NAGOYA UNIVERSITY

Lesson 11
Logarithm and Common Logarithm

11A

- Logarithm of a number
- Merits of logarithm
- Logarithmic Identities

Logarithm of a Number

$$M = a^p$$

Definition

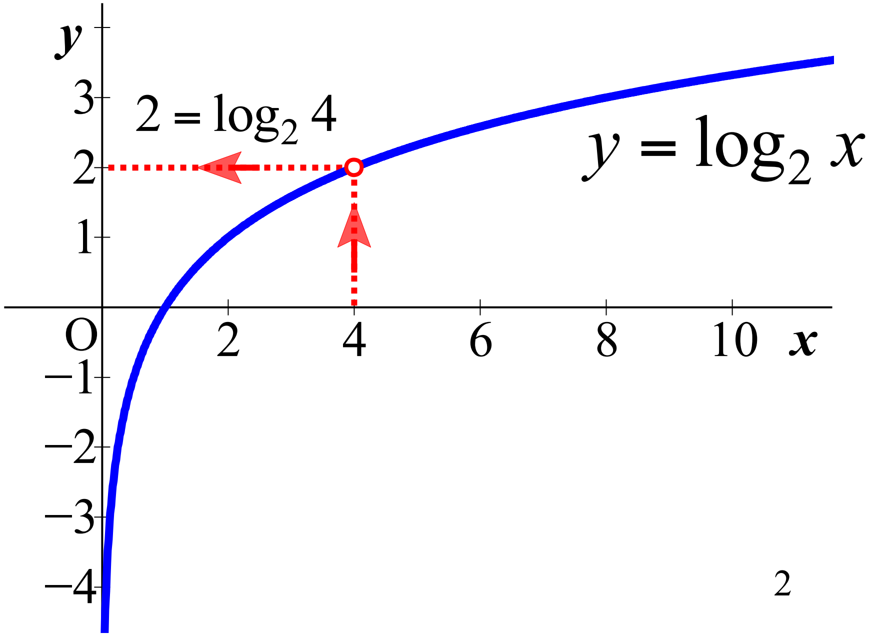
The **logarithm** of a number M is the **exponent** p by which another fixed value (the base) a has to be raised to produce that number, and it written as

$$p = \log_a M$$

.

In other words

○ The value p is the answer to the question "To what power must a be raised in order to yield M ?".



Merits of Logarithm (1)

Merit (1)

- Logarithm can represent a large number by a small number.

