NAGOYA UNIVERSITY

Lesson 9
Antiderivatives

9A

- Antiderivative

Inverse Problem

To find the function when its derivative is given.

[Ex.] velocity $v(t)\left(=\dfrac{ds(t)}{dt}\right)$ $\rightarrow$ position $s(t)$

Antiderivative

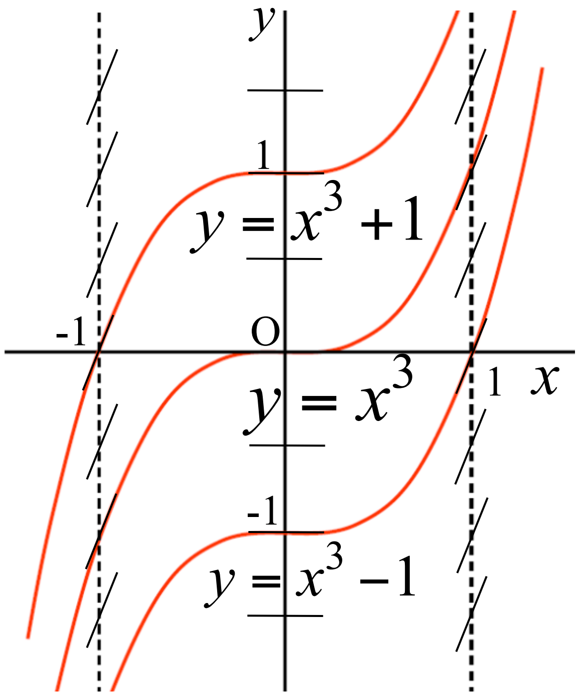
A function $F(x)$ is an antiderivative (or primitive integral) of $f(x)$ if $F'(x) = f(x)$.

[Ex.] $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$ because

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

Example:

$x^3,\ x^3-2,\ x^3+2$: antiderivatives of $3x^2$



Theorem

When $F'(x) = f(x)$, every other antiderivative is of the form

$$F(x) + C \qquad (\ C : \text{constant}\ )$$

Indefinite Integral

The collection of all antiderivatives of a function is called the general antiderivative or indefinite integral of $f(x)$ and denoted by $\int f(x)dx$

Namely,

$$\int f(x)dx = F(x) + C$$

Fundamental Indefinite Integrals

$$x^3 + C \quad \xrightarrow{\text{Differentiation}} \quad 3x^2$$

$$x^3 + C \quad \xleftarrow[\text{Integration}]{} \quad 3x^2$$

Algebraic Functions

< Differentiation >

$$\frac{d}{dx} x^{n+1} = (n+1)x^n$$

< Integration >

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{for} \quad n \neq -1$$

Example

$$\int x^5 dx = \frac{1}{6} x^6 + C$$

Note

For $n = -1$

$$\int x^{-1} dx = \frac{x^{-1+1}}{-1+1} + C = \frac{x^0}{0} + C$$

Meaningless

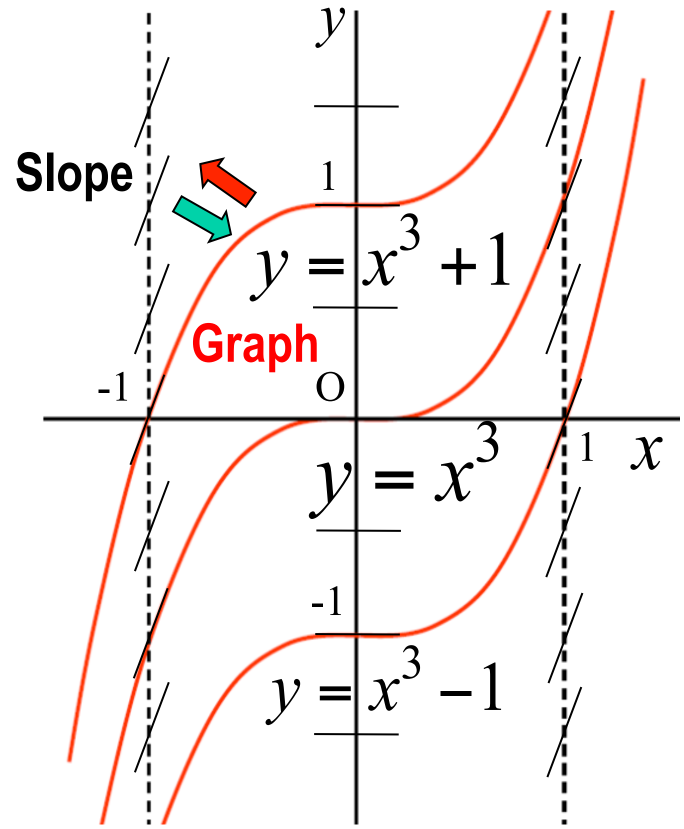
Graphical Interpretation.

