Course II

NAGOYA UNIVERSITY

Lesson 2
Derivative and Graphs

2A

- **Tangent Lines to a Graph of a Function**

Equation for a Tangent Line

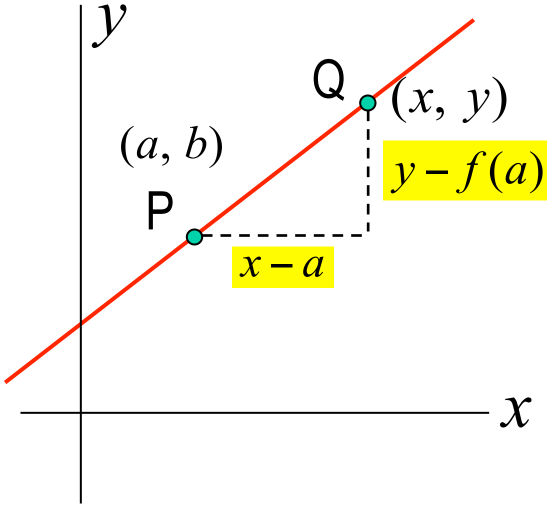
The straight line

The straight line through P(a, b) with slope m

$$m = \frac{y-b}{x-a}$$

Therefore

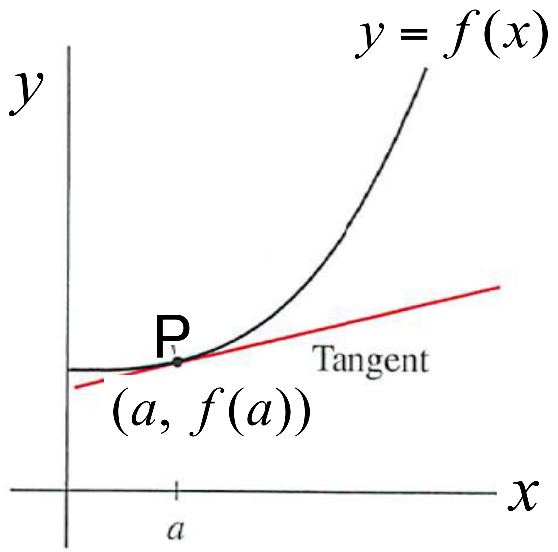
$$\boxed{y - b = m(x-a)}$$



The tangent line

The tangent line to the graph of $y = f(x)$ at point $(a,\ f(a))$ is

$$\boxed{y - f(a) = f'(a)(x-a)}$$



Examples

[Examples 2-1] Find an equation of the tangent line to the graph $y = f(x) = x^2 + 1$ at $x = 1$.

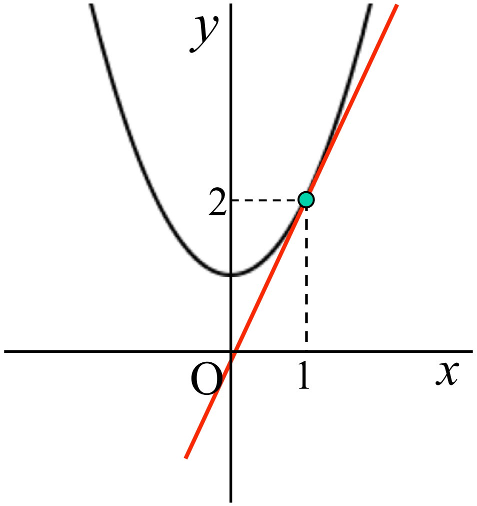
Ans. The y-coordinate at $x = 1$ is $f(1) = 2$

Since $f'(x) = 2x$ we have $f'(1) = 2$

Therefore,

$$y - 2 = 2(x - 1)$$

$$\therefore \quad y = 2x$$



Examples

[Examples 2-2] Find the equations and contact points of the tangent lines which contact with $y = x^2 + 4$ and pass (1, 1) .

Ans.