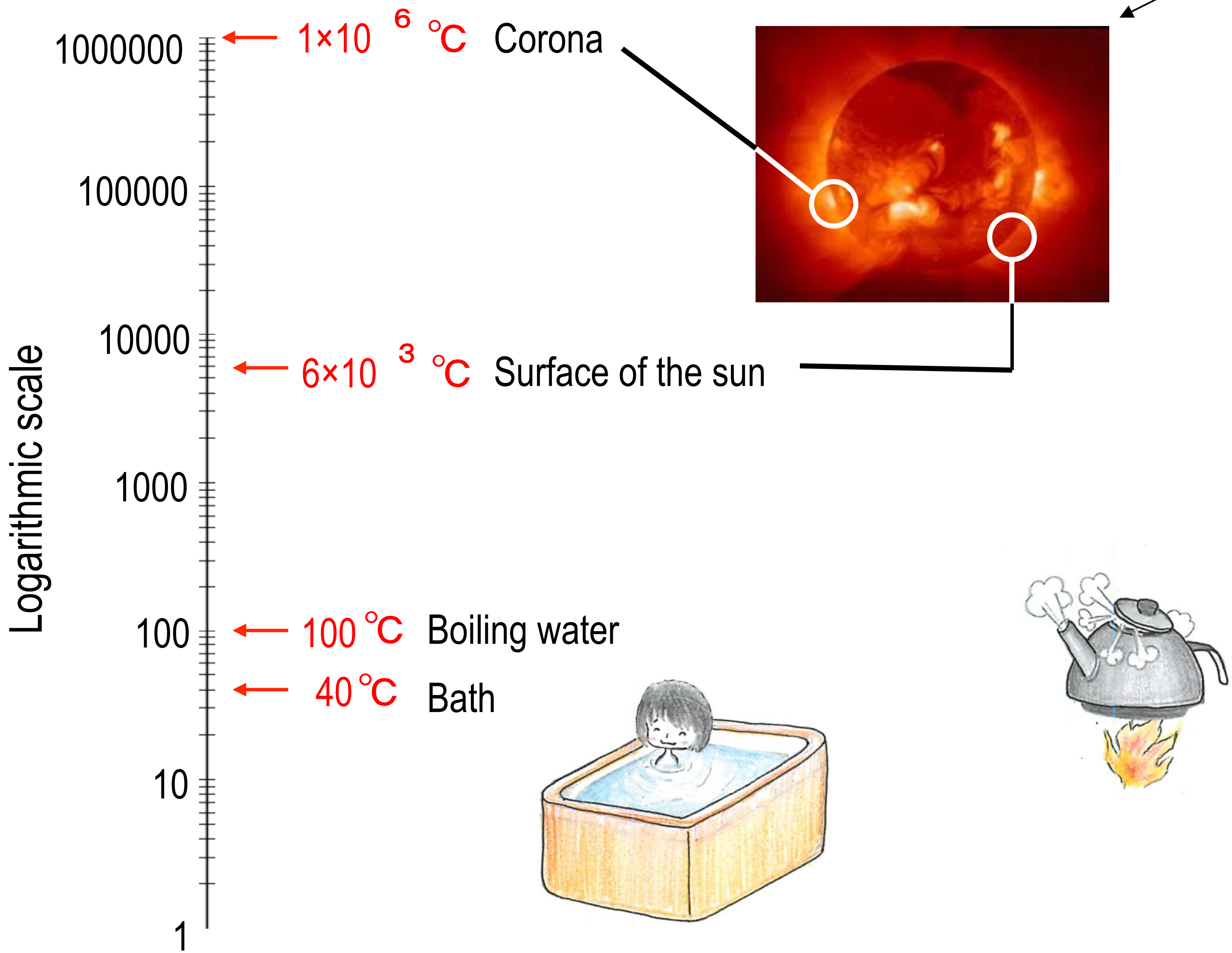
Ex. $x = 100000 \rightarrow \log_{10} x = 5$

- Logarithmic scale is helpful to represent wide range of values with different details.
  (See next slide for illustration)

That makes sense!

Logarithmic Scale

http://www.isas.jaxa.jp/home/solar/home/solar/yohkoh/Graph/Yohkoh full.gif



Logarithmic Identities

Product $\log_a MN = \log_a M + \log_a N$

Quotient $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

Power $\log_a M^k = k \log_a M$

[Example 11.1] Prove the product rule of logarithm.

Ans.

Let $M = a^p$ and $N = a^q$. Then $p = \log_a M$, $q = \log_a N$

By multiplication $M \cdot N = a^p \cdot a^q = a^{p+q}$

Therefore $\log_a M \cdot N = p + q = \log_a M + \log_a N$

Special Cases

Special Values

$$a^1 = a \longrightarrow \log_a a = 1$$

$$a^0 = 1 \longrightarrow \log_a 1 = 0$$

$$a^{-1} = \frac{1}{a} \longrightarrow \log_a \left(\frac{1}{a}\right) = -1$$

Special Identities

$$\log_a \frac{M}{N} = \log_a M - \log_a N \xrightarrow{M=1} \log_a \frac{1}{N} = -\log_a N$$

$$\log_a M^k = k \log_a M \xrightarrow{k=\frac{1}{n}} \log_a \sqrt[n]{M} = \frac{1}{n} \log_a M$$

Changing the Base of Logarithm

[Example 11.2] Prove the following formula : $\log_a b = \dfrac{\log_c b}{\log_c a}$

$a(\neq 1),\ b(\neq 1),\ c(\neq 1)$ are positive real numbers

Ans.

Let $p = \log_a b$, then $b = a^p$ holds.

If we make logarithms with base c of both sides, we have

$$\log_c b = \log_c a^p$$

$$\therefore\ \log_c b = p \log_c a$$

Since $a \neq 1$, we see $\log_c a \neq 0$

Therefore, $$p = \frac{\log_c b}{\log_c a}$$

Logarithmic Equation

[Example 11.3] Solve the following equations.

(1) $\log_2 x = 3$ (2) $\log_2(x+3) + \log_2(2x-1) = 2$

Ans.

