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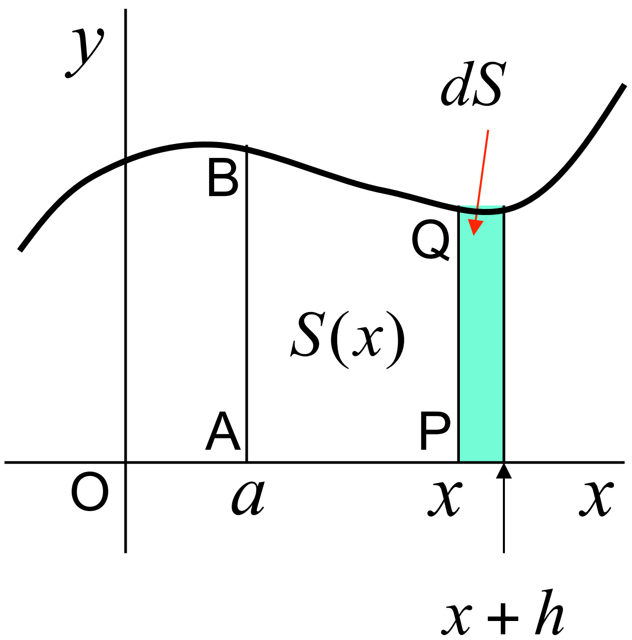
Lesson 10
Definite Integrals

10A

- Area and Antiderivatives

Derivative of Area

Let the area of ABQP be $S(x)$



When x increases by h ,
the area increases by

$$dS = S(x+h) - S(x) \approx hf(x)$$

Derivative of $S(x)$

$$\frac{dS}{dx} = \lim_{h \to 0} \frac{S(x+h) - S(x)}{h} \approx f(x)$$

$$S'(x) = f(x)$$

Area function $S(x)$ is one of antiderivatives of $f(x)$.

Area and Indefinite integral

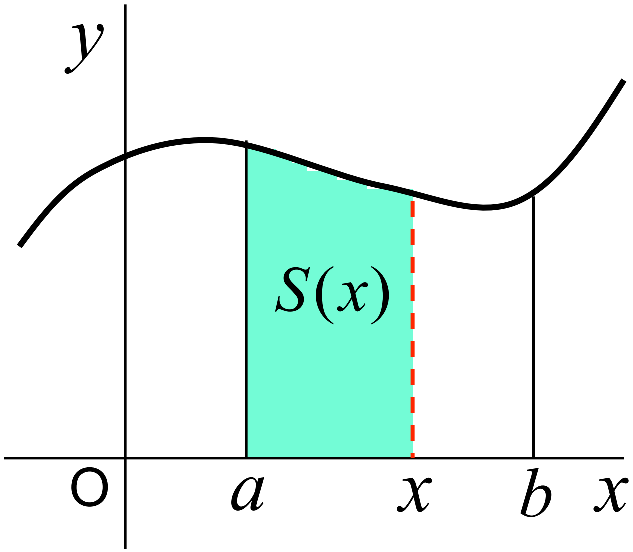
Let $F(x)$ be an arbitrary antiderivative.

$$S(x) = F(x) + C$$

From the definition $S(a) = 0$ at $x = a$

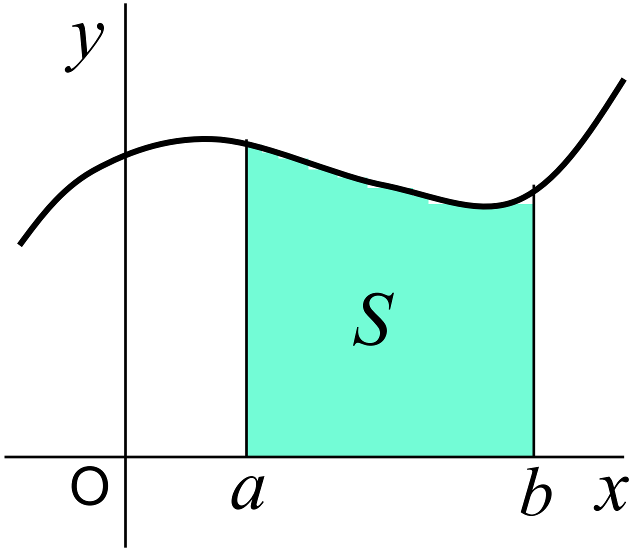
$$\therefore \quad 0 = F(a) + C \qquad \therefore \quad C = -F(a)$$

