NAGOYA UNIVERSITY

Lesson 05
Multiplication of Matrices

5A

- Multiplication of a row vector and a column vector
- Multiplication of a matrix and a matrix
- Basic rules of multiplication

Multiplication (Row Vector and Column Vector)

Scalar product

- Vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ ← With comma

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

Row vector and column vector

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

← No comma

Multiplication of a row vector and a column vector

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Example

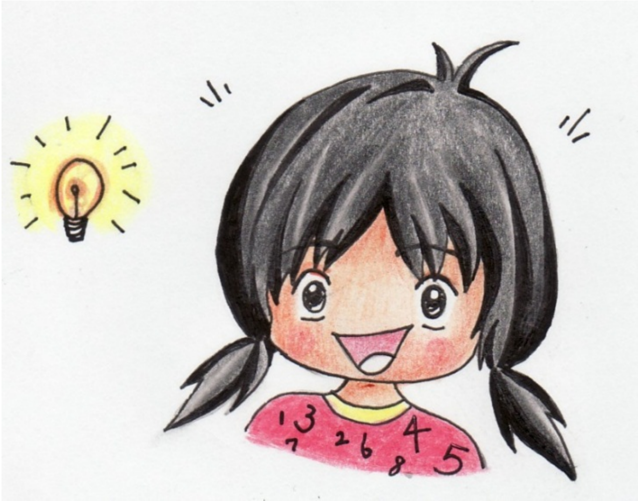
[Examples 5-1] Find the values of the following products.

(1) $\begin{pmatrix} 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}$

Ans.

(1) $\begin{pmatrix} 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} = 2 \times 8 + 6 \times 1 + 3 \times 4 = 34$

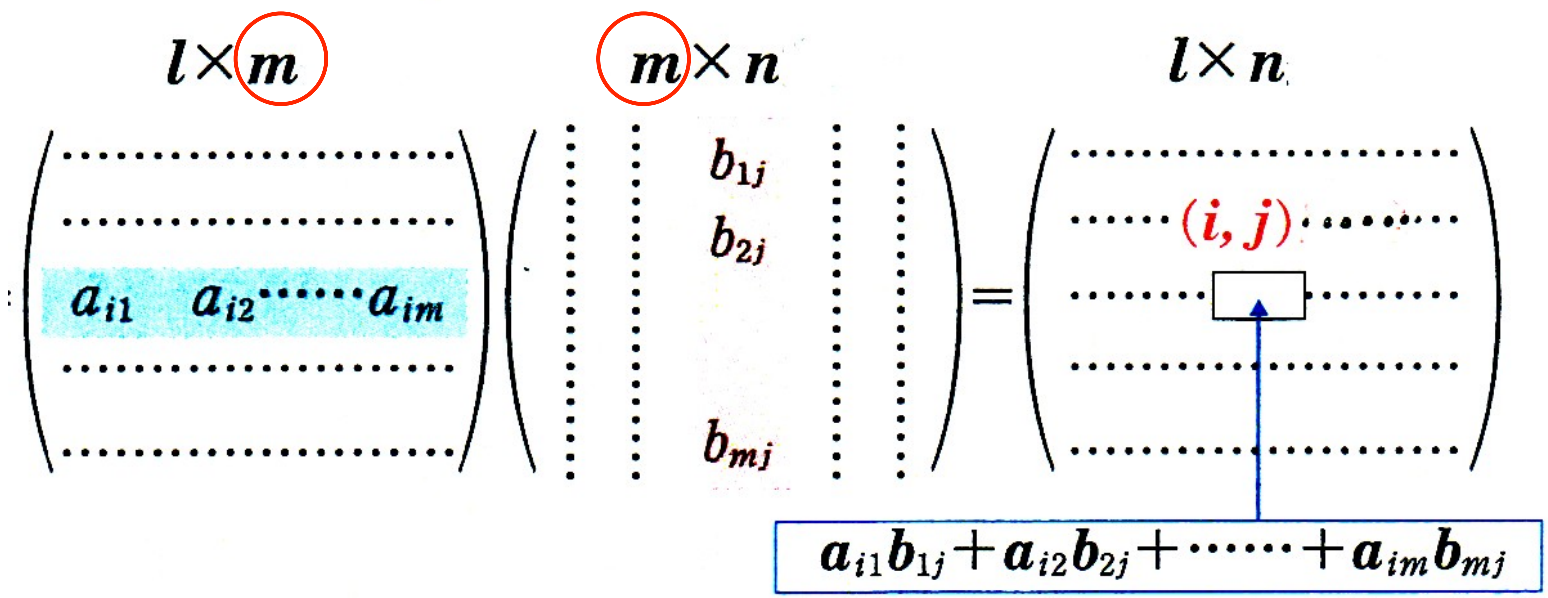
(2) This multiplication is not legal.



