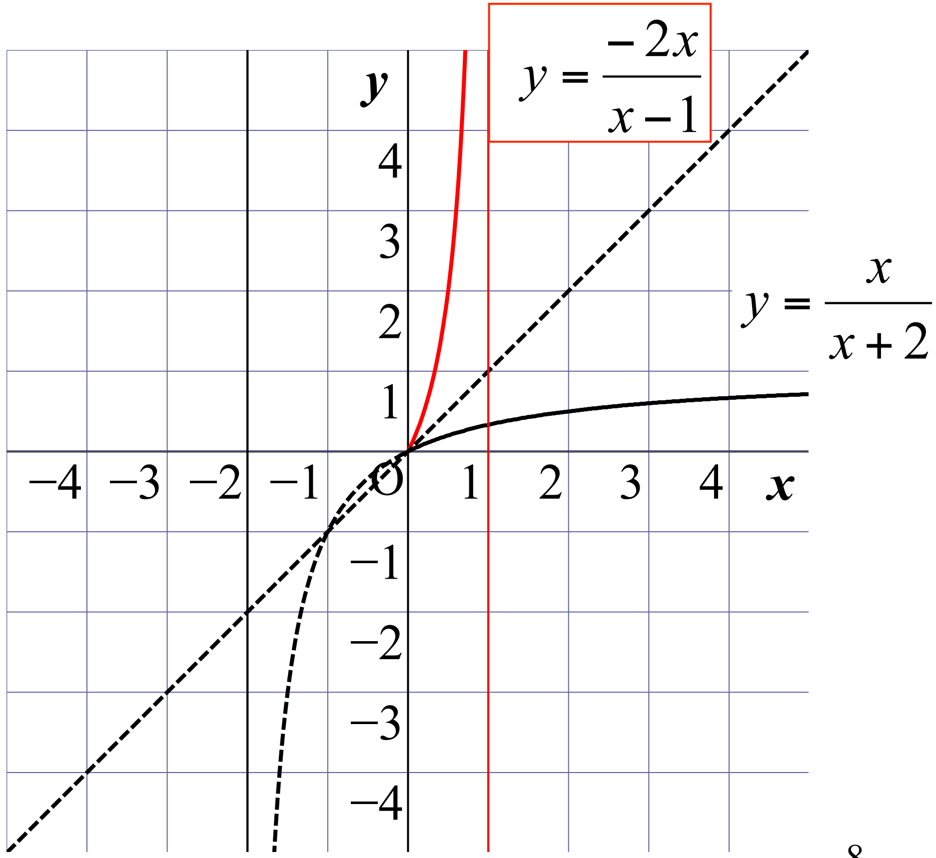
(2) . See the figure.

