NAGOYA UNIVERSITY

Course II : Calculus

What is Calculus ?

Calculus is a branch of mathematics.

• functions, • limits, • derivatives, • integrals, • power series

Calculus is the study of change.

cf. • *Geometry* is the study of shape.

• *Algebra* is the study of operation.

Calculus is a gateway to advanced mathematics.

• We must study and understand completely.

Calculus has wide applications in

• science, • engineering, •economics, •biology

Calculus has two branches

• differential calculus, • integral calculus,

Contents

Lesson 1
Limit of Functions and Derivatives

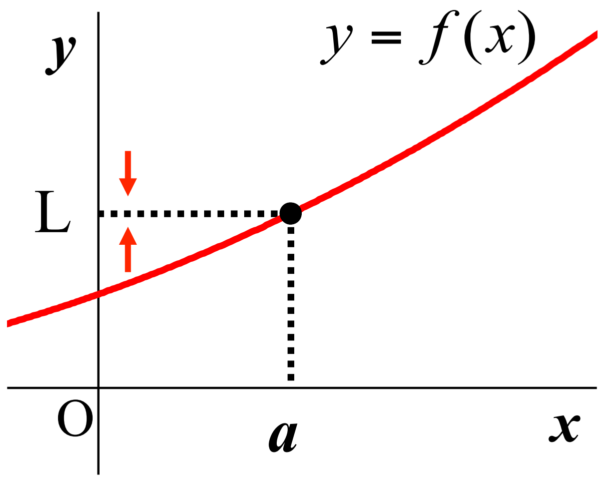
1A

- Limit of a function

Definition of a Limit

If a function $f(x)$ can be made to be as close to L as desired by making x sufficiently close to a, we say that
"the limit of $f(x)$, as x approaches a, is L"
and we write as follows

$$\lim_{x \to a} f(x) = L$$

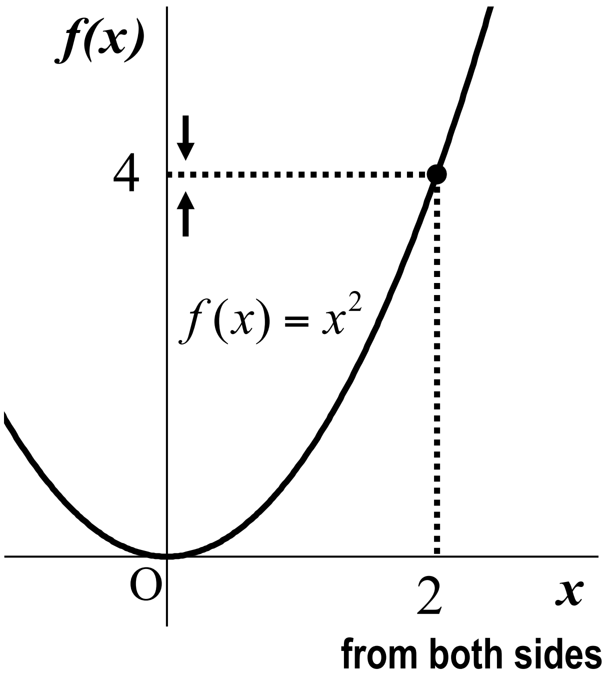


We can also write $f(x) \to L$ as $x \to a$
and read "$f(x)$ approaches L as x approaches a.

Limit of a Function

[Example] $f(x) = x^2$

x	$f(x)$	x	$f(x)$
1.0	1.000000	3.0	9.000000
1.5	2.250000	2.5	6.250000
1.8	3.240000	2.2	4.840000
1.9	3.610000	2.1	4.410000
1.99	3.960100	2.01	4.040100
1.999	3.996001	2.001	4.004001



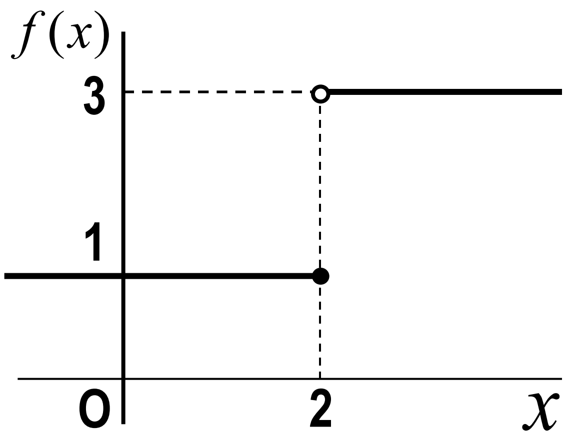
The limit of $f(x) = x^2$ as x approaches 2 is 4