(1) From the definition $x = 2^3 = 8$

(2) Because $x+3>0$ and $2x-1>0$

Therefore $x > \frac{1}{2}$

The given expression becomes

$$\log_2(x+3)(2x-1) = \log_2 4$$

$\therefore\ (x+3)(2x-1) = 4$ $\therefore\ 2x^2 + 5x - 7 = 0$

$\therefore\ (x-1)(2x+7) = 0$

$\therefore\ x = 1,\quad x = -\frac{7}{2}$

Since $x > \frac{1}{2}$, we have $x = 1$

Logarithmic Inequality

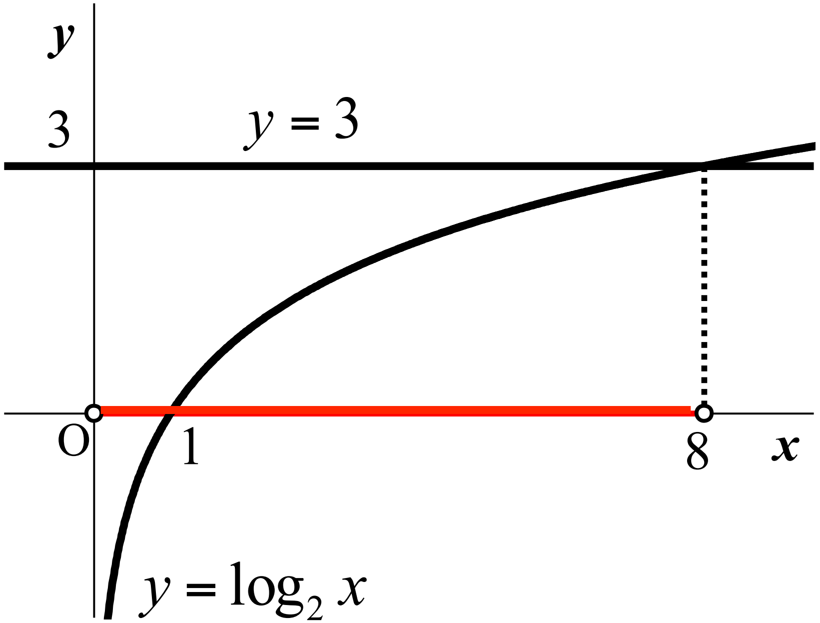
[Example 11.4] Solve the following inequality. $\log_2 x < 3$

Ans.

$$\log_2 x < 3 = \log_2 2^3 = \log_2 8$$

From the graph of $y = \log_2 x$, we have

$$0 < x < 8$$



Exercise

[Ex.11.1] Find the values of (1) $\log_5 125$ (2) $\log_{10} 10\sqrt{10}$

[Ex.11.2] Simplify the following expressions: (1) $\log_{10} 2 + \log_{10} 5$ (2) $\log_3 7 - \log_3 63$

Answer to the Exercise

[Ex.11.1] Find the values of (1) $\log_5 125$ (2) $\log_{10} 10\sqrt{10}$

(1)

We put $\log_5 125 = p$,

Then, we have $5^p = 125 = 5^3 \quad \therefore \; p = 3$

(2)

$$\log_{10} 10\sqrt{10} = \log_{10} 10^{\frac{3}{2}} = \frac{3}{2}\log_{10} 10 = \frac{3}{2}$$

[Ex.11.2] Simplify the following expressions: (1) $\log_{10} 2 + \log_{10} 5$ (2) $\log_3 7 - \log_3 63$

(1) $\log_{10} 2 + \log_{10} 5 = \log_{10}(2+5) = \log_{10} 10 = 1$

(2) $\log_3 7 - \log_3 63 = \log_3 \frac{7}{63} = \log_3 \frac{1}{9} = \log_3 3^{-2} = -2$

Exercise

[Ex.11.3] (1) Simplify the following expression $\left(\log_2 9 + \log_4 3\right)\left(\log_3 2 + \log_9 4\right)$

(Hint: Make a common base.)

(2) Solve the following logarithmic equation $\log_2 x + \log_2(x-7) = 3$

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.11.3] (1) Simplify the following expression

(Hint: Make a common base.)