Differentiation

Graph : $y = f(x)$ ➡ Slope : $\dfrac{df(x)}{dx}$

Integration

Slope : $\dfrac{df(x)}{dx}$ ➡ Graph : $y = f(x)$



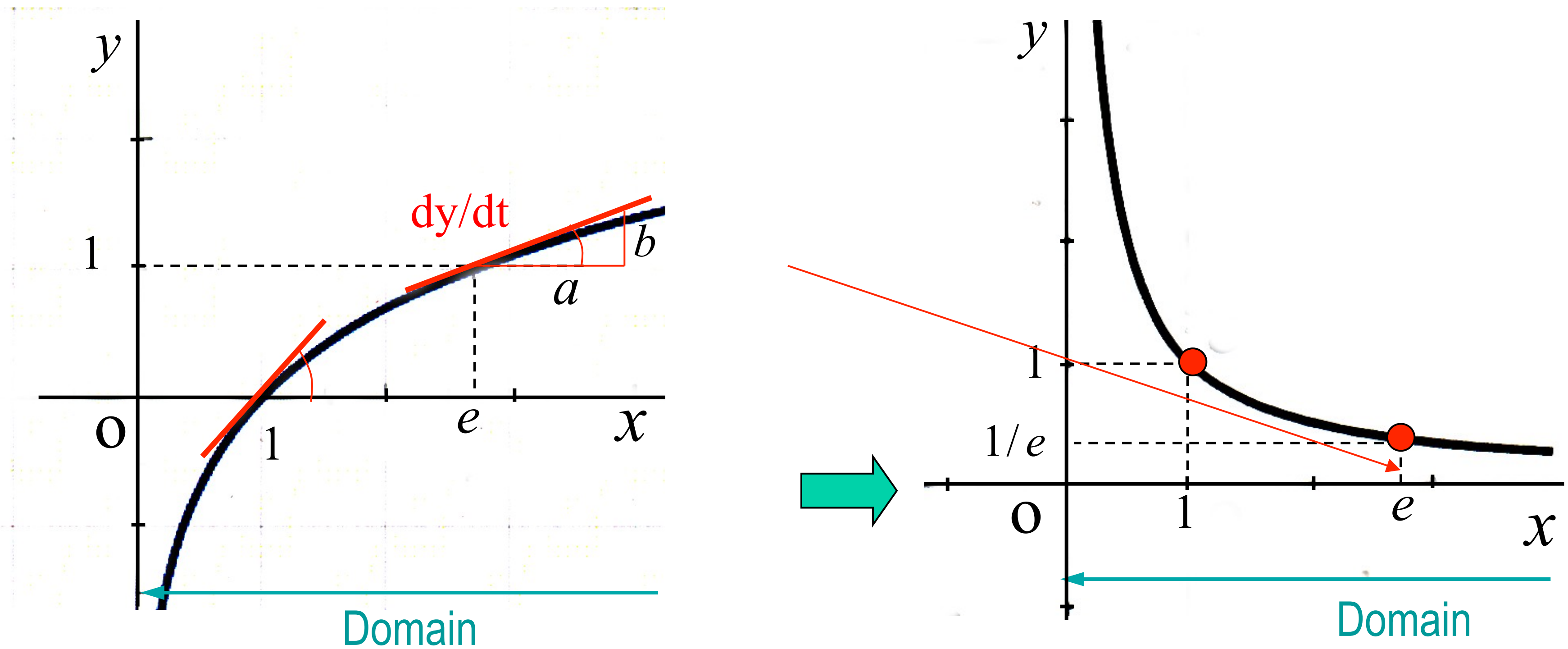
<Another interpretation>

Graph : $y = f(x)$ ➡ Area

（ We will study this later ）

Graphical Interpretation of Differentiation

How to find the derivative $y = \frac{1}{x}$ graphically from $y = \ln x$

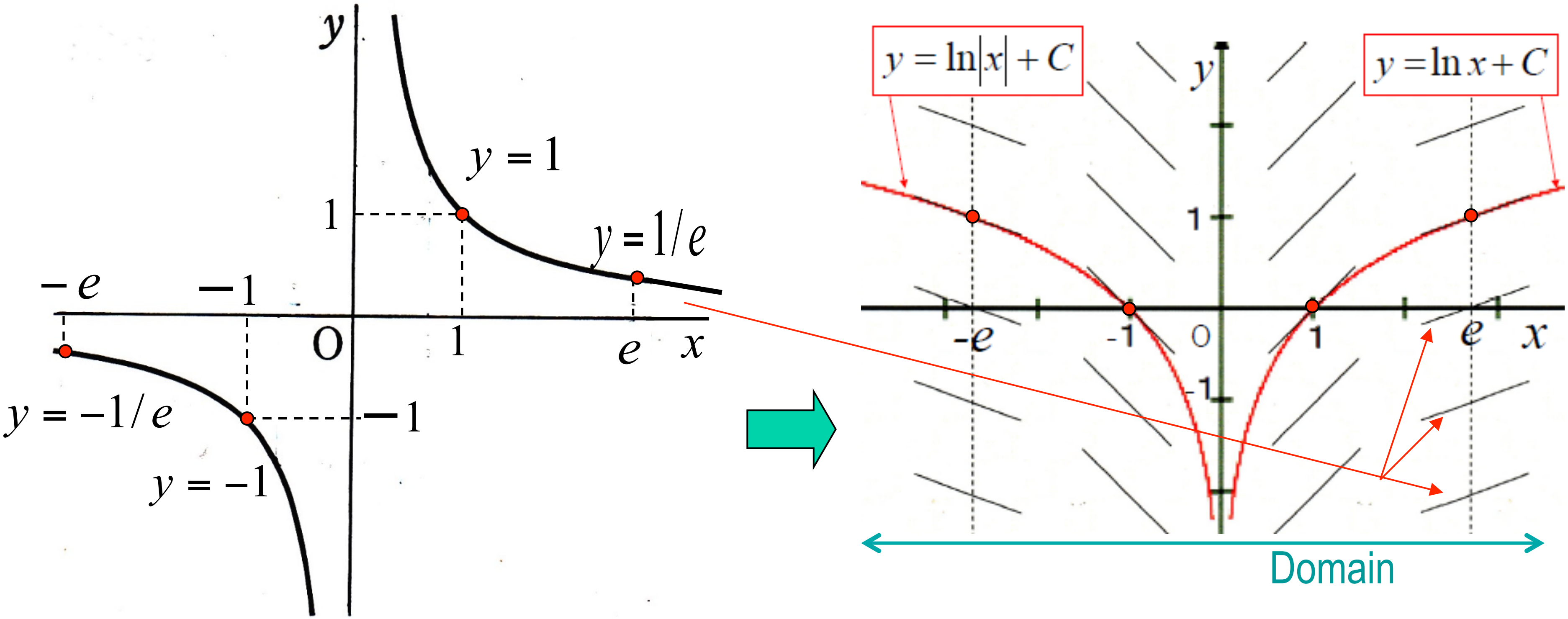


$$y = \ln x$$

$$y = \frac{d}{dx}\ln x = \frac{1}{x}$$

Graphical Interpretation of Integration

How to find the indefinite function graphically from $y = \dfrac{1}{x}$



$$y = \frac{1}{x}$$

$$y = \int \frac{1}{x} dx$$

Algebraic derivation

For $x > 0$ (Defined domain of $y = \ln x$)

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad \Longleftrightarrow \quad \int \frac{1}{x}\,dx = \ln x + C$$

For $x < 0$

From the graphical discussion, we assume $y = \ln|x|$

Put $x = -u \quad (u > 0)$

From the chain rule

$$\frac{dy}{dx} = \frac{d\left(\ln|x|\right)}{dx} = \frac{d\left(\ln u\right)}{du}\frac{du}{dx} = \frac{1}{u}\cdot(-1) = \frac{1}{x}$$

