NAGOYA UNIVERSITY

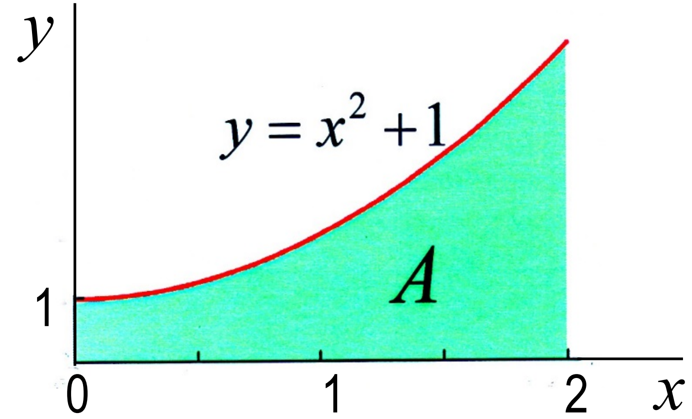
Lesson 11
Estimating Area by Rectangles

11A

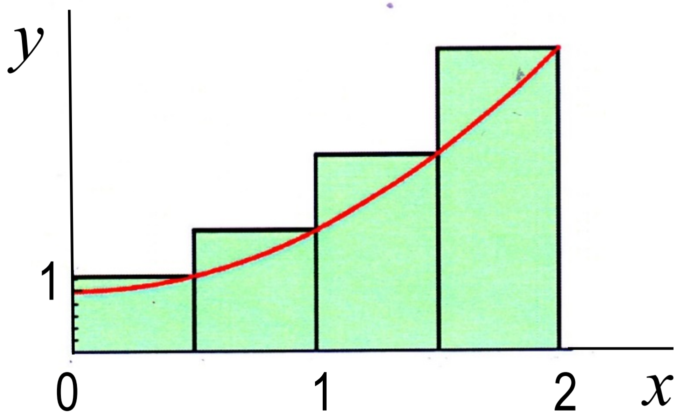
- Estimating Area by Rectangles

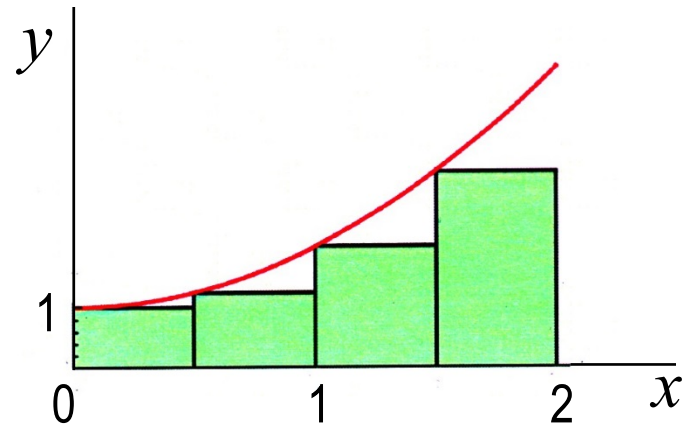
Estimation of Area by Rectangles

Calculate the area A between $y = x^2 + 1$ and $y = 0$ on $[0, 2]$.
(Suppose that we don't know the definite integral.)



Sprit [0, 2] into 4 subintervals.





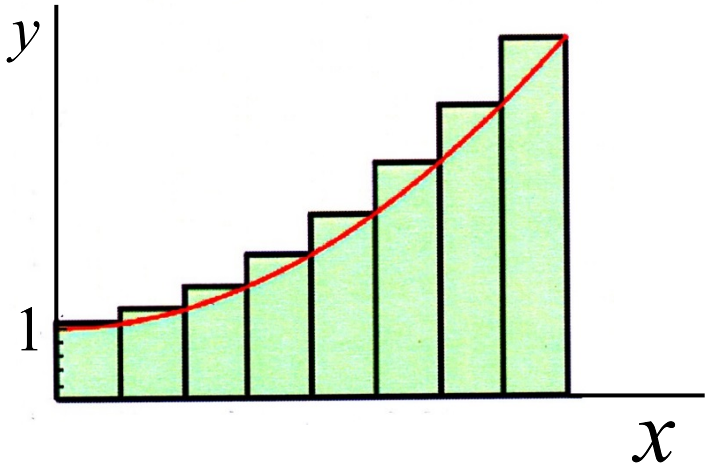
$$A_R = \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right) + \frac{1}{2} f(2)$$
$$= \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) + \frac{1}{2}(5) = 5.75$$

