SCATTERING AND WELL-POSEDNESS FOR THE ZAKHAROV SYSTEM AND THE KLEIN-GORDON-ZAKHAROV SYSTEM IN FOUR AND MORE SPATIAL DIMENSIONS

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Contents

1. Introduction

1.1. **Main results and main idea.** We study the Cauchy problem for the Klein-Gordon-Zakharov system and the Zakharov system in spatial dimension $d \geq 4$. The Klein-Gordon-Zakharov system is as follows.

$$\begin{cases} (\partial_t^2 - \Delta + 1)u = -nu, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (\partial_t^2 - c^2\Delta)n = \Delta |u|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (u, \partial_t u, n, \partial_t n)|_{t=0} = (u_0, u_1, n_0, n_1) & \\ \qquad \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d), & \end{cases} \tag{1.1}$$

where $u = u(t,x), n = n(t,x)$ are real valued functions, $c > 0$ and $c \neq 1$. The Zakharov system is as follows.

$$\begin{cases} (i\partial_t + \Delta)u = nu, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (\partial_t^2 - \Delta)n = \Delta |u|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ \big(u(0), n(0), \partial_t n(0)\big) = (u_0, n_0, n_1) \in H^k(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d) \times \dot{H}^{l-1}(\mathbb{R}^d), & \end{cases} \tag{1.2}$$

where $u = u(t,x)$ is complex valued and $n = n(t,x)$ is real valued function.

First, we consider (1.1). If we transform $u_\pm := \omega_1 u \pm i\partial_t u, n_\pm := n \pm i(c\omega)^{-1}\partial_t n$, $\omega_1 := (1-\Delta)^{1/2}, \omega := (-\Delta)^{1/2}$, then (1.1) is equivalent to the following.

$$\begin{cases} (i\partial_t \mp \omega_1)u_\pm = \pm(1/4)(n_+ + n_-)(\omega_1^{-1}u_+ + \omega_1^{-1}u_-), & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (i\partial_t \mp c\omega)n_\pm = \pm(4c)^{-1}\omega|\omega_1^{-1}u_+ + \omega_1^{-1}u_-|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (u_\pm, n_\pm)|_{t=0} = (u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d). & \end{cases} \tag{1.3}$$

Our main result is as follows.

Theorem 1.1. *(i) Let $d = 4$. Then (1.3) is locally well-posed in $H^{1/4}(\mathbb{R}^4) \times \dot{H}^{1/4}(\mathbb{R}^4)$.*
(ii) Let $d \geq 5$ and $s = (d^2 - 3d - 2)/2(d+1)$. Then (1.3) is locally well-posed in $H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$.
(iii) Let $d \geq 4, s = s_c = d/2 - 2$ and assume the initial data $(u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$ is small and radial. Then, (1.3) is globally well-posed in $H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$.

Corollary 1.2. *The solution obtained in Theorem 1.1 (iii) scatters as $t \to \pm\infty$.*

For more precise statement of Theorem 1.1 and Corollary 1.2, see Propositions 2.15, 2.16 in section 2.4. We consider both the radial case and the non-radial case. First, we consider the radial case. The scaling regularity of (1.3) is $s_c = d/2 - 2$. We have to recover a half derivative loss to derive the key bilinear estimates at the critical space. Bourgain introduced the Fourier restriction norm method to recover the derivative loss. However, it seems difficult to apply the method in the ctitical case. We apply the U^2, V^2 type spaces, which are introduced by Koch and Tataru [30]. Thanks to $c > 0$ and $c \neq 1$, if $|\xi| \gg |\xi'|$, then it holds that

$$M' := \max\{\big|\tau \pm c|\xi|\big|, |\tau' \pm \langle \xi' \rangle|, |\tau - \tau' \pm \langle \xi - \xi' \rangle|\} \gtrsim |\xi|. \tag{1.4}$$

Here, ξ (resp. ξ') denote frequency for the wave equation (resp. Klein-Gordon equation) and $\tau \pm c|\xi|$ (resp. $\tau' \pm \langle \xi' \rangle, \tau - \tau' \pm \langle \xi - \xi' \rangle$) denote the symbol of the linear part for the wave equation (resp. the Klein-Gordon equation). From (1.4) and by applying the U^2, V^2 type spaces, then we can recover the derivative loss. In the radial case, the Strichartz estimates hold for a more wider range of (q, r). More precisely, see Propostions 2.5, 2.6, 2.8 and 2.9. Therefore, we can derive the blinear estimates at the critical space by applying the U^2, V^2 type spaces and the radial Strichartz estimates. Next, we consider $d = 4$ and the non-radial case. When $d \leq 4$, the Lorentz regularity s_l is an important index as well as the scaling regularity for the well-posedness for the wave equation. When $d = 4$ with quadratic nonlinearity, the Lorentz regularity $s_l = 1/4$. On the other hand, $s_c = 0$, hence we need to consider $s \geq s_l = 1/4$. When $d = 4$, we obtain local well-posedness at $s = s_l = 1/4$ by applying U^2, V^2 type spaces. Finally, we consider $d \geq 5$ and the non-radial case. Since $s_c \geq s_l$ when $d \geq 5$, we expect the local well-posedness with $s = s_c$. However, we only obtain the local well-posedness with $s = s_c + 1/(d+1)$. It seems difficult to derive the bilinear estimates with $s = s_c$. The reason is as below. We observe the first equation of (1.3). We regard the nonlinearity as $n_\pm(\omega_1^{-1} u_\pm)$. Here, we consider the following cases. The case $|\xi| \lesssim |\xi'|$ and the case $|\xi| \gg |\xi'|$, where ξ, ξ' denote the frequency of $n_\pm, u_\pm$ respectively. For the case $|\xi| \lesssim |\xi'|$, the nonlinearity does not have the derivative loss, so we can show the bilinear estimate at the critical space only by applying the Strichartz estimates. However, for the case $|\xi| \gg |\xi'|$, we need to recover a half derivative loss by (1.4). Here, there are three cases in (1.4). The cases (a) $M' = \big|\tau \pm c|\xi|\big|$, (b) $M' = |\tau' \pm \langle \xi' \rangle|$ and (c) $M' = |\tau - \tau' \pm \langle \xi - \xi' \rangle|$. For the case (a) or (c), we apply (1.4) for $n_\pm$ and the Strichartz estimates for $\omega_1^{-1} u_\pm$. Then we can obtain the bilinear estimate at the critical space. Whereas for (b), we apply the Strichartz estimates for $n_\pm$ and apply (1.4) for $\omega_1^{-1} u_\pm$. In this case, we cannot prove the bilinear estimate at the critical space. As a result, we have to impose more regularity.

Next, we consider (1.2). Our main result is as follows.

Theorem 1.3. *Let $d \geq 4, k = (d-3)/2, l = (d-4)/2$. Then (1.2) is globally well-posed for small data in $H^k(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d)$. Moreover, the solution scatters in $H^k(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d)$.*

For more precise statement of Theorem 1.3, see Propositions 2.15, 3.12. (1.2) does not have scaling invariant transformation because of the difference of dilation transformations for the linear wave equation and the Schrödinger equation. However, in [10], Ginibre, Tsutsumi and Velo introduced a critical exponent for (1.2) which corresponds to the scaling criticality. The critical exponent is $(k, l) = \big((d-3)/2, (d-4)/2\big)$. More precisely, see subsection 3.1. We apply U^2, V^2 type spaces to obtain Theorem 1.2. However, it seems difficult to prove the bilinear estimates at the critical space only by applying U^2, V^2 type spaces. By the difference of the dilation scale of the Schödinger equation and the wave equation, the effect by oscillatory integral for the Schrödinger equation works more effective than that of the wave equation. Therefore, in our problem we have to use the endpoint Strichartz estimate for the Schrödinger equation, that is to say the case of $(p_1, q_1) = (2, 2d/(d-2))$ in Proposition 3.5. This causes the following problem: if we use the U^2 type function space and follow the argument by Hadac-Herr-Koch [19], then by the duality argument (see Proposition 1.10) we need to estimate $L_t^2 L_x^{2d/(d-2)}$ norm by the V^2 type norm. However, we can not get such estimate by Corollary 1.15 because the V^2 type norm is slightly weaker than U^2 type norm. For this reason, we need the function space weaker than the U^2 type and stronger than the V^2 type. For that purpose, we use an intersection space of V^2 type space and $E := L_t^2 L_x^{2d/(d-2)}$. See the definition of $\|u\|_{X_S^k}$ in Definition 4, which is the main idea. This is a joint work with Professor K. Tsugawa.

1.2. **Notations.** In this section, we prepare some lemmas, propositions and notations. Notations related to U^p and V^p spaces are based on the definition in [19] and [20]. $A \lesssim B$ means that there exists $C > 0$ such that $A \leq CB$. Also, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. Let $u = u(t, x)$. $\mathcal{F}_t u$, $\mathcal{F}_x u$ denote the Fourier transform of u in time, space, respectively. $\mathcal{F}_{t,x} u = \widehat{u}$ denotes the Fourier transform of u in space and time. Let $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \cdots < t_K = \infty$ and let $\mathcal{Z}_0$ be the set of finite partitions $-\infty < t_0 < t_1 < \cdots < t_K \leq \infty$.

