NAGOYA UNIVERSITY

Lesson 15
Linear Differential Equations (2)

15A

- First Order Linear Differential Equation

First Order Linear D.E.

First order linear D.E.

$$y' + P(x)y = Q(x)$$

: Nonhomogeneous D.E.

When $Q(x) = 0$

$$y' + P(x)y = 0$$

: Homogeneous D.E.

How to solve first order linear D.E.

Step 1: **Find the solution of homogeneous D.E.**
(Since this homogeneous is a separable D. E. , we can apply the previous process.)

Step 2: **Find the solution of nonhomogeneous D.E.**
utilizing the result of the corresponding homogeneous equation

Solving Homogeneous D.E.

Homogeneous first order linear D.E.

$$y' + P(x)y = 0$$

Separate the variables

$$\frac{1}{y}dy = -P(x)dx$$

By integration

$$\int \frac{1}{y}dy = -\int P(x)dx \quad \therefore \quad \ln|y| = -\int P(x)dx + C_1$$

Therefore, the

$$y = Ce^{-\int P(x)dx}$$

Solving Nonhomogeneous D.E.

Nonhomogeneous D.E.

$$y' + P(x)y = Q(x)$$

Assumption

$$y = u(x)e^{-\int P(x)dx}$$

Revise ←

Solution of the homogeneous D.E.

$$y = Ce^{-\int P(x)dx}$$

Substitution

$$\left[\frac{du}{dx}e^{-\int P(x)dx} - u(x)P(x)e^{-\int P(x)dx}\right] + P(x)u(x)e^{-\int P(x)dx} = Q(x)$$

$$\therefore \quad \frac{du}{dx} = Q(x)e^{\int P(x)dx} \qquad \therefore \quad u = \int\left(Q(x)e^{\int P(x)dx}\right)dx + C$$

General solution $\quad y = \left\{\int\left(Q(x)e^{\int P(x)dx}\right)dx + C\right\}e^{-\int P(x)dx}$

Example

[Examples 15-1] Find the solution of $y' - 2xy = 2x$

Ans. First step: $$\frac{dy}{dx} - 2xy = 0$$

$$\int \frac{dy}{y} = \int 2x\,dx \quad \therefore \quad \ln|y| = x^2 + C_1 \quad \therefore \quad y = e^{x^2 + C_1} = Ce^{x^2}$$

Second step: Put $y = u(x)e^{x^2}$

$$\left\{u'e^{x^2} + 2uxe^{x^2}\right\} - 2xy = 2x \quad \therefore \quad u'e^{x^2} = 2x \quad \therefore \quad u = \int 2xe^{-x^2}dx + C$$

After substitution, we have

$$\therefore \quad u = \left(\int 2xe^{-x^2}dx + C\right)e^{x^2} = \left(-e^{-x^2} + C\right)e^{x^2} = Ce^{x^2} - 1$$

Exercise

[Ex.15-1] An object of mass m is dropped from a bridge. It is assumed that the air resistance cv proportional to the velocity v works. From Newton's Law, the equation of motion is given by

$$m\frac{dv}{dt} = -cv + mg$$

Find the velocity v as a function of time t.

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.15-1] An object of mass m is dropped from a bridge. It is assumed that the air resistance cv proportional to the velocity v works. From Newton's Law, the equation of motion is given by $m\frac{dv}{dt} = -cv + mg$ Find the velocity v as a function of time t.

Ans.

Introducing $\gamma = \frac{c}{m}$, we have $\frac{dv}{dt} + \gamma v = g$

[Step 1]

$$\frac{dv}{dt} + \gamma v = 0 \qquad \therefore \int \frac{dv}{v} = -\gamma \int dt$$

$$\therefore \ln|v| = -\gamma t + c_1 \qquad \therefore v = e^{-\gamma t} e^{c_1} = Ce^{-\gamma t}$$

Answer to the Exercise - *Cont.*-

[Step 2] Assume the solution $v = u(t)e^{-\gamma t}$

Substituting this, we have

$$(u'e^{-\gamma t} - u\gamma e^{-\gamma t}) + \gamma u e^{-\gamma t} = g \quad \therefore \; u' = ge^{\gamma t} \qquad \therefore \; u = g\int e^{\gamma t}dt$$

General solution

$$v = \left(g\int e^{\gamma t}dt\right)e^{-\gamma t} = \left(\frac{g}{\gamma}e^{\gamma t} + C\right)e^{-\gamma t} = \frac{g}{\gamma} + Ce^{-\gamma t}$$

Apply boundary condition $v = 0$ at $t = 0$

$$0 = \frac{g}{\gamma} + Ce^{0} \qquad \therefore \; C = -\frac{g}{b_1}$$

The solution is $$v = \frac{g}{\gamma} - \frac{g}{\gamma}e^{-b_1 t} = \frac{gm}{c}(1 - e^{-\frac{c}{m}t})$$

Lesson 15

Second Order Differential Equations

15B

- Second Order D.E. with Constant Coefficients

Second Order Linear D.E.

Second order D.E.

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

The general solution

$$y = y_h + y_p$$

General solution of the adjoint homogeneous D.E.

Particular solution of the original D.E.

Finding Solutions of the Homogeneous D.E.

The corresponding homogeneous D.E.

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

Assumed solution $y = Ce^{\lambda t}$

Substituting this, we have $C(\lambda^2 + a\lambda + b)e^{\lambda t} = 0$

$$\therefore \quad \lambda^2 + a\lambda + b = 0 \qquad \therefore \quad \lambda = \frac{a \pm \sqrt{b^2 - 4ac}}{2} \quad (\equiv \lambda_1, \lambda_2)$$

General solution of a homogeneous equation

$$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

General Solutions of the Nonhomogeneous D.E.

The solution of nonhomogeneous D.E.

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

is given by

$$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + y_p$$

($y = y_p$ is a particular solution)

[Note] In order to understand the phenomena governed by the D.E., we must discuss the results corresponding to the following three cases.

(1) λ_1 and λ_2 are two different real toots

(2) λ_1 and λ_2 are two complex roots

(3) λ_1 and λ_2 are a double root

Example

[Examples 15-2] Find the general solution of $$\frac{d^2y}{dx^2} - 4y = \sin 3x$$

Ans. (1) Homogeneous equation is $\frac{d^2y}{dx^2} - 4y = 0$

Assume the solution as $y = Ce^{\lambda x}$

After substitution $C(\lambda^2 - 1)e^{\lambda x} = 0$ $\quad \therefore \quad \lambda = 2, -2$

Solution $y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

(2) Assume the solution as $y_p = A\sin 3x$

After substitution $(-9-4)A\sin 3x = \sin 3x$

The particular solution $y_p = -\frac{1}{13}\sin 3x$

(3) General solution

$$y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} - \frac{1}{13}\sin 3x$$