Let the contact point be $(a,\ a^2+4)$

The derivative function $f'(x) = 2x$

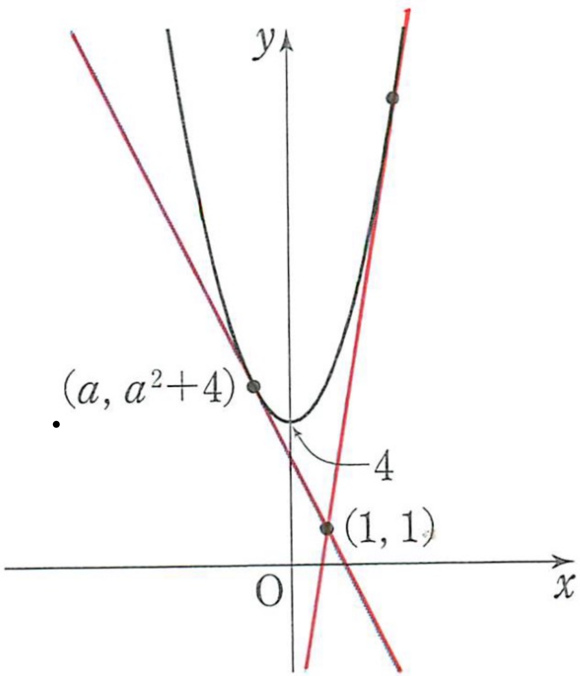
The tangent line is

$$y - (a^2+4) = 2a(x-a)$$

Since this line pass (1,1)

$$1 - (a^2+4) = 2a(1-a)$$

$$\therefore a^2 - 2a - 3 = 0,\ \therefore a = -1,\ 3$$



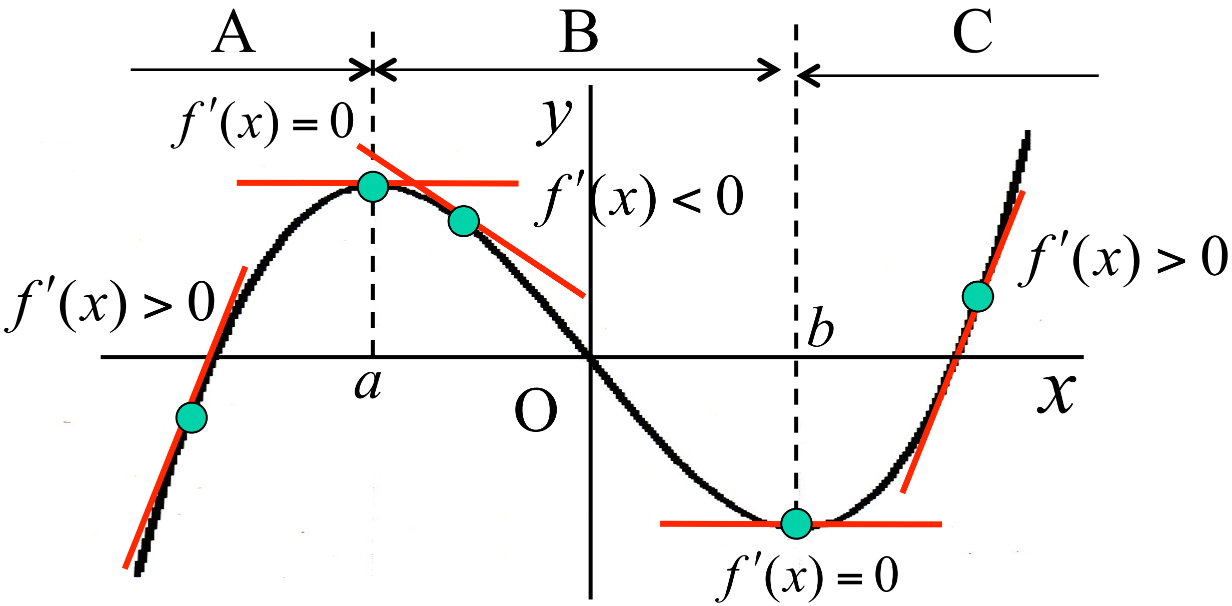
When $a = -1$

contact point $(-1,\ 5)$, tangent line $y = -2x + 3$

When $a = 3$

contact point $(3,\ 13)$, tangent line $y = -6x + 5$

Behavior of graphs based on its derivatives



In open domains A and C :

$f'(x) < 0 \longrightarrow$ $f(x)$ is increasing.

