Course I

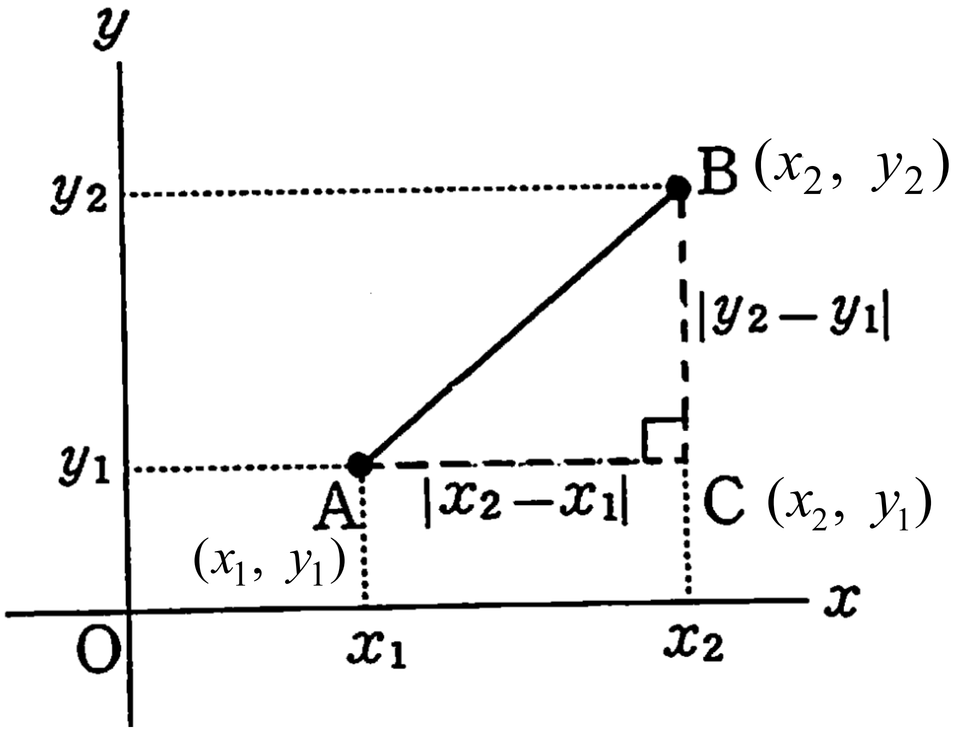
NAGOYA UNIVERSITY

Lesson 14
Graphs and Equations (I)

14A

- **Distance Between Two Points**
- **Division of a Line Segment**
- **Straight Lines**

Distance Formula



Distance Formula

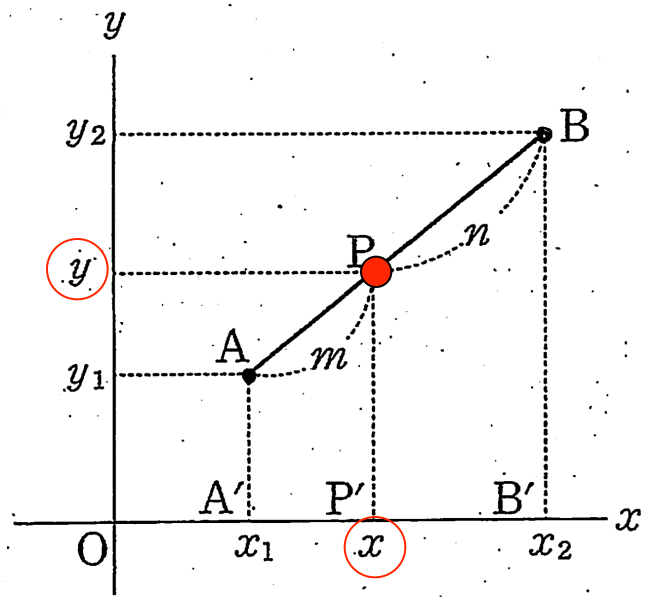
When two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are given, the distance between these points is given by

$$\text{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Division of a Line Segment

Two points $A(x_1,\ y_1)$ and $B(x_2,\ y_2)$ are given.