$$A_L = \frac{1}{2} f(0) + \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) + \frac{1}{2} f\left(\frac{3}{2}\right)$$
$$= \frac{1}{2}(1) + \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) = 3.75$$

$$3.75 < A < 5.75$$

Estimation of Area by Rectangles – *Cont*.

Sprit [0, 2] into 8 subintervals.



$$4.1875 < A < 5.1875$$

Exact value $A = \int_0^2 (x^2 + 1)dt = \left[\frac{x^3}{3} + x\right]_0^2 = \frac{14}{3} = 4.666$

Theorem

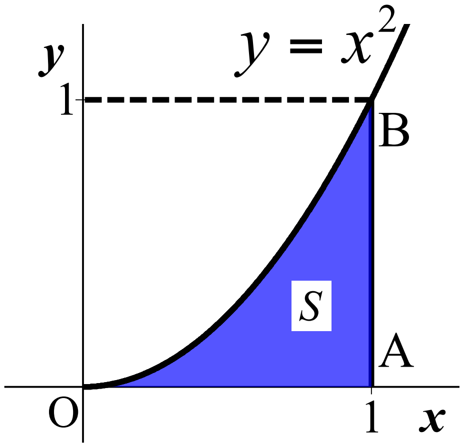
If $f(x)$ is continuous on $[a, b]$ then the right and left endpoints approximations approach one and the same limit as $N \to \infty$

$$\lim_{N\to\infty} A_R = \lim_{N\to\infty} A_L = L$$

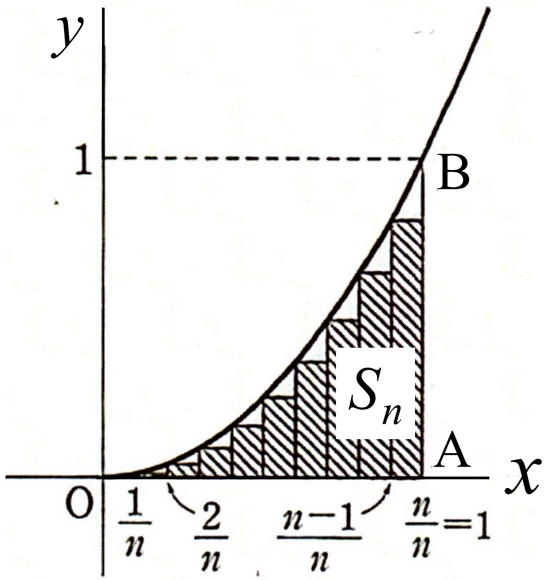
Example

Examples 11-1 Find the area between $y = x^2$ and $y = 0$ on [0, 1] by making $n \to \infty$. Use the following formula.

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(n+2)}{6}$$



Ans.



$$S_n = \frac{1}{n}\left\{0 + \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots\ldots + \left(\frac{n+1}{n}\right)^2\right\}$$

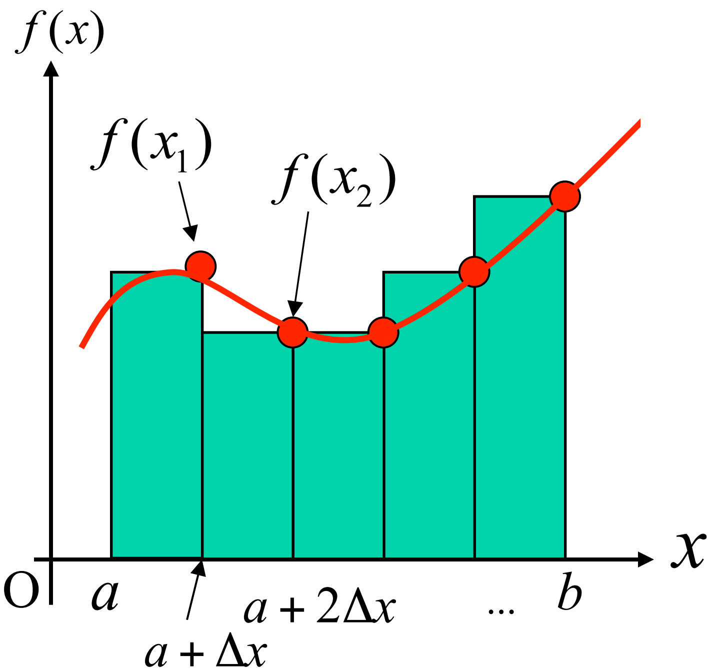
$$= \frac{1}{n^3}\left\{1^2 + 2^2 + \ldots\ldots + (n-1)^2\right\}$$