In open domain B :

$f'(x) > 0 \longrightarrow$ $f(x)$ is decreasing.

Example

[Examples 2-3] Investigate the change of the function $y = x^3 + 3x^2 - 2$. and illustrate the graph.

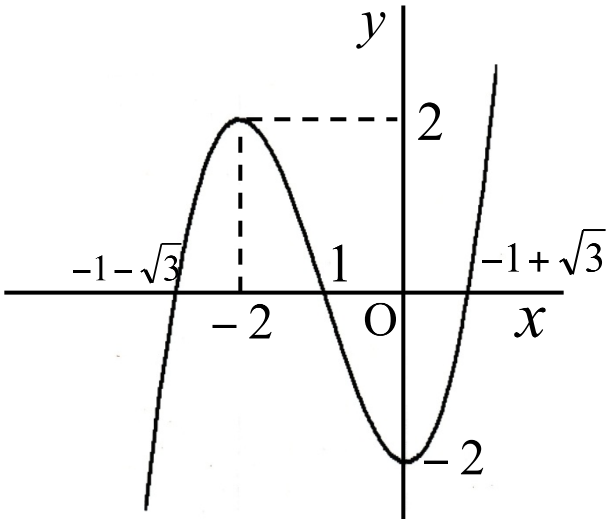
Derivative function $y' = 3x^2 + 6x$

Horizontal tangent line

$$y' = 3x(x+2) = 0 \qquad x = -2,\ 0$$

$$y' > 0 \quad \text{in} \quad x < -2,\ x > 0$$

$$y' < 0 \quad \text{in} \quad -2 < x < 0$$



x	……	-2	……	0	……
y'	+	0	−	0	+
y	↗	2	↘	-2	↗

y-intercept : $(0, 2)$

x-intercepts: $y = x^3 + 3x^2 - 2 = (x+1)(x^2 + 2x - 2) = 0$

$(-1, 0)$, $(-1+\sqrt{3}, 0)$ $(-1-\sqrt{3}, 0)$

Exercise

[Exercise 2-1] About the graph $y = x^3 - 2x^2 - 3x$, answer the following questions.

(1) Find the equation of the tangent line at (-1, 0).

(2) Find the cross-point of the graph and this tangent line.

(Hint : The cross-point of $y = f(x)$ and $y = ax + b$ is given by the roots of $f(x) = ax + b$)

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Exercise 2-1] About the graph $y = x^3 - 2x^2 - 3x$, answer the following questions.

(1) Find the equation of the tangent line at (-1, 0).

(2) Find the cross-point of the graph and this tangent line.

Ans. The derivative function $y' = 3x^2 - 4x - 3$ $\therefore\ f'(-1) = 4$

(1) Tangent line

$$y - 0 = 4(x+1) \qquad \therefore\ y = 4x + 4$$

(2) Cross point of the tangent line and the curve

$$x^3 - 2x^2 - 3x = 4x + 4$$

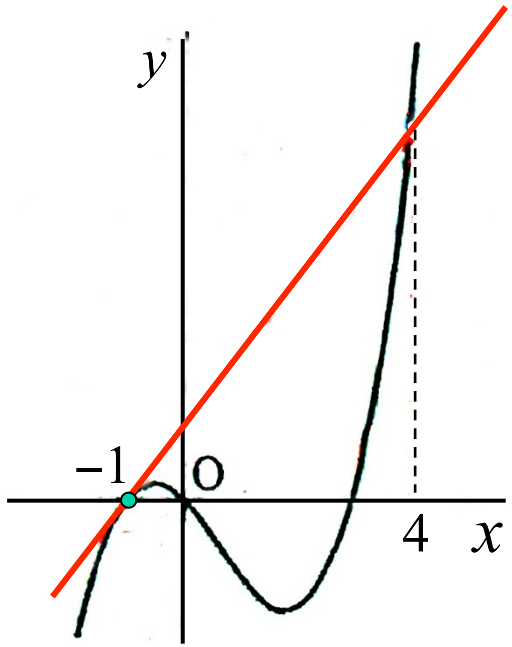
By factoring

$$(x+1)^2(x-4) = 0$$

Therefore,

$$x = 4$$

The cross-point is (4, 20).