Internal Division Point



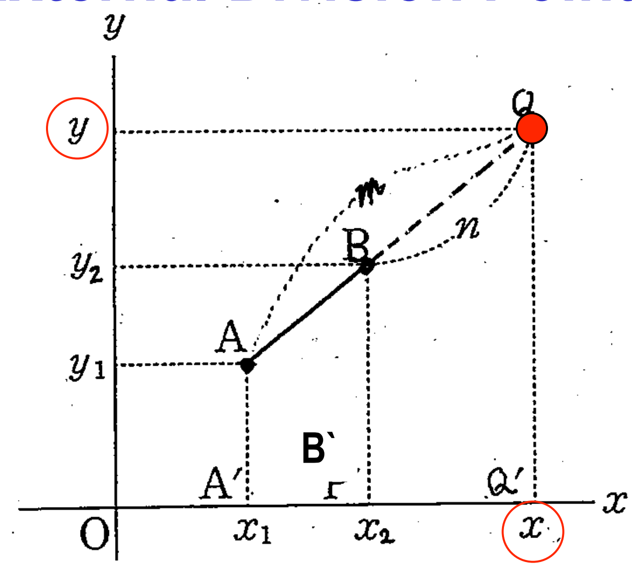
$$(x - x_1):(x_2 - x) = m:n$$

Therefore

$$x = \frac{nx_1 + mx_2}{m+n}$$

$$y = \frac{ny_1 + my_2}{m+n}$$

External Division Point



$$(x - x_1):(x - x_2) = m:n$$

Therefore

$$x = \frac{-nx_1 + mx_2}{m-n}$$

$$y = \frac{-ny_1 + my_2}{m-n}$$

Example

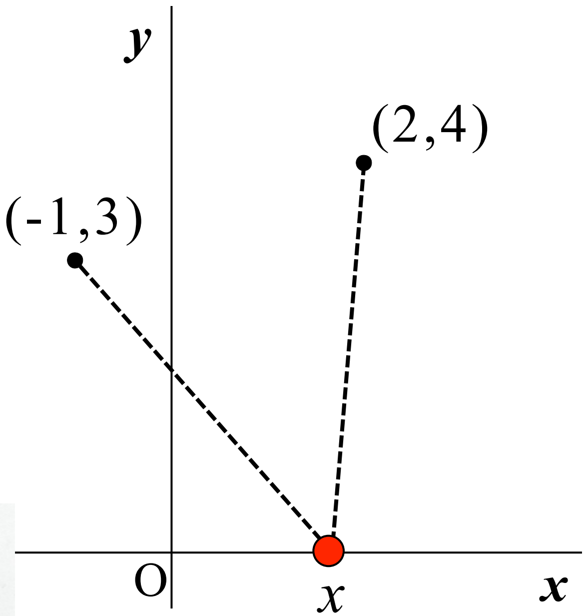
[Example 14.1] Find the point C which is located on the x-axis and has equal distances from points A(-1, 3) and B(2, 4).

Ans.

Let the coordinates of point C be $(x, 0)$.

Then

$$\sqrt{(x+1)^2+(0-3)^2}=\sqrt{(x-2)^2+(0-4)^2}$$



Square both sides

$$(x+1)^2+9=(x-2)^2+16$$

Therefore,

$$x=\frac{5}{3}$$

That was too easy!

