Global existence of solutions of the Navier-Stokes equations with the Coriolis force

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Abstract

The Cauchy problem for the Navier-Stokes equations with the Coriolis force is considered. It is proved that a similar a priori estimate, which is derived for the Navier-Stokes equations by Lei-Lin [12], holds under the effect of the Coriolis force. As an application existence of a unique global solution for arbitrary speed of rotation is proved, as well as its asymptotic behavior.

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Chapter 1

Introduction

In this thesis, we consider the initial value problem of the Navier-Stokes equations with the Coriolis force in $\mathbb{R}^3$,

$$\begin{cases} \partial_t u - \nu\Delta u + \Omega e_3 \times u + (u, \nabla)u + \nabla p = 0, & \text{in } (0,\infty)\times\mathbb{R}^3, \\ \operatorname{div} u = 0, & \text{in } (0,\infty)\times\mathbb{R}^3, \\ u|_{t=0} = u_0, & \text{in } \mathbb{R}^3, \end{cases} \tag{NS$_\Omega$}$$

where $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ denotes the unknown velocity field, and $p = p(t,x)$ denotes the unknown scalar pressure, while $u_0 = u_0(x) = (u_0^1(x), u_0^2(x), u_0^3(x))$ denotes the initial velocity field. The constant $\nu > 0$ denotes the viscosity coefficient of the fluid, and $\Omega \in \mathbb{R}$ represents the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$, which is called the Coriolis parameter.

Recently, this problem gained some attention due to its importance in applications to geophysical flows, see e.g. [4]. Mathematically, (NS$_\Omega$) also have a interesting feature that there exists a global solution for arbitrary large data provided the speed of rotation Ω is large enough, see e.g. [1, 4, 9]. There are another type of results which shows the existence of a global solution uniformly in Ω provided the data is sufficiently small, see e.g. [5, 8, 11, 10]. The purpose of this thesis is, concerning to the latter, to relax the smallness condition of the data, based on the idea for the Navier-Stokes equations, $\Omega = 0$ in (NS$_\Omega$), by [12].

Before stating our main results, we give a definition of function spaces. For $m \in \mathbb{R}$, we define

$$\chi^m(\mathbb{R}^3) := \big\{ f \in \mathcal{S}' \mid \widehat{f} \in L^1_{\mathrm{loc}},\ \|f\|_{\chi^m} := \int_{\mathbb{R}^3} |\xi|^m |\widehat{f}(\xi)|\, d\xi < \infty \big\}.$$

Recently, Lei and Lin introduced the space χ^{-1}, which is contained in BMO^{-1} and equivalent to the Fourier-Herz space $\dot{\mathcal{B}}_1^{-1}$. It is known that $H^s(\mathbb{R}^3) \subseteq \chi^{-1}$, if $s > \frac{1}{2}$, see Lemma 2.1. Moreover it is known that there is an example so that $H^{\frac{1}{2}}(\mathbb{R}^3) \not\subseteq \chi^{-1}$, see [12]. It is also known that $\chi^{-1} \not\subseteq H^{\frac{1}{2}}(\mathbb{R}^3)$, see [14].

Theorem 2.3. Let $u_0 \in \chi^{-1}$ satisfy $\operatorname{div} u_0 = 0$ and $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$. For $T > 0$, assume that $u \in C([0,T);\, \chi^{-1})$ is a solution to (NS_Ω) in the distribution sense satisfying

$$u \in L^1(0,T;\, \chi^1), \quad \partial_t u \in L^1(0,T;\, \chi^{-1}).$$

Then, u satisfies

$$\|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}) \int_0^t \|u(\tau)\|_{\chi^1}\, d\tau \leq \|u_0\|_{\chi^{-1}}, \quad 0 \leq t < T. \tag{1.0.1}$$

Remark 1.1. (1) This a priori estimate is first derived in the case $\Omega = 0$ in [12, Proof of Theorem 1.1]. Here, Theorem 2.3 states that the same estimate also holds under the effect of the Coriolis force.

(2) In this thesis, we define the Fourier transform of f by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) := \int e^{-ix\cdot\xi} f(x)\, dx.$$

The constant $(2\pi)^3$ in the theorem appears from the following formula:

$$\mathcal{F}[fg](\xi) = (2\pi)^{-3}(\widehat{f} * \widehat{g})(\xi),$$

where $f * g$ denotes the convolution of f and g.

As an application of Theorem 2.3 we obtain the global solution to (NS_Ω).

Theorem 2.5. Let $s > 3/2$ and $\Omega \in \mathbb{R}$. Assume that $u_0 \in H^s(\mathbb{R}^3)$ satisfy $\operatorname{div} u_0 = 0$ and $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$. Then, there exists a unique global solution $u \in C([0,\infty);\, H^s(\mathbb{R}^3))$ to (NS_Ω) satisfying

$$u \in AC([0,\infty);\, H^{s-1}(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}(0,\infty; H^{s+1}(\mathbb{R}^3))$$

and

$$\sup_{t>0}\Big\{\|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}) \int_0^t \|u(\tau)\|_{\chi^1}\, d\tau\Big\} \leq \|u_0\|_{\chi^{-1}}.$$

Remark 1.2. Since $s > 3/2$, we have $H^s \hookrightarrow \chi^{-1}$ by Lemma 2.1. For a interval I and a Banach space X, $AC(I; X)$ denotes the space of X-valued absolutely continuous functions. There are several results which treats the existence of a unique global solution to (NS_Ω), see [10] and reference therein. The advantage of this result is that the condition of the size of the data is merely $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$.

In chapter 3, we show the existence of a unique global solution for the data $u_0 \in \chi^{-1}$ with $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$. For the Navier-Stokes case $\Omega = 0$, see [14, Theorem 1.3].

Theorem 3.1. Let $u_0 \in \chi^{-1}$ and $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$. Then, there is a unique global in time solution $u \in C([0,\infty); \chi^{-1})$ of (NS_Ω) satisfying

$$u \in L^2(0,\infty;\chi^0) \cap L^1(0,\infty;\chi^1), \quad \partial_t u \in L^1(0,\infty;\chi^{-1}),$$

and

$$\sup_{t>0}\left\{\|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}})\int_0^t \|u(\tau)\|_{\chi^1}\,d\tau\right\} \le \|u_0\|_{\chi^{-1}}.$$

Remark 1.3. (1) There are several results which treats the existence of a unique global solution to (NS_Ω), see [10] and reference therein. In particular, the spaces FM_0^{-1}, which is considered by Giga, Inui, Mahalov, and Saal [5], and $\mathcal{B}^{-1}_{1,2}$ by [10], are larger than χ^{-1}. However, the advantage of this result is that the condition of the size of the data is merely $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$.

(2) In the Navier-Stokes equations, the case $\Omega = 0$, the corresponding result is proved in [12, Theorem 1.1]. We notice that there is also the another approach by [14, Theorem 1.3].

Chapter 2

H^s Theory

2.1 A Priori Estimate and Its Application to H^s Theory

In this thesis, we only use spaces χ^{-1}, χ^0, and χ^1 below, so we summarize elementary estimates concerning the spaces we will use later.

Lemma 2.1. (1) *For $m > -3/2$, and $s > m + 3/2$,*

$$\|f\|_{\chi^m(\mathbb{R}^3)} \leq C\|f\|_{L^2}^{1-\frac{1}{s}(m+\frac{3}{2})}\|f\|_{\dot{H}^s}^{\frac{1}{s}(m+\frac{3}{2})}.$$

(2) $\|f\|_{\chi^0} \leq \|f\|_{\chi^{-1}}^{1/2}\|f\|_{\chi^1}^{1/2}$.

(3) $\|\nabla f\|_{L^\infty} \leq (2\pi)^{-3}\|f\|_{\chi^1}$.