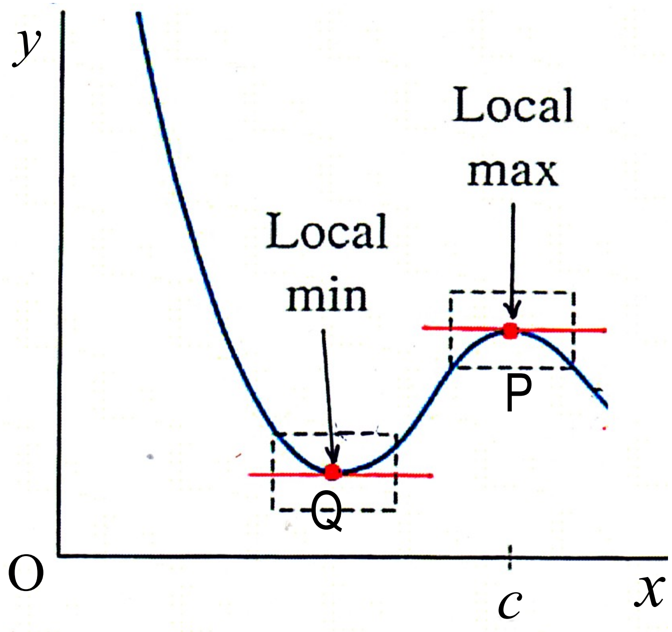


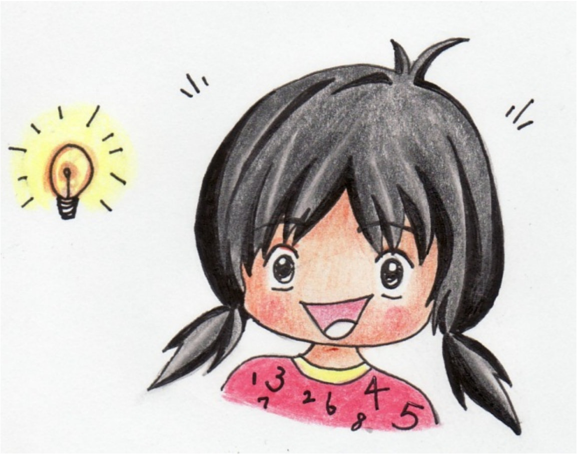
Lesson 2
Derivative and Graphs

2B

- Local Extrema
- Global Extrema
- Second derivative test

Local Exrema





Ahh! That's so easy!

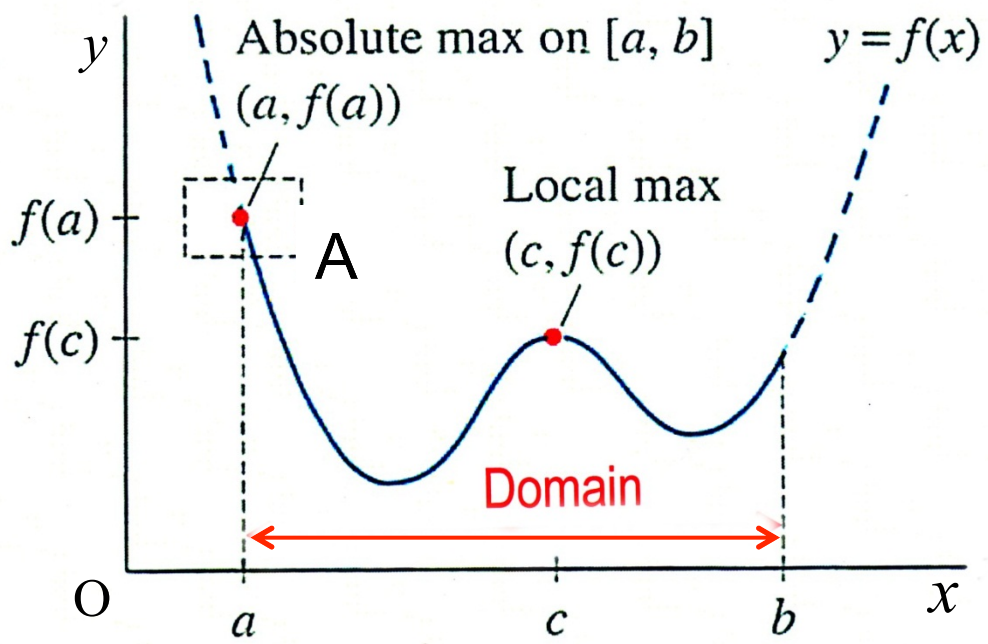
Local Extrema

- The function has a local maximum (local minimum) at $x = c$

  if $f(c)$ is the maximum (minimum) value in a neighborhood around c .

