NAGOYA UNIVERSITY

Lesson 6
Applications of Derivatives to Equations and Inequality

6A

- Graphs and Equations

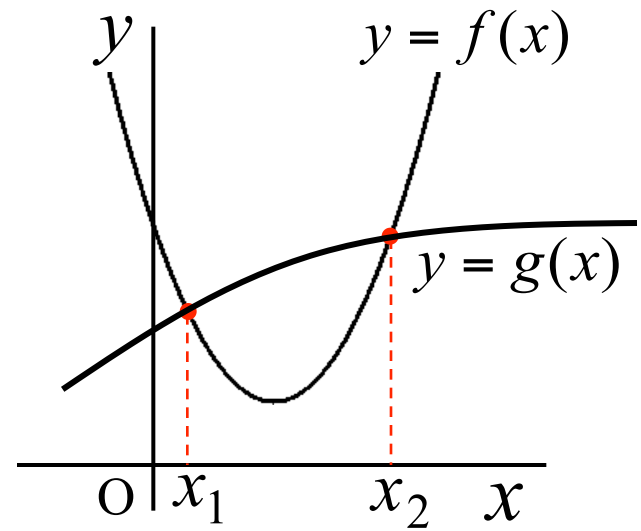
Graphs and Equations (Case with No Parameter)

Equation

$$f(x) = g(x)$$

The roots of this equation are given by the coordinates of the cross points of the following graphs.

$y = f(x)$ and $y = g(x)$

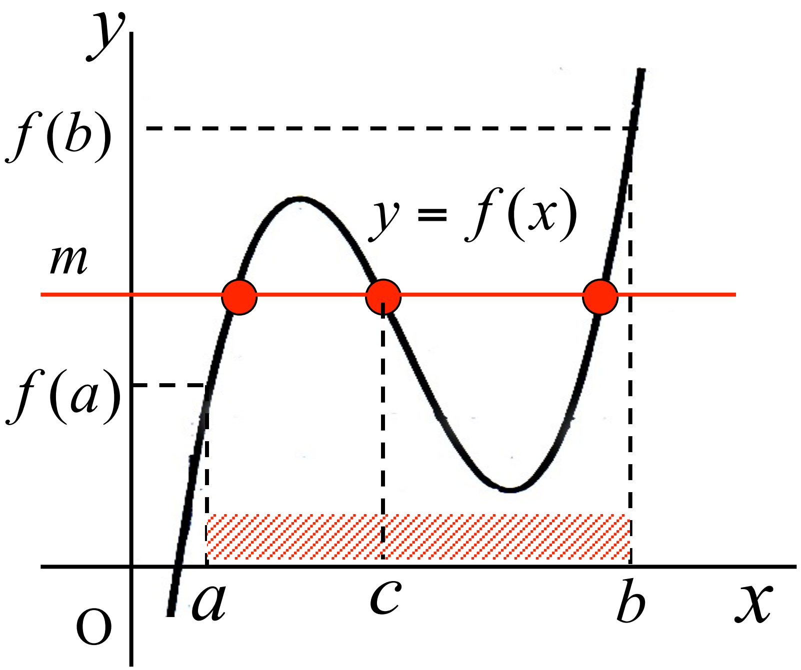


Technique

- When the equation $f(x) = g(x)$ does not include a parameter, we investigate the cross points of the graph $y = f(x) - g(x)$ and the x -axis.
- In other words, we solve for the zeros of $f(x) - g(x)$.

Intermediate Value Theorem

Let the curve represented by $y = f(x)$ be continuous on the interval $[a, b]$ and m be a number between $f(a)$ and $f(b)$.
Then, there must be at least one value c within $[a, b]$ such that $f(c) = m$.



[Note]
In case that the function increases or decreases monotonically in $[a, b]$, then the number c is unique.

