NAGOYA UNIVERSITY

Lesson 07
Linear Transformation

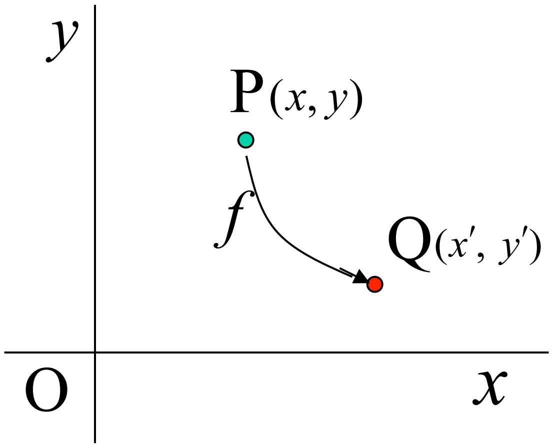
7A

- Linear transformation
- Reflections

Linear Transformation

Transformation

$$f:\ (x, y) \rightarrow (x', y')$$



Linear transformation

$$\left.\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}\right\} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

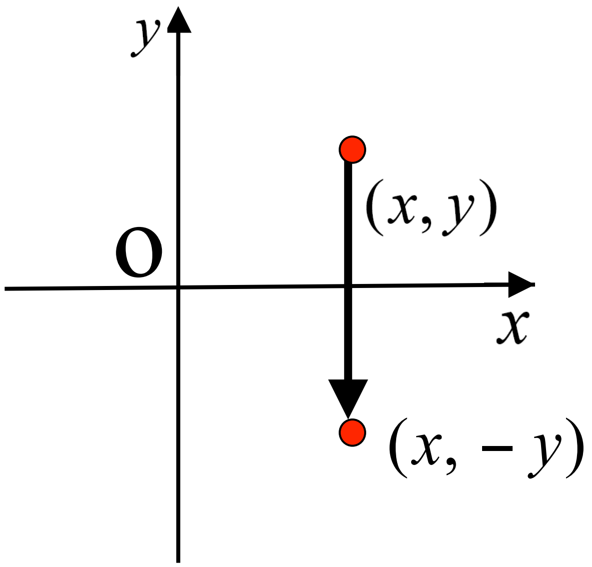
Return to the old coordinate

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

1. Reflection about the x-axis

$$f:\ (x, y) \rightarrow (x, -y)$$

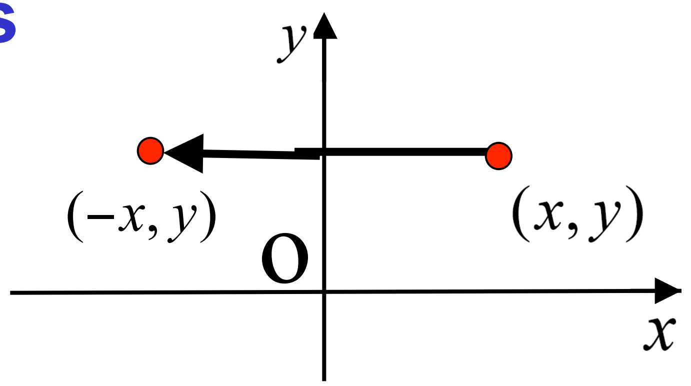
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



2. Reflection about the y-axis

$$f:\ (x, y) \rightarrow (-x, y)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