Straight Line

General form of a straight line

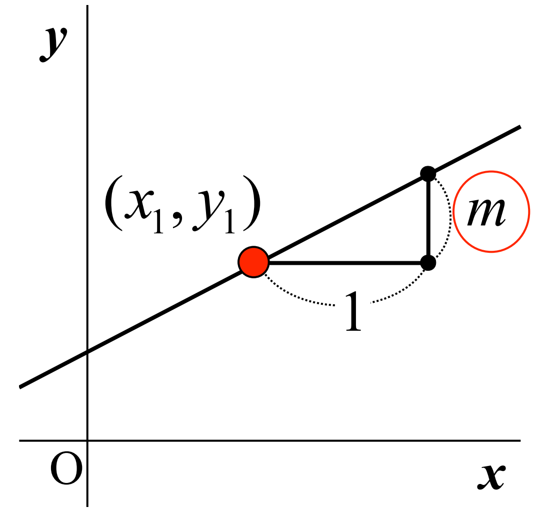
$$ax + by + c = 0$$

Various Forms

(1) Point-slope form

From point $(x_1,\ y_1)$ and slope m

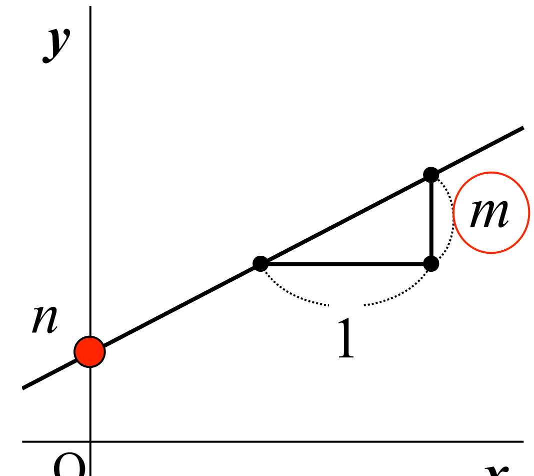
$$m = \frac{y - y_1}{x - x_1} \quad \rightarrow \quad \boxed{y - y_1 = m(x - x_1)}$$



(2) Slope-intercept form

From $y-$ intercept $(0,\ n)$ and slope m

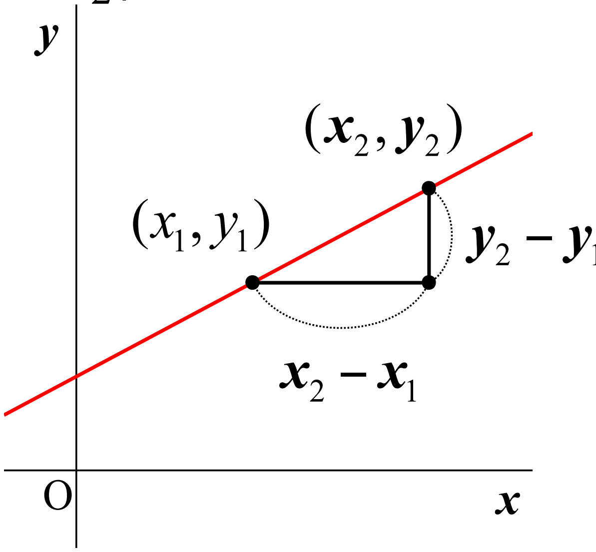
$$m = \frac{y - n}{x - 0} \quad \rightarrow \quad \boxed{y = mx + n}$$



(3) Line through two points (x_1, y_1) and (x_2, y_2)

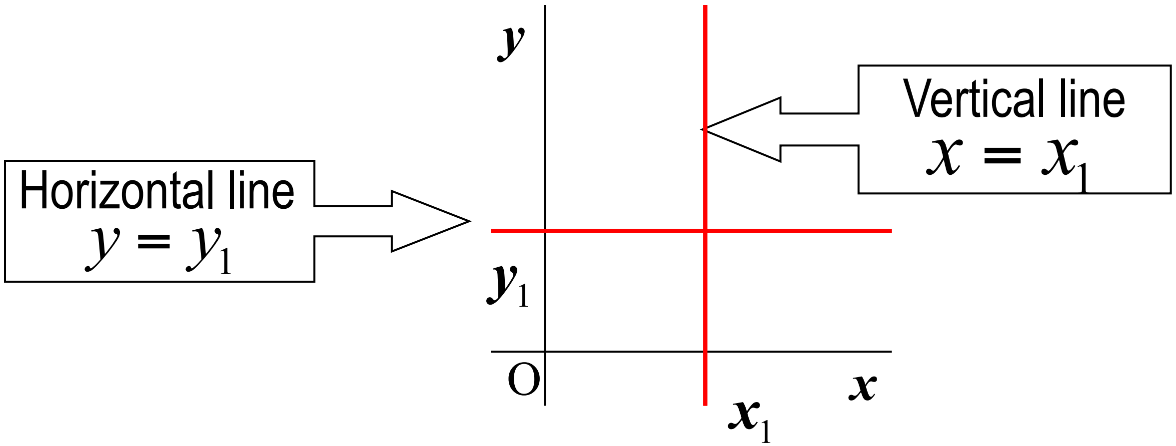
* $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$ (when $x_1 \neq x_2$)

Slope $$m = \frac{y_2 - y_1}{x_2 - x_1}$$



* $$x = x_1$$ (when $x_1 = x_2$)

(This represents a vertical line.)