Definition 1. Let $1 \le p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L_x^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{L_x^2}^p = 1$, we call the function $a : \mathbb{R} \to L_x^2$ given by

$$a = \sum_{k=1}^{K} \mathbf{1}_{[t_{k-1},\, t_k)} \phi_{k-1}$$

a U^p-atom. Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \,\middle|\, a_j : U^p\text{-atom}, \lambda_j \in \mathbb{C} \textit{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \,\middle|\, u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j : U^p\text{-atom} \right\}.$$

Proposition 1.4. *Let $1 \le p < q < \infty$.*
(i) U^p is a Banach space.
(ii) The embeddings $U^p \subset U^q \subset L_t^\infty(\mathbb{R}; L_x^2)$ are continuous.
(iii) For $u \in U^p$, it holds that $\lim_{t \to t_0+} \|u(t) - u(t_0)\|_{L_x^2} = 0$, i.e. every $u \in U^p$ is right-continuous.
(iv) The closed subspace U_c^p of all continuous functions in U^p is a Banach space.

The above proposition is in [19] (Proposition 2.2).

Definition 2. Let $1 \le p < \infty$. We define V^p as the normed space of all functions $v : \mathbb{R} \to L_x^2$ such that $\lim_{t \to \pm\infty} v(t)$ exist and for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L_x^2}^p \right)^{1/p}$$

is finite, where we use the convention that $v(-\infty) := \lim_{t \to -\infty} v(t)$ and $v(\infty) := 0$. Likewise, let V_-^p denote the closed subspace of all $v \in V^p$ with $\lim_{t \to -\infty} v(t) = 0$.

The definitions of V^p and V_-^p, see the erratum [20].

Proposition 1.5. *Let $1 \le p < q < \infty$.*
(i) Let $v : \mathbb{R} \to L_x^2$ be such that

$$\|v\|_{V_0^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}_0} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L_x^2}^p \right)^{1/p}$$

is finite. Then, it follows that $v(t_0^+) := \lim_{t \to t_0+} v(t)$ exists for all $t_0 \in [-\infty, \infty)$ and $v(t_0^-) := \lim_{t \to t_0-} v(t)$ exists for all $t_0 \in (-\infty, \infty]$ and moreover,

$$\|v\|_{V^p} = \|v\|_{V_0^p}.$$

(ii) We define the closed subspace V^p_{rc} $(V^p_{-,rc})$ *of all right-continuous* V^p *functions* $(V^p_-$ *functions). The spaces* V^p, V^p_{rc}, V^p_- *and* $V^p_{-,rc}$ *are Banach spaces.*
(iii) The embeddings $U^p \subset V^p_{-,rc} \subset U^q$ *are continuous.*
(iv) The embeddings $V^p \subset V^q$ *and* $V^p_- \subset V^q_-$ *are continuous.*

The proof of Proposition 1.5 is in [19] (Proposition 2.4 and Corollary 2.6). Let $\{\mathcal{F}^{-1}_\xi[\varphi_n](x)\}_{n\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^d)$ be the Littlewood-Paley decomposition with respect to x, that is to say

$$\begin{cases} \varphi(\xi) \geq 0, \\ \operatorname{supp}\varphi(\xi) = \{\xi \,|\, 2^{-1} \leq |\xi| \leq 2\}, \end{cases}$$

$$\varphi_n(\xi) := \varphi(2^{-n}\xi), \ \sum_{n=-\infty}^{\infty} \varphi_n(\xi) = 1 \ (\, \xi \neq 0), \ \psi(\xi) := 1 - \sum_{n=0}^{\infty} \varphi_n(\xi).$$

Let $N = 2^n$ $(n \in \mathbb{Z})$ be dyadic number. P_N and $P_{<1}$ denote

$$\mathcal{F}_x[P_N f](\xi) := \varphi(\xi/N)\mathcal{F}_x[f](\xi) = \varphi_n(\xi)\mathcal{F}_x[f](\xi),$$
$$\mathcal{F}_x[P_{<1} f](\xi) := \psi(\xi)\mathcal{F}_x[f](\xi).$$

Similarly, let Q_N be

$$\mathcal{F}_t[Q_N g](\tau) := \phi(\tau/N)\mathcal{F}_t[g](\tau) = \phi_n(\tau)\mathcal{F}_t[g](\tau),$$

where $\{\mathcal{F}^{-1}_\tau[\phi_n](t)\}_{n\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R})$ be the Littlewood-Paley decomposition with respect to t. Let $K_\pm(t) = \exp\{\mp it(1-\Delta)^{1/2}\} : L^2_x \to L^2_x$ be the Klein-Gordon unitary operators such that $\mathcal{F}_x[K_\pm(t)u_0](\xi) = \exp\{\mp it\langle\xi\rangle\}\,\mathcal{F}_x[u_0](\xi)$. Similarly, we define the wave unitary operators $W_{\pm c}(t) = \exp\{\mp ict(-\Delta)^{1/2}\} : L^2_x \to L^2_x$ such that $\mathcal{F}_x[W_{\pm c}(t)n_0](\xi) = \exp\{\mp ict|\xi|\}\,\mathcal{F}_x[n_0](\xi), W_\pm(t) = \exp\{\mp it(-\Delta)^{1/2}\} : L^2_x \to L^2_x$ such that $\mathcal{F}_x[W_\pm(t)n_0](\xi) = \exp\{\mp it|\xi|\}\,\mathcal{F}_x[n_0](\xi)$. Let $S(t) = \exp\{it\Delta\} : L^2_x \to L^2_x$ be the Schrödinger unitary operator such that $\mathcal{F}_x[S(t)u_0](\xi) = \exp\{-it|\xi|^2\}\,\mathcal{F}_x[u_0](\xi)$.

Definition 3. We define
(i) $U^p_{K_\pm} = K_\pm(\cdot)U^p$ with norm $\|u\|_{U^p_{K_\pm}} = \|K_\pm(-\cdot)u\|_{U^p}$,
(ii) $V^p_{K_\pm} = K_\pm(\cdot)V^p$ with norm $\|u\|_{V^p_{K_\pm}} = \|K_\pm(-\cdot)u\|_{V^p}$.
For dyadic number N, M,

$$Q^{K_\pm}_N := K_\pm(\cdot)Q_N K_\pm(-\cdot), \quad Q^{K_\pm}_{\geq M} := \sum_{N\geq M} Q^{K_\pm}_N, \quad Q^{K_\pm}_{<M} := Id - Q^{K_\pm}_{\geq M}.$$

Here summation over N means that summation over $n \in \mathbb{Z}$. Similarly, for $A \in \{S, W_\pm, W_{\pm c}\}$, we define $U^p_A, V^p_A, Q^A_N, Q^A_{\geq M}, Q^A_{<M}$.

Remark 1.1. For L^2_x unitary operator $A \in \{K_\pm, S, W_\pm, W_{\pm c}\}$,

$$U^2_A \subset V^2_{-,rc,A} \subset L^\infty(\mathbb{R}; L^2_x)$$

holds by Proposition 1.4 (ii) and Proposition 1.5 (iii).

Definition 4. For the Schrödinger equation, we define X^k_S as the closure of all $u \in C(\mathbb{R}; H^k_x(\mathbb{R}^d)) \cap \langle \nabla_x \rangle^{-k} V^2_{-,rc,S} \cap \langle \nabla_x \rangle^{-k} E$ with X^k_S norm, where

$$\|u\|_{X^k_S} := \|u\|_{Y^k_S} + \|u\|_{E^k} < \infty,\ \|u\|_{Y^k_S} := \|P_{<1} u\|_{V^2_S} + \Big(\sum_{N \geq 1} N^{2k} \|P_N u\|^2_{V^2_S} \Big)^{1/2},$$
$$\|u\|_{E^k} := \|P_{<1} u\|_E + \Big(\sum_{N \geq 1} N^{2k} \|P_N u\|^2_E \Big)^{1/2},$$

where $E := L^2_t L^{2d/(d-2)}_x$. For $A \in \{K_\pm, W_\pm, W_{\pm c}\}$, we define $\dot{Z}^l_A$ $(resp.\, Z^l_A, \dot{Y}^l_A, Y^l_A)$ as the closure of all $n \in C(\mathbb{R}; \dot{H}^l_x(\mathbb{R}^d)) \cap |\nabla_x|^{-l} U^2_A$ $(resp.\, n \in C(\mathbb{R}; H^l_x(\mathbb{R}^d)) \cap \langle \nabla_x \rangle^{-l} U^2_A,\ n \in C(\mathbb{R}; \dot{H}^l_x(\mathbb{R}^d)) \cap |\nabla_x|^{-l} V^2_A,\ n \in C(\mathbb{R}; H^l_x(\mathbb{R}^d)) \cap \langle \nabla_x \rangle^{-l} V^2_A)$ with

$$\|n\|_{\dot{Z}^l_A} := \Big(\sum_N N^{2l} \|P_N n\|^2_{U^2_A} \Big)^{1/2},\ \|n\|_{Z^l_A} := \|P_{<1} n\|_{U^2_A} + \Big(\sum_{N \geq 1} N^{2l} \|P_N n\|^2_{U^2_A} \Big)^{1/2},$$
$$\|n\|_{\dot{Y}^l_A} := \Big(\sum_N N^{2l} \|P_N n\|^2_{V^2_A} \Big)^{1/2},\ \|n\|_{Y^l_A} := \|P_{<1} n\|_{V^2_A} + \Big(\sum_{N \geq 1} N^{2l} \|P_N n\|^2_{V^2_A} \Big)^{1/2}.$$

Definition 5. For a Hilbert space H and a Banach space $X \subset C(\mathbb{R}; H)$, we define

$$B_r(H) := \{ f \in H \,|\, \|f\|_H \leq r \},$$
$$X([0,T)) := \{ u \in C([0,T); H) \,|\, {}^\exists \tilde{u} \in X, \tilde{u}(t) = u(t), t \in [0,T) \}$$

endowed with the norm $\|u\|_{X([0,T))} = \inf\{ \|\tilde{u}\|_X \,|\, \tilde{u}(t) = u(t), t \in [0,T) \}$.