Remark 2.2. Taking $m = -1, 1$ in Lemma 2.1 (1) respectively, we have for $s > 1/2$,

$$\|f\|_{\chi^{-1}(\mathbb{R}^3)} \leq C\|f\|_{L^2}^{1-\frac{1}{2s}}\|f\|_{\dot{H}^s}^{\frac{1}{2s}},$$

and for $s > 5/2$,

$$\|f\|_{\chi^1(\mathbb{R}^3)} \leq C\|f\|_{L^2}^{1-\frac{5}{2s}}\|f\|_{\dot{H}^s}^{\frac{5}{2s}}.$$

proof. (1) We take $R > 0$, which is determined later, to divide the integral

$$\begin{aligned}\|f\|_{\chi^m} &= \int_{|\xi|\le R} |\xi|^m |\widehat{f}(\xi)|\, d\xi + \int_{|\xi|>R} |\xi|^m |\widehat{f}(\xi)|\, d\xi \\ &\le \Big(\int_{|\xi|\le R} |\xi|^{2m}\, d\xi\Big)^{1/2} (2\pi)^{\frac{3}{2}} \|f\|_{L^2} + \Big(\int_{|\xi|>R} |\xi|^{2(m-s)}\, d\xi\Big)^{1/2} \|f\|_{\dot{H}^s} \\ &= |S^2|^{1/2}\Big(\frac{1}{\sqrt{2m+3}} R^{m+3/2} (2\pi)^{\frac{3}{2}} \|f\|_{L^2} + \frac{1}{\sqrt{2(s-m)-3}} R^{m-s+3/2} \|f\|_{\dot{H}^s}\Big).\end{aligned}$$

Then, choosing $R = \|f\|_{L^2}^{-1/s} \|f\|_{\dot{H}^2}^{1/s}$, we obtain the desired result.
(2) This estimate is easily derived by the Hölder inequality,

$$\|f\|_{\chi^0} = \int |\xi|^{-1/2} |\widehat{f}(\xi)|^{1/2} |\xi|^{1/2} |\widehat{f}(\xi)|^{1/2}\, d\xi \le \|f\|_{\chi^{-1}}^{1/2} \|f\|_{\chi^1}^{1/2}.$$

(3) This is also easily derived from the Fourier inversion formula and the Hausdorff-Young inequality. □

Now we state our main results.

Theorem 2.3. *Let $u_0 \in \chi^{-1}$ satisfy $\operatorname{div} u_0 = 0$ and $\|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu$. For $T > 0$, assume that $u \in C([0,T); \chi^{-1})$ is a solution to (NS_Ω) in the distribution sense satisfying*

$$u \in L^1(0,T;\, \chi^1), \quad \partial_t u \in L^1(0,T;\, \chi^{-1}).$$

Then, u satisfies

$$\|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3} \|u_0\|_{\chi^{-1}}) \int_0^t \|u(\tau)\|_{\chi^1}\, d\tau \le \|u_0\|_{\chi^{-1}}, \quad 0 \le t < T. \tag{2.1.1}$$

Remark 2.4. From the a priori estimate (2.1.1), we especially obtain

$$\|u\|_{L^\infty(0,T;\,\chi^{-1})} \le \|u_0\|_{\chi^{-1}}, \quad \|u\|_{L^1(0,T;\,\chi^1)} \le \frac{\|u_0\|_{\chi^{-1}}}{\nu - (2\pi)^{-3} \|u_0\|_{\chi^{-1}}}.$$

As an application of Theorem 2.3 we obtain the global solution to (NS_Ω).

Theorem 2.5. *Let $s > 3/2$ and $\Omega \in \mathbb{R}$. Assume that $u_0 \in H^s(\mathbb{R}^3)$ satisfy* $\operatorname{div} u_0 = 0$ *and $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$. Then, there exists a unique global solution $u \in C([0,\infty); H^s(\mathbb{R}^3))$ to* (NS_Ω) *satisfying*

$$u \in AC([0,\infty); H^{s-1}(\mathbb{R}^3)) \cap L^1(0,\infty; H^{s+1}(\mathbb{R}^3))$$

and

$$\sup_{t>0}\Big\{\|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}})\int_0^t \|u(\tau)\|_{\chi^1}\, d\tau\Big\} \le \|u_0\|_{\chi^{-1}}.$$

Remark 2.6. Since $s > 3/2$, we have $H^s \hookrightarrow \chi^{-1}$ by Lemma 2.1. For a interval I and a Banach space X, $AC(I; X)$ denotes the space of X-valued absolutely continuous functions. There are several results which treats the existence of a unique global solution to (NS_Ω), see [10] and reference therein. The advantage of this result is that the condition of the size of the data is merely $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$.

Next theorem states the asymptotic behavior of a given global solution to (NS_Ω).

Theorem 2.7. *Let $s > 1/2$ and $\Omega \in \mathbb{R}$. Assume that $u \in C([0,\infty); H^s(\mathbb{R}^3))$ is a global solution to* (NS_Ω) *satisfying*

$$u \in AC([0,\infty); H^{s-1}(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}([0,\infty); H^{s+1}(\mathbb{R}^3)).$$

Then, $\lim_{t\to\infty} \|u(t)\|_{\chi^{-1}} = 0$.

Remark 2.8. In the Navier-Stokes case $\Omega = 0$, this result corresponds to the result in [3]. In that result, the assumption is only $u \in C([0,\infty); \chi^{-1})$ is a global solution. Compared with that result, additional assumptions are imposed for the uniqueness of solutions.

As an application of Theorem 2.7 we obtain the following.

Corollary 2.9. *The global solution to* (NS_Ω) *derived in Theorem* 2.5 *satisfies*

$$\lim_{t\to\infty} \|u(t)\|_{\chi^{-1}} = 0.$$

This chapter is organized as follows. In Section 2.2 we give a proof of Theorem 2.3. In Section 2.3 we prove Theorem 2.5 as an application of Theorem 2.3. In Section 2.4 we give a proof of Theorem 2.7.

2.2 Proof of Theorem 2.3

In this section we give a proof of Theorem 2.3.

Proof of Theorem 2.3. By applying the Fourier transform to the equation, we have

$$\partial_t \widehat{u} + \nu|\xi|^2 \widehat{u} + \Omega e_3 \times \widehat{u} + \mathcal{F}\big[(u,\nabla)u\big] + i\xi\widehat{p} = 0.$$

Thus, we obtain

$$\begin{aligned}\partial_t|\widehat{u}|^2 &= 2\mathrm{Re}(\partial_t \widehat{u}\cdot\overline{\widehat{u}})\\ &= -2\nu|\xi|^2|\widehat{u}|^2 - 2\Omega\,\mathrm{Re}\big[(e_3\times\widehat{u})\cdot\overline{\widehat{u}}\big] - 2\mathrm{Re}\big\{\mathcal{F}\big[(u,\nabla)u\big]\cdot\overline{\widehat{u}}\big\} - 2\mathrm{Re}\big[(i\xi\widehat{p})\cdot\overline{\widehat{u}}\big].\end{aligned}$$

Here, since

$$(e_3\times\widehat{u})\cdot\overline{\widehat{u}} = -\widehat{u}_2\overline{\widehat{u}_1} + \widehat{u}_1\overline{\widehat{u}_2} = 2i\,\mathrm{Im}\big[\widehat{u}_1\overline{\widehat{u}_2}\big],$$

we observe that $\mathrm{Re}[(e_3\times\widehat{u})\cdot\overline{\widehat{u}}] = 0$. Also, we have $(i\xi\widehat{p})\cdot\overline{\widehat{u}} = 0$, since $\operatorname{div} u = 0$. Moreover, we notice that

$$\begin{aligned}\mathcal{F}\big[(u,\nabla)u\big]_j(\xi) &= \sum_{k=1}^{3}(2\pi)^{-3}\widehat{u}_k * \widehat{\partial_k u_j}(\xi)\\ &= \sum_{k=1}^{3}(2\pi)^{-3}\int \widehat{u}_k(\xi-\eta)\, i\eta_k\widehat{u}_j(\eta)\,d\eta\\ &= \sum_{k=1}^{3}(2\pi)^{-3} i\xi_k\int \widehat{u}_k(\xi-\eta)\,\widehat{u}_j(\eta)\,d\eta,\end{aligned}$$

since $\sum_{k=1}^{3}(\xi_k-\eta_k)\widehat{u}_k(\xi-\eta) = 0$. Therefore, we obtain

$$\begin{aligned}\partial_t|\widehat{u}|^2 + 2\nu|\xi|^2|\widehat{u}|^2 &\le 2(2\pi)^{-3}\sum_{j,k=1}^{3}|\xi_k|\,(|\widehat{u}_k| * |\widehat{u}_j|)\,|u_j|\\ &\le 2(2\pi)^{-3}|\xi|\,|\widehat{u}|\,(|\widehat{u}| * |\widehat{u}|).\end{aligned}$$

