Course III

Nagoya University

Lesson 06
Inverse Matrix and Simultaneous Equations

6A

- Inverse Matrix

Inverse Matrix

Inverse of a number

Original number a → The inverse number $x = \frac{1}{a} = a^{-1}$

$xa = 1$

Inverse of a matrix

Original square matrix A

→ If X satisfies $AX = XA = I$, then X is an inverse of matrix

Notation : A^{-1}

How to Find the Inverse Matrix (2x2-matrix)

For 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The inverse matrix is given as follows

When $|A| = ad - bc \neq 0$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

When $|A| = ad - bc = 0$

A^{-1} does not exist

Proof : Refer to the next example

Example

[Example 6-1] Find the inverse matrix of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Ans. Let the inverse matrix be $X = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$

From $AX = I$, we have

$$\begin{pmatrix} a & b \\ c & w \end{pmatrix}\begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} ax+by & az+bw \\ cx+dy & cz+dw \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore $ax + by = 1 \quad az + dw = 0$ then $x(ad-bc) = d \quad z(ad-bc) = -b$

$cx + dy = 0 \quad cz + dw = 1 \quad y(ad-bc) = -c \quad w(ad-bc) = a$

When $(ad - bc) \neq 0$

$$x = \frac{d}{(ad-bc)}, \quad y = -\frac{c}{(ad-bc)}, \quad z = -\frac{b}{(ad-bc)}, \quad w = \frac{a}{(ad-bc)}$$

When $(ad - bc) = 0$

$a = b = c = d = 0$ This does not satisfy $AX = I$

Example

[Example 6-2] Find the inverse matrix of

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix}$$

Ans. The determinant of matrix A

$$|A| = \begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} = 2 \cdot 4 - 1 \cdot 4 = 4$$

The inverse matrix

$$A^{-1} = \frac{1}{4} \begin{vmatrix} 4 & -1 \\ -4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1/4 \\ -1 & 1/2 \end{vmatrix}$$

Exercise

[Ex.6-1] Find the inverse of the following matrices.

(1) $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$

(2) $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.6-1] Find the inverse of the following matrices.

(1) $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ (2) $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

Ans.

(1) The determinant is $|A| = \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = 3 \times 3 - 4 \times 2 = 1 \neq 0$

Therefore $A^{-1} = \frac{1}{1} \begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix}$

(2) The determinant is $|A| = \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 2 \times 6 - 4 \times 3 = 0$

Therefore, the inverse does not exist

Lesson 06
Inverse Matrix and Simultaneous Equations

6B

- Simultaneous equation
-

Simultaneous Equations

Simultaneous equations

$$\left.\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}\right\} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Using notations $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $P = \begin{pmatrix} p \\ q \end{pmatrix}$, we have

$$AX = P$$

If $ad - bc \neq 0$, we have $A^{-1}AX = A^{-1}P$ $\quad\therefore\ X = A^{-1}P$

If $ad - bc = 0$, there exist infinite number of solutions

or there is no solution (refer to the example)

Example

[Examples 6-2] Solve the following simultaneous equations.

$$\left.\begin{aligned} 3x - y &= 3 \\ -2x + y &= 5 \end{aligned}\right\}$$

Ans. (1) The matrix form

$$\begin{pmatrix} 3 & 1 \\ 9 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

The determinant $|A| = \begin{vmatrix} 3 & 1 \\ 9 & 4 \end{vmatrix} = 3$

The inverse is

$$\begin{pmatrix} 3 & 1 \\ 9 & 4 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -9 & 3 \end{pmatrix} = \begin{pmatrix} 4/3 & -1/3 \\ -3 & 1 \end{pmatrix}$$

Premultiplying this inverse

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/3 & -1/3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Therefore $x = 2,\ y = 3$

Example

[Examples 6-3] Solve the following simultaneous equations.

(1) $$\left.\begin{array}{r} 2x - y = -5 \\ -6x + 3y = 15 \end{array}\right\}$$ (2) $$\left.\begin{array}{r} 2x + y = 2 \\ 4x + 2y = 3 \end{array}\right\}$$

Ans. (1) The determinant of coefficients $|A| = \begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} = 0$

Dividing the second equation by (-3), $2x - y = -5$

This is the same as the first equation.

There are infinite number of solution (x, y) which satisfy $2x - y = -5$

(2) The determinant of coefficients $|A| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$

Multiplying the first equation by 2, we have $4x + 2y = 4$

This contradicts to the second equation.

So, there is no solution

Exercise

[Ex. 6-2] Solve the following simultaneous equation.

$$\left.\begin{aligned} 3x - y &= 3 \\ -2x + y &= 5 \end{aligned}\right\}$$

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex. 6-2] Solve the following simultaneous equation.

$$\left.\begin{aligned} 3x - y &= 3 \\ -2x + 3y &= 5 \end{aligned}\right\}$$

Ans. The matrix form is $\begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

The determinant of coefficients $|A| = \begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} = 7 \neq 0$

The inverse is

$$\begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix}^{-1} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3/7 & 1/7 \\ 2/7 & 3/7 \end{pmatrix}$$

By premultiplying this inverse matrix, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/7 & 1/7 \\ 2/7 & 3/7 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Exercise

[Ex. 6-3] Solve the following simultaneous equation.

(1) $\left.\begin{array}{r} 2x - y = -5 \\ -6x + 3y = 15 \end{array}\right\}$

(2) $\left.\begin{array}{r} 2x + y = 2 \\ 4x + 2y = 3 \end{array}\right\}$

Pause the video and solve the problem by yourself.

Exercise

[Ex. 6-3] Solve the following simultaneous equation.

(1) $\left.\begin{array}{r} 2x - y = -5 \\ -6x + 3y = 15 \end{array}\right\}$ (2) $\left.\begin{array}{r} 2x + y = 2 \\ 4x + 2y = 3 \end{array}\right\}$

Ans. (1) The determinant of coefficients $|A| = \begin{vmatrix} 2 & -1 \\ -6 & 3 \end{vmatrix} = 0$

Dividing the second equation by (-3), $2x - y = -5$

This is the same as the first equation.

There are infinite number of solution (x, y) which satisfy $2x - y = -5$

(2) The determinant of coefficients $|A| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$

Multiplying the first equation by 2, we have $4x + 2y = 4$

This contradicts to the second equation.

So, there is no solution