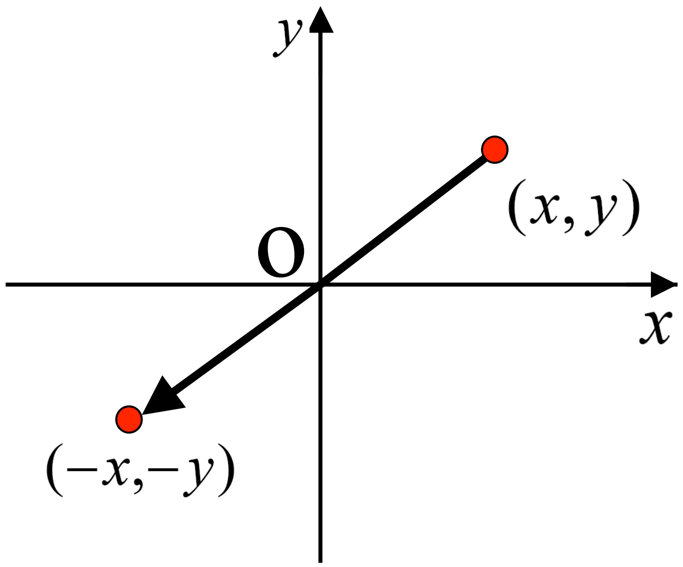


Representative Reflections *Contd.*

3. Reflection about the origin

$$f:\ (x, y) \to (-x, -y)$$

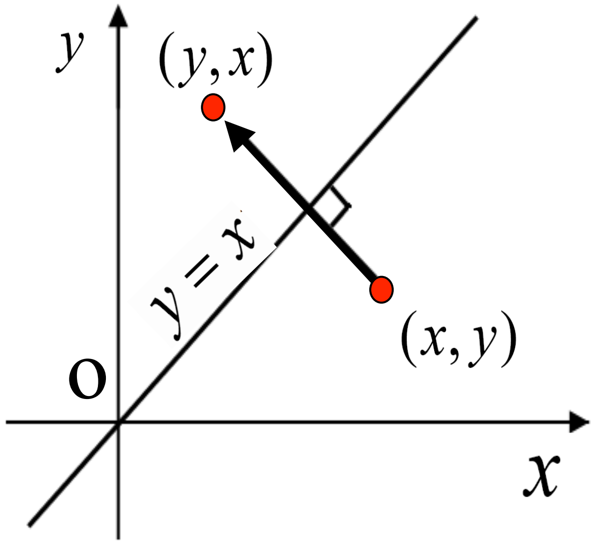
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



4. Reflection about the line y=x

$$f:\ (x, y) \to (y, x)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Example

[Example 7-1] By the linear transformation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, points (1, 1) and (2, 1) are transformed to the points (3, 2) and (7, 0), respectively. Find the transformation matrix A .

Ans.

Transformations : $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $\begin{pmatrix} 7 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

Since

$$\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \times 1 - 2 \times 1 = -1 \neq 0$$

Inverse matrix $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \dfrac{1}{(-1)}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

Postmultiplying

$$\begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -2 & 4 \end{pmatrix}$$

Exercise

[Ex.7-1] Three points P(-2, -2), Q(0, 2) and R(2, -2) are the vertices of a triangle. Consider the following linear transformation. $x' = x + y$, $y' = x - y$

Illustrate the original triangle and the transformed triangle P'Q'R'.

Ans.

Pause the video and solve the problem by yourself.

Answer to the Exercise

[Ex.7-1] Three points P(-2, -2), Q(0, 2) and R(2, -2) are the vertices of a triangle. Consider the following linear transformation. x'=x+y , y'=x-y

Illustrate the original triangle and the transformed triangle P'Q'R'.

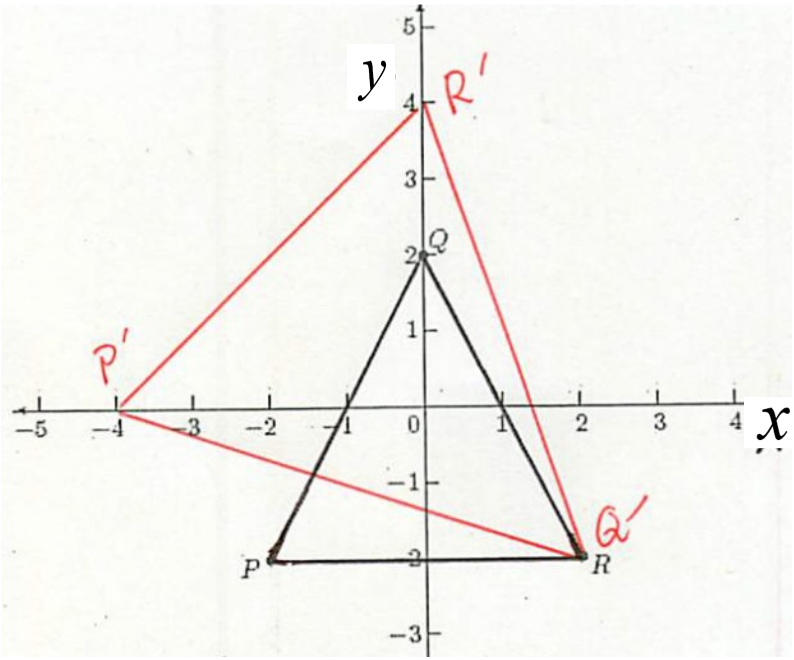
Ans.