Then, for $\varepsilon > 0$, we observe that

$$\begin{aligned}\partial_t(|\widehat{u}|^2+\varepsilon)^{1/2} &= \frac{\partial_t|\widehat{u}|^2}{2(|\widehat{u}|^2+\varepsilon)^{1/2}}\\ &\le -\frac{\nu|\xi|^2|\widehat{u}|^2}{(|\widehat{u}|^2+\varepsilon)^{1/2}} + (2\pi)^{-3}\frac{|\xi|\,|\widehat{u}|}{(|\widehat{u}|^2+\varepsilon)^{1/2}}\,(|\widehat{u}| * |\widehat{u}|).\end{aligned}$$

Integrating with respect to t, we obtain

$$(|\widehat{u}(t,\xi)|^2+\varepsilon)^{1/2}+\int_0^t \frac{\nu|\xi|^2|\widehat{u}(\tau,\xi)|^2}{(|\widehat{u}(\tau,\xi)|^2+\varepsilon)^{1/2}}\,d\tau$$
$$\le (|\widehat{u}_0(\xi)|^2+\varepsilon)^{1/2}+(2\pi)^{-3}\int_0^t \frac{|\xi|\,|\widehat{u}(\tau,\xi)|}{(|\widehat{u}(\tau,\xi)|^2+\varepsilon)^{1/2}}\,(|\widehat{u}(\tau)|*|\widehat{u}(\tau)|)(\xi)\,d\tau.$$

Then, letting $\varepsilon\to 0$, we get

$$|\widehat{u}(t,\xi)|+\int_0^t \nu|\xi|^2|\widehat{u}(\tau,\xi)|\,d\tau \le |\widehat{u}_0(\xi)|+(2\pi)^{-3}\int_0^t |\xi|\,(|\widehat{u}(\tau)|*|\widehat{u}(\tau)|)(\xi)\,d\tau.$$

Finally, dividing by $|\xi|$, and then integrating over $\mathbb{R}^n$, we obtain

$$\|u(t)\|_{\chi^{-1}}+\nu\int_0^t \|u(\tau)\|_{\chi^1}\,d\tau \le \|u_0\|_{\chi^{-1}}+(2\pi)^{-3}\int_0^t \|u(\tau)\|_{\chi^0}^2\,d\tau.$$

By applying Lemma 2.1 (2), we obtain,

$$\|u(t)\|_{\chi^{-1}}+\nu\|u\|_{L^1((0,t);\,\chi^1)} \le \|u_0\|_{\chi^{-1}}+(2\pi)^{-3}\|u\|_{L^\infty((0,t);\,\chi^{-1})}\|u\|_{L^1((0,t);\,\chi^1)}. \tag{2.2.1}$$

To derive the desired estimate (2.1.1), it suffices to prove that

$$\|u\|_{L^\infty((0,t);\,\chi^{-1})}\le \|u_0\|_{\chi^{-1}},\quad 0\le t<T.$$

For the proof, we first show that

$$\|u(t)\|_{\chi^{-1}}<(2\pi)^3\nu,\quad 0\le t<T \tag{2.2.2}$$

holds by contradiction. From the assumption $\|u_0\|_{\chi^{-1}}<(2\pi)^3\nu$ and $u\in C([0,T);\chi^{-1})$, we observe that there exists $\delta>0$ such that (2.2.2) holds on $[0,\delta)$. Now assume that there exists $t_0\in(0,T)$ such that $\|u(t)\|_{\chi^{-1}}<(2\pi)^3\nu$ for $0<t<t_0$ and

$$\|u(t_0)\|_{\chi^{-1}}=(2\pi)^3\nu,$$

then by (2.2.1) we reach the contradiction

$$(2\pi)^3\nu=\|u(t_0)\|_{\chi^{-1}}\le\|u_0\|_{\chi^{-1}}<(2\pi)^3\nu,$$

since $\|u\|_{L^\infty((0,t_0);\,\chi^{-1})}=(2\pi)^3\nu$. Therefore, we obtain (2.2.2). Finally, applying (2.2.2) to estimate on the right hand side of (2.2.1), we obtain

$$\|u(t)\|_{\chi^{-1}}<\|u_0\|_{\chi^{-1}},\quad 0\le t<T.$$

This completes the proof. □

2.3 Proof of Theorem 2.5

Below we fix $\Omega \in \mathbb{R}$. For the existence of local solutions, we employ the following result.

Proposition 2.10. *Let $s > 3/2$. For $u_0 \in H^s(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, there exists $T = T(|\Omega|, s, \|u_0\|_{H^s}) > 0$ such that* (NS$_\Omega$) *admits a unique strong solution $u \in C([0,T];\, H^s(\mathbb{R}^3))$ satisfying*

$$u \in AC([0,T];\, H^{s-1}(\mathbb{R}^3)) \cap L^1(0,T;\, H^{s+1}(\mathbb{R}^3)).$$

Remark 2.11. (1) For the proof, we refer to [13, Lemma 3.1]. We notice that the condition in [13, Lemma 3.1] is $s > 3/2 + 1$, because their main subject is the Euler equation. For the above statement, $s > 3/2$ is sufficient.

(2) In this proposition, the size of T is characterized by

$$C_0|\Omega|T + C_1\|u_0\|_{H^s}(T + T^{1/2}\nu^{-1/2}) \leq \frac{1}{2}. \tag{2.3.1}$$

(3) Since $s > 3/2$, the solution constructed by Proposition 2.10 satisfies the assumptions in Theorem 2.3. In particular, since

$$\partial_t u = \nu\Delta u - \Omega\mathbb{P}(e_3 \times u) - \mathbb{P}(u, \nabla u) \quad \text{in } H^{s-1}$$

holds for a.e. $t \in (0,T)$, where $\mathbb{P} = (\delta_{ij} + R_iR_j)_{i,j}$ is the Helmholtz projection, we easily observe that $\partial_t u \in L^1(0,T;\, \chi^{-1})$.

We will use the following energy estimate.

Proposition 2.12. *Let $s \geq 0$ and $T > 0$. Assume that $u \in C([0,T);\, H^s(\mathbb{R}^3))$ is a solution to* (NS$_\Omega$) *satisfying*

$$u \in AC([0,T);\, H^{s-1}(\mathbb{R}^3)) \cap L^1(0,T; H^{s+1}(\mathbb{R}^3)).$$

Then, u satisfies

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} e^{C\int_0^T \|\nabla u(\tau)\|_{L^\infty}\, d\tau}, \quad 0 \leq t < T.$$

Remark 2.13. For the proof of this proposition, we also refer to [13, Proof of Theorem 4.1]. There, we easily observe that

$$\frac{d}{dt}\|u(t)\|_{H^s} \leq C\|\nabla u(t)\|_{L^\infty}\|u(t)\|_{H^s}$$

holds for $s \geq 0$. We notice that the term concerning $\Omega e_3 \times u$ vanishes due to its property of the skew symmetry in H^s.

Now we are in a position to prove Theorem 2.5.

Proof of Theorem 2.5. Let T^* be the maximal existence time of a unique solution derived by applying Proposition 2.10 repeatedly. Now assume $T^* < \infty$. Then, by (2.3.1), we must have

$$\lim_{t \to T^*} \|u(t)\|_{H^s} = \infty. \tag{2.3.2}$$

Since this solution satisfies the energy estimate in Proposition 2.12, we have

$$\|u(t)\|_{H^s} \le \|u_0\|_{H^s} e^{C \int_0^{T^*} \|\nabla u(\tau)\|_{L^\infty} \, d\tau}, \quad 0 \le t < T^*.$$

Then, since $\|u_0\|_{\chi^{-1}} < (2\pi)^3 \nu$, applying Theorem 2.3 we obtain

$$\int_0^{T^*} \|\nabla u(\tau)\|_{L^\infty} \, d\tau \le \|u\|_{L^1(0,T^*;\chi^1)} \le \frac{\|u_0\|_{\chi^{-1}}}{\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}}}.$$

This implies $\sup_{0<t<T^*} \|u(t)\|_{H^s} < \infty$, which contradicts to (2.3.2). □

2.4 Proof of Theorem 2.7

In this section we give a proof of Theorem 2.7.