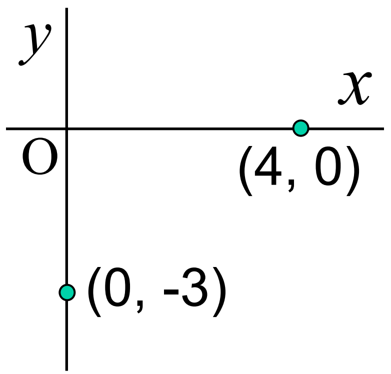


Example (Intercept-Intercept Form)

[Example 14.2] Find the expression of the straight line which passes two points (4, 0) and (0, -3).

Ans. From the expression mentioned above, we have

$$y - 0 = \frac{-3 - 0}{0 - 4}(x - 4)$$



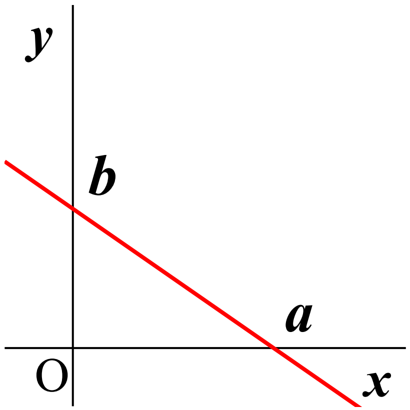
Then we have

$$3x - 4y = 12 \quad \text{or} \quad \frac{x}{4} + \frac{y}{(-3)} = 1$$

(4) Intercept-intercept form

When a straight line intersects the x-axis at a and the y-axis at b, it is expressed as

$$\frac{x}{a} + \frac{y}{b} = 1$$



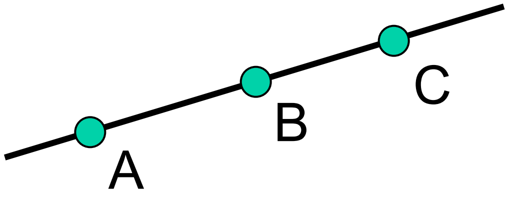
Exercise

[Exercise 14.1] Three points A(-1, 1), B $(3,\ a)$ and C $(a+3,\ 7)$ are on the straight line l in this order. Find the value of a and the equation of l in the following steps.

(1) Assume the line as $y = mx + n$ and find the conditions that points A, B and C are on the line.

(2) Find which satisfy these conditions.

(3) Select appropriate values.



Ans.

Pause the video and solve the problem by yourself.

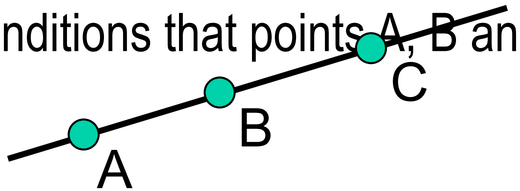
Exercise

[Exercise 14.1] Three points A(-1, 1), B $(3,\ a)$ and C $(a+3,\ 7)$ are on the straight line l in this order. Find the value of a and the equation of l in the following steps.

(1) Assume the line as $y = mx + n$ and find the conditions that points A, B and C are on the line.

(2) Find which satisfy these conditions.

(3) Select appropriate values.



Ans. (1) Because the line l passes points A, B, and C, we have

$$1 = -m + n \quad \text{(i)}, \qquad a = 3m + n \quad \text{(ii)}, \qquad 7 = m(a+3) + n \quad \text{(iii)}$$

(2) From (ii)-(i) and (iii)-(i), we have $a = 4m + 1$ and $6 = (a+4)m$.

By eliminating a, we have $(m+2)(4m-3) = 0$

Therefore, we obtain $(m,\ a,\ n) = (-2, -7, -1)$ and $(\frac{3}{4},\ 4,\ \frac{7}{4})$.

