Course III

NAGOYA UNIVERSITY

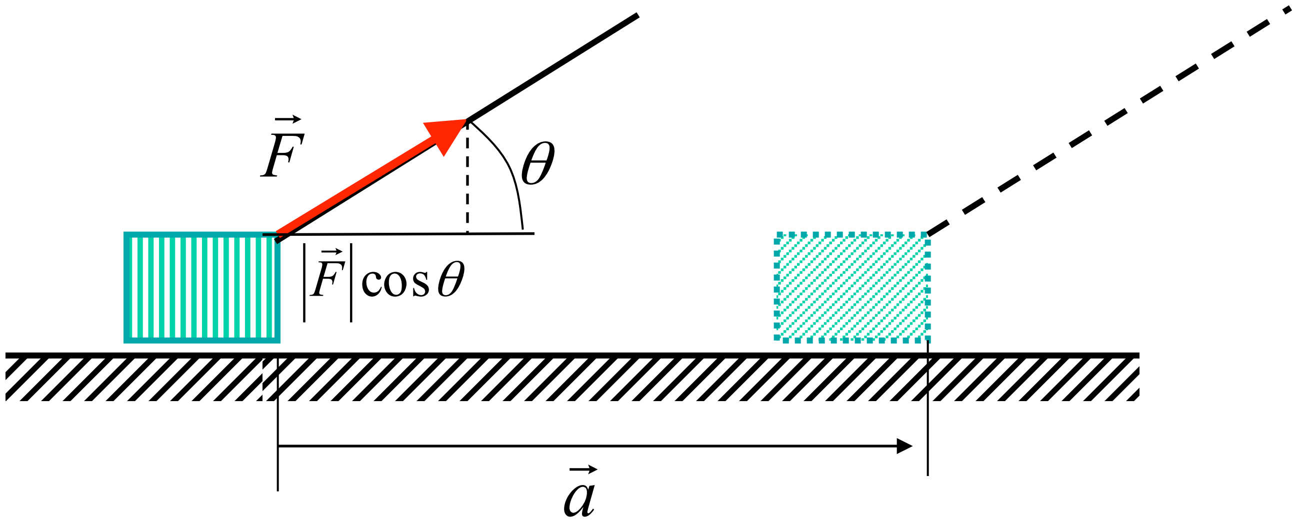
Lesson 02
Scalar Product

2A

- Definitions of the scalar product
- Basic rules

Example : Work

Movement by a constant force

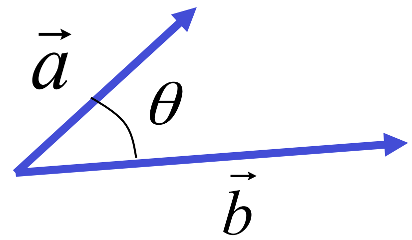


Work $W = |\vec{F}||\vec{a}|\cos\theta$

Scalar product $\vec{F} \cdot \vec{a}$

Scalar Product

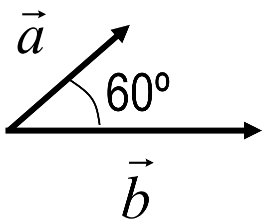
$$\vec{a} \cdot \vec{b} = |a||b|\cos\theta$$



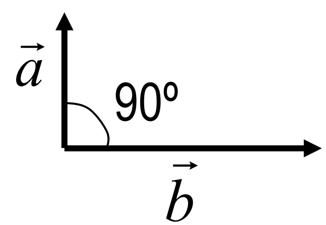
(where $0 \le \theta \le \pi$)

[Examples 2-1] Find the following dot products where $|\vec{a}| = 2$ and $|\vec{b}| = 3$.

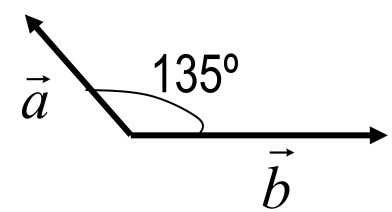
(1)



(2)



(3)



(1) $\vec{a} \cdot \vec{b} = 3 \cdot 2\cos 60° = 3$

(2) $\vec{a} \cdot \vec{b} = 3 \cdot 2\cos 90° = 0$

(3) $\vec{a} \cdot \vec{b} = 3 \cdot 2\cos 135° = -3\sqrt{2}$

Properties and Rules

Properties

1. Perpendicular condition $\vec{a}\cdot\vec{b}=0$
2. Parallel condition $\vec{a}\cdot\vec{b}=\pm|a||b|$
3. Unit vectors $\vec{i}\cdot\vec{j}=0$ $\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=1$