For all $x \neq 0$ $\displaystyle \int \frac{1}{x}\,dx = \ln|x| + C$

Fundamental Indefinite Integrals – *Cont.*

Trigonometric Integrals

$$\frac{d}{dx}\sin x = \cos x \quad \Rightarrow \quad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}\cos x = -\sin x \quad \Rightarrow \quad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} \quad \Rightarrow \quad \int \frac{1}{\cos^2 x} dx = \tan x + C$$

Fundamental Indefinite Integrals – *Cont.*

Exponential Integrals

$$\frac{d}{dx}e^x = e^x \quad \Rightarrow \quad \int e^x\,dx = e^x + C$$

$$\frac{d}{dx}a^x = a^x \ln a \quad \Rightarrow \quad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

Basic Rules of Integration

Sum Rule

$$\int \{f(x) + g(x)\}dx = \int f(x)dx + \int g(x)dx$$

Constant Multiple Rule

$$\int cf(x)dx = c\int f(x)dx$$

Example

Examples 9-2 Find the indefinite integrals of the following function.

(1) $(x+1)(x-2)$

(2) $\sin x + 2\cos x$

Ans.

(1) $$\int (x+1)(x-2)dx = \int \left(x^2 - x - 2\right)dx = \int x^2 dx - \int x dx - 2\int dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} - 2x + C$$

(2) $$\int (\sin x + 2\cos x)dx = \int \sin x + 2\int \cos x dx$$

$$= -\cos x + 2\sin x + C$$

Exercise

Exercise. 9-1 The slope of the curve $y = f(x)$ is given by $3x^2$ and this curve passes the point (1,2). Determine the equation of this curve.

Ans.

Pause the video and solve by yourself.

Answer to the Exercises

Exercise 12-1] The slope of the curve $y = f(x)$ is given by $3x^2$ and this curve passes the point (1,2). Determine the equation of this curve.

Ans.

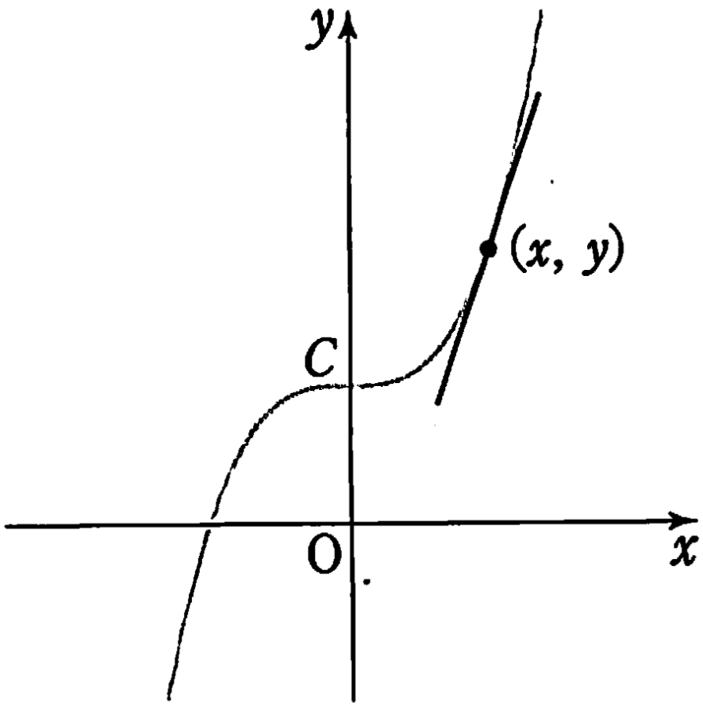
$$f'(x) = 3x^2$$

$$\therefore \quad f(x) = \int 3x^2 dx = x^3 + C$$

Since this line passes the point (1,2),

$$f(1) = 1 + C = 2 \quad \therefore C = 1$$

$$\therefore \quad y = x^3 + 1$$



Lesson 9
Antiderivatives

9B

- Substitution Method
- Integration by Parts (IBP)