Lemma 1.6. *Let $a \geq 0$. Then for $A \in \{K_\pm, S, W_\pm, W_{\pm c}\}$, it holds that*

$$\|\langle \nabla_x \rangle^a f\|_{V^2_A} \lesssim \|f\|_{Y^a_A}.$$

Proof. By L_x^2 orthogonality, we have

$$
\begin{aligned}
\|\langle \nabla_x \rangle^a f\|_{V_A^2}^2 &\lesssim \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^{I} (\|P_{<1}(A(-t_i)f(t_i) - A(-t_{i-1})f(t_{i-1}))\|_{L_x^2}^2 \\
&\quad + \sum_{N \geq 1} N^{2a} \|P_N(A(-t_i)f(t_i) - A(-t_{i-1})f(t_{i-1}))\|_{L_x^2}^2) \\
&\lesssim \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^{I} \|A(-t_i)P_{<1}f(t_i) - A(-t_{i-1})P_{<1}f(t_{i-1})\|_{L_x^2}^2 \\
&\quad + \sum_{N \geq 1} N^{2a} \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^{I} \|A(-t_i)P_N f(t_i) - A(-t_{i-1})P_N f(t_{i-1})\|_{L_x^2}^2 \\
&\lesssim \|f\|_{Y_A^a}^2.
\end{aligned}
$$

□

Remark 1.2. Similarly, we see

$$\||\nabla_x|^a f\|_{V_A^2} \lesssim \|f\|_{\dot{Y}_A^a}.$$

For the proof of the following propositions, see Proposition 2.7, Theorem 2.8 and Proposition 2.10 in [19].

Proposition 1.7. *Let p, p' satisfy $1/p + 1/p' = 1$. For $u \in U^p$ and $v \in V^{p'}$ and a partition $t := \{t_i\}_{i=0}^I \in \mathcal{Z}$ we define*

$$B_t(u, v) := \sum_{i=1}^{I} \langle u(t_{i-1}), v(t_i) - v(t_{i-1}) \rangle_{L_x^2}.$$

There is a unique number $B(u, v)$ with the property that for all $\varepsilon > 0$ there exists $t \in \mathcal{Z}$ such that for every $t' \supset t$ it holds

$$|B_{t'}(u, v) - B(u, v)| < \varepsilon,$$

and the asociated bilinear form

$$B : U^p \times V^{p'} \ni (u, v) \mapsto B(u, v) \in \mathbb{C}$$

satisfies the estimate

$$|B(u, v)| \leq \|u\|_{U^p} \|v\|_{V^{p'}}.$$

Proposition 1.8. *Let $1 < p < \infty$. We have*

$$(U^p)^* = V^{p'}$$

in the sense that

$$T : V^{p'} \to (U^p)^*, \quad T(v) := B(\cdot, v)$$

is an isometric isomorphism.

Proposition 1.9. *Let $1 < p < \infty$, $u \in V^1_-$ be absolutely continuous on compact intervals and $v \in V^{p'}$. Then,*

$$B(u, v) = -\int_{-\infty}^{\infty} \langle u'(t), v(t)\rangle_{L^2}\, dt.$$

By Propositions 1.8, 1.9, we have the following proposition (see also Remark 2.11 in [19]).

Proposition 1.10. *$u \in V^1_- \subset U^2$ be absolutely continuous on compact intervals. Then, $\|u\|_{U^2} = \sup_{v\in V^2,\, \|v\|_{V^2}=1} \Big|\int_{-\infty}^{\infty} \langle u'(t), v(t)\rangle_{L^2_x} dt\Big|$.*

By the proposition above, we immediately have the following corollary.

Corollary 1.11. *Let $A \in \{K_\pm, S, W_\pm, W_{\pm c}\}$ and $u \in V^1_{-,A} \subset U^2_A$ be absolutely continuous on compact intervals. Then,*

$$\|u\|_{U^2_A} = \sup_{v\in V^2_A,\, \|v\|_{V^2_A}=1} \Big|\int_{-\infty}^{\infty} \langle A(t)\big(A(-\cdot)u\big)'(t), v(t)\rangle_{L^2_x} dt\Big|.$$

For the following remark, see Remark 2.12 in [19].

Remark 1.3. For $v \in V^2$, it holds that

$$\|v\|_{V^p} = \sup_{u; U^2\text{-atom}} |B(u, v)|.$$

For the proof of the following Lemma, see section 2.2 and section 3.2.

Lemma 1.12. *Let $c > 0, c \neq 1$ and $\tau_3 = \tau_1 - \tau_2$, $\xi_3 = \xi_1 - \xi_2$. If $|\xi_1| \gg \langle \xi_2\rangle$ or $\langle \xi_1\rangle \ll |\xi_2|$, then it holds that*

$$\max\big\{\big|\tau_1 \pm \langle\xi_1\rangle\big|, \big|\tau_2 \pm \langle\xi_2\rangle\big|, \big|\tau_3 \pm c|\xi_3|\big|\big\} \gtrsim \max\{|\xi_1|, |\xi_2|\}, \tag{1.5}$$

$$\max\big\{\big|\tau_1 + |\xi_1|^2\big|, \big|\tau_2 + |\xi_2|^2\big|, \big|\tau_3 \pm |\xi_3|\big|\big\} \gtrsim \max\{|\xi_1|^2, |\xi_2|^2\}. \tag{1.6}$$

Proposition 1.13. *We have*

$$\|Q^S_M u\|_{L^2_{t,x}(\mathbb{R}^{1+d})} \lesssim M^{-1/2}\|u\|_{V^2_S}, \qquad \|Q^S_{\geq M} u\|_{L^2_{t,x}(\mathbb{R}^{1+d})} \lesssim M^{-1/2}\|u\|_{V^2_S}, \tag{1.7}$$

$$\|Q^S_{<M} u\|_{V^2_S} \lesssim \|u\|_{V^2_S}, \qquad \|Q^S_{\geq M} u\|_{V^2_S} \lesssim \|u\|_{V^2_S},$$

$$\|Q^S_{<M} u\|_{U^2_S} \lesssim \|u\|_{U^2_S}, \qquad \|Q^S_{\geq M} u\|_{U^2_S} \lesssim \|u\|_{U^2_S}.$$

The same estimates hold by replacing the Schrödinger operator S with the wave operators $W_{\pm}, W_{\pm c}$, the Klein-Gordon operators $K_{\pm}$.

For the proof of Proposition 1.13, see Corollary 2.18 in [19].

Remark 1.4. If we define M as the left-hand side of (1.5), a half derivative loss is recovered by Lemma 1.12 and (1.7) in Proposition 1.13. Similarly, if we define M as the left-hand side of (1.6), the first derivative loss is recovered by Lemma 1.12 and (1.7) in Proposition 1.13.

Proposition 1.14. *Let $T_0 : L^2_x \times \cdots \times L^2_x \to L^1_{loc}(\mathbb{R}^d;\mathbb{C})$ be a n-linear operator and $A \in \{K_{\pm}, S, W_{\pm}, W_{\pm c}\}$. Assume that for some $1 \le p,\ q \le \infty$, it holds that*

$$\|T_0(A(\cdot)\phi_1, \dots, A(\cdot)\phi_n)\|_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))} \lesssim \prod_{i=1}^{n} \|\phi_i\|_{L^2_x}.$$

Then, there exists $T : U^p_A \times \cdots \times U^p_A \to L^p_t(\mathbb{R}; L^q_x(\mathbb{R}^d))$ satisfying

$$\|T(u_1, \dots, u_n)\|_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))} \lesssim \prod_{i=1}^{n} \|u_i\|_{U^p_A},$$

such that $T(u_1, \dots, u_n)(t)(x) = T_0(u_1(t), \dots, u_n(t))(x)$ a.e.

See Proposition 2.19 in [19] for the proof of the above proposition. See section 2.2 and 3.2 for the Strichartz estimates for the Klein-Gordon equation, the wave equation and the Schrödinger equation. Combining the Strichartz estimate, Propositions 1.5 and 1.14, we have the following corollary.

Corollary 1.15. *Let (p_1, q_1) satisfy the assumption in Proposition 3.5 and $p \ge p_1$. Then, U^p_S is continuously embedded in $L^{p_1}_t L^{q_1}_x$.*

Lemma 1.16. *If f, g are measurable functions and $A \in \{K_{\pm}, S\}, Q^A \in \{Q^A_{<M}, Q^A_{\ge M}\}$, then it holds that*

$$\int_{\mathbb{R}^{1+d}} f(t,x)\overline{Q^A g(t,x)}dxdt = \int_{\mathbb{R}^{1+d}} \big(Q^A f(t,x)\big)\overline{g(t,x)}dxdt.$$

For the proof of Lemma 1.16, see section 3.2.