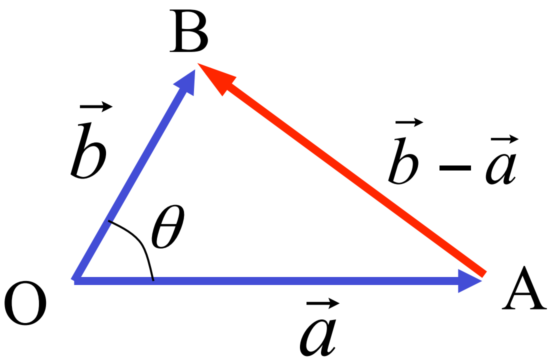
Fundamental Rules

1. Commutative law $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
2. Distributive law $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
3. Scalar multiplication $k(\vec{a}\cdot\vec{b})=k\vec{a}\cdot\vec{b}=\vec{a}\cdot k\vec{b}$

Representation by Components

When $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$



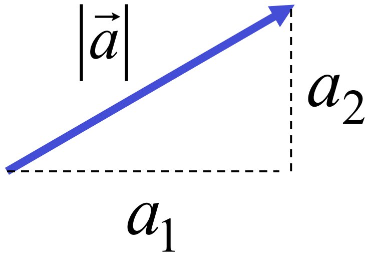
[Proof]

From the Law of Cosine

$$\mathrm{AB}^2 = \mathrm{OA}^2 + \mathrm{OB}^2 - 2\mathrm{OA} \times \mathrm{OB} \times \cos\theta$$

By using vectors, we have

$$|\vec{b} - \vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$$



Applying Pythagorean theorem

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 = (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2\vec{a} \cdot \vec{b}$$

After rearrangement

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

Example

[Examples 2-1] Find the scalar product and the angle of the following two vectors.

(1) $\vec{a}=(2,\ 1),\ \ \vec{b}=(-3,\ 1)$ (2) $\vec{a}=(-1,\ 3),\ \ \vec{b}=(6,\ 2)$

Ans. (1) Inner product

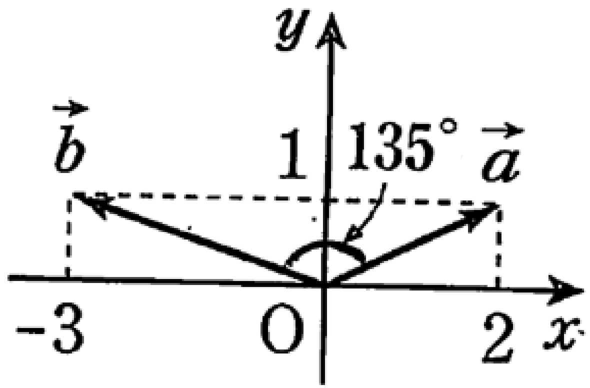
$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2=2\cdot(-3)+1\cdot1=-5$$

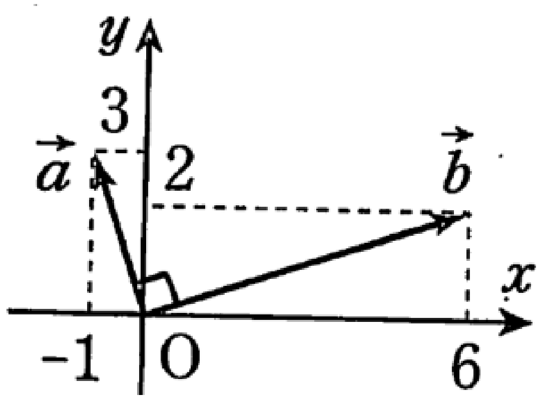
Let the angle be θ, then

$$\sqrt{2^2+1^2}\cdot\sqrt{(-3)^2+1^2}\cos\theta=-5$$

Therefore

$$\cos\theta=\frac{-5}{\sqrt{5}\sqrt{10}}=-\frac{1}{\sqrt{2}}\qquad \theta=135^\circ$$