- If $f(c)$ is a local extrema then $f'(c) = 0$

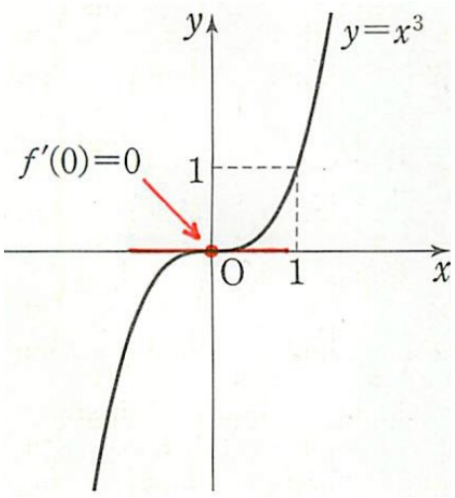
Global Extrema



- Absolute maximum (or global maximum) is
  the maximum value in the whole domain [a, b].
- Absolute minimum (or global minimum) is
  the minimum value in the whole domain [a, b].

Comments

(1) That the slope is zero $f'(c) = 0$ does not necessarily mean that the point is a local max./min. point.



(2) A number c in the domain of $f(x)$ is called a critical point if either $f'(c) = 0$ or $f'(c)$ does not exist.

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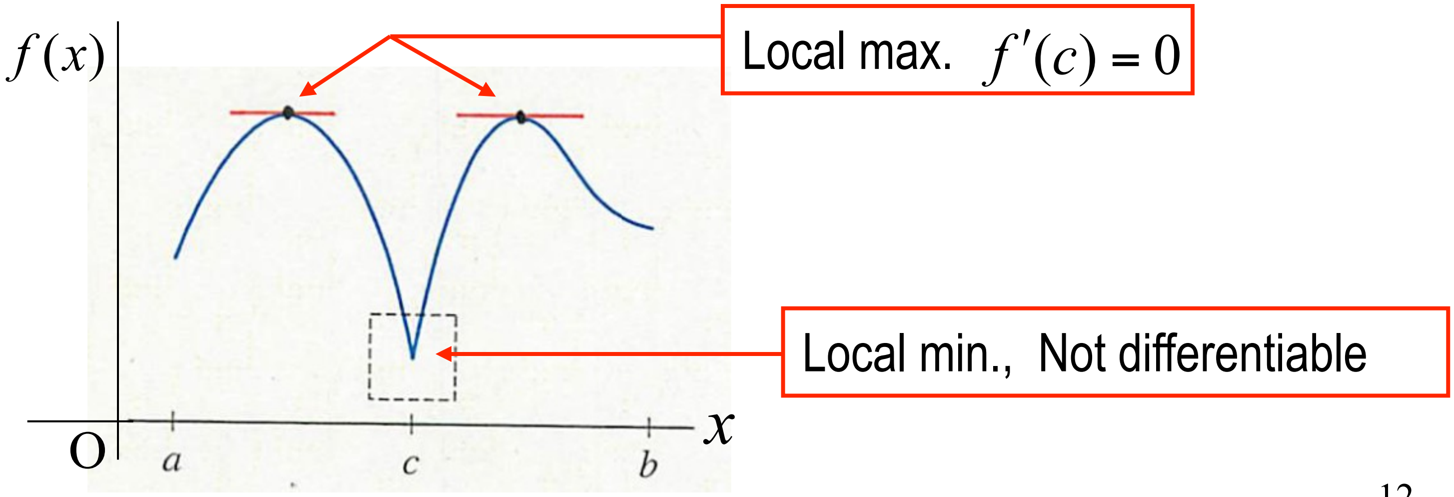


Example

[Examples 2-4] Find the maximum and minimum values of $f(x) = 2x^3 + 3x^2 - 12x + 1$ in the domain $-1 \le x \le 2$.