Therefore $S(x) = F(x) - F(a)$



When $x = b$, the area is given by

$$S = F(b) - F(a)$$



Definite Integral

$F(b)-F(a)$ is called **the definite integral** of $f(x)$ over $[a, b]$

and written by $\int_a^b f(x)dx$ or $\left[F(x)\right]_a^b$

Definite integral

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Since

$$\begin{aligned}\left[F(x)+C\right]_a^b &= \{F(b)+C\}-\{F(a)+C\} \\ &= F(b)-F(a) \\ &= \left[F(x)\right]_a^b\end{aligned}$$

$F(x)$ is an arbitrary antiderivative.

Properties of Definite Integrals

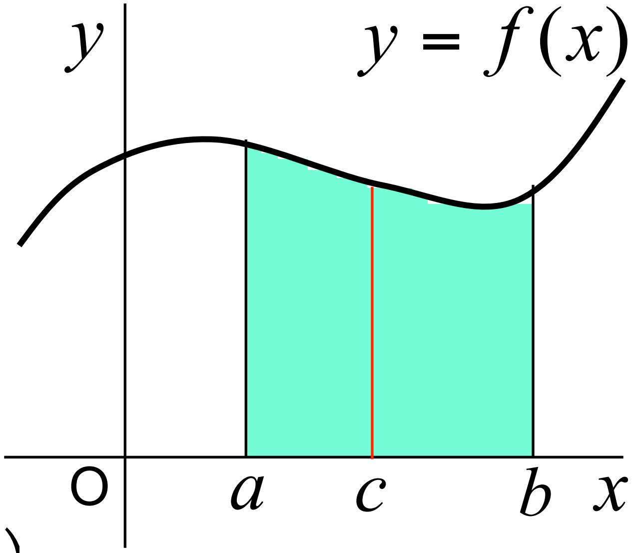
Linearity of the definite integrals

(1) $$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

(2) $$\int_a^b \{f(x)+g(x)\}dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Reversing the integration

(3) $$\int_b^a f(x)dx = -\int_a^b f(x)dx$$



Additivity for adjacent integrals

(4) $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example

Examples 13-1 Find the following definite integral

$$\int_{-2}^{4}(x^2-3x+1)\,dx$$

Ans.

$$\int_{-2}^{4}(x^2-3x+1)\,dx=\int_{-2}^{4}x^2\,dx-3\int_{-2}^{4}x\,dx+\int_{-2}^{4}dx$$

$$=\left[\frac{x^3}{3}\right]_{-2}^{4}-3\left[\frac{x^2}{2}\right]_{-2}^{4}+\left[x\right]_{-2}^{4}$$

$$=\frac{4^3-(-2)^3}{3}-\frac{3\{4^2-(-2)^2\}}{2}+\{4-(-2)\}$$

$$=12$$

Example

Examples 13-2 Find the following definite integrals.

(1) $\int_{-1}^{3} x\,dx - \int_{-1}^{1} x\,dx$ (2) $\int_{0}^{\frac{\pi}{2}} \sin 2\theta d\theta$

Ans.