(2) Inner product

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2=(-1)\cdot6+3\cdot2=0$$

Therefore

$$\theta=90^\circ$$

Exercise

[Ex.2-1] Find unit vectors which are perpendicular to the vector

$$\vec{a} = (\sqrt{3}, -1)$$

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

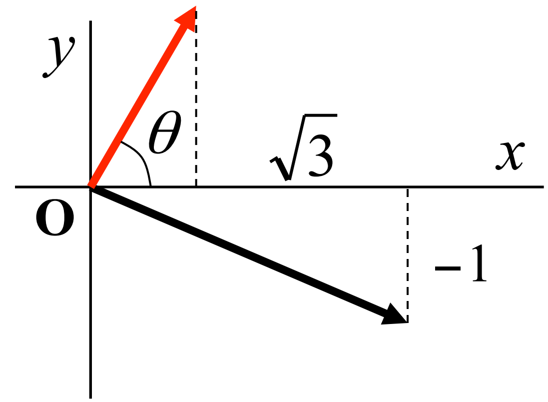
[Ex.2-1] Find a unit vectors which are perpendicular to the vector

$$\vec{a} = (\sqrt{3},\ -1)$$

Ans.

(1) Let the unit vector be

$$\vec{e} = (\cos\theta,\ \sin\theta)$$

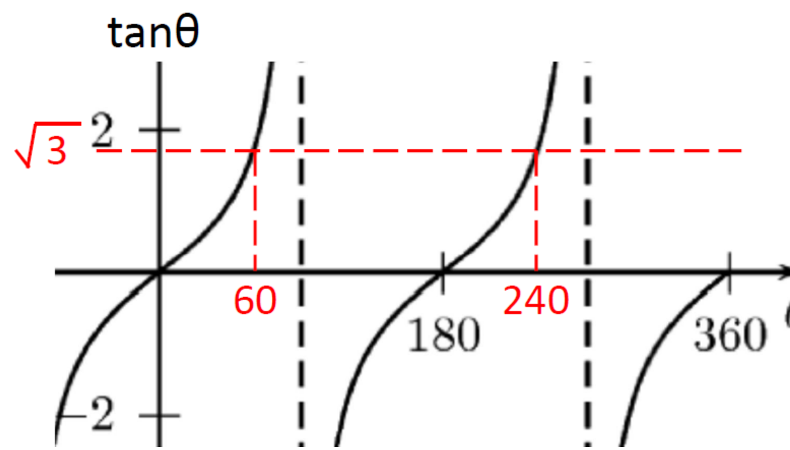


From the given condition,

$$\vec{e}\cdot\vec{a} = \sqrt{3}\cos\theta - \sin\theta = 0$$

$$\therefore \quad \tan\theta = \sqrt{3}$$

$$\therefore \quad \theta = 60°,\ 240°$$



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Lesson 02
Basic Rules of Vectors

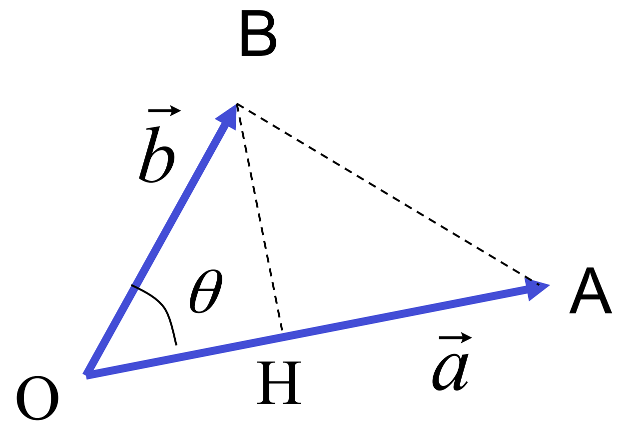
2B

- Application of the scalar product

Area

Perpendicular line : BH

BH= $\left|\vec{b}\right|\sin\theta$



Therefore, the area is

$$S=\frac{1}{2}\left|\vec{a}\right|\mathrm{BH}=\frac{1}{2}\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta=\frac{1}{2}\left|\vec{a}\right|\left|\vec{b}\right|\sqrt{1-\cos^2\theta}$$

$$=\frac{1}{2}\sqrt{\left(\left|\vec{a}\right|\left|\vec{b}\right|\right)^2-\left(\left|\vec{a}\right|\left|\vec{b}\right|\right)^2\cos^2\theta}=\frac{1}{2}\sqrt{\left|\vec{a}\right|^2\left|\vec{b}\right|^2-\left(\vec{a}\cdot\vec{b}\right)^2}$$

$$S=\frac{1}{2}\sqrt{\left|\vec{a}\right|^2\left|\vec{b}\right|^2-\left(\vec{a}\cdot\vec{b}\right)^2}$$