Several Comments about the Limit

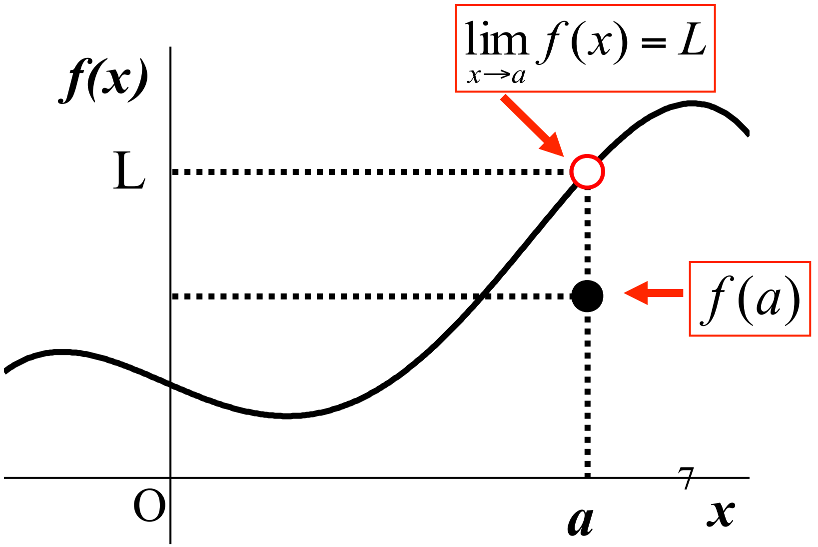
○ For the limit of a function to exists, the left-hand and right-hand limits must be equal, that is

$$\lim_{x \to a-} f(x) = L \quad \text{and} \quad \lim_{x \to a+} f(x) = L$$

Limit does not exit at $x = 2$



○ $\lim_{x \to a} f(x) = L$ is not always equal to $f(a)$



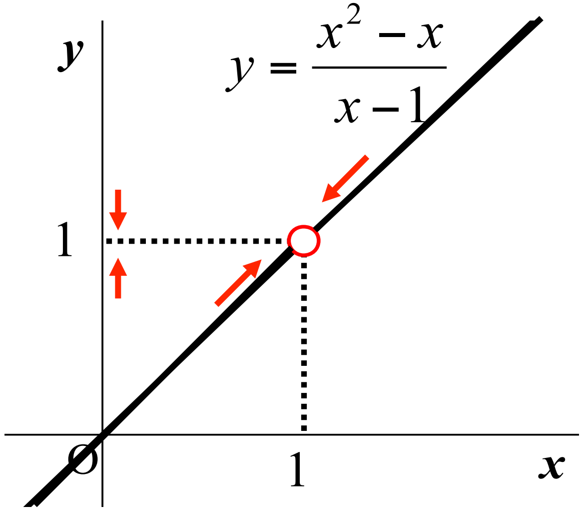
Example

Example 1.1 Find the limit value of the following function.

$$\lim_{x \to 1} \frac{x^2 - x}{x - 1}$$

Ans.

$$\lim_{x \to 1} \frac{x^2 - x}{x - 1} = \lim_{x \to 1} \frac{x(x-1)}{x - 1} = \lim_{x \to 1} x = 1$$



○ Even if the function has not a value at $x = a$, the limit may exist .

Indeterminate Form

The forms $0^0, \frac{0}{0}, 1^\infty, \infty - \infty, \frac{\infty}{\infty}, 0 \times \infty, \infty^0$, etc. are called indeterminate forms because they do not give enough information to determine values.

Example

Example 1.2 Find the limit value of the following function.

$$\lim_{x \to 1} \frac{\sqrt{x-1}-1}{x}$$

Ans.

$$\lim_{x \to 1} \frac{\sqrt{x+1}-1}{x} = \lim_{x \to 1} \frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{x\left(\sqrt{x+1}+1\right)} = \lim_{x \to 1} \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)}$$

$$= \lim_{x \to 1} \frac{1}{\left(\sqrt{x+1}+1\right)} = \frac{1}{2}$$

Means to find a limit of an Indeterminate Form 0/0

(1) Case of Polinomial $\longrightarrow$ Factor them

(2) Case of Irrational Function $\longrightarrow$ Multiply the conjugate

Example

Example 1.3 Determine the values of a and b so that the following expression holds.

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2+ax+b}=1$$

Ans.