(3) Considering the order, we have $a + 3 > 3$

Therefore, the latter satisfies the request of the problem.

Finally, we obtain $a = 4$ and $y = \frac{3}{4}x + \frac{7}{4}$

Lesson 14
Graphs and Equations (I)

14B

- Parallel Straight Lines
- Perpendicular Straight Lines
- Reflected Images
- Distance to a Straight Line

Parallel Straight Lines

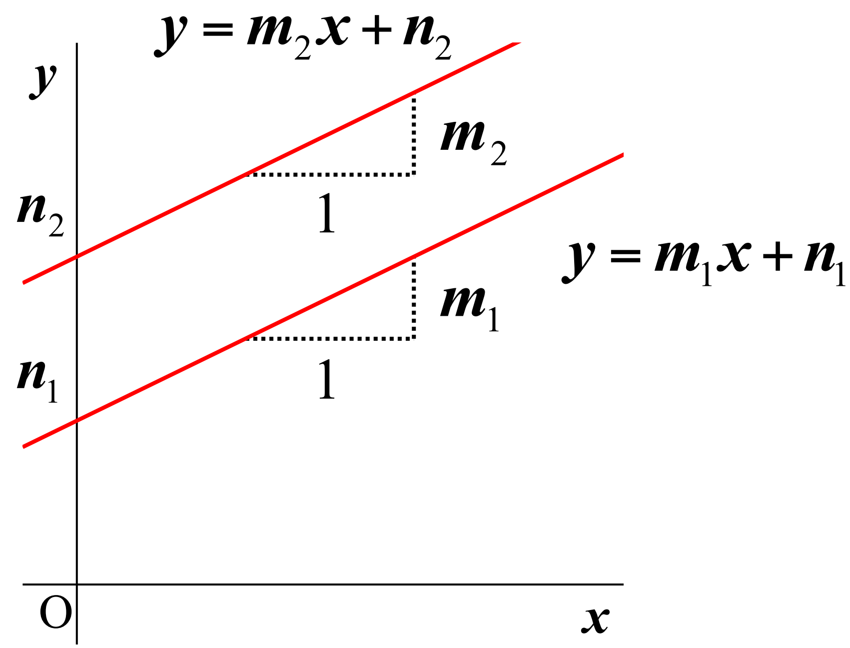
For two straight lines

$$y = m_1 x + n_1$$

$$y = m_2 x + n_2$$

Parallel Lines Condition

$$m_1 = m_2$$



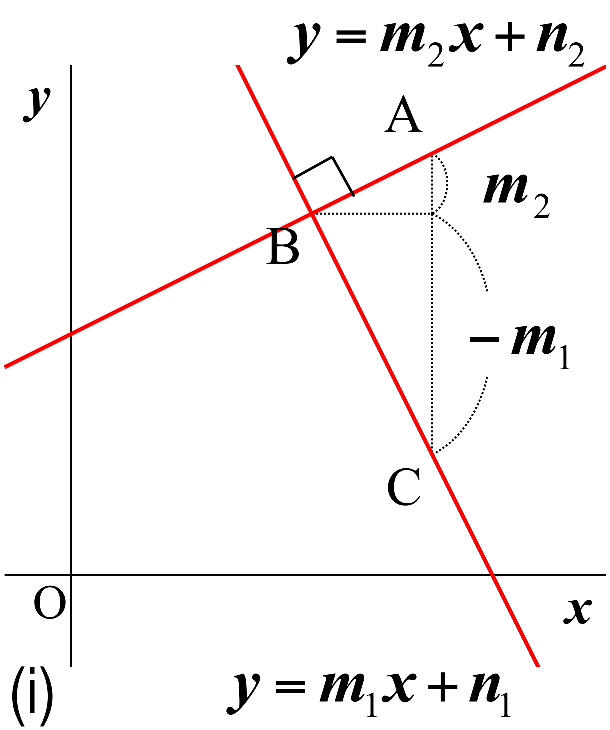
Perpendicular Straight Lines

For two straight lines

$$y = m_1 x + n_1 \qquad y = m_2 x + n_2$$

Perpendicular Lines Condition

$$m_1 m_2 = -1$$



,

Proof:

For ΔABC: AB^2+BC^2=AC^2 (i)

For ΔABD: AB^2=BD^2+AD^2= $1 + m_2^2$ (ii)

For ΔBCD: BC^2= $1 + (-m_1)^2$ (iii)

Therefore, from (i)

$$1 + m_2^2 + 1 + (-m_1)^2 = (m_2 - m_1)^2$$

$$\therefore \; m_1 m_2 = -1$$

[Example 14.3] Find the reflected image of point P(2, 3) about the mirror line $y = 2x + 5$.