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2. Well-posedness for the Cauchy problem of the Klein-Gordon-Zakharov system in four and more spatial dimensions

2.1. Introduction.

We consider the Cauchy problem of the Klein-Gordon-Zakharov system:

$$\begin{cases} (\partial_t^2 - \Delta + 1)u = -nu, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (\partial_t^2 - c^2\Delta)n = \Delta|u|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (u, \partial_t u, n, \partial_t n)|_{t=0} = (u_0, u_1, n_0, n_1) & \\ \qquad \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d), & \end{cases} \tag{2.1}$$

where u, n are real valued functions, $d \geq 4, c > 0$ and $c \neq 1$. The physical model of (2.1) is the interaction of the Langmuir wave and the ion acoustic wave in a plasma. In the physical model, c satisfies $0 < c < 1$. When $d = 3$, Ozawa, Tsutaya and Tsutsumi [41] proved that (2.1) is globally well-posed in the energy space $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times \dot{H}^{-1}(\mathbb{R}^3)$. They applied the Fourier restriction norm method to obtain the local well-posedness. Then by the local well-posedness and the energy method, they obtained the global well-posedness. For $d = 3$, Guo, Nakanishi and Wang [17] proved scattering in the energy class with small, radial initial data. They applied the normal form reduction and the radial Strichartz estimates. If we transform $u_\pm := \omega_1 u \pm i\partial_t u, n_\pm := n \pm i(c\omega)^{-1}\partial_t n, \omega_1 := (1-\Delta)^{1/2}, \omega := (-\Delta)^{1/2}$, then (2.1) is equivalent to the following.

$$\begin{cases} (i\partial_t \mp \omega_1)u_\pm = \pm(1/4)(n_+ + n_-)(\omega_1^{-1}u_+ + \omega_1^{-1}u_-), & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (i\partial_t \mp c\omega)n_\pm = \pm(4c)^{-1}\omega|\omega_1^{-1}u_+ + \omega_1^{-1}u_-|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (u_\pm, n_\pm)|_{t=0} = (u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d). & \end{cases} \tag{2.2}$$

Our main result is as follows.

Theorem 2.1. *(i) Let $d = 4$. Then (2.2) is locally well-posed in $H^{1/4}(\mathbb{R}^4) \times \dot{H}^{1/4}(\mathbb{R}^4)$.*
(ii) Let $d \geq 5$ and $s = (d^2 - 3d - 2)/2(d+1)$. Then (2.2) is locally well-posed in $H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$.
(iii) Let $d \geq 4, s = s_c = d/2 - 2$ and assume the initial data $(u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$ is small and radial. Then, (2.2) is globally well-posed in $H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$.

Corollary 2.2. *The solution obtained in Theorem 2.1 (iii) scatters as $t \to \pm\infty$.*

For more precise statement of Theorem 2.1 and Corollary 2.2, see Propositions 2.15, 2.16. The scaling regularity of (2.2) is $s_c = d/2 - 2$. We consider both the radial case and the non-radial case. First, we consider the radial case. In the radial case, the Strichartz estimates hold for a more wider range of (q, r). More precisely, see Propostions 2.8, 2.9. On the other hand, we have to recover a half derivative loss to derive the key bilinear estimates at the critial space. Thanks to $c > 0$ and $c \neq 1$, if $|\xi| \gg |\xi'|$, then it holds that

$$M' := \max\{\big|\tau \pm c|\xi|\big|, |\tau' \pm \langle \xi' \rangle|, |\tau - \tau' \pm \langle \xi - \xi' \rangle|\} \gtrsim |\xi|. \tag{2.3}$$

Here, ξ (resp. ξ') denote frequency for the wave equation (resp. Klein-Gordon equation) and $\tau \pm c|\xi|$ (resp. $\tau' \pm \langle \xi' \rangle, \tau - \tau' \pm \langle \xi - \xi' \rangle$) denote the symbol of the linear part for the wave equation (resp. the Klein-Gordon equation). From (2.3) and by applying the U^2, V^2 type spaces, then we can recover the derivative loss. Therefore, we can obtain the bilinear estimates at the critical space by applying the radial Strichartz estimates and U^2, V^2 type spaces. Next, we consider $d = 4$ and the non-radial case. When $d \leq 4$, the Lorentz regularity s_l is an important index as well as the scaling regularity for the well-posedness for the wave equation. When $d = 4$ with quadratic nonlinearity, the Lorentz regularity $s_l = 1/4$. On the other hand, $s_c = 0$, so we need to consider $s \geq s_l = 1/4$. When $d = 4$, we obtain local well-posedness at $s = s_l = 1/4$ by applying U^2, V^2 type spaces. Finally, we consider $d \geq 5$ and the non-radial case. Since $s_c \geq s_l$ when $d \geq 5$, we expect the local well-posedness with $s = s_c$. However, we only obtain the local well-posedness with $s = s_c + 1/(d+1)$. It seems difficult to prove the bilinear estimate with $s = s_c$. The reason is as below. We observe the first equation of (2.2). We regard the nonlinearity as $n_\pm(\omega_1^{-1} u_\pm)$. Here, we consider the following cases. The case $|\xi| \lesssim |\xi'|$ and the case $|\xi| \gg |\xi'|$, where ξ, ξ' denote the frequency of $n_\pm, u_\pm$ respectively. For the case $|\xi| \lesssim |\xi'|$, the nonlinearity does not have the derivative loss, so we can derive the bilinear estimate at the critical space only by applying the Strichartz estimates. However, for the case $|\xi| \gg |\xi'|$, we need to recover a half derivative loss by (2.3). Here, there are three cases in (2.3). The cases (a) $M' = \big|\tau \pm c|\xi|\big|$, (b) $M' = |\tau' \pm \langle \xi' \rangle|$ and (c) $M' = |\tau - \tau' \pm \langle \xi - \xi' \rangle|$. For the case (a) or (c), we apply (2.3) for $n_\pm$ and the Strichartz estimates for $\omega_1^{-1} u_\pm$. Then we can obtain the bilinear estimate at the critical space. Whereas for (b), we apply the Strichartz estimates for $n_\pm$ and apply (2.3) for $\omega_1^{-1} u_\pm$. In this case, we cannot prove the bilinear estimate at the critical space. As a result, we have to impose more regularity.

In section 2.2, we prepare some notations and lemmas with respect to U^p, V^p, in section 2.3, we prove the bilinear estimates and in section 2.4, we prove the main result.

2.2. **Notations and preliminary lemmas.** We denote the Duhamel term

$$I_{T,K_\pm}(n,v) := \pm \int_0^t \mathbf{1}_{[0,T]}(t')K_\pm(t-t')n(t')(\omega_1^{-1}v(t'))dt',$$
$$I_{T,W_{\pm c}}(u,v) := \pm \int_0^t \mathbf{1}_{[0,T]}(t')W_{\pm c}(t-t')\omega\big((\omega_1^{-1}u(t'))(\overline{\omega_1^{-1}v(t')})\big)dt'$$

for the Klein-Gordon equation and the wave equation respectively.

Lemma 2.3. *Let $c>0, c\neq 1$ and $\tau_3=\tau_1-\tau_2,\ \xi_3=\xi_1-\xi_2$. If $|\xi_1|\gg\langle\xi_2\rangle$ or $\langle\xi_1\rangle\ll|\xi_2|$, then it holds that*

$$\max\big\{\big|\tau_1\pm\langle\xi_1\rangle\big|,\big|\tau_2\pm\langle\xi_2\rangle\big|,\big|\tau_3\pm c|\xi_3|\big|\big\}\gtrsim \max\{|\xi_1|,|\xi_2|\}. \tag{2.4}$$

Proof. We only prove the case $|\xi_1|\gg\langle\xi_2\rangle$ since the case $\langle\xi_1\rangle\ll|\xi_2|$ is proved by the same manner.

$$\text{(l.h.s)}\gtrsim|(\tau_1\pm(1+|\xi_1|))-(\tau_2\pm(1+|\xi_2|))-(\tau_3\pm c|\xi_3|)| \tag{2.5}$$

If $0<c<1$, then we take ε_c such that $0<\varepsilon_c<(1-c)/(1+c), |\xi_2|\le\varepsilon_c|\xi_1|$. Then, the right hand side of (2.5) is bounded by

$$(1+|\xi_1|)-(1+|\xi_2|)-c|\xi_1-\xi_2|\ge|\xi_1|-\varepsilon_c|\xi_1|-c(1+\varepsilon_c)|\xi_1|\gtrsim|\xi_1|.$$

If $c>1$, then we take $\tilde{\varepsilon}_c$ such that $0<\tilde{\varepsilon}_c<(c-1)/(c+3), |\xi_2|\le\tilde{\varepsilon}_c|\xi_1|, |\xi_1|\ge 1/\tilde{\varepsilon}_c$. Then, the right hand side of (2.5) is bounded by

$$c|\xi_1-\xi_2|-(1+|\xi_1|)-(1+|\xi_2|)\ge c(1-\tilde{\varepsilon}_c)|\xi_1|-(1+\tilde{\varepsilon}_c)|\xi_1|-2\tilde{\varepsilon}_c|\xi_1|\gtrsim|\xi_1|.$$

□

Lemma 2.4. *Let $M>0$ and $Q\in\{Q^{K_\pm}_{<M},Q^{K_\pm}_{\ge M}\}$. For $1\le p\le\infty$ and $f\in V^2_{K_\pm}$, it holds that*

$$\|Q(\mathbf{1}_{[0,T]}f)\|_{L^p_tL^2_x}\lesssim T^{1/p}\|f\|_{V^2_{K_\pm}}. \tag{2.6}$$

Proof. By scaling, we only need to prove (2.6) for $M=1$. We will show (2.6) for $Q=Q^{K_\pm}_{<1}$. Put $g:=K_\pm(-\cdot)f$. Then (2.6) is equivalent to

$$\|Q^{K_\pm}_{<1}(\mathbf{1}_{[0,T]}K_\pm(\cdot)g)\|_{L^p_tL^2_x}\lesssim T^{1/p}\|g\|_{V^2}. \tag{2.7}$$

By the unitarity of $K_\pm$, we have

$$\begin{aligned}\|Q^{K_\pm}_{<1}(\mathbf{1}_{[0,T]}K_\pm(\cdot)g)\|_{L^p_tL^2_x} &= \Big\|\sum_{N<1} K_\pm(\cdot)Q_N K_\pm(-\cdot)(\mathbf{1}_{[0,T]}K_\pm(\cdot)g)\Big\|_{L^p_tL^2_x}\\ &= \Big\|\sum_{N<1} Q_N(\mathbf{1}_{[0,T]}g)\Big\|_{L^p_tL^2_x}\\ &= \|Q_{<1}(\mathbf{1}_{[0,T]}g)\|_{L^p_tL^2_x}. \end{aligned}\tag{2.8}$$