When $x\to 1$, then $x^2+x-2\to 0$ and $x^2+ax+b\to 1+a+b$

In order for limit to exist, $1+a+b$ must be zero. $\therefore\ b=-a-1$

Substituting this, we have

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2+ax+b}=\lim_{x\to 1}\frac{(x-1)(x+2)}{(x-1)(x+a+1)}=\lim_{x\to 1}\frac{(x+2)}{(x+a+1)}=\frac{3}{a+2}$$

Therefore,

$$\frac{3}{a+2}=1 \quad \therefore \quad a=1 \text{ and } b=-2$$

Exercise

[Exercise 1.1] Determine the values of a and b so that the following relationship holds.

$$\lim_{x \to 1} \frac{ax^2 + bx + 1}{x - 1} = 3$$

Ans.

Pause the video and try to solve by yourself

Answer to the Exercise

[Exercise 1.1] Determine the values of a and b so that the following relationship holds.

$$\lim_{x \to 1} \frac{ax^2 + bx + 1}{x - 1} = 3$$

Ans.

When $x \to 1$, then $x - 1 \to 0$ and $ax^2 + bx + 1 \to a + b + 1$

In order to exist a limit value 1, $a + b + 1 = 0$ $\therefore\ b = -a - 1$

Substituting this, we have

$$\lim_{x \to 1} \frac{ax^2 + bx + 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(ax-1)}{x - 1} = \lim_{x \to 1}(ax - 1) = a - 1$$

Therefore,

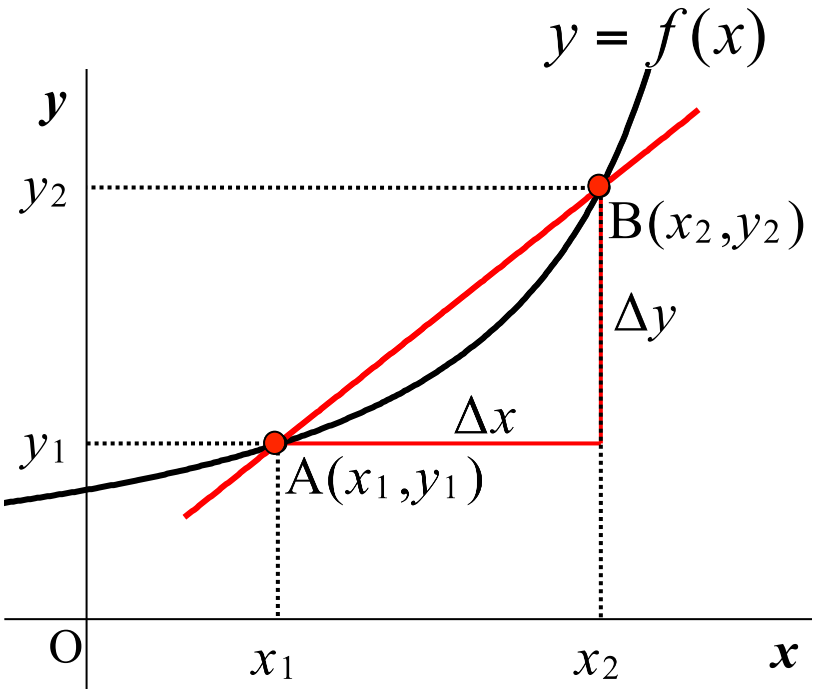
$a - 1 = 3$. Namely, $a = 4$ and $b = -5$