We take $\varepsilon > 0$ arbitrary small. Since $u_0 \in H^s \hookrightarrow \chi^{-1}$, we are able to take $R_0 > 0$ such that

$$\int_{|\xi| > R_0} |\xi|^{-1} |\widehat{u}_0(\xi)| \, d\xi < \frac{\varepsilon}{2}.$$

Now we set

$$v_0 = \mathcal{F}^{-1}[\chi_{\{|\xi| \le R_0\}} \widehat{u_0}], \quad w_0 = \mathcal{F}^{-1}[\chi_{\{|\xi| > R_0\}} \widehat{u_0}].$$

Then, we observe that $v_0 \in H^\infty$, $w_0 \in H^s$, $u_0 = v_0 + w_0$, and

$$\|w_0\|_{\chi^{-1}} < \frac{\varepsilon}{2}.$$

By applying Theorem 2.5 for the initial data w_0 we obtain the solution (w, p_w) to (NS_Ω). Then, $w \in C([0,\infty); H^s) \cap L^1(0,\infty; H^{s+1})$ satisfies

$$\|w(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|w_0\|_{\chi^{-1}}) \int_0^t \|w(\tau)\|_{\chi^1} \, d\tau \le \|w_0\|_{\chi^{-1}} < \frac{\varepsilon}{2}, \quad t > 0. \tag{2.4.1}$$

Now we set $v := u - w$. Then, $v \in C([0,\infty);\, H^s)$ satisfies

$$v \in AC([0,\infty);\, H^{s-1}(\mathbb{R}^3)) \cap L^1(0,\infty; H^{s+1}(\mathbb{R}^3))$$

and

$$\begin{cases} \partial_t v + \nu\Delta v + \Omega e_3 \times v + (v,\nabla)v + (w,\nabla)v + (v,\nabla)w + \nabla(p - p_w) = 0, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0. \end{cases}$$

Taking L^2-inner product with v, the equation becomes

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 + \nu\|\nabla v(t)\|_{L^2}^2 = \langle (v,\nabla)w, v\rangle_{L^2}.$$

Since

$$\langle (v,\nabla)w, v\rangle_{L^2} = -\langle w, (v,\nabla)v\rangle_{L^2},$$

we obtain

$$\begin{aligned} |\langle (v,\nabla)w, v\rangle_{L^2}| &\le \|w\|_{L^\infty}\|v\|_{L^2}\|\nabla v\|_{L^2} \\ &\le C\|w\|_{\chi^0}\|v\|_{L^2}\|\nabla v\|_{L^2} \\ &\le C_\nu\|w\|_{\chi^0}^2\|v\|_{L^2}^2 + \frac{\nu}{2}\,\|\nabla v\|_{L^2}^2 \end{aligned}$$

Therefore, we obtain

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 + \frac{\nu}{2}\,\|\nabla v(t)\|_{L^2}^2 = C_\nu\|w(t)\|_{\chi^0}^2\|v(t)\|_{L^2}^2.$$

Then, by Gronwall's inequality,

$$\|v(t)\|_{L^2}^2 + \frac{\nu}{2}\int_0^t \|\nabla v(t)\|_{L^2}^2 \le \|v(0)\|_{L^2}^2 e^{C_\nu \int_0^t \|w(\tau)\|_{\chi^0}^2\, d\tau}. \tag{2.4.2}$$

Here, by (2.4.1) we have

$$\int_0^t \|w(\tau)\|_{\chi^0}^2\, d\tau \le \|w\|_{L^\infty((0,t);\,\chi^{-1})}\|w\|_{L^1((0,t);\,\chi^1)} \le \frac{\|w_0\|_{\chi^{-1}}^2}{\nu - (2\pi)^{-3}\|w_0\|_{\chi^{-1}}}. \tag{2.4.3}$$

Therefore, by Lemma 2.1 (1), (2.4.2), (2.4.3) we obtain

$$\int_0^\infty \|v(t)\|_{\chi^{-1}}^4\, d\tau \le \int_0^\infty \|v(t)\|_{L^2}^2\|\nabla v(t)\|_{L^2}^2 \le \frac{2}{\nu}\,\|v_0\|_{L^2}^4\, \exp\Big(\frac{C_\nu\,\|w_0\|_{\chi^{-1}}^2}{\nu - (2\pi)^{-3}\|w_0\|_{\chi^{-1}}}\Big).$$

Since $v \in C([0,\infty);\, \chi^{-1})$, we observe that there exists $t_0 > 0$ such that $\|v(t_0)\|_{\chi^{-1}} < \varepsilon/2$, and thus we have $\|u(t_0)\|_{\chi^{-1}} \le \|v(t_0)\|_{\chi^{-1}} + \|w(t_0)\|_{\chi^{-1}} < \varepsilon$. So, applying Theorem 2.5 for the data $u(t_0)$ we obtain

$$\|u(t)\|_{\chi^{-1}} \le \|u(t_0)\|_{\chi^{-1}} < \varepsilon, \quad t > t_0,$$

which implies $\lim_{t\to\infty} \|u(t)\|_{\chi^{-1}} = 0$.

Here, we notice that in the final part of the proof we need the uniqueness of solutions, which is assured in our class of solutions. In fact, if u_1, and $u_2 \in C([0,\infty);\, H^s)$ are two solutions to (NS_Ω) satisfying

$$u_1,\ u_2 \in AC([0,\infty);\, H^{s-1}(\mathbb{R}^3)) \cap L^1_{\mathrm{loc}}(0,\infty; H^{s+1}(\mathbb{R}^3)),$$

then, $\widetilde{u} := u_1 - u_2$ satisfies $\mathrm{div}\widetilde{u} = 0$ and

$$\partial_t \widetilde{u} + \nu\Delta\widetilde{u} + \Omega e_3 \times \widetilde{u} + (\widetilde{u},\nabla)\widetilde{u} + (u_1,\nabla)\widetilde{u} + (\widetilde{u},\nabla)u_2 + \nabla(p_1 - p_2) = 0,$$

and thus we obtain

$$\frac{d}{dt}\|\widetilde{u}(t)\|_{L^2}^2 + \frac{\nu}{2}\|\nabla\widetilde{u}(t)\|_{L^2}^2 = |\langle(\widetilde{u},\nabla)u_2, \widetilde{u}\rangle_{L^2}| \le \|\nabla u_2(t)\|_{L^\infty}^2 \|\widetilde{u}(t)\|_{L^2}^2.$$

Therefore, we have

$$\frac{d}{dt}\|\widetilde{u}(t)\|_{L^2}^2 = C\|u_2(t)\|_{H^{s+1}}\|\widetilde{u}(t)\|_{L^2}^2$$

and Gronwall's inequality implies $\widetilde{u}(t) = 0$ for $t > 0$.

Chapter 3

χ^{-1} Theory

3.1 Main Theorem in χ^{-1} Theory

In this section, we state our main result and representation of the solution of linearized equation of (NS_Ω).

Theorem 3.1. *Let $u_0 \in \chi^{-1}$ and $\|u_0\|_{\chi^{-1}} < (2\pi)^3\nu$. Then, there is a unique global in time solution $u \in C([0,\infty);\chi^{-1})$ of (NS_Ω) satisfying*

$$u \in L^2(0,\infty;\chi^0) \cap L^1(0,\infty;\chi^1), \quad \partial_t u \in L^1(0,\infty;\chi^{-1}),$$

and

$$\sup_{t>0}\left\{\|u(t)\|_{\chi^{-1}} + (\nu - (2\pi)^{-3}\|u_0\|_{\chi^{-1}})\int_0^t \|u(\tau)\|_{\chi^1}\,d\tau\right\} \le \|u_0\|_{\chi^{-1}}.$$

Remark 3.2. In the Navier-Stokes equations, the case $\Omega = 0$, the corresponding result is proved in [12, Theorem 1.1]. We notice that there is also another approach by [14, Theorem 1.3]. The argument below is based on the latter.