Example

[Examples 6-1] Find the number of the real roots of the following equation.

$$x^3 = 3x - 1$$

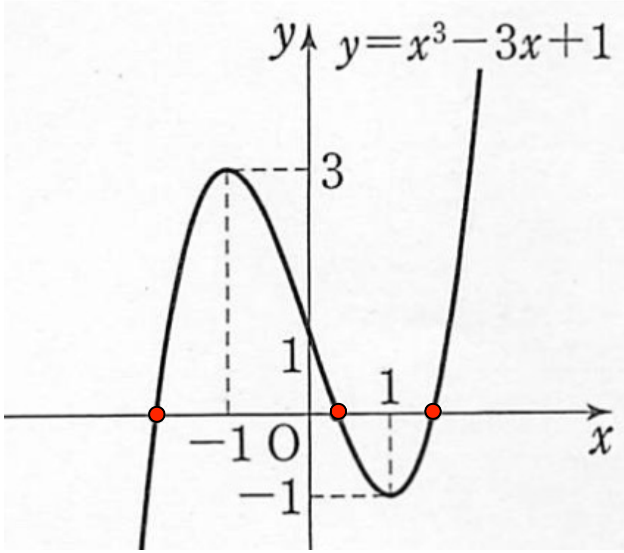
Ans.

We put $y = x^3 - 3x + 1$

Then $y' = 3x^2 - 3 = 3(x+1)(x-1)$

$y' = 0$ at $x = -1, +1$

x	$\cdots$	-1	$\cdots$	$+1$	$\cdots$
y'	$+$	0	$-$	0	$+$
y	↗	3	↘	-1	↗

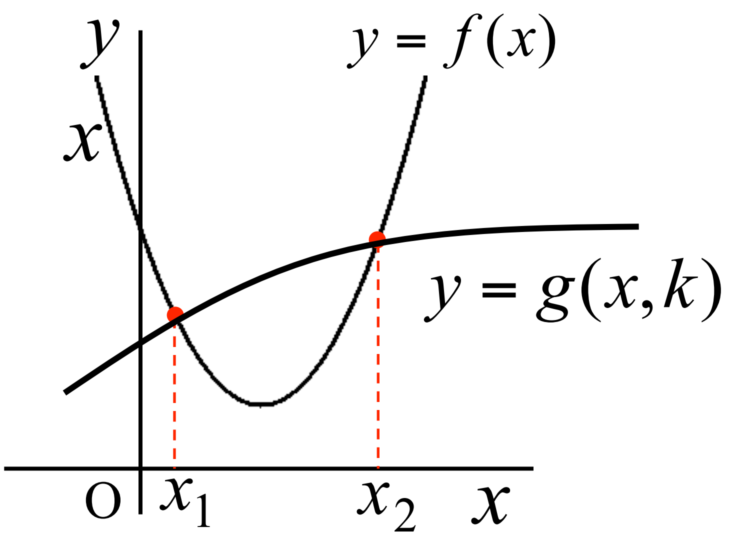


This graph crosses x-axis at three points.
Therefore, this equation has three real roots

Graphs and Equations (Case with a Parameter)

Equation

$$f(x) = g(x,k)$$



Technique

Suppose that the equation has the form

$$f(x) = k \cdot g(x) \quad \xrightarrow{\text{Rearrange}} \quad -\frac{f(x)}{g(x)} = k \quad \left(\; g(x) \neq 0 \; \right)$$

Illustrate graphs

$$\left.\begin{aligned} y &= -\frac{f(x)}{g(x)} \\ y &= k \end{aligned}\right\} \quad \text{Find cross-points}$$

Example

[Examples 6-2] Find the range of parameter a when the following a cubic equation has three real roots. $x^3+3x^2-a=0$