Lesson 1
Limit of Functions and Derivatives

1B

- Derivatives of Functions

Average Rate of Change



Increments

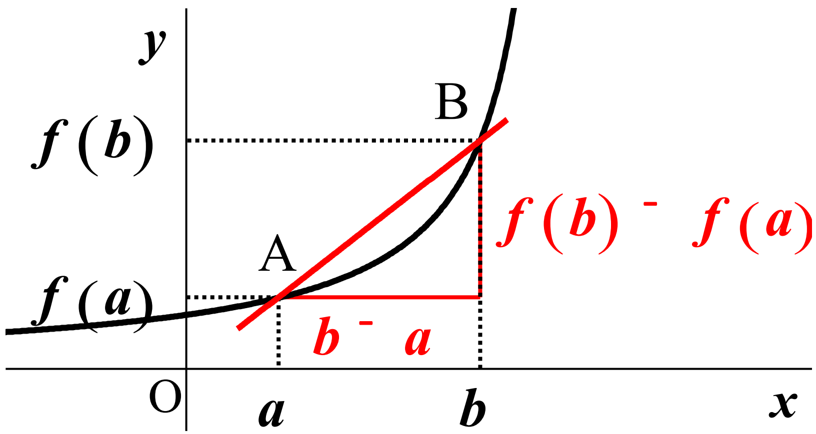
$$\Delta x = x_2 - x_1$$

$$\Delta y = y_2 - y_1$$

The slope $\dfrac{\Delta y}{\Delta x}$

Average Rate of Change

$$\frac{\Delta y}{\Delta x} \quad \text{or} \quad \frac{f(b) - f(a)}{b - a}$$

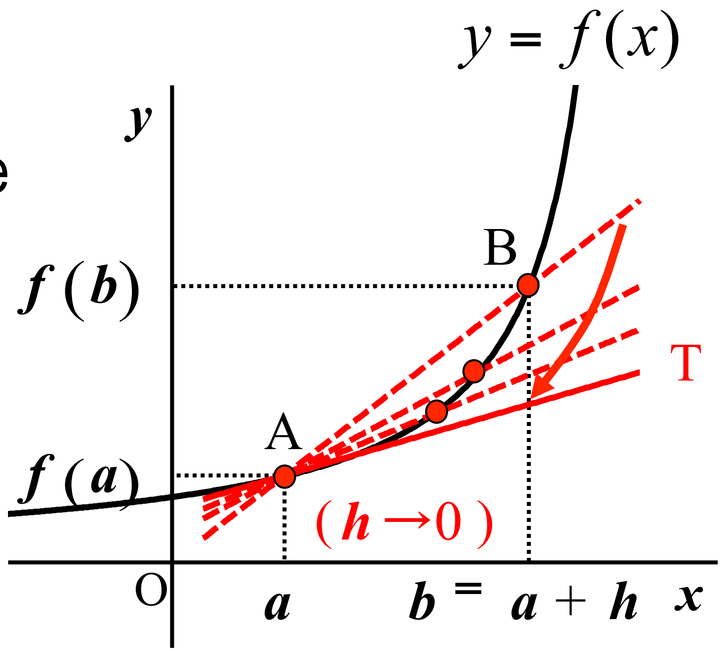


Derivative

The slope at point A (the tangent line T) can be obtained by making point B approach point A .

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a} = f'(a)$$

This is called the derivative of $f(x)$ at a



Putting $b = a + h$, we also have

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a)$$

That makes sense!

How to Find the Derivative

[Example 1-4] About the function $f(x) = x^2$

(1) Find the average rate of change between $x = 1$ and $x = 2$.

(2) Find the instantaneous rate of change at $x = a$.

(3) Find the point where the instantaneous rate of change is equal to the average rate of change between $x = 1$ and $x = 2$.

Ans. (1) $\dfrac{f(2) - f(1)}{2 - 1} = 4 - 1 = 3$

(2) $f'(a) = \lim_{h \to 0} \dfrac{(a+h)^2 - a^2}{h} = \lim_{h \to 0}(2a + h)$

$= 2a$

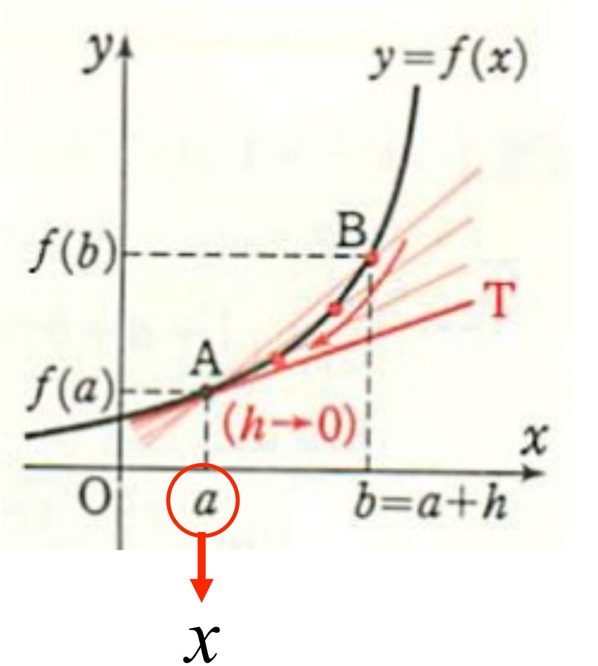
(3) Using the results of (1) and (2), we put $2a = 3$

$$\therefore \quad a = \frac{3}{2}$$

Derivative as a Function

Let the number a varies and replace it by x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$f'(x)$ is called the derivative of $f(x)$
or the derivative function of $f(x)$
(because it has been “derived” from $f(x)$.)