(1) $$\int_{-1}^{3} x\,dx - \int_{-1}^{1} x\,dx = \int_{-1}^{3} x\,dx + \int_{1}^{-1} x\,dx$$

$$= \int_{1}^{3} x\,dx = \left[\frac{x^2}{2}\right]_{1}^{3} = 4$$

(2) $$\int_{0}^{\frac{\pi}{2}} \sin 2\theta d\theta = \left[\frac{-\cos 2\theta}{2}\right]_{0}^{\frac{\pi}{2}} = \frac{-(-1)}{2} - \frac{(-1)}{2} = 1$$

Exercises

Ex.10-1 Find the following definite integral.

$$\int_0^1 (x^2 - x)\,dx + \int_1^3 (x^2 - x)\,dx$$

Ans.

Pause the video and solve by yourself.

Ex.10-1 Find the following definite integrals

$$\int_0^1 (x^2-x)\,dx+\int_1^3 (x^2-x)\,dx$$

Ans.

$$\int_0^1 (x^2-x)\,dx+\int_1^3 (x^2-x)\,dx=\int_0^3 (x^2-x)\,dx$$

$$=\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_0^3=\frac{9}{2}$$

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Lesson 10
Definite Integrals (1)

10B

- Integration by substitution
- Integration by parts

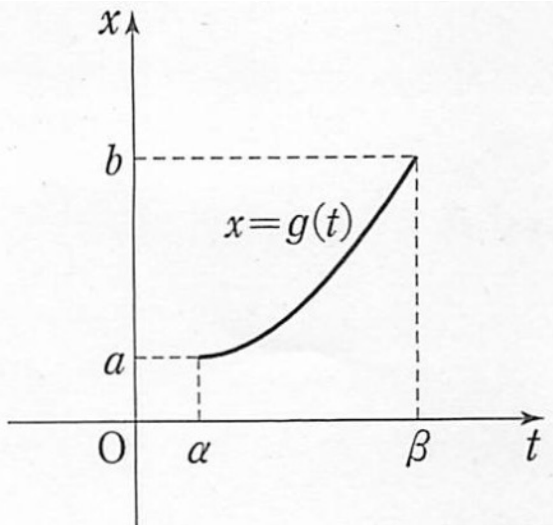
Integration by Substitution

Integration by Substitution (1)

For the function $x = g(t)$

If $a = g(\alpha)$ and $b = g(\beta)$

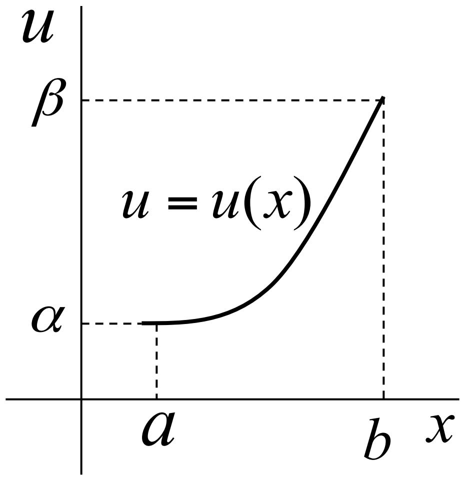
then $$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt$$



Integration by Substitution (2)