Ans. After rearrarngement

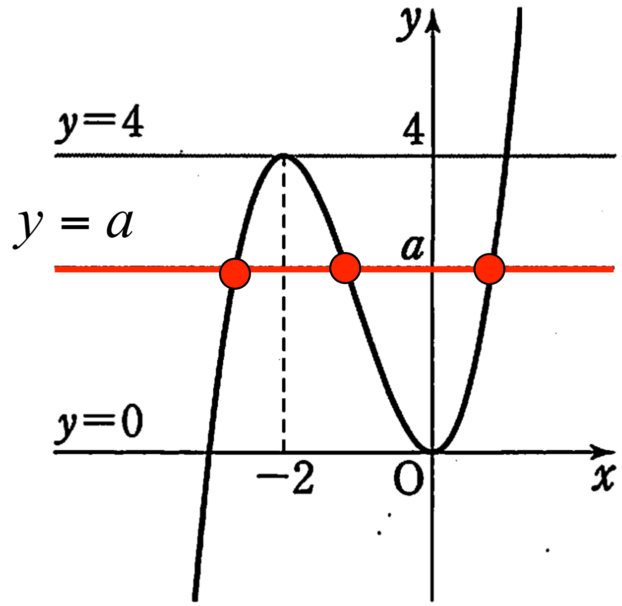
$$x^3+3x^2=a$$

About the function

$$y=x^3+3x^2$$

$$y'=3x^2+6x=3x(x+2)$$

x	……	-2	……	0	……
y'	+	0	−	0	+
y	↗	L.Max 4	↘	L.Min. 0	↗



From the graph, we have $0<a<4$

Example

[Examples 6-3] Investigate the number of the roots of the following equation.

$$e^x = x + a$$

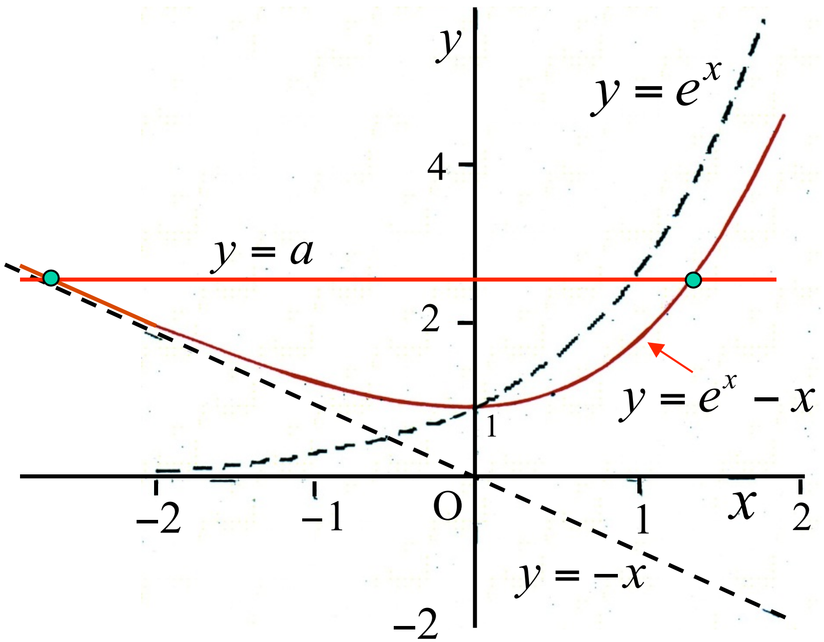
Ans.

$$y = e^x - x$$

$$\therefore \quad y' = e^x - 1$$

$$y' = 0 \text{ at } x = 0$$

x	$\cdots$	0	$\cdots$
y'	$-$	0	$+$
y	↘	$+1$	↗



Number of roots

Zero when $a < 0$

One when $a = 1$

Two when $a > 1$

[Ex.6-1] Let m be a real constant. Investigate the relationship between m and the number of the cross points of $y = x^3 - x - 1$ and $y = 2x + m$

Ans.

Pause the video and solve the problem by yourself.