Alternative notation

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

[note]

The definition $\frac{dy}{dx}$ is read as : “*the derivative with respect to* x”, “dy by dx”, “dy over dx” or simply “$dy\ dx$”

How to Find a Derivative Function

[Example 1-4] Find the derivative function of

(1) $f(x) = x$ (2) $f(x) = x^2$ (3) $f(x) = x^3.$

Ans. Definition $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(1) $$f'(x) = \lim_{h \to 0} \frac{(x+h) - x}{h} = \lim_{h \to 0}(1) = 1$$

(2) $$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0}(2x + h) = 2x$$

(3) $$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0}(3x^2 + 3xh + h^2) = 3x^2$$

Formula

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Higher Derivatives

Since $f'(x)$ is a function, it also has its own derivative which is denoted by

$$\frac{d^2y}{dx^2} = f''(x) = f^{(2)}(x)$$: The second derivative function

We can continue

$$\frac{d^3y}{dx^3} = f'''(x) = f^{(3)}(x)$$: The third derivative function

The process of finding a derivative function is called differentiation.

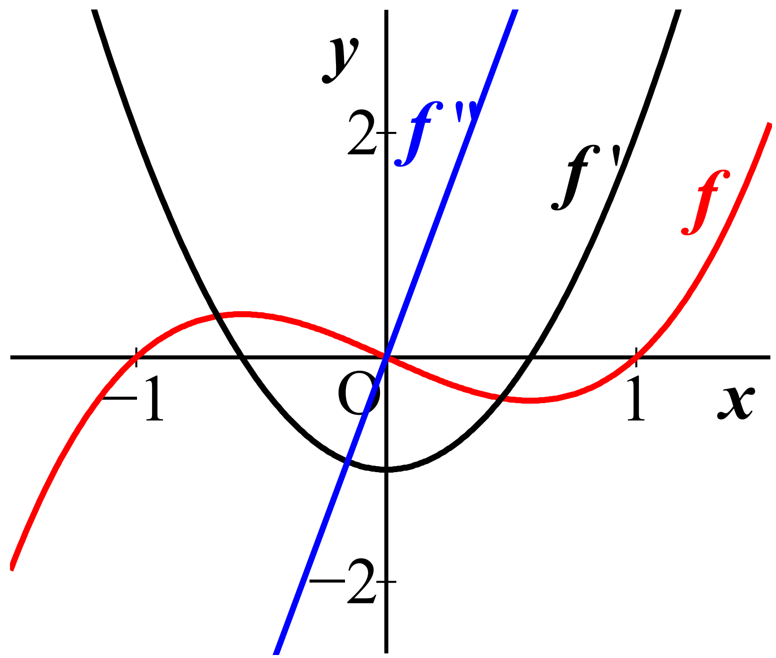
Example

[Example 1-5] If $f(x) = x^3 - x$, find and interpret $f''(x)$

Using the formula $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$, we get

$$f'(x) = 3x^2 - 1$$

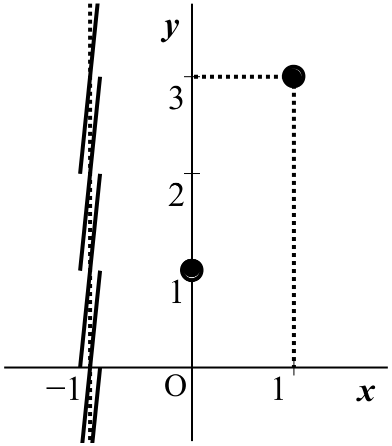
$$f''(x) = 6x$$



These derivatives are illustrated in the Right-hand side.

- $f''(x)$ is the slope of the curve $y = f'(x)$
- $f''(x)$ is the rate of change of $y = f'(x)$

[Exercise 1.2] Function $f(x) = x^3 + ax^2 + bx + c$ Satisfy the conditions $f(1) = 3$, $f(0) = 1$ and $f'(-1) = 16$. Find the constants a, b and c .



Ans.

Pause the video and try to solve by yourself

Answer to the Exercise

[Exercise 1.2] Function $f(x) = x^3 + ax^2 + bx + c$
Satisfy the conditions $f(1) = 3$, $f(0) = 1$ and $f'(-1) = 16$. Find the constants a, b and c.

Ans.

The derivative function is

$$f'(x) = 3x^2 + 2ax + b$$

Given condition

$$f(1) = 1 + a + b + c = 3$$

$$f(0) = c = 1$$

$$f'(-1) = 3 - 2a + b = 16$$

From these equations

$$a = -4,\ b = 5,\ c = 1$$