(3) From the figure

Domain : $0 \leq x < 1$

Range: $y \geq 0$

(4) $x = \dfrac{y}{y+2} \quad \therefore \; y = \dfrac{-2x}{x-1}$



NAGOYA UNIVERSITY

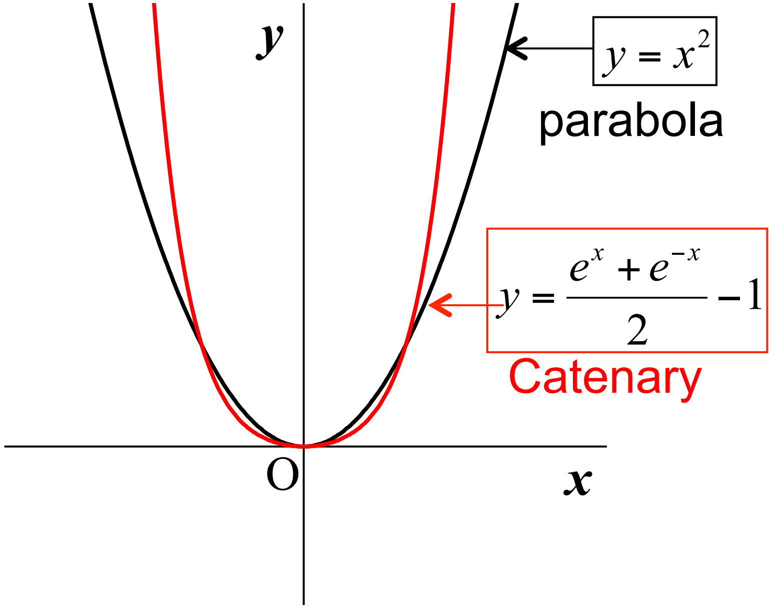
Lesson 10
Hyperbolic Functions

10B

- Catenary
- Hyperbolic Function
- Why Do We Call Them “Hyperbolic” ?

Catenary

Hanging chain
(Catenary)



Hyperbolic Function

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Fundamental Characteristics of Hyperbolic Functions

Trigonometric functions

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cos^2 x + \sin^2 x = 1$$

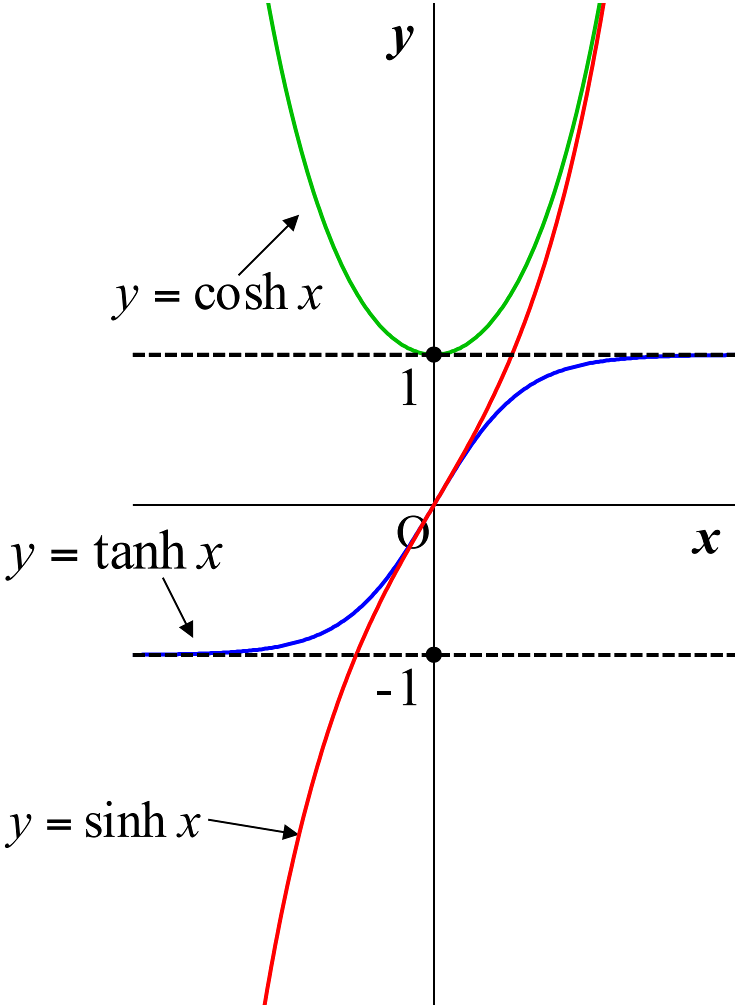
$$1 + \tan^2 x = \frac{1}{\cos^2 x}$$

Hyperbolic functions

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \frac{1}{\cosh^2 x}$$



Graphs

Why Do We Call Them “Hyperbolic” ?