[Ex.6-1] Let m be a real constant. Investigate the relationship between and the number of the cross points of $y = x^3 - x - 1$ and $y = 2x + m$

Ans. Put $x^3 - x - 1 = 2x + m$

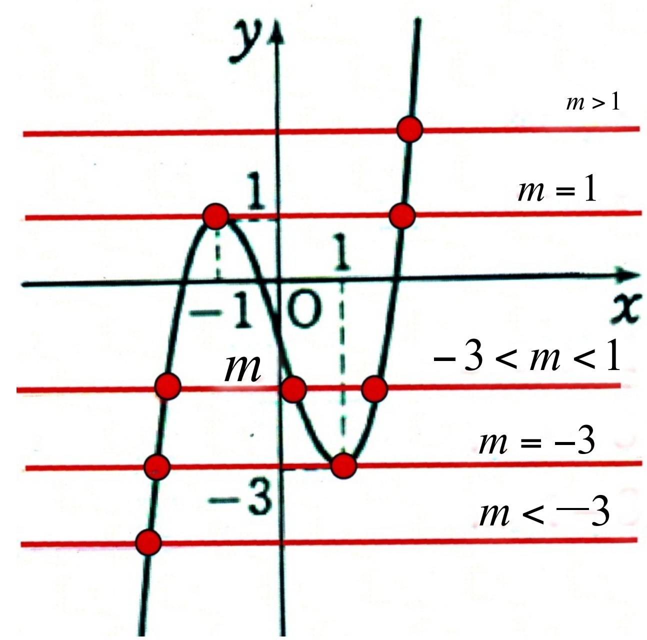
$$\therefore \quad x^3 - 3x - 1 = m$$

We consider two functions.

$$y = x^3 - 3x - 1 \quad \text{and} \quad y = m$$

From the former

$$y' = 3x^2 - 3 = 3(x-1)(x+1)$$



x	$\cdots$	-1	$\cdots$	$+1$	$\cdots$
y'	$+$	0	$-$	0	$+$
y	$\nearrow$	$+1$	$\searrow$	-3	$\nearrow$

Number of roots

One when $m < -3, \ 1 < m$

Two when $m = -3, \ 1$

Three when $-3 < m < 1$

Lesson 6
Applications of Derivatives to Equations and Inequality

6B

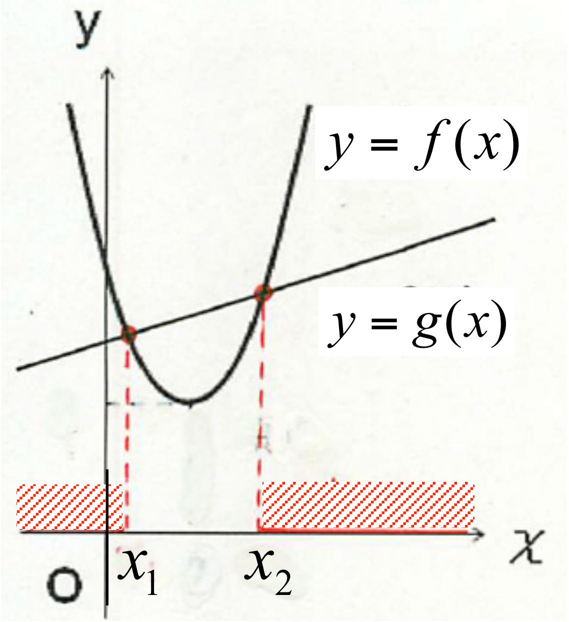
- Graphs and Inequalities

Graphs and Inequality (Case with No Parameter)

Inequality

$$f(x) > g(x)$$

The solutions of this inequality are given by the domain in the x-axis where the graphs of $y = f(x)$ is larger than that of $y = g(x)$.



Technique

Illustrate the graph of $y = f(x) - g(x)$ and find the domain where $y > 0$ holds.

Example

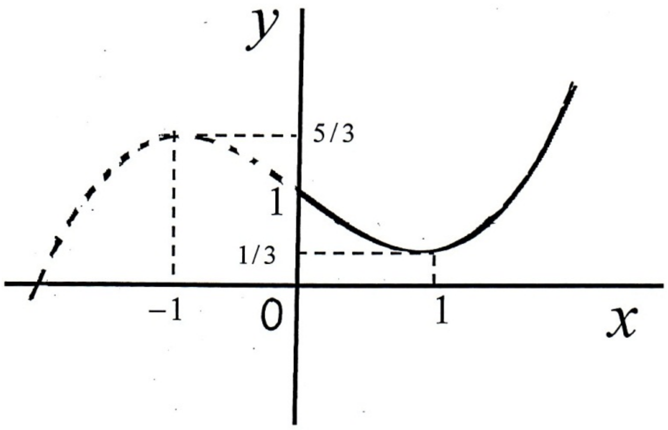
[Examples 6-5] Prove the following inequality.