$$= \frac{1}{n^3} \cdot \frac{1}{6}(n-1)n(2n-1)$$

Therefore $\lim_{n \to \infty} S_n = \frac{1}{3}$

<Compare> $S = \int_0^1 x^2 dt = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

Historical Definition of the Definite Integral



$$a = x_0,\ x_1,\ x_2, \ldots\ldots,\ x_n = b$$

Width $\quad \Delta x = (b-a)/n$

The right-endpoint approximation:

$$A_{Rn} = f(x_1)\Delta x + f(x_2)\Delta x + \ldots\ldots + f(x_n)\Delta x$$

$$= \sum_{k=1}^{n} f(x_k)\Delta x$$

(Riemann sum of f over $[a, b]$).

Historical definition of the definite integral

Left-endpoint approx.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\,\Delta x$$

Right-endpoint approx.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

Lesson 11
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11B

- Finding of the Limit Value of a Series

Finding the Limit Value of a Series by an Integral

If we put $a = 0, \quad b = 1$, we have

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$$

Left-endpoint approx.

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

Right-endpoint approx.

This relationship is sometimes used to find the sum of infinite series.

Example

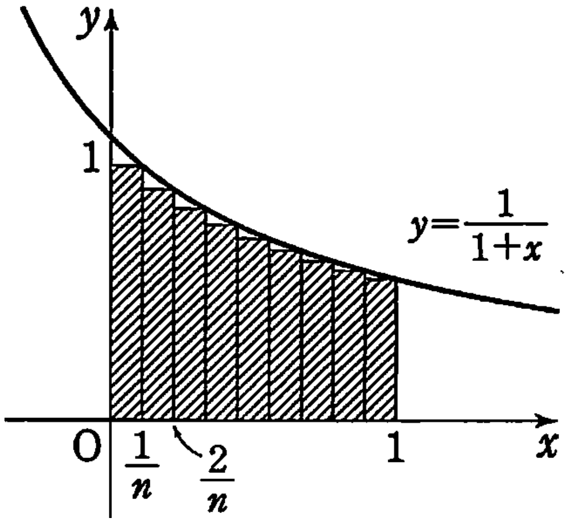
Examples 11-2 Find the limit value of

$$S=\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{2n}\right)$$

Ans. Take out quantity $\frac{1}{n}$ from all terms.

$$S=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\frac{1}{1+\frac{3}{n}}+\cdots\cdots+\frac{1}{1+\frac{n}{n}}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{1+\frac{k}{n}}\right)$$



Replace the part $\frac{k}{n}$ by x and make the definite integral

$$S=\int_0^1\frac{dx}{1+x}=\Big[\log(1+x)\Big]_0^1=\log 2$$

Exercise

Ex.11-1

Find the value of the following series.

$$S = \lim_{n \to \infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots\cdots + \sin\frac{n\pi}{n}\right)$$

Ans.

Pause the video and solve the problem by yourself.

Ex.11-1

Find the value of

$$S=\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots\cdots+\sin\frac{n\pi}{n}\right)$$

Ans.

$$S=\lim_{h\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin\left(\frac{k}{n}\pi\right)=\int_0^1\sin(\pi x)dx=\left[-\frac{\cos(\pi x)}{\pi}\right]_0^1$$

$$=-\left(\frac{\cos\pi-1}{\pi}\right)=\frac{2}{\pi}$$