Ans. Put the image point be Q $(x_q,\ y_q)$.

Line PQ is perpendicular to the mirror line:

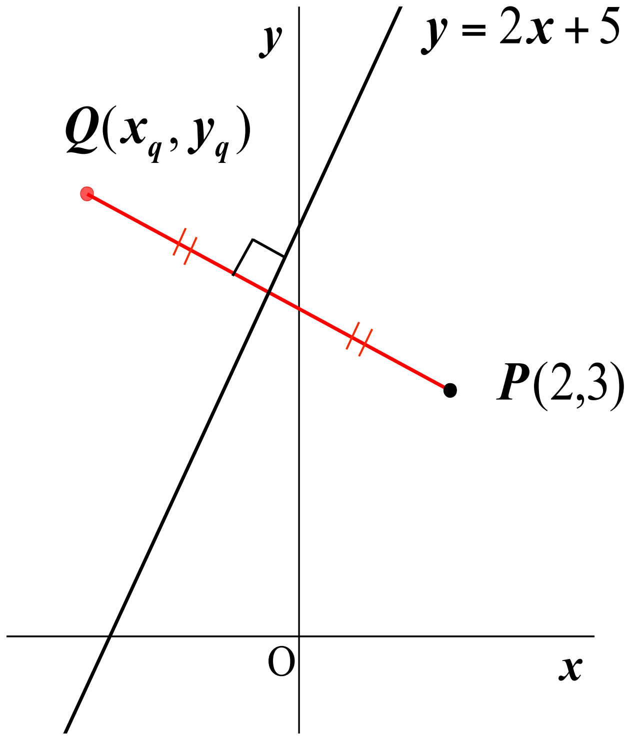
$$2 \cdot \left(\frac{y_q - 3}{x_q - 2} \right) = -1 \qquad \text{(i)}$$

The middle point of segment PQ is on the mirror line:

$$\frac{3 + y_q}{2} = 2 \cdot \left(\frac{2 + x_q}{2} \right) + 5 \qquad \text{(ii)}$$

From (i) and (ii), we have .

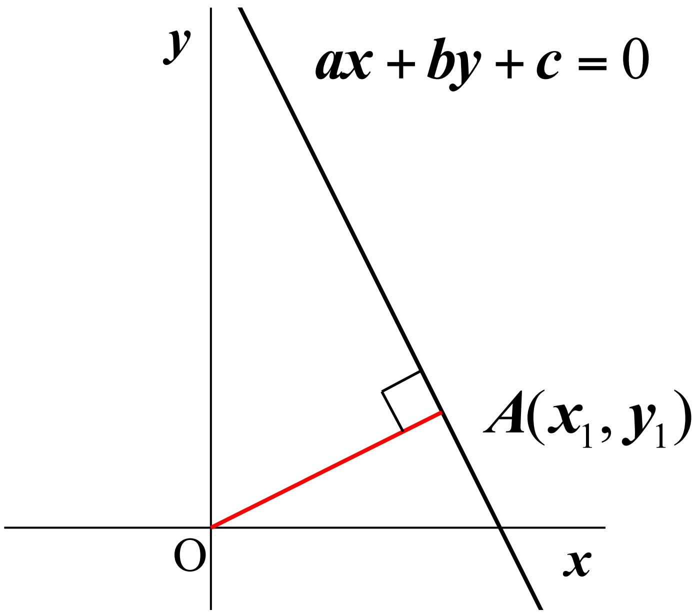
$$x_q = -\frac{14}{5},\ y_q = \frac{27}{5}$$



Distance To the Straight Line

[Example 14.4] Find the shortest distance between the origin and the straight line $ax + by + c = 0$

Ans. This line is rewritten as $y = -\frac{a}{b}x - \frac{c}{b}$.