By the definition $Q_{<1}$, there exists a Schwartz function ϕ, it holds that

$$Q_{<1}h = \phi *_t h.$$

Hence by the Young inequality and the Hölder inequality, we have

$$\begin{aligned}\|Q_{<1}(\mathbf{1}_{[0,T]}g)\|_{L^p_tL^2_x} &\lesssim \|\phi\|_{L^1_t}\|\mathbf{1}_{[0,T]}g\|_{L^p_tL^2_x}\\ &\lesssim \|\mathbf{1}_{[0,T]}\|_{L^p_t}\|g\|_{L^\infty_tL^2_x}\\ &\lesssim T^{1/p}\|g\|_{V^2}. \end{aligned}\tag{2.9}$$

Collecting (2.8)–(2.9), we obtain (2.7). For the proof of (2.6) for $Q = Q^{K_\pm}_{\geq 1}$, we use $Q^{K_\pm}_{\geq 1} = Id - Q^{K_\pm}_{<1}$ and $\|Id(\mathbf{1}_{[0,T]}g)\|_{L^p_tL^2_x} \lesssim T^{1/p}\|g\|_{V^2}$. □

Proposition 2.5. *Let $d \geq 3, 2 \leq r < \infty, 2/q = (d-1)(1/2-1/r), (q,r) \neq (2, 2(d-1)/(d-3))$ and $s = 1/q - 1/r + 1/2$. Then it holds that*

$$\|W_{\pm c}(t)f\|_{L^q_t\dot{W}^{-s,r}_x(\mathbb{R}^{1+d})} \lesssim \|f\|_{L^2_x(\mathbb{R}^d)}.$$

For the proof of Proposition 2.5, see [23], [11].

Proposition 2.6. *Let $d \geq 3, 2 \leq r < \infty, 2/q = (d-1)(1/2-1/r), (q,r) \neq (2, 2(d-1)/(d-3))$ and $s = 1/q - 1/r + 1/2$. Then, it holds that*

$$\|K_\pm(t)f\|_{L^q_tW^{-s,r}_x(\mathbb{R}^{1+d})} \lesssim \|f\|_{L^2_x(\mathbb{R}^d)}.$$

For the proof of Proposition 2.6, see [37]. Combining Proposition 1.5, Proposition 2.5, Proposition 2.6 and Proposition 1.14, we have the following proposition.

Proposition 2.7. *Let $d \geq 3, 2 \leq r < \infty, 2/q = (d-1)(1/2-1/r), (q,r) \neq (2, 2(d-1)/(d-3))$ and $s = 1/q - 1/r + 1/2$. If $p < q$, then it holds that*

$$\|f\|_{L^q_tW^{-s,r}_x(\mathbb{R}^{1+d})} \lesssim \|f\|_{V^p_{K_\pm}}, \qquad \|f\|_{L^q_t\dot{W}^{-s,r}_x(\mathbb{R}^{1+d})} \lesssim \|f\|_{V^p_{W_{\pm c}}}.$$

Proposition 2.8. *Let $d \geq 3$. Then, for all radial functions $f \in L^2_x(\mathbb{R}^d)$, it holds that*

$$\|W_{\pm c}(t)P_Nf\|_{L^q_tL^r_x(\mathbb{R}^{1+d})} \lesssim N^{d(1/2-1/r)-1/q}\|P_Nf\|_{L^2_x(\mathbb{R}^d)}, \tag{2.10}$$

if and only if

$$(q,r)=(\infty,2) \quad or \quad 2\le q\le\infty, \quad 1/q<(d-1)(1/2-1/r). \tag{2.11}$$

See Theorem 1.5 (a) in [18] for the proof of Proposition 2.8.

Proposition 2.9. *Let $d\ge 2$. If $2d/(d-1)<q\le\infty, N\ge 1$ and $f\in L^2_x(\mathbb{R}^d)$ is radial function, then it holds that*

$$\|K_\pm(t)P_Nf\|_{L^q_{t,x}(\mathbb{R}^{1+d})}\lesssim N^{d/2-(d+1)/q}\|P_Nf\|_{L^2_x(\mathbb{R}^d)}. \tag{2.12}$$

If $q\ge(4d+2)/(2d-1), N<1$ and $f\in L^2_x(\mathbb{R}^d)$ is radial function, then it holds that

$$\|K_\pm(t)P_Nf\|_{L^q_{t,x}(\mathbb{R}^{1+d})}\lesssim N^{d/2-(d+2)/q}\|P_Nf\|_{L^2_x(\mathbb{R}^d)}. \tag{2.13}$$

See (3.13) in [18] for the proof of Proposition 2.9.

Proposition 2.10. *Let $d\ge 2$ and (q,r) satisfy $2\le r\le 2(d+1)/(d-1), r\le q, (1/2)(d-1)(1/2-1/r)\le 1/q<(d-1)(1/2-1/r)$ and $N\ge 1$. Then, for all radial function $f\in L^2_x(\mathbb{R}^d)$, it holds that*

$$\|K_\pm(t)P_Nf\|_{L^q_tL^r_x(\mathbb{R}^{1+d})}\lesssim N^{d(1/2-1/r)-1/q}\|P_Nf\|_{L^2_x(\mathbb{R}^d)}. \tag{2.14}$$

Proof. When $q=r$, (2.14) follows from (2.12). Interpolating $L^q_{t,x}$ with $L^\infty_tL^2_x$, we obtain (2.14). □

Proposition 2.11. *(i) Let $d\ge 3, (q,r)$ satisfy (2.11) in Proposition 2.8 and $s=d(1/2-1/r)-1/q$. If $p<q$, then for all spherically symmetric function u, it holds that*

$$\|P_Nu\|_{L^q_t\dot{W}^{-s,r}_x(\mathbb{R}^{1+d})}\lesssim\|P_Nu\|_{V^p_{W_{\pm c}}}.$$

(ii) Let $d\ge 2$ and (q,r) satisfy the condition in Proposition 2.10. If $p<q, N\ge 1$ and $s_1=d(1/2-1/r)-1/q$, then for all spherically symmetric function u, it holds that

$$\|P_Nu\|_{L^q_t\dot{W}^{-s_1,r}_x(\mathbb{R}^{1+d})}\lesssim\|P_Nu\|_{V^p_{K_\pm}}. \tag{2.15}$$

(iii) Let $d\ge 2$. If $p<q, N<1$ and $s_2=d/2-(d+2)/q$, then for all spherically symmetric function u, it holds that

$$\|P_Nu\|_{L^q_t\dot{W}^{-s_2,q}_x(\mathbb{R}^{1+d})}\lesssim\|P_Nu\|_{V^p_{K_\pm}}. \tag{2.16}$$

Combining Proposition 1.5, Proposition 2.8 and Proposition 1.14, we have Proposition 2.11 (i). Combining Proposition 1.5, Proposition 2.10 and Proposition 1.14, we have Proposition 2.11 (ii). Combining Proposition 1.5, (2.13) and Proposition 1.14, we have Proposition 2.11 (iii).

Proposition 2.12. *(i) Let $T>0$ and $u\in Y^s_{K_\pm}([0,T]), u(0)=0$. Then, there exists $0\le T'\le T$ such that $\|u\|_{Y^s_{K_\pm}([0,T'])}<\varepsilon$.*
(ii) Let $T>0$ and $n\in \dot{Y}^s_{W_{\pm c}}([0,T]), n(0)=0$. Then, there exists $0\le T'\le T$ such that $\|n\|_{\dot{Y}^s_{W_{\pm c}}([0,T'])}<\varepsilon$.

For the proofs of (i) and (ii), see Proposition 2.24 in [19].

Lemma 2.13. *Let $\tilde{u}_{N_1}:=\mathbf{1}_{[0,T)}P_{N_1}u, \tilde{v}_{N_2}:=\mathbf{1}_{[0,T)}P_{N_2}v, \tilde{n}_{N_3}:=\mathbf{1}_{[0,T)}P_{N_3}n, Q_1,Q_2\in\{Q^{K_\pm}_{<M},Q^{K_\pm}_{\ge M}\}, Q_3\in\{Q^{W_{\pm c}}_{<M},Q^{W_{\pm c}}_{\ge M}\}, s':=(d^2-3d-2)/2(d+1), s_c:=d/2-2$. Then the following estimates hold for sufficiently small $T>0$ if $\theta>0$, and hold for all $0<T<\infty$ if $\theta=0$ or spherically symmetric (u,v,n):*
(i) If $N_3\lesssim N_2\sim N_1$, then

$$
\begin{aligned}
|I_1|&:=\Big|\int_{\mathbb{R}^{1+d}}(\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}})dxdt\Big|\\
&\lesssim T^\theta N_3^s\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}},
\end{aligned}
$$

where $(\theta,s)=(1/4,1/4)$ for $d=4$ and $(\theta,s)=(0,s_c),(1/(d+1),s')$ for $d\ge5$. Moreover, if (u,v,n) are spherically symmetric, then for $d\ge4$,

$$|I_1|\lesssim\langle N_2\rangle^{(d-8)/3}N_3^{(d+4)/6}\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}.$$