(2) Solve the following equation. $\log_2 x + \log_2(x-7) = 3$

(1) $$(\log_2 9 + \log_4 3)(\log_3 2 + \log_9 4) = \left(\log_2 9 + \frac{\log_2 3}{\log_2 4}\right)\left(\frac{\log_2 2}{\log_2 3} + \frac{\log_2 4}{\log_2 9}\right)$$

$$= \left(2\log_2 3 + \frac{\log_2 3}{2}\right)\left(\frac{1}{\log_2 3} + \frac{2}{2\log_2 3}\right)$$

$$= \frac{5}{2}\log_2 3 \cdot \frac{2}{\log_2 3} = 5$$

(2) $$\log_2 x + \log_2(x-7) = 3$$

Since, $x > 0$ and $x - 7 > 0$, therefore $x > 7$

$$\log_2 x(x-7) = \log_2 2^3 \quad \therefore \quad x(x-7) = 8$$

$$\therefore \ (x+1)(x-8) = 0$$

From ①, the answer is $x = 8$

Lesson 11
Logarithm and Common Logarithm

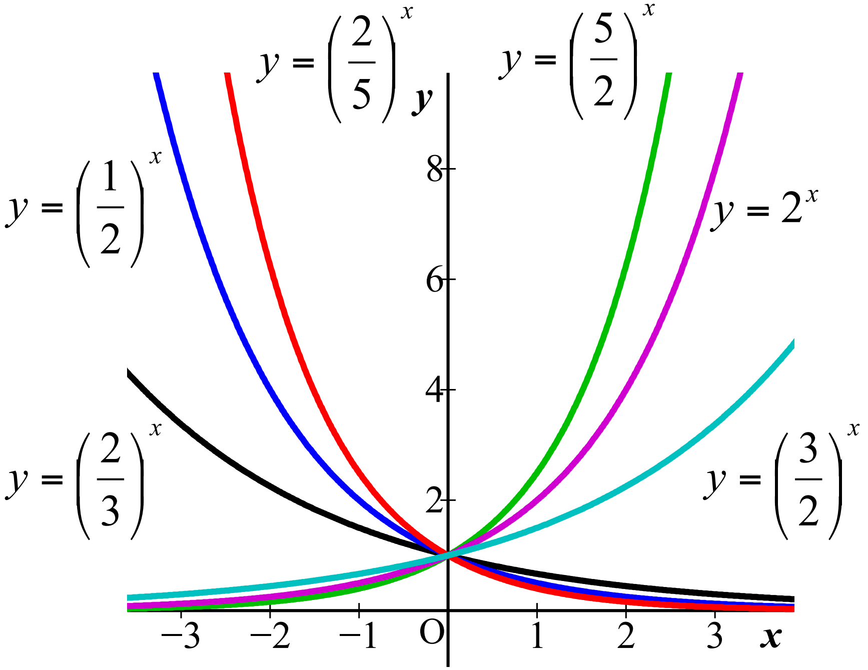
11B

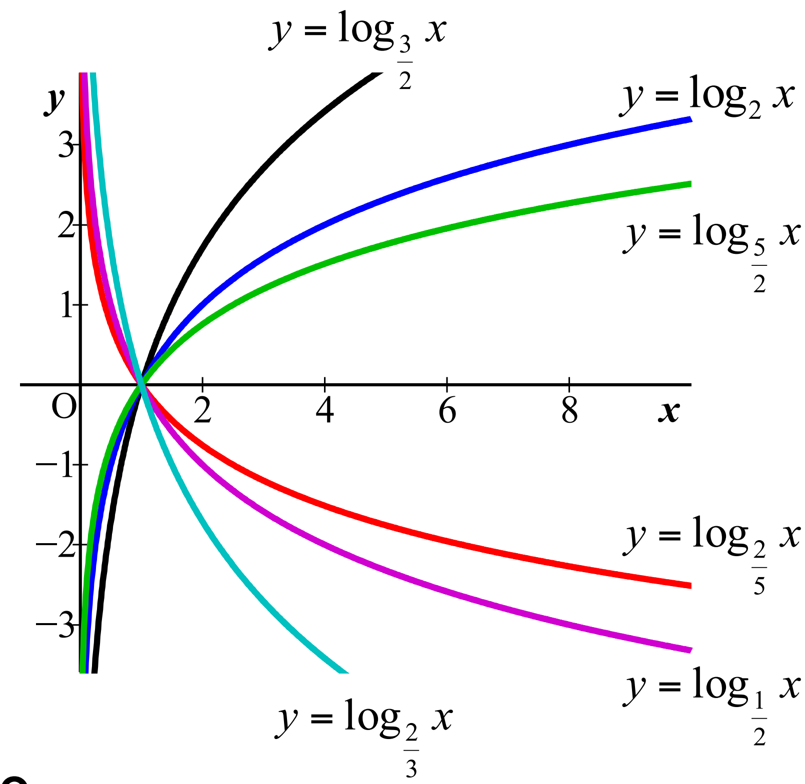
- Special Logarithm
- Common Logarithm

Various Bases

Various exponential function

Various logarithmic function





Then, which one shall we use ?