Example

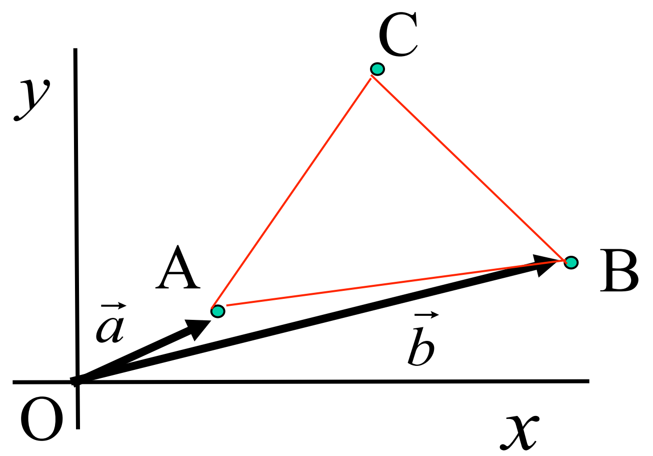
[Examples 2-2]

(1) Find the area made by two vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$.

(2) There are three points A(2, 1), B(7, 2), C(4, 5). Find the area of △ABC

Ans. (1) Magnitudes of vectors are

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2} \qquad |\vec{b}| = \sqrt{b_1^2 + b_2^2}$$



The scalar product is $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

Then

$$S = \frac{1}{2}\sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}$$

$$= \frac{1}{2}\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2}$$

$$= \frac{1}{2}\sqrt{(a_1^2 b_2^2 + a_2^2 b_1^2) - 2a_1 b_1 a_2 b_2} = \frac{1}{2}\sqrt{(a_1 b_2 - a_2 b_1)^2} = \frac{1}{2}|a_1 b_2 - a_2 b_1|$$

(2) $\vec{a} = \vec{AB} = (7,2) - (2,1) = (5,1) \quad \vec{b} = \vec{AC} = (4,5) - (2,1) = (2,4)$

$$S = \frac{1}{2}|a_1 b_2 - a_2 b_1| = \frac{1}{2}|5 \cdot 4 - 1 \cdot 2| = 9$$

Exercise

[Ex2-2] There are three points A(4, 1), B(5, 4) and C(2, 3) on a plane. Answer the following questions. (1) Let ∠BAC= θ . Find $\cos\theta$ and $\sin\theta$. (2) Find the area of △ABC.

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

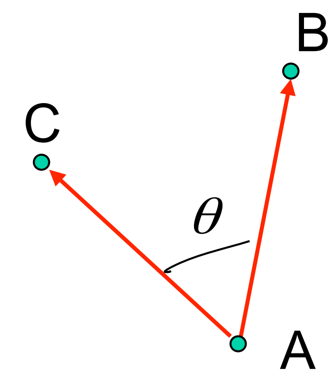
[Ex2-2] There are three points A(4, 1), B(5, 4) and C(2, 3) on a plane. Answer the following questions. (1) Let $\angle$BAC=θ . Find $\cos\theta$ and $\sin\theta$.. (2) Find the area of $\triangle$ABC.

Ans. (1) $\vec{a} = \vec{AB} = (5,4) - (4,1) = (1,3)$ $\vec{b} = \vec{AC} = (2,3) - (4,1) = (-2,2)$

Therefore, $|\vec{AB}| = \sqrt{1^2 + 3^2} = \sqrt{10},$ $|\vec{AC}| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$

$$\vec{AB} \cdot \vec{AC} = 1 \cdot (-2) + 3 \cdot 2 = 4$$

From the formula, $\cos\theta = \dfrac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|} = \dfrac{4}{\sqrt{10}\,2\sqrt{2}} = \dfrac{1}{\sqrt{5}}$



Since $\sin\theta > 0$, we have

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{1}{5}} = \frac{2}{\sqrt{5}}$$

(2) $\triangle$ABC $= \dfrac{1}{2}|\vec{AB}||\vec{AC}|\sin\theta = \dfrac{1}{2} \cdot \sqrt{10} \cdot 2\sqrt{2} \cdot \dfrac{2}{\sqrt{5}} = 4$