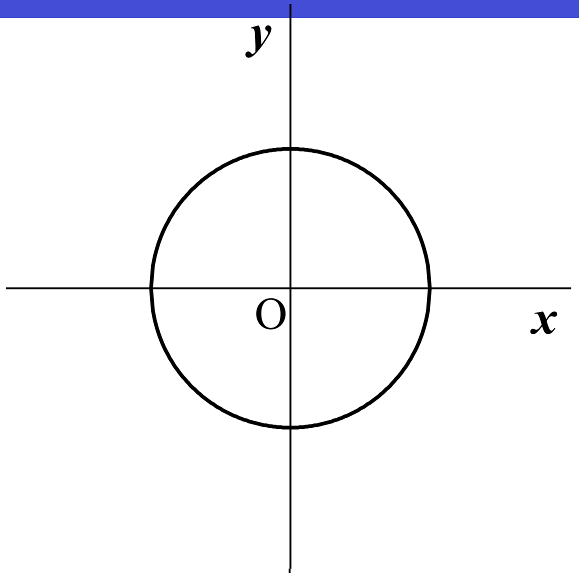
Comparison of Two Parametric Expressions

$$(x,y) = (\cos t,\ \sin t)$$

Eliminating the parameter, we have

$$x^2 + y^2 = 1$$

This expression represents a circle.

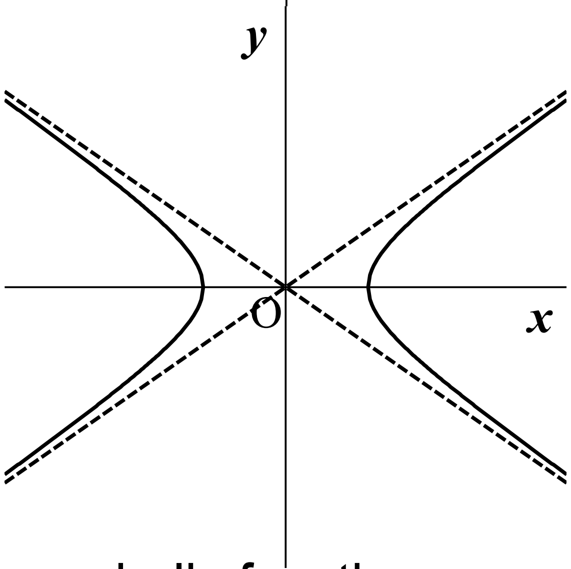


$$(x,y) = (\cosh t,\ \sinh t)$$

From the previous slide, we have

$$x^2 - y^2 = 1$$

This expression represents a hyperbola.



- ▸ Based on the similarity in shape, we call them hyperbolic functions.
- ▶ Based on the similarity in character, we use the symbols $\sinh x$ etc.

Example 2. Prove the following addition formula.

$$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$$

Ans.

$$\cosh x\cosh y+\sinh x\sinh y=\left(\frac{e^{x}+e^{-x}}{2}\right)\left(\frac{e^{y}+e^{-y}}{2}\right)+\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{y}-e^{-y}}{2}\right)$$

$$=\frac{1}{4}\left(e^{x}e^{y}+e^{x}e^{-y}+e^{-x}e^{y}+e^{-x}e^{-y}\right)+\frac{1}{4}\left(e^{x}e^{y}-e^{x}e^{-y}-e^{-x}e^{y}+e^{-x}e^{-y}\right)$$

$$=\frac{1}{2}\left(e^{x+y}+e^{-x-y}\right)=\cosh(x+y)$$

Exercise

Exercise 2. Prove the following formulae.

(1) $$\cosh^2 x - \sinh^2 x = 1$$

(2) $$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

Ans.

Pause the video and solve the problem.

Answer to the Exercise

Exercise 2. Prove the following formulae.

(1) $\cosh^2 x - \sinh^2 x = 1$

(2) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

Ans.

(1) $$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) = 1$$

(2) $$\sinh x \cosh y + \cosh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{1}{4}\left(e^x e^y + e^x e^{-y} - e^{-x} e^y - e^{-x} e^{-y}\right) + \frac{1}{4}\left(e^x e^y - e^x e^{-y} + e^{-x} e^y - e^{-x} e^{-y}\right)$$

$$= \frac{1}{2}\left(e^{x+y} - e^{-x-y}\right) = \sinh(x+y)$$