$\rightarrow$ We are familiar to decimal number.

$$2356 = 2\times10^3 + 3\times10^2 + 5\times10 + 6$$

Special Logarithms

Common Logarithm

$a = 10$ $\quad y = \log_{10} x$ or $y = \log x$

- This has applications in science and engineering.

Natural Logarithm

$a = e = 2.718...$ $\quad y = \log_e x$ or $y = \ln x$

- This is also written as $y = \ln x$
- This is used in pure mathematics.
- This will be explained later in the lesson on differentiation.

Binary Logarithm

$a = 2$ $\quad y = \log_2 x$

- This is used in computer science.

Common Logarithm

- Values of common logarithm are given in the logarithmic table.
- The table gives the value for 1< M <9.99.
- For $M = a \times 10^n \quad (1 \le a < 10)$, the logarithmic value is obtained by

$$\log_{10} M = \log_{10}(a \times 10^n) = n + \log_{10} a$$

Table of Common Logarithm

	0	1	2	3	4	5	6	7	8	9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996

Examples

[Example 11.5] Find the common logarithm of the following numbers.

$$0.01,\quad 0.1,\quad 1,\quad 10,\quad 100$$

Ans.

$$\log_{10} 0.01 = \log_{10} 10^{-2} = -2$$

$$\log_{10} 0.1 = -1$$

$$\log_{10} 1 = 0$$

$$\log_{10} 10 = 1$$

$$\log_{10} 100 = 2$$

[Example 11.6] Find the common logarithm of 11910 from the logarithmic table.

Ans.

$$\log_{10} 11910 = \log_{10}(1.191 \times 10^4) \approx 4 + \log_{10} 1.19 = 4 + 0.0755 = 4.0755$$

Examples

[Example 11.7] Find the number of digits of 6^{25} . Use the following values. .

$$\log_{10}2 = 0.3010, \quad \log_{10}3 = 0.4771$$

Ans.

Put $N = 6^{25}$ and take the common logarithm of both sides.

$$\log_{10} N = \log_{10} 6^{25} = 25 \times (\log_{10}2 + \log_{10}3) = 25 \times (0.3010 + 0.4771)$$
$$= 19.4525$$

Then $N = 10^{19.4525}$

Therefore $10^{19} < N < 10^{20}$

This means that 6^{25} is a 20 digits number

.

Merit of Logarithm (2)

By this product rule $\log_a MN = \log_a M + \log_a N$, we can replace the laborious product by an easy addition.

[Example 11.8] $x = 123111$, $y = 9663189$ are given. Find the approximate value of xy using a logarithmic table in the previous slide.

Ans.

$$x = 1.23111 \times 10^5 \quad \therefore \quad \log_{10} x = \log_{10} 1.23111 + \log_{10} 10^5 \approx \log_{10} 1.23 + 5$$

$$y = 9.663189 \times 10^6 \quad \therefore \quad \log_{10} y = \log_{10} 9.663189 + \log_{10} 10^6 = \log_{10} 9.66 + 6$$

From table $\log_{10} 1.23 = 0.0899$, $\log_{10} 9.66 = 0.9850$

Therefore

$$\log_{10} xy = \log_{10} x + \log_{10} y \approx (\log_{10} 1.23 + 5) + (\log_{10} 9.66 + 6)$$

$$\approx 5.0899 + 6.9850 \approx 12 + 0.0749 \approx 12 + \log_{10} 1.19 = \log_{10}(1.19 \times 10^{12})$$