Ans. (1) Find extrema: $f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1)$

$x = -2,\ 1$ $f(1) = -6$

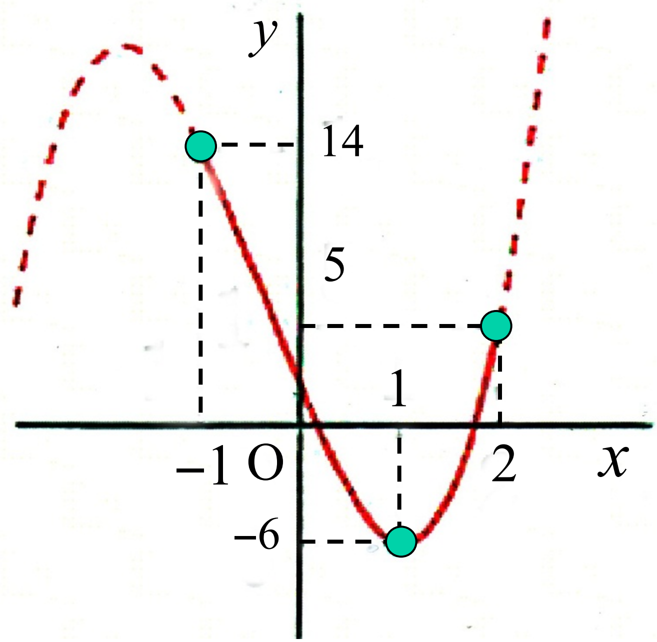
(2) Find the values at the boundaries :

$f(-1) = 14$ $f(2) = 5$

(3) Find y − intercept : $y = f(0) = 1$

(4) Make a table

x	-1	$\cdots$	1	$\cdots$	2
$f'(x)$	$-$	$-$	0	$+$	$+$
$f(x)$	14	↘	Local Min. -6	↗	5



(5) Illustrate the graph

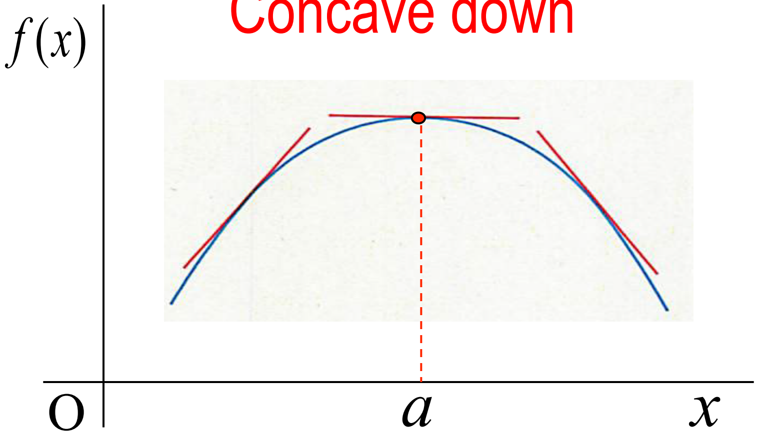
(6) Find the max. and min. values :

Max. value 14 (at x=-1), Min. value -6 (at x=1)

Shape of a graph

Local maximum

Concave down



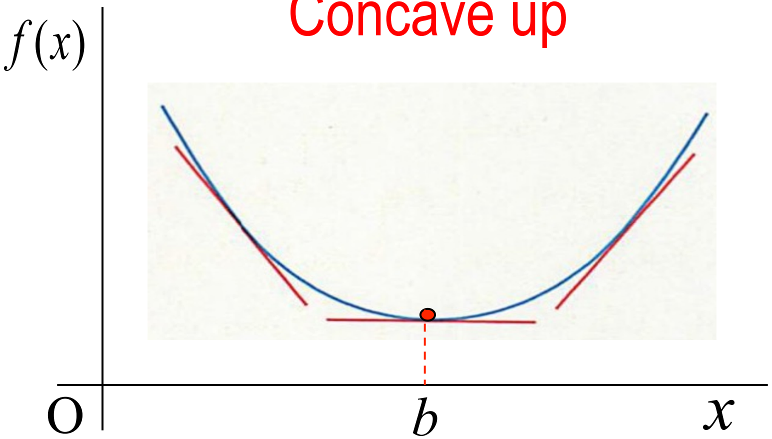
$$f'(x) > 0 \rightarrow f'(a) = 0 \rightarrow f'(x) < 0$$

