NAGOYA UNIVERSITY

Lesson 3
Differentiation Formulas

3A

- Power Rule
- Linearity Rule
- Product Rule
- Quotient Rule

The Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

- This rule holds for any real number.

[Examples 3-1] Differentiate the following functions.

(1) $f(x) = \dfrac{1}{x^2}$ (2) $f(x) = \sqrt{x^3}$

(1) $f'(x) = \dfrac{d}{dt}\left(x^{-2}\right) = -2x^{-2-1} = -2x^{-3} = -\dfrac{2}{x^3}$

(2) $f'(x) = \dfrac{d}{dt}\left(\sqrt{x^3}\right) = \dfrac{d}{dt}\left(x^{\frac{3}{2}}\right) = \dfrac{3}{2}x^{\frac{1}{2}} = \dfrac{3}{2}\sqrt{x}$

Linearity Rules

Assume that $f(x)$ and $g(x)$ are differentiable functions.

- Constant Multiple rule : $(cf)' = cf'$
- Sum Rule : $(f+g)' = f' + g'$
- Difference Rule : $(f-g)' = f' - g'$

[Proof of Sum Rule]

By definition

$$\{f(x)+g(x)\}' = \lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h}$$

After rearrangement, we have

$$\{f(x)+g(x)\}' = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)$$

Product Rule

Product Rule

Assume that $f(x)$ and $g(x)$ are differentiable functions.

$$(fg)' = f'g + fg'$$

$$-f(x)g(x+h)+f(x)g(x+h) = 0 \qquad \text{(ii)}$$

$$\{f(x)g(x)\}' = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \qquad \text{(i)}$$

$$= \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\cdot g(x+h)+f(x)\cdot\frac{g(x+h)-g(x)}{h}\right\} \qquad \text{(iii)}$$

$$= f'(x)g(x)+f(x)g'(x)$$

[Examples 3-3] Find the derivative function of $h(x) = 3x^2(5x+1)$

Ans. $h'(x) = 6x(5x+1) + 3x^2(5) = 45x^2 + 6x$

The Quotient Rule

Quotient Rule

$$\left\{\frac{f}{g}\right\}' = \frac{f'g - fg'}{g^2} \qquad \text{In particular} \qquad \left\{\frac{1}{g}\right\}' = -\frac{g'}{g^2}$$

The second one is proved as

$$\left\{\frac{1}{g(x)}\right\}' = \lim_{h\to 0}\frac{1}{h}\cdot\left\{\frac{1}{g(x+h)} - \frac{1}{g(x)}\right\} \qquad \text{(i)}$$

$$= \lim_{h\to 0}\left\{-\frac{g(x+h)-g(x)}{h}\cdot\frac{1}{g(x+h)g(x)}\right\} = -\frac{g'(x)}{\{g(x)\}^2} \qquad \text{(ii)}$$

Using this, we have

$$\left\{\frac{f(x)}{g(x)}\right\}' = \left\{f(x)\cdot\frac{1}{g(x)}\right\}' = \frac{f'(x)}{g(x)} + f(x)\cdot\frac{-g'(x)}{\{g(x)\}^2} = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2}$$

[Examples 3-4] Compute the derivative function of $f(x) = \dfrac{x}{x+1}$

Ans.

$$f'(x) = \frac{1(x+1) - x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

Exercise

[Exercise 3-1] Calculate the derivatives of the following functions in two ways. First use the Quotient Rule, then rewrite the function algebraically and apply the Power Rule directly.

(1) $g(x) = \dfrac{x^3 + 2x^2 + 3x^{-1}}{x}$ (2) $h(t) = \dfrac{t^2 - 1}{t - 1}$

Pause the video and solve the problem by yourself.

Excercise

[Ex.3-1] Calculate the derivatives of the following functions in two ways. First use the Quotient Rule, then rewrite the function algebraically and apply the Power Rule directly.

(1) $g(x) = \dfrac{x^3 + 2x^2 + 3x^{-1}}{x}$ (2) $h(t) = \dfrac{t^2 - 1}{t - 1}$

Ans. (1) Quotient Rule

$$g'(x) = \frac{(3x^2 + 4x - 3x^{-2})x - (x^3 + 2x^2 + 3x^{-1}) \cdot 1}{x^2} = \frac{2x^3 + 2x^2 - 6x^{-1}}{x^2} = 2x + 2 - 6x^{-3}$$

Power Rule

$$g'(x) = (x^2 + 2x + 3x^{-2})' = 2x + 2 - 6x^{-3}$$

(2) Quotient Rule

$$h'(t) = \frac{2t(t-1) - (t^2 - 1) \cdot 1}{(t-1)^2} = \frac{t^2 - 2t + 1}{(t-1)^2} = \frac{(t-1)^2}{(t-1)^2} = 1$$

Power Rule

$$h'(t) = (t+1)' = 1$$

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Lesson 3
Differentiation Formulas

3B

- Chain Rule
- Derivative of Implicit Functions

The Chain Rule

Composite function $y = f(g(x))$

y=f(u) and u=g(x)

The Chain Rule

If $f(x)$ and $g(x)$ are differentiable, the next relationship holds.

$$\frac{df(g(x))}{dx} = \frac{df(g)}{dg}\frac{dg(x)}{dx}$$

Setting $u = g(x)$, we may also write this as

$$\frac{dy}{dx} = f'(u)\frac{du}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example

[Examples 3-5] Calculate the derivative of $y = \sqrt{x^3 + 1}$

Ans.