$$\therefore \quad xy \approx 1.19 \times 10^{12}$$

Slide Rule

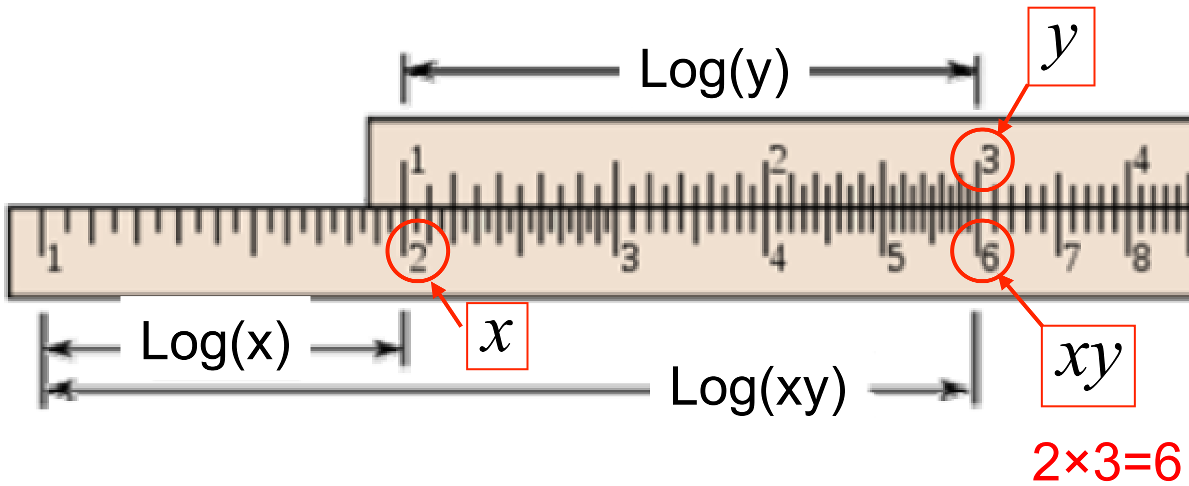


Fig.1 Principle

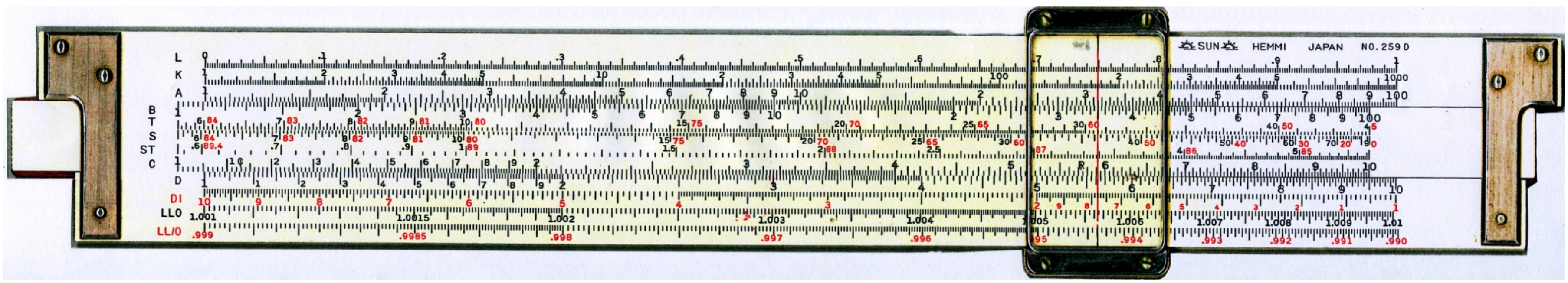


Fig.2 Overview

Example

[Example 11.7] When we keep 1 million yen in the bank with the rate of interest 5%, how much can we get after 5 years? Use the table in the former slide.

Ans.

$$y = 1000000 \times 1.05^5$$

$$\log_{10} y = 6 + \log_{10} 1.05^5 = 6 + 5 \times \log_{10} 1.05 = 6 + 5 \times 0.0212 = 6 + 0.106$$

From the table $0.106 \approx \log_{10} 1.28$

$$\log_{10} y \approx 6 + \log_{10} 1.28 = \log_{10}(1.28 \times 10^6) = \log_{10}(128 \times 10^4)$$

We can get about 1.28 million yen

The precise value is $1000000 \times 1.05^5 = 1276281.5$

Exercise

[Ex.11.4] Let $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$

Find the values of (1) $\log_{10} 125,$ (2) $\log_2 5$

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.11.4] Let $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$

Find the value of (1) $\log_{10} 125$, (2) $\log_2 5$

(1) $$\log_{10} 125 = \log_{10} 5^3 = 3\log_{10} 5 = 3\log_{10}(10/2) = 3(1 - \log_{10} 2)$$
$$= 3 \times (1 - 0.3010) = 2.0970$$

(2)
$$\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = \frac{1 - \log_{10} 2}{\log_{10} 2} = \frac{1 - 0.3010}{0.3010} \approx 2.322$$