For the proof of Theorem 3.1 we consider the integral equation

$$u(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)ds, \tag{3.1.1}$$

where $\mathbb{P} = (\delta_{ij} + R_iR_j)_{i,j}$ denotes the Helmholtz projection, $R_j = \mathcal{F}^{-1}\frac{i\xi_j}{|\xi|}\mathcal{F}$ denotes the Riesz transforms, and $\nabla\cdot(u\otimes u) = (\sum_j \partial_j(u_iu_j))_{i=1,2,3}$. Here

$\mathcal{S}(t)$ represents the semigroup corresponding to the linear problem

$$\begin{cases} \partial_t v - \nu \Delta v + \Omega e_3 \times v + \nabla q = 0, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0, \end{cases} \tag{3.1.2}$$

which is given explicitly by

$$\widehat{\mathcal{S}(t)v_0}(\xi) = \cos\left(\Omega \frac{\xi_3}{|\xi|} t\right) e^{-\nu|\xi|^2 t} I \widehat{v_0}(\xi) + \sin\left(\Omega \frac{\xi_3}{|\xi|} t\right) e^{-\nu|\xi|^2 t} R(\xi) \widehat{v_0}(\xi),$$

where I is the 3×3 identity matrix and

$$R(\xi) = \begin{pmatrix} 0 & \frac{\xi_3}{|\xi|} & -\frac{\xi_2}{|\xi|} \\ -\frac{\xi_3}{|\xi|} & 0 & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & -\frac{\xi_1}{|\xi|} & 0 \end{pmatrix}.$$

For its derivation, see e.g. [8]. The integral equation (3.1.1) formally derived as follows. We first apply $\mathbb{P}$ to the equation, then we have

$$\partial_t u - \nu \Delta u + \Omega \mathbb{P} e_3 \times u + \mathbb{P} \nabla \cdot (u \otimes u) = 0, \tag{3.1.3}$$

where $(u, \nabla u) = \nabla \cdot (u \otimes u)$ holds since $\operatorname{div} u = 0$. Here, we notice that

$$\Omega \mathbb{P} e_3 \times u = \Omega e_3 \times u + \nabla q \tag{3.1.4}$$

holds, where q denotes the pressure to the linear problem (3.1.2). Indeed, taking div to the first equation of (3.1.2), we have

$$\operatorname{div} \partial_t u - \nu \operatorname{div} \Delta u + \Omega \operatorname{div}(e_3 \times u) + \Delta q = 0.$$

Since $\operatorname{div} u = 0$, it follows that

$$\begin{aligned} \Delta q &= -\Omega \operatorname{div}(e_3 \times u) \\ &= \Omega(\partial_1 u_2 - \partial_2 u_1). \end{aligned}$$

Thus we have

$$\widehat{\nabla q} = i\xi \left(-\Omega \left(\frac{i\xi_1}{|\xi|^2} \widehat{u_2} - \frac{i\xi_2}{|\xi|^2} \widehat{u_1} \right) \right).$$

Since $\mathbb{P} = (\delta_{ij} + R_i R_j)_{i,j}$, we find that

$$\begin{aligned}
\mathbb{P}e_3 \times u &= \mathbb{P}\begin{pmatrix} -u_2 \\ u_1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -u_2 \\ u_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -R_1^2 u_2 + R_1 R_2 u_1 \\ -R_2 R_1 u_2 + R_2^2 u_1 \\ -R_3 R_1 u_2 + R_3 R_2 u_1 \end{pmatrix} \\
&= e_3 \times u + \frac{1}{\Omega}\nabla q,
\end{aligned}$$

by definition of R_j. Thus, we obtain the equation

$$\begin{cases} \partial_t u - \nu\Delta u + \Omega e_3 \times u + \nabla q = -\mathbb{P}(u, \nabla)u, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{3.1.5}$$

Therefore, by Duhamel's principle, we obtain (3.1.1).

3.2 Existence of Solutions for Any Time Interval

In this section, we state there exists a local solution of (NS_Ω) if $\|u_0\|_{\chi^{-1}} \leq 4\pi^3\nu$. First we prove the following lemma about $\mathcal{S}(t)$.

Lemma 3.3. *For $u \in \mathcal{S}'$ and $\widehat{u} \in L^1_{\mathrm{loc}}$, we have*

$$|\widehat{\mathbb{P}u}(\xi)| \leq |\widehat{u}(\xi)|,$$

and

$$|\widehat{\mathcal{S}(t)u}(\xi)| \leq e^{-\nu|\xi|^2 t}|\widehat{u}(\xi)|.$$

proof. By definition of $\mathbb{P}$, we have

$$|\widehat{\mathbb{P}u}(\xi)| = \left|\mathcal{F}\begin{pmatrix} u_1 + R_1\sum_{j=1}^3 R_j u_j) \\ u_2 + R_2\sum_{j=1}^3 R_j u_j) \\ u_3 + R_3\sum_{j=1}^3 R_j u_j) \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} \widehat{u_1} + \frac{i\xi_1}{|\xi|}\sum_{j=1}^3 \frac{i\xi_j}{|\xi|}\widehat{u_j}) \\ \widehat{u_2} + \frac{i\xi_2}{|\xi|}\sum_{j=1}^3 \frac{i\xi_j}{|\xi|}\widehat{u_j}) \\ \widehat{u_3} + \frac{i\xi_3}{|\xi|}\sum_{j=1}^3 \frac{i\xi_j}{|\xi|}\widehat{u_j}) \end{pmatrix}\right|$$
$$= \left|\begin{pmatrix} \widehat{u_1} - \frac{\xi_1}{|\xi|^2}(\xi\cdot\widehat{u}) \\ \widehat{u_2} - \frac{\xi_2}{|\xi|^2}(\xi\cdot\widehat{u}) \\ \widehat{u_3} - \frac{\xi_3}{|\xi|^2}(\xi\cdot\widehat{u}) \end{pmatrix}\right|$$
$$= \sqrt{|\widehat{u}|^2 - \frac{2}{|\xi|^2}(\xi\cdot\widehat{u})^2 + \frac{\xi_1^2+\xi_2^2+\xi_3^2}{|\xi|^4}(\xi\cdot\widehat{u})^2}$$
$$= \sqrt{|\widehat{u}|^2 - \frac{1}{|\xi|^2}(\xi\cdot\widehat{u})^2} \le |\widehat{u}(\xi)|.$$

By definition of $R(\xi)$, we have

$$R(\xi)u = \frac{\xi}{|\xi|}\times u.$$

It follows that

$$e^{2\nu|\xi|^2 t}|\widehat{\mathcal{S}(t)u}(\xi)|^2 = \left|\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)\widehat{u}(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)\frac{\xi}{|\xi|}\times\widehat{u}(\xi)\right|^2$$
$$= \cos^2\left(\Omega\frac{\xi_3}{|\xi|}t\right)|\widehat{u}(\xi)|^2 + \sin^2\left(\Omega\frac{\xi_3}{|\xi|}t\right)\left|\frac{\xi}{|\xi|}\times\widehat{u}(\xi)\right|^2$$
$$+ 2\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)\sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)\widehat{u}(\xi)\cdot\left(\frac{\xi}{|\xi|}\times\widehat{u}(\xi)\right)$$
$$= \cos^2\left(\Omega\frac{\xi_3}{|\xi|}t\right)|\widehat{u}(\xi)|^2 + \sin^2\left(\Omega\frac{\xi_3}{|\xi|}t\right)\left\{|\widehat{u}(\xi)|^2 - \left(\frac{\xi}{|\xi|}\cdot\widehat{u}(\xi)\right)^2\right\}$$
$$= |\widehat{u}(\xi)|^2 - \sin^2\left(\Omega\frac{\xi_3}{|\xi|}t\right)\left(\frac{\xi}{|\xi|}\cdot\widehat{u}(|\xi|)\right)^2$$
$$\le |\widehat{u}(\xi)|^2.$$

$\square$

Proposition 3.4. *For any* $T > 0$*, we define* $\mathcal{B} : L^2([0,T];\chi^0)\times L^2([0,T];\chi^0) \to L^2([0,T];\chi^0)$ *as*

$$\mathcal{B}(u,v) = \int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes v)ds, \quad u,v\in L^2([0,T];\chi^0).$$

Then, $\mathcal{B}$ *is the continuous bilinear map, and*

$$\|\mathcal{B}\| := \sup_{\|u\|\le 1,\|v\|\le 1} \|\mathcal{B}(u,v)\|_{L^2_T\chi^0} \le \frac{1}{\sqrt{2\nu}(2\pi)^3}.$$

proof. For $T > 0$ and $u, v \in L^2([0,T];\chi^0)$, we have from Lemma 3.3

$$\begin{aligned}
|\widehat{\mathcal{B}(u,v)}(t,\xi)| &\le \int_0^t |\mathcal{F}[\mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes v)](s,\xi)|ds \\
&\le \int_0^t e^{-\nu(t-s)|\xi|^2}|\mathcal{F}[\mathbb{P}\nabla\cdot(u\otimes v)](s,\xi)|ds \\
&\le \int_0^t e^{-\nu(t-s)|\xi|^2}|\mathcal{F}[\nabla\cdot(u\otimes v)](s,\xi)|ds \\
&\le \frac{1}{(2\pi)^3}\int_0^t e^{-\nu(t-s)|\xi|^2}|\xi||\widehat{u}|*|\widehat{v}|(s,\xi)ds.
\end{aligned}$$