This is a composite function in the form

$$f(u) = \sqrt{u} \quad \text{and} \quad u = g(x) = x^3 + 1$$

Since $f'(u) = \frac{1}{2}u^{-1/2}$ and $g'(x) = 3x^2$, we have

$$\frac{d}{dx}\sqrt{x^3 + 1} = \frac{1}{2}u^{-1/2}(3x^2) = \frac{1}{2}(x^3 + 1)^{-1/2}(3x^2) = \frac{3x^2}{2\sqrt{x^3 + 1}}$$

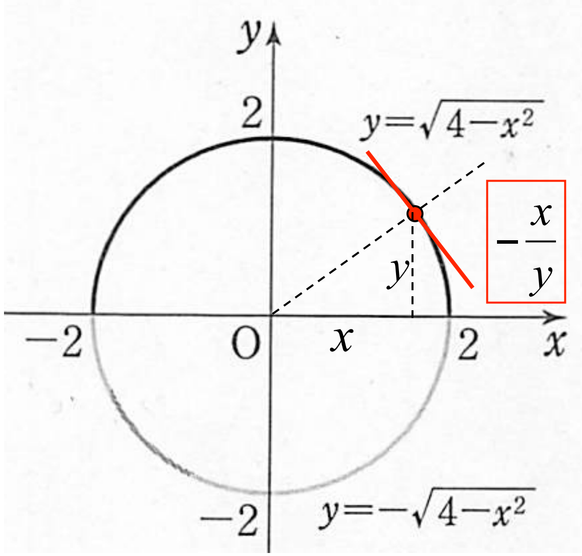
Two kinds of function

- Explicit function : $y = f(x)$ [Ex.] $y = x^2$
- Implicit function : $f(x, y) = 0$ [Ex.] $x^2 + y^2 = 1$

Two ways to calculate the derivative.

Ex: circle $x^2 + y^2 = 4$

(1) Solve for y and then differentiate.



$$y = \pm\sqrt{4 - x^2}$$

$$\therefore \quad \frac{dy}{dx} = \left(\pm \frac{x}{\sqrt{4 - x^2}} \right) = -\frac{x}{y}$$

Derivative of Implicit Functions (2)

(2) Take derivative of each term and apply the chain rule.

[Ex.] A circle $x^2 + y^2 = 1$

Take the derivative of both sides

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

$$\therefore\ 2x + \frac{d}{dx}(y^2) = 0$$

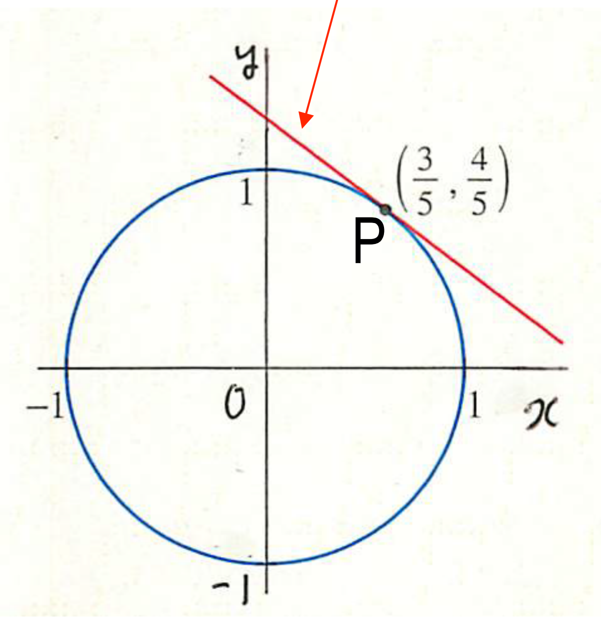
Applying the chain rule

$$2x + \frac{d}{dy}(y^2)\frac{dy}{dx} = 0$$

$$\therefore\ 2x + 2y\frac{dy}{dx} = 0$$

Then $\dfrac{dy}{dx} = -\dfrac{x}{y}$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y} = -\left(\frac{3}{5}\right)\Big/\left(\frac{4}{5}\right) = -\frac{3}{4}}$$



Exercise

[Exercise 3-2] Calculate the derivatives of the following functions

(1) $y = (x^2 + 2x - 1)^2$

(2) $y = \dfrac{1}{(3x-2)^2}$

Pause the video and solve the problem by yourself.

Excercise

[Ex.3-3] Calculate the derivatives of the following functions

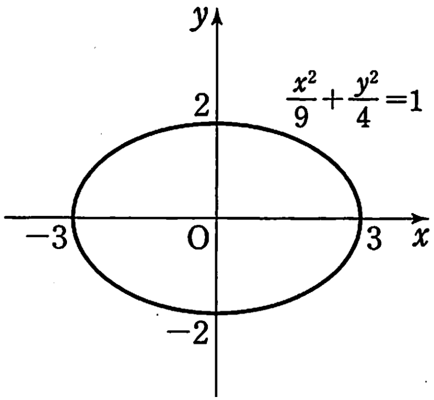
(1) $y = (x^2 + 2x - 1)^2$

(2) $y = \dfrac{1}{(3x-2)^2}$

(1) $$y' = 2(x^2 + 2x - 1)(2x + 2) = 4(x^2 + 2x - 1)(x + 1)$$

(2) $$y' = \left\{(3x-2)^{-2}\right\}' = -2(3x-2)\cdot 3 = -6(3x-2)$$

[Ex.3-4] Find the derivative $\dfrac{dy}{dx}$ of the function $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$



Answer to the Excercise

[Ex.3-4] Find the derivative $\dfrac{dy}{dx}$ of the function $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Taking the derivative of both sides with respect to x , we have

$$\frac{2x}{9} + \frac{2y}{4}\frac{dy}{dx} = 0$$

Therefore, when $y \neq 0$, we have

$$\frac{dy}{dx} = -\frac{4x}{9y}$$