$$x - 1 < \frac{1}{3}x^3 \quad \text{where} \quad x > 0$$

Ans. Put $y = \frac{1}{3}x^3 - x + 1$

$$\therefore \quad y' = x^2 - 1 = (x-1)(x+1)$$

x	0	$\cdots$	$+1$	$\cdots$
y'	$-$	$-$	0	$+$
y	1	↘	$\frac{1}{3}$	↗



This function $y = \frac{1}{3}x^3 - x + 1$ has the minimum value $\frac{1}{3}$ at $x = 1$.

Since $y > 0$ in the domain $x > 0$, we find

$$x - 1 < \frac{1}{3}x^3 \quad \text{in} \quad x > 0$$

[Examples 6-6] Prove the following inequality.

$$e^x > 1 + x \qquad \text{where} \qquad x > 0$$

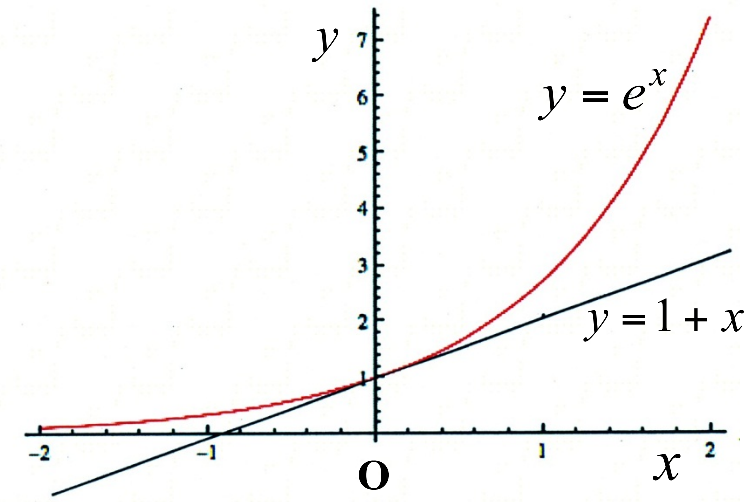
Ans. Put $y = e^x - (1 + x)$

$\therefore\ y' = e^x - 1 > 0$ for $x > 0$

y increases monotonically.

$y = 0$ at $x = 0$

x	0	$\cdots$
y'	0	$+$
y	0	↗



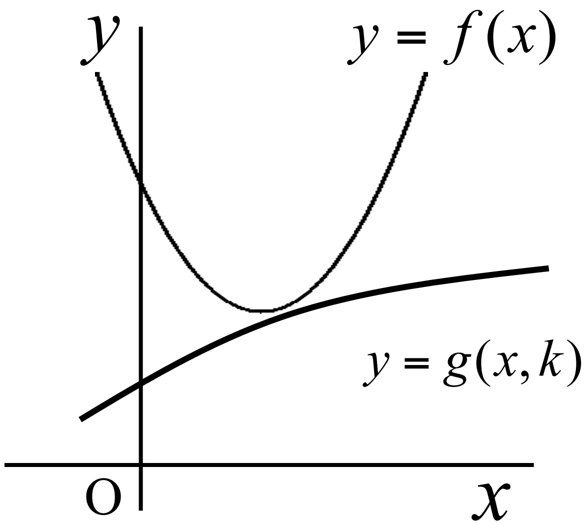
From this table $e^x > 1 + x$ in $x > 0$

Graphs and Equations (Case with a Parameter)

Problem and Equation

$$f(x) > g(x,k)$$

PROBLEM: Find value of k which satisfy this equation.