If $\alpha = u(a)$ and $\beta = u(b)$

then $$\int_a^b f(u(x))u'(x)dx = \int_\alpha^\beta f(u)du$$



Example

[Examples 13-4] Find the following definite integral by putting $2x+1=t$

$$\int_0^1 (2x+1)^3 dx$$

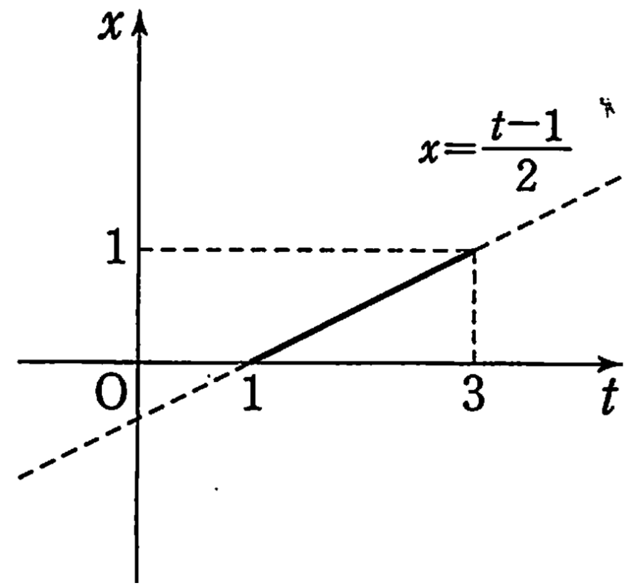
Ans.

Put $2x+1=t$

Then

$$x=\frac{t-1}{2},\quad dx=\frac{1}{2}dt$$

x	$0 \longrightarrow 1$
t	$1 \longrightarrow 3$



Therefore

$$\int_0^1 (2x+1)^3 dx=\int_1^3 t^3\cdot\frac{1}{2}dt=\frac{1}{2}\int_1^3 t^3 dt=\frac{1}{2}\left[\frac{t^4}{4}\right]_1^3=10$$

Example

Examples 10-5 Evaluate the following integral

$$\int_0^2 x^2\sqrt{x^3+1}\,dx$$

Ans.

We put $u = x^3 + 1$, then $du = 3x^2 dx$

Therefore,

$$\int_0^2 x^2\sqrt{x^3+1}\,dx = \int_1^9 \sqrt{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\int_1^9 u^{\frac{1}{2}}du = \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^9 = \frac{52}{9}$$

Integration by Parts

We learned

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

In the case of definite integral

$$\int_a^b f(x)g'(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)dx$$

Example

$$\int_0^{\frac{\pi}{2}} \underbrace{x}_{\boxed{f(x)}}\,\underbrace{\cos x}_{\boxed{g'(x)}}\,dx = \int_0^{\frac{\pi}{2}} x(\sin x)'\,dx = \left[x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (x)'\,\underbrace{\sin x}_{\boxed{g(x)}}\,dx$$

$$= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\pi}{2} - \left[-\cos x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

Exercises

Ex.13-2 Find the following definite integrals

$$\int_{-3}^{-1} \frac{dx}{(2x+1)^3}$$

Ans.

Pause the video and solve by yourself.

Ex.13-2 Find the following definite integrals

$$\int_{-3}^{-1} \frac{dx}{(2x+1)^3}$$

Ans. We can solve this as follows. Put $2x+1=t$

then $x = \dfrac{t-1}{2}, \quad dx = \dfrac{1}{2}dt$

. The new limits of integration are

$$t(-3) = -5 \quad \text{and} \quad t(-1) = -1$$

. Therefore,

$$\int_{-3}^{-1} \frac{dx}{(2x+1)^3} = \int_{-5}^{-1} t^{-3} \cdot \frac{1}{2} dt = \frac{1}{2}\int_{-5}^{-1} t^{-3} dt = \frac{1}{2}\left[\frac{t^{-2}}{-2}\right]_{-5}^{-1} = -\frac{6}{25}$$

Ex.13-3 Find the following definite integrals

$$\int_0^2 x^2 e^x dx$$

Ans.

Pause the video and solve by yourself

Ex.13-3 Find the following definite integrals

$$\int_0^2 x^2 e^x dx$$

Ans.

In this exercise, we apply the formula twice as follows.

$$\int_0^2 x^2 e^x dx = \left[x^2 e^x\right]_0^2 - \int_0^2 2xe^x dx = 4e^2 - 2\left(\left[xe^x\right]_0^2 - \int_0^2 e^x dx\right)$$

$$= 4e^2 - 2\left(2e^2 - \left[e^x\right]_0^2\right) = 4e^2 - 4e^2 + 2\left(e^2 - 1\right) = 2\left(e^2 - 1\right)$$