Using Minkowski's integral inequality and Young's inequality, we have

$$
\begin{aligned}
\|\mathcal{B}(u,v)\|_{L^2([0,T];\chi^0)} &= \left(\int_0^T \|\mathcal{B}(u,v)\|_{\chi^0}^2 dt\right)^{\frac{1}{2}} = \left(\int_0^T \left(\int_{\mathbb{R}^3} |\widehat{\mathcal{B}(u,v)}(t,\xi)|d\xi\right)^2 dt\right)^{\frac{1}{2}} \\
&\leq \frac{1}{(2\pi)^3}\left(\int_0^T \left(\int_{\mathbb{R}^3}\int_0^t e^{-\nu(t-s)|\xi|^2}|\xi||\widehat{u}|*|\widehat{v}|dsd\xi\right)^2 dt\right)^{\frac{1}{2}} \\
&= \frac{1}{(2\pi)^3}\left(\int_0^T \left(\int_{\mathbb{R}^3}\int_0^T \chi_{[0,t]}(s)e^{-\nu(t-s)|\xi|^2}|\xi||\widehat{u}|*|\widehat{v}|dsd\xi\right)^2 dt\right)^{\frac{1}{2}} \\
&\leq \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_0^T \left(\int_s^T e^{-2\nu(t-s)|\xi|^2}|\xi|^2\left(|\widehat{u}|*|\widehat{v}|(s,\xi)\right)^2 dt\right)^{\frac{1}{2}} dsd\xi \\
&= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_0^T |\xi||\widehat{u}|*|\widehat{v}|(s,\xi)\left(\int_s^T e^{-2\nu(t-s)|\xi|^2}dt\right)^{\frac{1}{2}} dsd\xi \\
&\leq \frac{1}{\sqrt{2\nu}(2\pi)^3}\int_{\mathbb{R}^3}\int_0^T |\widehat{u}|*|\widehat{v}|(s,\xi)dsd\xi \\
&\leq \frac{1}{\sqrt{2\nu}(2\pi)^3}\int_0^T \|\widehat{u}(s,\cdot)\|_{L^1}\|\widehat{v}(s,\cdot)\|_{L^1}ds \\
&\leq \frac{1}{\sqrt{2\nu}(2\pi)^3}\|u\|_{L^2([0,T];\chi^0)}\|v\|_{L^2([0,T];\chi^0)},
\end{aligned}
$$

where $\chi_{[0,t]}$ is characteristic function on $[0,t]$. Thus we conclude

$$
\|\mathcal{B}\| \leq \frac{1}{\sqrt{2\nu}(2\pi)^3}.
$$

□

Theorem 3.5. *Let u_0 be in χ^{-1} and $\|u_0\|_{\chi^{-1}} \leq 4\pi^3\nu$. For any $T > 0$, there is a unique solution $u \in L^2([0,T];\chi^0)$ of (NS_Ω) such that $\|u\|_{L^2([0,T];\chi^0)} \leq 2\pi^3\sqrt{2\nu}$.*

Now we use the following lemma to prove this.

Lemma 3.6 ([2])**.** *Let E be a Banach space, $\mathcal{B}$ a continuous bilinear map from $E \times E \to E$, and a positive real number α such that $\alpha < \frac{1}{4\|\mathcal{B}\|}$. For any* *a in the ball $B(0,\alpha) = \{x \in E; \|x\|_E \leq \alpha\}$, then there exists a unique x in $B(0,2\alpha)$ such that $x = a + B(x,x)$.*

Using Lemma 3.6, we can prove Theorem.

proof. Using $\|\mathcal{B}\| \leq \frac{1}{\sqrt{2\nu}(2\pi)^3}$, we can get for any $T > 0$,

$$\begin{aligned}\|\mathcal{S}(t)u_0\|_{L^2([0,T];\chi^0)} &\leq \left(\int_0^T \left(\int_{\mathbb{R}^3} e^{-\nu|\xi|^2 t}|\widehat{u_0}(\xi)|d\xi\right)^2 dt\right)^{\frac{1}{2}} \\ &\leq \int_{\mathbb{R}^3}\left(\int_0^T e^{-2\nu|\xi|^2 t}|\widehat{u_0}(\xi)|^2 dt\right)^{\frac{1}{2}} d\xi \\ &\leq \int_{\mathbb{R}^3} \frac{|\widehat{u_0}(\xi)|}{(2\nu)^{\frac{1}{2}}|\xi|}d\xi = \frac{1}{(2\nu)^{\frac{1}{2}}}\|u_0\|_{\chi^{-1}}.\end{aligned}$$

Since $\|u_0\|_{\chi^{-1}} \leq 4\pi^3\nu$, we have $\frac{1}{(2\nu)^{\frac{1}{2}}}\|u_0\|_{\chi^{-1}} \leq \frac{\sqrt{2\nu}(2\pi)^3}{4} \leq \frac{1}{4\|\mathcal{B}\|}$.

So using Lemma 3.6 for $\alpha = \sqrt{2\nu}\pi^3$, $E = L^2([0,T];\chi^0)$ and $a = \mathcal{S}(t)u_0$, we can conclude there exists a unique u in $B(0,2\alpha)$ such that $u = \mathcal{S}(t)u_0 + \mathcal{B}(u,u)$. Moreover, we have $\|u\|_{L^2([0,T];\chi^0)} \leq 2\cdot\sqrt{2\nu}\pi^3 = 2\pi^3\sqrt{2\nu}$. □

3.3 Existence of Local Solutions for Any Initial Data

In this section we prove existence of local solutions for any initial data in χ^{-1}.

Theorem 3.7. *For any $u_0 \in \chi^{-1}$, there exists a positive number $\rho = \rho_{u_0} > 0$ and $T = T(\nu, \|u_0\|_{\chi^{-1}}, \rho) > 0$ such that (NS_Ω) has a unique solution $u \in C([0,T];\chi^{-1})$ satisfying*

$$u \in L^2(0,T;\chi^0) \cap L^1(0,T;\chi^1), \quad \partial_t u \in L^1(0,T;\chi^{-1}).$$

Remark 3.8. T is determined by

$$T = \frac{\pi^6\nu}{2\rho_{u_0}^2\|u_0\|_{\chi^{-1}}^2}.$$

proof. We fix some positive number $\rho_{u_0} > 0$ such that

$$\int_{|\xi|\geq\rho_{u_0}} \frac{|\widehat{u_0}(\xi)|}{|\xi|} d\xi \leq \pi^3\nu.$$

Defining $u_0^\flat = \mathcal{F}^{-1}(\chi_{B(0,\rho_{u_0})}(\xi)\widehat{u_0}(\xi))$, we get

$$\begin{aligned}
\|\mathcal{S}(t)u_0^\flat\|_{L^2([0,T];\chi^0)} &= \left(\int_0^T \left(\int_{\mathbb{R}^3} |\mathcal{F}[\mathcal{S}(t)u_0^\flat](\xi)|d\xi\right)^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^T \left(\int_{|\xi|\leq\rho_{u_0}} |\widehat{u_0^\flat}(\xi)|d\xi\right)^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^T \left(\int_{|\xi|\leq\rho_{u_0}} |\xi|\cdot\frac{1}{|\xi|}|\widehat{u_0}(\xi)|d\xi\right)^2 dt\right)^{\frac{1}{2}} \\
&\leq \rho_{u_0}\|u_0\|_{\chi^{-1}}T^{\frac{1}{2}}.
\end{aligned}$$

So using Minkowski inequality, we deduce that

$$\begin{aligned}
\|\mathcal{S}(t)u_0\|_{L^2([0,T];\chi^0)} &\leq \|\mathcal{S}(t)(u_0-u_0^\flat)\|_{L^2([0,T];\chi^0)} + \|\mathcal{S}(t)u_0^\flat\|_{L^2([0,T];\chi^0)} \\
&\leq \left(\int_0^T \left(\int_{|\xi|\geq\rho_{u_0}} e^{-\nu|\xi|^2t}|\widehat{u_0}(\xi)|d\xi\right)^2 dt\right)^{\frac{1}{2}} + \rho_{u_0}\|u_0\|_{\chi^{-1}}T^{\frac{1}{2}} \\
&\leq \int_{|\xi|\geq\rho_{u_0}} \left(\int_0^T e^{-2\nu|\xi|^2t}|\widehat{u_0}(\xi)|^2 dt\right)^{\frac{1}{2}} d\xi + \rho_{u_0}\|u_0\|_{\chi^{-1}}T^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2\nu}}\int_{|\xi|\geq\rho_{u_0}} \frac{|\widehat{u_0}(\xi)|}{|\xi|}d\xi + \rho_{u_0}\|u_0\|_{\chi^{-1}}T^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2\nu}}\cdot\pi^3\nu + \rho_{u_0}\|u_0\|_{\chi^{-1}}T^{\frac{1}{2}}.
\end{aligned}$$

So if

$$T = \frac{\pi^6\nu}{2\rho_{u_0}^2\|u_0\|_{\chi^{-1}}^2}, \tag{3.3.1}$$

we get

$$\|\mathcal{S}(t)u_0\|_{L^2([0,T];\chi^0)} \leq \sqrt{2\nu}\pi^3.$$

By Lemma 3.6, this implies that (NS_Ω) has a unique solution u in $L^2([0,T];\chi^0)$.