(ii) It holds that

$$|I_2|:=\Big|\int_{\mathbb{R}^{1+d}}\tilde{n}(\omega_1^{-1}\tilde{v})(\overline{P_{<1}\tilde{u}})dxdt\Big|\lesssim T^\theta\|n\|_{\dot{Y}^s_{W_{\pm c}}}\|v\|_{Y^s_{K_\pm}}\|P_{<1}u\|_{V^2_{K_\pm}},$$

where $(\theta,s)=(1,1/4)$ for $d=4$ and $(\theta,s)=(0,s_c),(1/(d+1),s')$ for $d\ge5$. Moreover, if (u,v,n) are spherically symmetric, then for $d=4$,

$$|I_2|\lesssim\|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}}\|v\|_{Y^{s_c}_{K_\pm}}\|P_{<1}u\|_{V^2_{K_\pm}}.$$

(iii) If $N_1\sim N_2$, then

$$|I_3|:=\Big|\int_{\mathbb{R}^{1+d}}\Big(\sum_{N_3\lesssim N_2}\tilde{n}_{N_3}\Big)(\omega_1^{-1}\tilde{v}_{N_2})\overline{\tilde{u}_{N_1}}dxdt\Big|\lesssim T^\theta\|n\|_{\dot{Y}^s_{W_{\pm c}}}\|v_{N_2}\|_{V^2_{K_\pm}}\|u_{N_1}\|_{V^2_{K_\pm}},$$

where $(\theta,s)=(1/4,1/4)$ for $d=4$ and $(\theta,s)=(0,s_c),(1/(d+1),s')$ for $d\ge5$. Moreover, if (u,v,n) are spherically symmetric, then for $d=4$,

$$|I_3|\lesssim\|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}}\|v_{N_2}\|_{V^2_{K_\pm}}\|u_{N_1}\|_{V^2_{K_\pm}}.$$

(iv) If $N_1\sim N_3, N_1\gg1, M=\varepsilon N_1$ and $\varepsilon>0$ is sufficiently small, then

$$|I_i|\lesssim T^\theta\|n_{N_3}\|_{V^2_{W_{\pm c}}}\|v\|_{Y^s_{K_\pm}}\|u_{N_1}\|_{V^2_{K_\pm}},$$

where $(\theta,s)=(1/4,1/4)$ *for* $d=4, i=4,5,6$ *and* $(\theta,s)=(0,s_c)$ *for* $d\geq 4, i=4,6$ *and* $(\theta,s)=(1/(d+1),s')$ *for* $d\geq 5, i=4,5,6$. *Moreover, if* (u,v,n) *are spherically symmetric, then for* $d\geq 4$*, it holds that*

$$|I_5|\lesssim \|n_{N_3}\|_{V^2_{W_{\pm c}}}\|v\|_{Y^{s_c}_{K_\pm}}\|u_{N_1}\|_{V^2_{K_\pm}},$$

where

$$I_4:=\int_{\mathbb{R}^{1+d}}(Q^{W_{\pm c}}_{\geq M}\tilde{n}_{N_3})\Big(\sum_{N_2\ll N_1}Q_2\omega_1^{-1}\tilde{v}_{N_2}\Big)(\overline{Q_1\tilde{u}_{N_1}})dxdt,$$

$$I_5:=\int_{\mathbb{R}^{1+d}}(Q_3\tilde{n}_{N_3})\Big(\sum_{N_2\ll N_1}Q^{K_\pm}_{\geq M}\omega_1^{-1}\tilde{v}_{N_2}\Big)(\overline{Q_1\tilde{u}_{N_1}})dxdt,$$

$$I_6:=\int_{\mathbb{R}^{1+d}}(Q_3\tilde{n}_{N_3})\Big(\sum_{N_2\ll N_1}Q_2\omega_1^{-1}\tilde{v}_{N_2}\Big)(\overline{Q^{K_\pm}_{\geq M}\tilde{u}_{N_1}})dxdt.$$

Proof. We show (i) first. For $f\in V^2_A,\ A\in\{K_\pm,W_{\pm c}\}$, we see

$$\|\mathbf{1}_{[0,T)}f\|_{V^2_A}\lesssim\|f\|_{V^2_A}. \tag{2.17}$$

First, we show it for $d=4$. We apply the Hölder inequality, Proposition 2.7 (2.17) and $N_3\lesssim N_1\sim N_2$, then we have

$$\begin{aligned}|I_1|&\lesssim\|\mathbf{1}_{[0,T)}\|_{L^4_t}\|\omega_1^{-1}\tilde{u}_{N_1}\|_{L^{20/3}_tL^{5/2}_x}\|\omega_1^{-1}\tilde{v}_{N_2}\|_{L^{20/3}_tL^{5/2}_x}\|\omega\tilde{n}_{N_3}\|_{L^{20/9}_tL^5_x}\\&\lesssim T^{1/4}\|\langle\nabla_x\rangle^{1/4}\omega_1^{-1}\tilde{u}_{N_1}\|_{V^2_{K_\pm}}\|\langle\nabla_x\rangle^{1/4}\omega_1^{-1}\tilde{v}_{N_2}\|_{V^2_{K_\pm}}\||\nabla_x|^{3/4}\omega\tilde{n}_{N_3}\|_{V^2_{W_{\pm c}}}\\&\lesssim T^{1/4}\langle N_1\rangle^{1/4-1}\|u_{N_1}\|_{V^2_{K_\pm}}\langle N_2\rangle^{1/4-1}\|v_{N_2}\|_{V^2_{K_\pm}}N_3^{3/4+1}\|n_{N_3}\|_{V^2_{W_{\pm c}}}\\&\lesssim T^{1/4}N_3^{1/4}\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}.\end{aligned}$$

For $d\geq 5$, we apply the Hölder inequality to have

$$|I_1|\lesssim\|\omega_1^{-1}\tilde{u}_{N_1}\|_{L^{2(d+1)/(d-1)}_{t,x}}\|\omega_1^{-1}\tilde{v}_{N_2}\|_{L^{2(d+1)/(d-1)}_{t,x}}\|\omega\tilde{n}_{N_3}\|_{L^{(d+1)/2}_{t,x}}. \tag{2.18}$$

We apply Proposition 2.7, (2.17) and the Sobolev inequality, then we have

$$\|\omega_1^{-1}\tilde{f}_N\|_{L^{2(d+1)/(d-1)}_{t,x}}\lesssim\langle N\rangle^{1/2-1}\|f_N\|_{V^2_{K_\pm}}=\langle N\rangle^{-1/2}\|f_N\|_{V^2_{K_\pm}}, \tag{2.19}$$

$$\begin{aligned}\|\omega\tilde{n}_{N_3}\|_{L^{(d+1)/2}_{t,x}}&\lesssim\||\nabla_x|^{d(d-5)/2(d-1)}\omega\tilde{n}_{N_3}\|_{L^{(d+1)/2}_tL^{2(d^2-1)/(d^2-9)}_x}\\&\lesssim\||\nabla_x|^{d/2-2}\omega\tilde{n}_{N_3}\|_{V^2_{W_{\pm c}}} &(2.20)\\&\lesssim N_3^{s_c+1}\|n_{N_3}\|_{V^2_{W_{\pm c}}} &(2.21)\end{aligned}$$

Collecting (2.18), (2.19), (2.21) and $N_3\lesssim N_1\sim N_2$, we obtain

$$|I_1|\lesssim N_3^{s_c}\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}.$$

In (2.18), if we apply the Hölder inequality, the Sobolev inequality and Proposition 2.7, then we have

$$\begin{aligned}\|\omega\tilde{n}_{N_3}\|_{L^{(d+1)/2}_{t,x}} &\lesssim \|\mathbf{1}_{[0,T)}\|_{L^{d+1}_t}\|\omega n_{N_3}\|_{L^{d+1}_t L^{(d+1)/2}_x}\\ &\lesssim T^{1/(d+1)}\||\nabla_x|^{d(d^2-4d-1)/2(d^2-1)}\omega n_{N_3}\|_{L^{d+1}_t L^{2(d^2-1)/(d^2-5)}_x}\\ &\lesssim T^{1/(d+1)}\||\nabla_x|^{(d^2-3d-2)/2(d+1)}\omega n_{N_3}\|_{V^2_{W_{\pm c}}} \qquad (2.22)\\ &\lesssim T^{1/(d+1)}N_3^{s'+1}\|n_{N_3}\|_{V^2_{W_{\pm c}}}. \qquad (2.23)\end{aligned}$$

Collecting (2.18), (2.19), (2.23) and $N_3 \lesssim N_1 \sim N_2$, we obtain

$$|I_1| \lesssim T^{1/(d+1)}N_3^{s'}\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}.$$

Next, we prove it for $d \geq 4$ and spherically symmetric functions (u, v, n). We apply the Hölder inequality to have

$$|I_1| \lesssim \|\omega_1^{-1}\tilde{u}_{N_1}\|_{L^3_{t,x}}\|\omega_1^{-1}\tilde{v}_{N_2}\|_{L^3_{t,x}}\|\omega\tilde{n}_{N_3}\|_{L^3_{t,x}}. \qquad (2.24)$$

We apply Proposition 2.11, (2.17) and $N_3 \lesssim N_2 \sim N_1$, then we have

$$\|\omega_1^{-1}\tilde{u}_{N_1}\|_{L^3_{t,x}} \lesssim \langle N_1\rangle^{(d-2)/6-1}\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \langle N_2\rangle^{(d-8)/6}\|u_{N_1}\|_{V^2_{K_\pm}}, \qquad (2.25)$$

$$\|\omega_1^{-1}\tilde{v}_{N_2}\|_{L^3_{t,x}} \lesssim \langle N_2\rangle^{(d-8)/6}\|v_{N_2}\|_{V^2_{K_\pm}}, \qquad (2.26)$$

$$\|\omega\tilde{n}_{N_3}\|_{L^3_{t,x}} \lesssim \||\nabla_x|^{(d-2)/6}\omega\tilde{n}_{N_3}\|_{V^2_{W_{\pm c}}} \lesssim N_3^{(d+4)/6}\|n_{N_3}\|_{V^2_{W_{\pm c}}}. \qquad (2.27)$$