Point A $(x_1,\ y_1)$ on the given line is nearest point from the origin.

Point A is on the line :

$$ax_1 + by_1 + c = 0 \qquad \text{(i)}$$

Segment OA is perpendicular to the line :

$$\left(-\frac{a}{b}\right) \cdot \frac{y_1}{x_1} = -1 \qquad \text{(ii)}$$

From (i) and (ii), we obtain $x_1 = \frac{-ac}{a^2 + b^2}, \quad y = \frac{-bc}{a^2 + b^2}$

Therefore, the distance is

$$d = \sqrt{x_1^2 + y_1^2} = \frac{|c|}{\sqrt{a^2 + b^2}}$$

Exercise

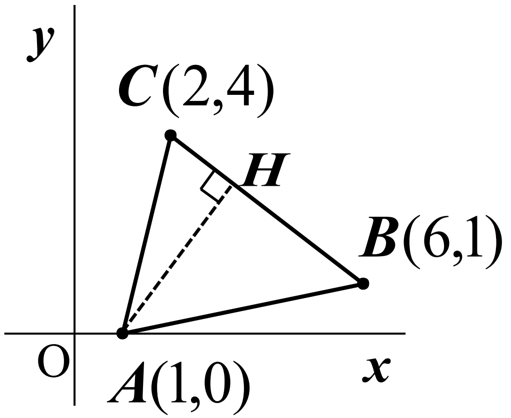
[Exercise 14.2] Find the area of the triangle in the following steps.

(1) Shift the triangle by 1 leftward and find the coordinates of the new position of points A, B, and C.

(2) Find the length AH using the result of Example 14.4.

(3) Find the length AH.

(4) Find the area of triangle ABC.



Ans.

Pause the video and solve the problem by yourself.

Exercise

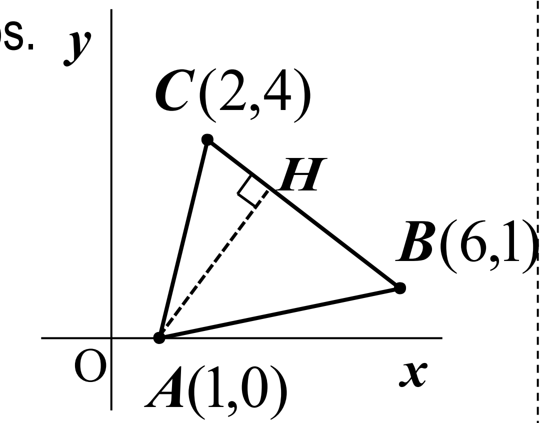
[Exercise 14.2] Find the area of the triangle in the following steps.

(1) Shift the triangle by 1 leftward and find the coordinates of the new position of points A, B, and C.

(1) Find the length AH using the result of Example 14.4.

(3) Find the length AH.

(4) Find the area of triangle ABC.

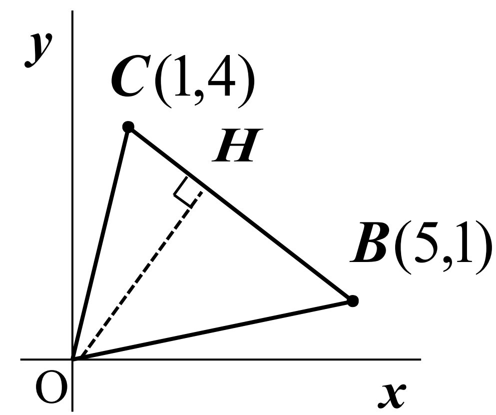


Ans. (1) The new positions are A(0, 0), B(5, 1) and C(1, 4).

(2) The equation for the straight line CB is

$$y-4=\frac{1-4}{5-1}(x-1) \qquad \therefore\ 3x+4y-19=0$$

(3) From Ex.14.4, length OH is $d=\frac{|-19|}{\sqrt{3^2+4^2}}=\frac{19}{5}$



(4) BC= $\sqrt{(5-1)^2+(1-4)^2}=5$

Therefore, the area is

$$\frac{1}{2}\times\frac{19}{5}\times 5=\frac{19}{2}$$