Now we show $u \in L^1([0,T];\chi^1)$.

$$\begin{aligned}
\|S(t)u_0\|_{L^1_T\chi^1} &= \int_0^T \|S(t)u_0\|_{\chi^1}dt \\
&= \int_0^T \int_{\mathbb{R}^3} |\xi||\widehat{S(t)u_0}(\xi)|d\xi dt \\
&\leq \int_{\mathbb{R}^3} |\xi||\widehat{u_0}(\xi)| \int_0^T e^{-\nu|\xi|^2 t}dtd\xi \\
&\leq \int_{\mathbb{R}^3} \frac{|\xi|}{\nu|\xi|^2}|\widehat{u_0}(\xi)|d\xi \\
&= \frac{1}{\nu}\|u_0\|_{\chi^{-1}}.
\end{aligned}$$

Similarly, we see that by Lemma 3.3

$$\begin{aligned}
&\left\| \int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)ds \right\|_{L^1_T\chi^1} \\
&\leq \int_0^T \int_{\mathbb{R}^3} |\xi| \int_0^t |\mathcal{F}[\mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)](s,\xi)|\, dsd\xi dt \\
&\leq \int_0^T \int_{\mathbb{R}^3} |\xi| \int_0^t e^{-\nu|\xi|^2(t-s)}|\mathcal{F}[\mathbb{P}\nabla\cdot(u\otimes u)](s,\xi)|dsd\xi dt \\
&\leq \int_0^T \int_{\mathbb{R}^3} \int_0^t e^{-\nu|\xi|^2(t-s)}\frac{|\xi|^2}{(2\pi)^3}|\widehat{u}|*|\widehat{u}|(s,\xi)dsd\xi dt \\
&\leq \int_0^T \int_{\mathbb{R}^3} \frac{1}{\nu(2\pi)^3}|\widehat{u}|*|\widehat{u}|(s,\xi)d\xi ds \\
&\leq \frac{1}{\nu(2\pi)^3}\int_0^T \|u(s)\|^2_{\chi^0}ds.
\end{aligned}$$

Therefore we obtain that $u \in L^1([0,T];\chi^1)$.

We see $u \in L^\infty_T \chi^{-1}$. Indeed, we have

$$\begin{aligned}
\|\mathcal{S}(t)u_0\|_{L^\infty_T\chi^{-1}} &= \sup_{0\le t\le T} \|\mathcal{S}(t)u_0\|_{\chi^{-1}} \\
&= \sup_{0\le t\le T} \int_{\mathbb{R}^3} |\xi|^{-1} |\widehat{\mathcal{S}(t)u_0}(\xi)| d\xi \\
&\le \sup_{0\le t\le T} \int_{\mathbb{R}^3} |\xi|^{-1} e^{-\nu|\xi|^2 t} |\widehat{u_0}(\xi)| d\xi \\
&\le \int_{\mathbb{R}^3} |\xi|^{-1} |\widehat{u_0}(\xi)| d\xi = \|u_0\|_{\chi^{-1}}.
\end{aligned}$$

Moreover using Lemma 3.3 and Hausdorff-Young inequality, we obtain

$$\begin{aligned}
&\left\| \int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)ds \right\|_{L^\infty_T\chi^{-1}} \\
&\sup_{0\le t\le T} \left\| \int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)ds \right\|_{\chi^{-1}} \\
&= \sup_{0\le t\le T} \int_{\mathbb{R}^3} |\xi|^{-1} \left| \mathcal{F}\left[\int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)ds \right](\xi) \right| d\xi \\
&= \sup_{0\le t\le T} \int_{\mathbb{R}^3} |\xi|^{-1} \left| \int_0^t \mathcal{F}\left[\mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)\right](s,\xi) ds \right| d\xi \\
&\le \sup_{0\le t\le T} \int_{\mathbb{R}^3} |\xi|^{-1} \int_0^t \left|\mathcal{F}\left[\mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)\right](s,\xi)\right| ds d\xi \\
&\le \sup_{0\le t\le T} \int_{\mathbb{R}^3} |\xi|^{-1} \int_0^t e^{-\nu|\xi|^2(t-s)} \left|\mathcal{F}\left[\mathbb{P}\nabla\cdot(u\otimes u)(s)\right](s,\xi)\right| ds d\xi \\
&\le \sup_{0\le t\le T} \int_{\mathbb{R}^3} |\xi|^{-1} \int_0^t e^{-\nu|\xi|^2(t-s)} \left|\mathcal{F}\left[\nabla\cdot(u\otimes u)(s)\right](s,\xi)\right| ds d\xi \\
&\le \sup_{0\le t\le T} \int_{\mathbb{R}^3} |\xi|^{-1} \int_0^t e^{-\nu|\xi|^2(t-s)} \frac{1}{(2\pi)^3} |\xi| |\widehat{u}| * |\widehat{u}|(s,\xi) ds d\xi \\
&\le \frac{1}{(2\pi)^3} \sup_{0\le t\le T} \int_0^t \int_{\mathbb{R}^3} |\widehat{u}| * |\widehat{u}|(s,\xi)(s,\xi) d\xi ds \\
&\le \frac{1}{(2\pi)^3} \sup_{0\le t\le T} \int_0^t \|u(s)\|^2_{\chi^0} ds \\
&\le \frac{1}{(2\pi)^3} \int_0^T \|u(s)\|^2_{\chi^0} ds.
\end{aligned}$$

It follows that $u \in L^\infty_T \chi^{-1}$.
Next we prove $\partial_t u \in L^1([0,T];\chi^{-1})$, which implies $u \in C([0,T];\chi^{-1})$. We have

$$\partial_t u(t) = \nu\Delta u(t) - \Omega \mathbb{P} e_3 \times u(t) - \mathbb{P}\nabla\cdot(u\otimes u)$$

in the distribution sense.

Then we see that

$$\int_0^T \|\Delta u(t)\|_{\chi^{-1}} dt \le \int_0^T \int_{\mathbb{R}^3} |\xi||\widehat{u}(t,\xi)| d\xi dt = \|u\|_{L^1_T\chi^1}$$

and

$$\begin{aligned}\int_0^T \|\mathbb{P}\nabla\cdot(u\otimes u)\|_{\chi^{-1}} dt &\le \int_0^T \int_{\mathbb{R}^3} \frac{1}{|\xi|} |\mathcal{F}[\nabla\cdot(u\otimes u)](t,\xi)| d\xi dt \\ &\le \int_0^T \int_{\mathbb{R}^3} |\widehat{u}| * |\widehat{u}| d\xi dt \\ &\le \frac{1}{(2\pi)^3}\|u\|_{L^2_T\chi^0}.\end{aligned}$$

Since $u \in L^\infty_T\chi^{-1}$, we see

$$\int_0^T \|\Omega\mathbb{P}e_3 \times u(t)\|_{\chi^{-1}} dt \le \Omega T \|u\|_{L^\infty_T\chi^{-1}}.$$

Thus we have $\partial_t u \in L^1([0,T];\chi^{-1})$. Finally, we prove uniqueness of the solution to (NS_Ω) in $L^\infty_T\chi^{-1} \cap L^1_T\chi^1$. For $u, v \in L^\infty_T\chi^{-1} \cap L^1_T\chi^1$, we set $w := u - v$. Then, we observe

$$\begin{aligned} w(t) &= \left\{ \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)(s)ds \right\} \\ &\quad - \left\{ \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(v\otimes v)(s)ds \right\} \\ &= -\int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u - v\otimes v)(s)ds. \end{aligned}$$