At $x = a$, slope decreases

$$f''(a) < 0$$

Local minimum

Concave up



$$f'(x) < 0 \rightarrow f'(b) = 0 \rightarrow f'(x) > 0$$

At $x = a$, slope increases

$$f''(b) > 0$$

Second derivative test

Suppose that $f'(c) = 0$ at $x = c$.

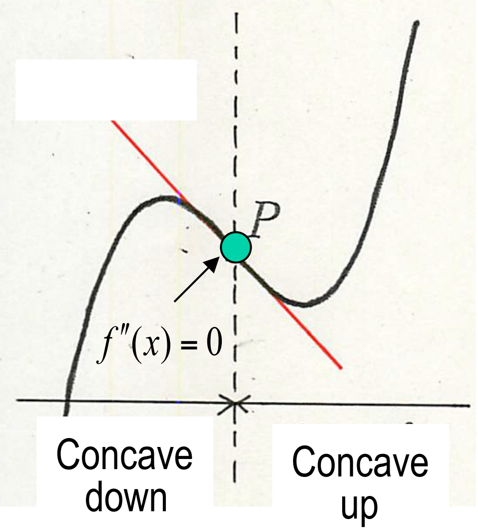
- If $f''(c) > 0$ then $f(x)$ is a local minimum at $x = c$.
- If $f''(c) < 0$ then $f(x)$ is a local maximum at $x = c$.
- If $f''(x) = 0$ then the situation is inconclusive.

$f''(0) = 0$

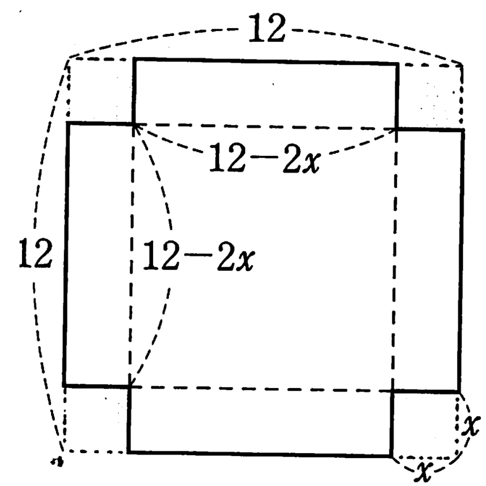
$y = x^4$	L.Min.
$y = -x^4$	L.Max.
$y = x^3$	Inflec.

Inflection Point

The point where the concavity of the graph changes from up to down or vice versa is called an inflection point.



[Exercise 3.2] A square cup was made by cutting the four corners of a thick square paper as shown in the Figure. When dos the volume of this cup becomes maximum ?



Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

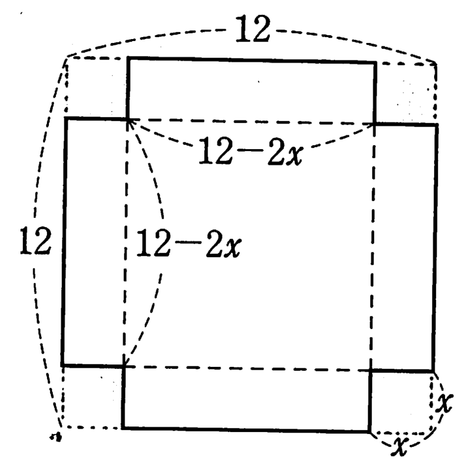
[Exercise 3.2] A square cup was made by cutting the four corners of a thick square paper as shown in the Figure. When dos the volume of this cup becomes maximum ?

Ans. The side length of the squares cut off from the corner : x

Then,

- the volume
- Condition $0 < x < 6$
- The derivative $V' = 12(x-2)(x-6)$

x	0	……	2	……	6
V'		+	0	−	
V		↗	128	↘	



$$V = (12-2x)^2 x = 4(x^3 - 12x^2 + 36x)$$