The transformed vertices are

P' $$\begin{pmatrix} x_P \\ y_P \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

Q' $$\begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

R' $$\begin{pmatrix} x_R \\ y_R \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$



NAGOYA UNIVERSITY

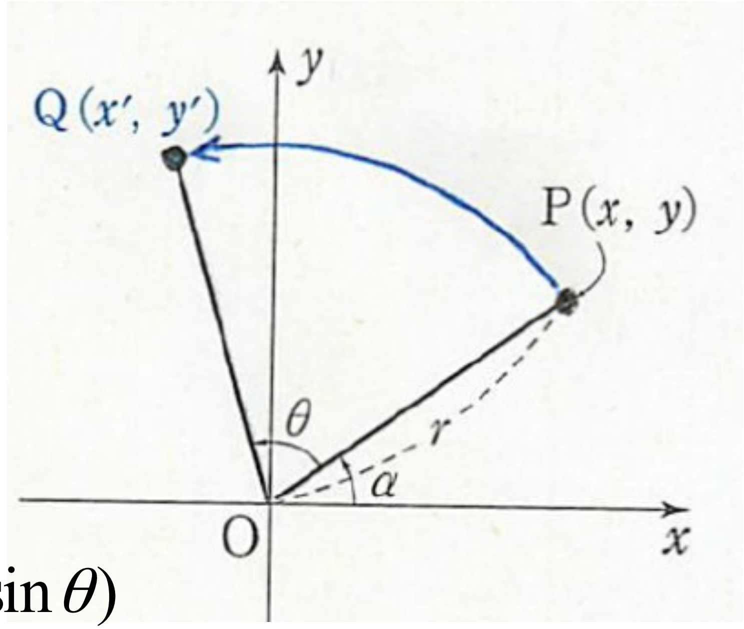
Lesson 07
Application: Linear Transformation

7B

- Rotation
-

Rotation

Rotation around the origin by θ



Original point P

$$x = r\cos\alpha \qquad y = r\sin\alpha \qquad (1)$$

Transferred point Q

$$x' = r\cos(\alpha+\theta) = r(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$
$$y' = r\sin(\alpha+\theta) = r(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

Substituting Eq.(1), we have

$$\left.\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}\right\}$$

Using matrix, we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Example

[Examples 7-2]

Find the new positions of each point after given rotation counterclockwise.

- Position (2, 4) , 60°   (2) Position $(\sqrt{3}, 1)$ 120°

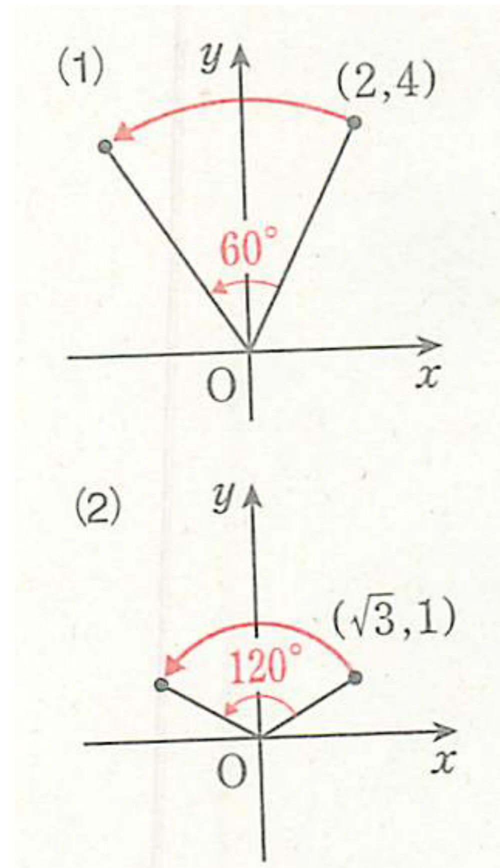
Ans. From the formula of rotation, we have

(1) $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1-2\sqrt{3} \\ \sqrt{3}+2 \end{pmatrix}$$

(2) $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$



End

Yeah！ I did it！

We appreciate your patience ar

Author