So by Lemma 3.3, we see

$$\begin{aligned}
|\widehat{w}(t,\xi)| &= \left| \mathcal{F}\left[\int_0^t \mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u - v\otimes v)(s)ds \right](\xi) \right| \\
&\le \int_0^t \left| \mathcal{F}\left[\mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u - v\otimes v) \right](s,\xi) \right| ds \\
&\le \int_0^t e^{-\nu|\xi|^2(t-s)} \left| \mathcal{F}\left[\mathbb{P}\nabla\cdot(u\otimes u - v\otimes v) \right](s,\xi) \right| ds \\
&\le \int_0^t e^{-\nu|\xi|^2(t-s)} \left| \mathcal{F}\left[\nabla\cdot(u\otimes(u-v) + (u-v)\otimes v) \right](s,\xi) \right| ds \\
&\le \frac{1}{(2\pi)^3} \int_0^t e^{-\nu|\xi|^2(t-s)} |\xi|(|\widehat{u}| * |\widehat{w}|(s,\xi) + |\widehat{w}| * |\widehat{v}|(s,\xi))ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|w(t)\|_{\chi^{-1}} &\le \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_0^t \{(|\widehat{u}| * |\widehat{w}|)(s,\xi) + (|\widehat{w}| * |\widehat{v}|)(s,\xi)\} ds d\xi \\
&\le \frac{1}{(2\pi)^3} \int_0^t (\|u(s)\|_{\chi^0} + \|v(s)\|_{\chi^0}) \|w(s)\|_{\chi^0} ds.
\end{aligned}$$

Using Lemma 2.1, for $\varepsilon > 0$, there exists a positive number C_ε such that

$$\begin{aligned}
\|u(s)\|_{\chi^0}\|w(s)\|_{\chi^0} &\le \|u(s)\|_{\chi^1}^{\frac{1}{2}} \|u(s)\|_{\chi^{-1}}^{\frac{1}{2}} \cdot \|w(s)\|_{\chi^1}^{\frac{1}{2}} \|w(s)\|_{\chi^{-1}}^{\frac{1}{2}} \\
&\le C_\varepsilon \|u(s)\|_{\chi^1}\|w(s)\|_{\chi^{-1}} + \varepsilon\|u(s)\|_{\chi^{-1}}\|w(s)\|_{\chi^1}.
\end{aligned}$$

Thus we have

$$\|w\|_{L^\infty_T\chi^{-1}} \le \varepsilon\|u\|_{L^\infty_T\chi^{-1}}\|w\|_{L^1_T\chi^1} + C_\varepsilon\|u\|_{L^1_T\chi^1}\|w\|_{L^\infty_T\chi^{-1}}.$$

Similarly we see

$$\nu\|w\|_{L^1_T\chi^1} \le \varepsilon\|u\|_{L^\infty_T\chi^{-1}}\|w\|_{L^1_T\chi^1} + C_\varepsilon\|u\|_{L^1_T\chi^1}\|w\|_{L^\infty_T\chi^{-1}}.$$

Here we take sufficiently small $\varepsilon > 0$ such that

$$\varepsilon\|u\|_{L^\infty_T\chi^{-1}} < \frac{\nu}{4}.$$

Furthermore, if we take $\delta > 0$ such that

$$C_\varepsilon\|u\|_{L^1_\delta\chi^1} < \frac{1}{4},$$

we have

$$\begin{aligned}\|w\|_{L^\infty_\delta \chi^{-1}} + \nu\|w\|_{L^1_\delta \chi^1} \leq & 2\varepsilon\|u\|_{L^\infty_\delta \chi^{-1}}\|w\|_{L^1_\delta \chi^1} + 2C_\varepsilon\|u\|_{L^1_\delta \chi^1}\|w\|_{L^\infty_\delta \chi^{-1}} \\ \leq & \frac{1}{2}(\|w\|_{L^\infty_\delta \chi^{-1}} + \nu\|w\|_{L^1_\delta \chi^1}).\end{aligned}$$

Thus we deduce $\|w\|_{L^\infty_\delta \chi^{-1}} + \nu\|w\|_{L^1_\delta \chi^1} = 0$. Therefore , we have

$$w(t) = 0, \quad t \in [0, \delta].$$

Repeating this argument, we see uniqueness of the solution. □

3.4 Proof of Theorem 3.1

In this section, we prove Theorem 3.1.

proof. Let T^* be the maximal existence time of a solution of (NS_Ω), derived by applying Theorem 3.7 repeatedly. Suppose $T^* < \infty$. By (3.3.1), we must have

$$\lim_{t\to T^*} \|u(t)\|_{\chi^{-1}} = \infty,$$

or

$$\lim_{t\to T^*} \rho(t) = \infty, \tag{3.4.1}$$

where $\rho(t)$ is determined by

$$\rho(t) = \inf\left\{\rho > 0; \int_{|\xi|\geq\rho} \frac{|\widehat{u}(t,\xi)|}{|\xi|} d\xi \leq \pi^3\nu\right\}.$$

We easily observe that $\sup_{0<t<T^*} \|u(t)\|_{\chi^{-1}} \leq \|u_0\|_{\chi^{-1}}$ by Theorem 2.3. So, it suffices to show that (3.4.1) would never happen. For $0 < t < T^*$, we find that

$$\begin{aligned}|\widehat{u}(t,\xi)| \leq & |\widehat{\mathcal{S}(t)u_0}(\xi)| + \int_0^t |\mathcal{F}[\mathcal{S}(t-s)\mathbb{P}\nabla\cdot(u\otimes u)](s,\xi)|ds \\ \leq & e^{-\nu|\xi|^2 t}|\widehat{u_0}(\xi)| + \int_0^t e^{-\nu(t-s)|\xi|^2}|\mathcal{F}[\nabla\cdot(u\otimes u)](s,\xi)|ds \\ \leq & |\widehat{u_0}(\xi)| + \int_0^t e^{-\nu(t-s)|\xi|^2}|\mathcal{F}[\nabla\cdot(u\otimes u)](s,\xi)|ds \\ \leq & |\widehat{u_0}(\xi)| + \int_0^t e^{-\nu(t-s)|\xi|^2}\frac{|\xi|}{(2\pi)^3}|\widehat{u}| * |\widehat{u}|(s,\xi)ds.\end{aligned}$$

Hence we see that

$$\int_{\mathbb{R}^3} \sup_{0\le t\le T^*} |\widehat{u}(t,\xi)| \frac{1}{|\xi|} d\xi \le \|u_0\|_{\chi^{-1}} + \int_{\mathbb{R}^3} \int_0^{T^*} \frac{1}{(2\pi)^3} |\widehat{u}| * |\widehat{u}|(s,\xi) ds d\xi$$
$$\le \|u_0\|_{\chi^{-1}} + \frac{1}{(2\pi)^3} \int_0^{T^*} \|u(s)\|_{\chi^0}^2 ds$$
$$\le \|u_0\|_{\chi^{-1}} + \frac{1}{(2\pi)^3} \|u\|^2_{L^2([0,T^*);\chi^0)}.$$

Applying Theorem 2.3, we obtain that

$$\|u\|^2_{L^2([0,T^*);\chi^0)} = \int_0^{T^*} \|u(t)\|^2_{\chi^0} dt$$
$$\le \int_0^{T^*} \|u(t)\|_{\chi^{-1}} \|u(t)\|_{\chi^1} dt$$
$$\le \|u_0\|_{\chi^{-1}} \cdot \frac{\|u_0\|_{\chi^{-1}}}{\nu - \frac{1}{(2\pi)^3}\|u_0\|_{\chi^{-1}}}.$$

Thus we have there exists some $M > 0$, such that

$$\int_{\mathbb{R}^3} \sup_{0\le t\le T^*} |\widehat{u}(t,\xi)| \frac{1}{|\xi|} d\xi < M.$$

This implies that we are able to take $\rho > 0$ such that

$$\int_{|\xi|>\rho} \sup_{0\le t\le T^*} |\widehat{u}(t,\xi)| \frac{1}{|\xi|} d\xi < \pi^3 \nu,$$

we get for any $0 < t < T^*$,

$$\int_{|\xi|>\rho} |\widehat{u}(t,\xi)| \frac{1}{|\xi|} d\xi < \int_{|\xi|>\rho} \sup_{0\le t\le T^*} \frac{|\widehat{u}(t,\xi)|}{|\xi|} d\xi < \pi^3 \nu.$$

This contradicts to (3.4.1). □

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