Substitution Method （1）

Applicable when the integrand is of the form $f(u(x))u'(x)$

Example: $(x^2+1)^5 x$ $\sin^5 x \cos x$

$u = x^2+1$ $u' = 2x$ $u = \sin x$ $u' = \cos x$

Let $F'(u) = f(u)$ or $F(u) = \int f(u)du$ ①

From the Chain Rule $\frac{d}{dx}F(u(x)) = F'(u)u'(x) = f(u)u'(x)$

By integration, we have $F(u(x)) = \int f(u)u'(x)dx$ ②

From ① and ②, we have

Substitution Method

$$\int f(u(x))u'(x)dx = \int f(u)du$$

Substitution Using Differentials

Differential

$$u = u(x) \longrightarrow \frac{du}{dx} = u'(x) \longrightarrow du = u'(x)dx$$

$$du = \frac{du}{dx}dx$$

< Example > If $u = x^3$, then $du = 3x^2dx$

$$\int f(u)u'(x)dx = \int f(u)du$$

$$du = \frac{du}{dx}dx$$

Example

Examples 9-4 Find the indefinite integral of the following functions.

(1) $\int (x^2+1)^5 x dx$ (2) $\int \sin^5 x \cos x dx$

Ans. (1) We put $x^2+1=u$

Then $2x\frac{dx}{du}=1$ $\therefore\ xdx=\frac{1}{2}du$

Substitution

$$\int (x^2+1)^5 x dx = \int u^5 \frac{1}{2}du = \frac{1}{12}u^6 + C = \frac{1}{12}(x^2+1)^6 + C$$

(2) We put $\sin x = u$

Then $\cos x \frac{dx}{du} = 1$ $\therefore\ \cos x dx = \frac{1}{\cos x}du$

Substitution

$$\int \sin^5 x \cos x dx = \int u^5 du = \frac{1}{6}u^6 + C = \frac{1}{6}\sin^6 x + C$$

Substitution Method （2）

Substitution Method

$$\int f(x)dx = \int f(x(t))x'(t)dt$$

This is also written as follows.

$$\int f(x)dx = \int f(x(t))\frac{dx}{dt}dt$$

Example

Examples 9-5 Find the indefinite integral of the following functions.

$$\int x\sqrt{1-x}\,dx$$

Ans. We put $\sqrt{1-x} = t$. Then $x = 1-t^2$ and $\dfrac{dx}{dt} = -2t$.

Therefore

$$\int x\sqrt{1-x}\,dx = \int (1-t^2)t(-2t)dt = \int (1-t^2)t(-2t)dt$$

$$= 2\int (t^4 - t^2)dt = 2\left(\frac{t^5}{5} - \frac{t^3}{3}\right) + C = -\frac{2}{15}(2+3x)(1-x)\sqrt{1-x} + C$$

Integration by Parts

We studied the derivative of product

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$$

After rearrangement, we have

$$f(x)g'(x) = \{f(x)g(x)\}' - f'(x)g(x)$$

Therefore, we obtain the following formula

Integration by Parts

$$\int f(x)g(x)'dx = f(x)g(x) - \int f(x)'g(x)dx$$

Example

Examples 9-5 Evaluate the following Integral.

$$\int x\cos x\,dx$$

Ans.

$$\int \underbrace{x}_{f}\,\underbrace{\cos x}_{g'}\,dx = \underbrace{x}_{f}\,\underbrace{\sin x}_{g} - \int \underbrace{1}_{f'}\cdot\underbrace{\sin x}_{g}\,dx = x\sin x + \cos x + C$$

Exercise

Exercise 9-2 Evaluate the following.

(1) $\int \cos^2 x \sin x dx$ (2) $\int xe^{6x} dx$

Ans.

Pause the video and solve by yourself.

Answers to the Exercise

Exercise 9-2 Evaluate the followings.

(1) $\int \cos^2 x \sin x dx$ (2) $\int x e^{6x} dx$

Ans. (1) Since $(\cos x)' = -\sin x$, we put $\cos x = u$

$$\therefore -\sin x = \frac{du}{dx}$$

Therefore $\int \cos^2 x \sin x dx = \int u^2 (-du) = -\frac{1}{3} u^3 + C = -\frac{1}{3} \cos^3 x + C$

(2)

$$\int \underbrace{x}_{f} \underbrace{e^{6x}}_{g'} dx = x\left(\frac{e^{6x}}{6}\right) - \int 1 \cdot \left(\frac{e^{6x}}{6}\right) dx = \frac{x}{6} e^{6x} - \frac{x}{36} e^{6x} + C$$