Technique

In case of $f(x) > kg(x)$ Rearrange $\dfrac{f(x)}{g(x)} > k \quad (g(x) > 0)$

Illustrate graphs of

$$\left.\begin{array}{l} y = \dfrac{f(x)}{g(x)} \\ y = k \end{array}\right\}$$

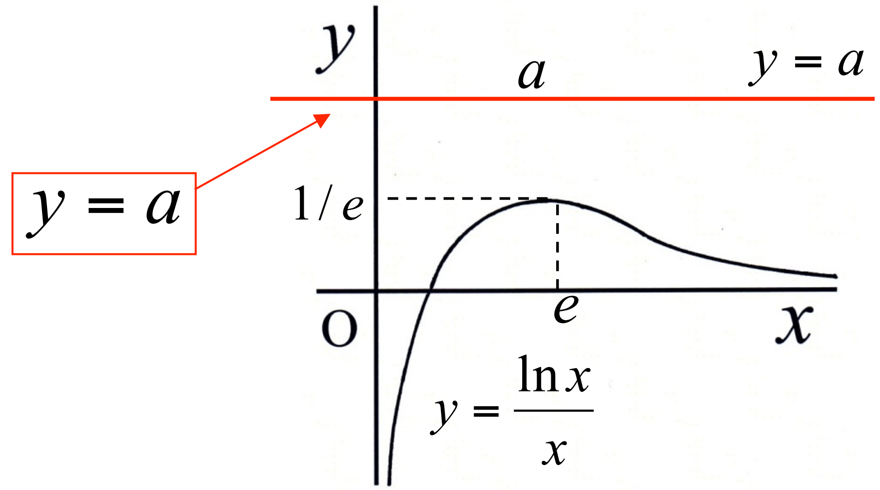
Determine k which satisfy $\dfrac{f(x)}{g(x)} > k$

Example

[Examples 6-7] Find the range of the parameter a which satisfy the following inequality. $ax \geq \ln x \quad (x>0)$

Ans. Rearrangement. $a \geq \dfrac{\ln x}{x}$

Plot $y = \dfrac{\ln x}{x}$



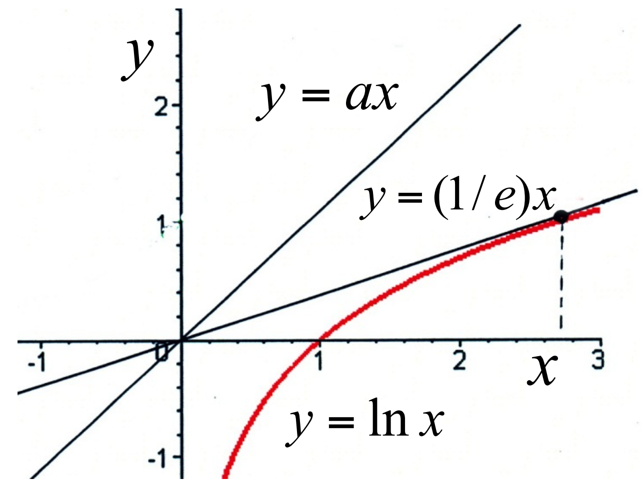
$$\therefore \quad y' = \frac{\frac{1}{x}x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$y' = 0$ at $x = e$

x	$0<x<e$	e	$e<x$
y'	$+$	0	$-$
y	↗	$1/e$	↘

Answer

$$a \geq \frac{1}{e}$$



Exercises

[Ex.6-2] Prove the following inequality

$$x^3 - 9x + 27 \geq 3x^2 \quad \text{where} \quad x \geq 0$$

Ans.

Pause the video and solve the problem by yourself.

[Ex.6-2] Prove the following inequality

$$x^3 - 9x + 27 \geq 3x^2 \quad \text{where} \quad x \geq 0$$

Ans.

We put $y = x^3 - 3x^2 - 9x + 27$

Then $y' = 3x^2 - 6x - 9 = 3(x+1)(x-3)$

$y' = 0$ at $x = -1, 3$

x	0	・・・	3	・・・
y'	−	−	0	+
y	27	↘	0	↗

From this table $x^3 - 3x^2 - 9x + 27 \geq 0$, that is, $x^3 - 9x + 27 \geq 3x^2$
in the domain $x \geq 0$