From (2.24)–(2.27), we obtain

$$|I_1| \lesssim \langle N_2\rangle^{(d-8)/3}N_3^{(d+4)/6}\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}.$$

Next, we prove (ii). For $d = 4$, we apply the Hölder inequality, the Sobolev inequality, Remark 1.1, (2.17), Remark 1.2, discarding ω_1^{-1} and Lemma 1.6, we obtain

$$\begin{aligned}|I_2| &\lesssim \|\mathbf{1}_{[0,T)}\|_{L^1_t}\|\tilde{n}\|_{L^\infty_t L^{16/7}_x}\|\omega_1^{-1}\tilde{v}\|_{L^\infty_t L^{16/7}_x}\|P_{<1}\tilde{u}\|_{L^\infty_t L^8_x}\\ &\lesssim T\||\nabla_x|^{1/4}\tilde{n}\|_{L^\infty_t L^2_x}\|\langle\nabla_x\rangle^{1/4}\omega_1^{-1}\tilde{v}\|_{L^\infty_t L^2_x}\||\nabla_x|^{3/2}P_{<1}\tilde{u}\|_{L^\infty_t L^2_x}\\ &\lesssim T\||\nabla_x|^{1/4}\tilde{n}\|_{V^2_{W_{\pm c}}}\|\langle\nabla_x\rangle^{1/4}\omega_1^{-1}v\|_{V^2_{K_\pm}}\|P_{<1}u\|_{V^2_{K_\pm}}\\ &\lesssim T\|n\|_{\dot{Y}^{1/4}_{W_{\pm c}}}\|v\|_{Y^{1/4}_{K_\pm}}\|P_{<1}u\|_{V^2_{K_\pm}}.\end{aligned}$$

For $d \geq 5$, by the Hölder inequality to have

$$|I_2| \lesssim \|\tilde{n}\|_{L^{(d+1)/2}_{t,x}}\|\omega_1^{-1}\tilde{v}\|_{L^{2(d+1)/(d-1)}_{t,x}}\|P_{<1}\tilde{u}\|_{L^{2(d+1)/(d-1)}_{t,x}}. \qquad (2.28)$$

From Proposition 2.7, (2.20), Remark 1.2 and Lemma 1.6, we obtain

$$\|\tilde{n}\|_{L^{(d+1)/2}_{t,x}} \lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}}, \tag{2.29}$$

$$\|\omega_1^{-1}\tilde{v}\|_{L^{2(d+1)/(d-1)}_{t,x}} \lesssim \|\langle\nabla_x\rangle^{-1/2} v\|_{V^2_{K_\pm}} \lesssim \|\langle\nabla_x\rangle^{s_c} v\|_{V^2_{K_\pm}} \lesssim \|v\|_{Y^{s_c}_{K_\pm}}, \tag{2.30}$$

$$\|P_{<1}\tilde{u}\|_{L^{2(d+1)/(d-1)}_{t,x}} \lesssim \|\langle\nabla_x\rangle^{1/2} P_{<1}u\|_{V^2_{K_\pm}} \lesssim \|P_{<1}u\|_{V^2_{K_\pm}}. \tag{2.31}$$

Collecting (2.28)–(2.31), we obtain

$$|I_2| \lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|_{Y^{s_c}_{K_\pm}} \|P_{<1}u\|_{V^2_{K_\pm}}.$$

Also for $d \geq 5$, from (2.22), Remark 1.2, (2.28), (2.30) and (2.31) to have

$$|I_2| \lesssim T^{1/(d+1)} \|n\|_{\dot{Y}^{s'}_{W_{\pm c}}} \|v\|_{Y^{s'}_{K_\pm}} \|P_{<1}u\|_{V^2_{K_\pm}}.$$

We prove it for $d = 4$ and spherically symmetric functions (u, v, n). Due to the operator $P_{<1}$,

$$\begin{aligned} |I_2| &\lesssim \Big| \int_{\mathbb{R}^{1+4}} \Big(\sum_{N_3 \lesssim 1} \tilde{n}_{N_3} \Big) \Big(\sum_{N_2 < 1} \omega_1^{-1} \tilde{v}_{N_2} \Big) (\overline{P_{<1}\tilde{u}}) dx dt \Big| \\ &\quad + \sum_{N_2 \geq 1} \Big| \int_{\mathbb{R}^{1+4}} \Big(\sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big) (\omega_1^{-1} \tilde{v}_{N_2}) (\overline{P_{<1}\tilde{u}}) dx dt \Big| \\ &=: I_{2,1} + I_{2,2}. \end{aligned}$$

First, we estimate $I_{2,2}$. We apply the Hölder inequality to have

$$|I_{2,2}| \lesssim \sum_{N_2 \geq 1} \Big\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{L^3_{t,x}} \|\omega_1^{-1} \tilde{v}_{N_2}\|_{L^3_{t,x}} \|P_{<1}\tilde{u}\|_{L^3_{t,x}}. \tag{2.32}$$

By Proposition 2.11, (2.17), $N_3 \lesssim N_2 \sim N_1$ and the Cauchy-Schwarz inequality, then we have

$$\begin{aligned} \Big\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{L^3_{t,x}} &\lesssim \Big\| |\nabla_x|^{1/3} \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{V^2_{W_{\pm c}}} \\ &\lesssim \Big(\sum_{N_3 \lesssim N_2} N_3^{2/3} \Big)^{1/2} \Big(\sum_{N_3 \lesssim N_2} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \Big)^{1/2} \\ &\lesssim N_2^{1/3} \|n\|_{\dot{Y}^0_{W_{\pm c}}}. \end{aligned} \tag{2.33}$$

From (2.25) and (2.26), we see

$$\|P_{<1}\tilde{u}\|_{L^3_{t,x}} \lesssim \|P_{<1}u\|_{V^2_{K_\pm}}, \qquad \|\omega_1^{-1}\tilde{v}_{N_2}\|_{L^3_{t,x}} \lesssim \langle N_2 \rangle^{-2/3} \|v_{N_2}\|_{V^2_{K_\pm}}. \tag{2.34}$$

Collecting (2.32)–(2.34), $N_2 \geq 1$ and applying the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|I_{2,2}| &\lesssim \sum_{N_2 \geq 1} N_2^{-1/3} \|n\|_{\dot{Y}^0_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \|P_{<1}u\|_{V^2_{K_\pm}} \\
&\lesssim \|n\|_{\dot{Y}^0_{W_{\pm c}}} \|v\|_{Y^0_{K_\pm}} \|P_{<1}u\|_{V^2_{K_\pm}}. \qquad (2.35)
\end{aligned}
$$

Next, we estimate $I_{2,1}$. By the Hölder inequality to have

$$
|I_{2,1}| \lesssim \Big\| \sum_{N_3 \lesssim 1} \tilde{n}_{N_3} \Big\|_{L^3_{t,x}} \Big\| \sum_{N_2 < 1} \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^3_{t,x}} \|P_{<1}\tilde{u}\|_{L^3_{t,x}}. \qquad (2.36)
$$

From $d = 4$, (2.33), (2.34) and discarding ω_1^{-1}, we see

$$
\Big\| \sum_{N_3 \lesssim 1} \tilde{n}_{N_3} \Big\|_{L^3_{t,x}} \lesssim \|n\|_{\dot{Y}^0_{W_{\pm c}}}, \qquad \Big\| \sum_{N_2 < 1} \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^3_{t,x}} \lesssim \|P_{<1}v\|_{V^2_{K_\pm}}. \qquad (2.37)
$$

Collecting (2.34), (2.36) and (2.37), we have

$$
|I_{2,1}| \lesssim \|n\|_{\dot{Y}^0_{W_{\pm c}}} \|P_{<1}v\|_{V^2_{K_\pm}} \|P_{<1}u\|_{V^2_{K_\pm}} \lesssim \|n\|_{\dot{Y}^0_{W_{\pm c}}} \|v\|_{Y^0_{K_\pm}} \|P_{<1}u\|_{V^2_{K_\pm}}. \qquad (2.38)
$$

From (2.35) and (2.38), we obtain $|I_2| \lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|_{Y^{s_c}_{K_\pm}} \|P_{<1}u\|_{V^2_{K_\pm}}$. We prove (iii) for $d = 4$ below. We apply the Hölder inequality, Proposition 2.7 and (2.17), then we have

$$
\begin{aligned}
|I_3| &\lesssim \|\mathbf{1}_{[0,T)}\|_{L^4_t} \Big\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{L^{20/9}_t L^5_x} \|\omega_1^{-1}\tilde{v}_{N_2}\|_{L^{20/3}_t L^{5/2}_x} \|\tilde{u}_{N_1}\|_{L^{20/3}_t L^{5/2}_x} \\
&\lesssim T^{1/4} \Big\| |\nabla_x|^{3/4} \sum_{N_3 \lesssim N_2} n_{N_3} \Big\|_{V^2_{W_{\pm c}}} \langle N_2 \rangle^{1/4-1} \|v_{N_2}\|_{V^2_{K_\pm}} \langle N_1 \rangle^{1/4} \|u_{N_1}\|_{V^2_{K_\pm}}. \qquad (2.39)
\end{aligned}
$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\Big\| |\nabla_x|^{3/4} \sum_{N_3 \lesssim N_2} n_{N_3} \Big\|_{V^2_{W_{\pm c}}} &\lesssim \sum_{N_3 \lesssim N_2} N_3^{3/4} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \\
&\lesssim \Big(\sum_{N_3 \lesssim N_2} N_3 \Big)^{1/2} \Big(\sum_{N_3 \lesssim N_2} N_3^{1/2} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \Big)^{1/2} \\
&\lesssim N_2^{1/2} \|n\|_{\dot{Y}^{1/4}_{W_{\pm c}}}. \qquad (2.40)
\end{aligned}
$$