Ahh! That's so easy!

Multiplication of Two Matrices

- When matrix A and matrix B are multiplied, the number of columns of A ($l \times m$) has to be equal to the number of rows of B ($m \times n$).
- Product element c_{ij} = multiplication of the i-th row of A by the j-th column of B.



Example

[Examples 5-2] Calculate the following multiplication

(1) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & u \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 8 & 1 \\ 3 & 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 7 \\ 9 & -2 \\ 6 & 3 \end{pmatrix}$

Ans. (1) $\begin{pmatrix} ax+bz & ay+bu \\ cx+dz & cy+du \end{pmatrix}$

(2) $\begin{pmatrix} 2\times1+8\times9+1\times6 & 2\times7+8\times(-2)+1\times3 \\ 3\times1+6\times9+4\times6 & 3\times7+6\times(-2)+4\times3 \end{pmatrix} = \begin{pmatrix} 80 & 1 \\ 81 & 21 \end{pmatrix}$

Basic Rules of Multiplication

The following rules hold

1. About scalar multiplication $(kA)B = A(kB) = k(AB)$
2. The associative law $(AB)C = A(BC)$
3. The distributive law $(A+B)C = AC + BC$
   $A(B+C) = AB + AC$

But the commutative law does not hold

$$AB \neq BA$$

$l \times m \quad m \times n \qquad m \times n \quad l \times m$

Example

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 4 & 8 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 6 & 6 \end{pmatrix}$$

Identity matrix (Unit matrix)

A square matrix with ones on the diagonal element and zeros elsewhere.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zero matrix

A matrix with all its entries being zero.

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Properties

1. $AI = IA = A$

2. $AO = OA = O$

3. [note] Product of non-zero matrices may become zero.

Example

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$AB = O$$

Exercise

[Ex.5-1] Calculate the following multiplication.

(1) $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.5-1] Calculate the following multiplication.

(1) $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Ans.

(1) $$\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 4\cdot(-1)+2\cdot 5 & 4\cdot 0+2\cdot(-1) \\ 3\cdot(-1)+1\cdot 5 & 3\cdot 0+1\cdot(-1) \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 2 & -1 \end{pmatrix}$$

(2) $$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0+0 & 1+1+0 & 1+1+1 \\ 0+0+0 & 0+1+0 & 0+1+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Lesson 05

Multiplication of Matrices and Determinant

5B

- Definition of the determinant
- Calculating the determinant

Definition of a Determinant

- The determinant of a matrix is a special number that can be calculated from a matrix.

- It is useful in the study of a system of linear equations, that of a linear transformation, and others.

- The determinant is defined for a square matrix.

Notation

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \quad |A| \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \det A$$

Calculation of Determinant for 2×2-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The determinant is $|A| = ad - bc$

Memorize this calculation as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Example $\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = +2 \cdot 5 - 3 \cdot 1 = 7$

Calculation of Determinant for 3×3-matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The determinant is

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Memorize this calculation as

$$+\ a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} \ -\ b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} \ +\ c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Minor and Cofactor

For a n×n matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{14} \\ a_{21} & a_{22} & \cdots & a_{24} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Minor

Minor M_{ij} is the determinant of the $(n-1)\times(n-1)$–matrix obtained by deleting the i-th row and the j-th column.

(i, j)

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{14} \\ a_{21} & a_{22} & \cdots & a_{24} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

Cofactor

The cofactor C_{ij} is a number which has such a sign to the minor.

$$C_{11} = M_{11},\ C_{12} = -M_{12},\ C_{13} = M_{13},\ \cdots$$

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Determinant

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots$$

Example

[Examples 5-2] Caclulate the determinant of this matrix.

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -5 & 0 & -2 \end{pmatrix}$$

Ans.

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 4 & 3 \\ -5 & 0 & -2 \end{vmatrix} = 2\begin{vmatrix} 4 & 3 \\ 0 & -2 \end{vmatrix} - \begin{vmatrix} 0 & 3 \\ -5 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 4 \\ -5 & 0 \end{vmatrix}$$

$$= 2 \cdot (-8) - 1 \cdot 15 + (-1) \cdot 20 = -51$$

Exercise

[Ex5-2] Calculate the determinants of these matrices.

(1)

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

(2)

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex5-2] Calculate the determinants of these matrices.

(1)
$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

(2)
$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ans.

(1)
$$|A| = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = 2 \cdot 3 - 1 \cdot (-1) = 7$$

(2)
$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 4 \\ -5 & 0 \end{vmatrix}$$

$$= 2 \cdot 3 - 1 \cdot 1 + 0 = 5$$