Collecting (2.39), (2.40) and $N_1 \sim N_2$, we obtain

$$
|I_3| \lesssim T^{1/4} \|n\|_{\dot{Y}^{1/4}_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}.
$$

We prove it for $d \geq 5$. We apply the Hölder inequality, (2.20), (2.19), Remark 1.2 and $N_1 \sim N_2$, then we have

$$\begin{aligned} |I_3| &\lesssim \Big\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{L^{(d+1)/2}_{t,x}} \|\omega_1^{-1} \tilde{v}_{N_2}\|_{L^{2(d+1)/(d-1)}_{t,x}} \|\tilde{u}_{N_1}\|_{L^{2(d+1)/(d-1)}_{t,x}} \\ &\lesssim \Big\| |\nabla_x|^{s_c} \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{V^2_{W_{\pm c}}} \langle N_2 \rangle^{-1/2} \|v_{N_2}\|_{V^2_{K_\pm}} \langle N_1 \rangle^{1/2} \|u_{N_1}\|_{V^2_{K_\pm}} \\ &\lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}. \end{aligned} \tag{2.41}$$

Similar to (2.22) and Remark 1.2, we have

$$\Big\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{L^{(d+1)/2}_{t,x}} \lesssim T^{1/(d+1)} \|n\|_{\dot{Y}^{s'}_{W_{\pm c}}}. \tag{2.42}$$

Collecting (2.19), (2.41) and (2.42), we obtain

$$|I_3| \lesssim T^{1/(d+1)} \|n\|_{\dot{Y}^{s'}_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}.$$

When $d = 4$ and (u, v, n) are spherically symmetric functions, we apply the Hölder inequality to have

$$|I_3| \lesssim \Big\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{L^3_{t,x}} \|\omega_1^{-1} \tilde{v}_{N_2}\|_{L^3_{t,x}} \|\tilde{u}_{N_1}\|_{L^3_{t,x}}. \tag{2.43}$$

From (2.25), (2.26), we see

$$\|\tilde{u}_{N_1}\|_{L^3_{t,x}} \lesssim \langle N_2 \rangle^{1/3} \|u_{N_1}\|_{V^2_{K_\pm}}, \qquad \|\omega_1^{-1} \tilde{v}_{N_2}\|_{L^3_{t,x}} \lesssim \langle N_2 \rangle^{-2/3} \|v_{N_2}\|_{V^2_{K_\pm}}. \tag{2.44}$$

Collecting (2.43), (2.33) and (2.44), we obtain

$$|I_3| \lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}.$$

We prove (iv). The estimate for I_6 is obtained from the same manner as the estimate for I_4, so we only estimate I_4, I_5. First, we estimate I_4 for $d = 4$. We apply the Hölder inequality, Proposition 1.13, the Sobolev inequality, Lemma 2.4, Proposition 2.7, (2.17), $\langle N_1 \rangle \sim N_1 \gg 1$ and Lemma 1.6 to have

$$\begin{aligned} |I_4| &\lesssim \|Q^{W_{\pm c}}_{\geq M} \tilde{n}_{N_3}\|_{L^2_{t,x}} \Big\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^4_t L^{16/3}_x} \|Q_1 \tilde{u}_{N_1}\|_{L^4_t L^{16/5}_x} \\ &\lesssim N_1^{-1/2} \|\tilde{n}_{N_3}\|_{V^2_{W_{\pm c}}} \Big\| \langle \nabla_x \rangle^{5/4} \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^4_t L^2_x} \|\langle \nabla_x \rangle^{1/12} Q_1 \tilde{u}_{N_1}\|_{L^4_t L^3_x} \\ &\lesssim N_1^{-1/2} \|n_{N_3}\|_{V^2_{W_{\pm c}}} T^{1/4} \Big\| \langle \nabla_x \rangle^{5/4} \sum_{N_2 \ll N_1} \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{V^2_{K_\pm}} \langle N_1 \rangle^{1/2} \|u_{N_1}\|_{V^2_{K_\pm}} \\ &\lesssim T^{1/4} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v\|_{Y^{1/4}_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}. \end{aligned}$$

Next, we prove it for $d \geq 4$ and non-radial case. We apply the Hölder inequality to have

$$|I_4| \lesssim \|Q_{\geq M}^{W_{\pm c}} \tilde{n}_{N_3}\|_{L^2_{t,x}} \Big\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^{d+1}_{t,x}} \|Q_1 \tilde{u}_{N_1}\|_{L^{2(d+1)/(d-1)}_{t,x}}. \tag{2.45}$$

By Proposition 1.13, (2.19) and (2.17), we have

$$\|Q_{\geq M}^{W_{\pm c}} \tilde{n}_{N_3}\|_{L^2_{t,x}} \lesssim N_1^{-1/2} \|n_{N_3}\|_{V^2_{W_{\pm c}}}, \tag{2.46}$$

$$\|Q_1 \tilde{u}_{N_1}\|_{L^{2(d+1)/(d-1)}_{t,x}} \lesssim \langle N_1 \rangle^{1/2} \|u_{N_1}\|_{V^2_{K_\pm}}. \tag{2.47}$$

We apply the Sobolev inequality, Proposition 2.7, Proposition 1.13, (2.17) and Lemma 1.6, we have

$$\begin{aligned} \Big\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^{d+1}_{t,x}} &\lesssim \Big\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)} \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^{d+1}_t L^{2(d^2-1)/(d^2-5)}_x} \\ &\lesssim \Big\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)+1/(d-1)-1} \sum_{N_2 \ll N_1} \tilde{v}_{N_2} \Big\|_{V^2_{K_\pm}} \\ &\lesssim \|v\|_{Y^{s_c}_{K_\pm}}. \end{aligned} \tag{2.48}$$

Collecting (2.45)–(2.48) and $N_1 \gg 1$, we obtain

$$|I_4| \lesssim \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v\|_{Y^{s_c}_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}.$$

For $d \geq 5$ and non-radial case, in (2.45), if we apply the Sobolev inequality, Proposition 1.13, Lemma 2.4 and Lemma 1.6, then we have

$$\begin{aligned} \Big\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^{d+1}_{t,x}} &\lesssim \Big\| \langle \nabla_x \rangle^{d(d-1)/2(d+1)} \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^{d+1}_t L^2_x} \\ &\lesssim T^{1/(d+1)} \Big\| \langle \nabla_x \rangle^{(d^2-3d-2)/2(d+1)} \sum_{N_2 \ll N_1} \tilde{v}_{N_2} \Big\|_{V^2_{K_\pm}} \\ &\lesssim T^{1/(d+1)} \|v\|_{Y^{s'}_{K_\pm}}. \end{aligned} \tag{2.49}$$

Collecting (2.45)–(2.47), (2.49) and $N_1 \gg 1$, we obtain

$$|I_4| \lesssim T^{1/(d+1)} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v\|_{Y^{s'}_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}.$$

Next, we prove I_5. When $d = 4$, by the Hölder inequality, the Sobolev inequality, Lemma 2.4, Proposition 2.7, (2.17), Proposition 1.13, $N_1 \sim N_3$ and Lemma 1.6, we have

$$\begin{aligned}|I_5| &\lesssim \|Q_3\tilde{n}_{N_3}\|_{L_t^4L_x^{16/5}}\Big\|\sum_{N_2\ll N_1}Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{v}_{N_2}\Big\|_{L_t^2L_x^{16/3}}\|Q_1\tilde{u}_{N_1}\|_{L_t^4L_x^2}\\ &\lesssim \||\nabla_x|^{1/12}Q_3\tilde{n}_{N_3}\|_{L_t^4L_x^3}\Big\|\langle\nabla_x\rangle^{5/4}\sum_{N_2\ll N_1}Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{v}_{N_2}\Big\|_{L_{t,x}^2}T^{1/4}\|u_{N_1}\|_{V_{K_\pm}^2}\\ &\lesssim N_3^{1/2}\|n_{N_3}\|_{V_{W_{\pm c}}^2}N_1^{-1/2}\Big\|\langle\nabla_x\rangle^{5/4}\sum_{N_2\ll N_1}\omega_1^{-1}\tilde{v}_{N_2}\Big\|_{V_{K_\pm}^2}T^{1/4}\|u_{N_1}\|_{V_{K_\pm}^2}\\ &\lesssim T^{1/4}\|n_{N_3}\|_{V_{W_{\pm c}}^2}\|v\|_{Y_{K_\pm}^{1/4}}\|u_{N_1}\|_{V_{K_\pm}^2}.\end{aligned}$$

For $d \geq 5$, by the Hölder inequality, we have

$$|I_5| \lesssim \|Q_3\tilde{n}_{N_3}\|_{L_{t,x}^{2(d+1)/(d-1)}}\Big\|\sum_{N_2\ll N_1}Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{v}_{N_2}\Big\|_{L_t^2L_x^{d+1}}\|Q_1\tilde{u}_{N_1}\|_{L_t^{d+1}L_x^2}. \tag{2.50}$$

Similar to (2.47), $N_3 \sim N_1$ and Lemma 2.4, we have

$$\|Q_3\tilde{n}_{N_3}\|_{L_{t,x}^{2(d+1)/(d-1)}} \lesssim \langle N_1\rangle^{1/2}\|n_{N_3}\|_{V_{W_{\pm c}}^2}, \tag{2.51}$$

$$\|Q_1\tilde{u}_{N_1}\|_{L_t^{d+1}L_x^2} \lesssim T^{1/(d+1)}\|u_{N_1}\|_{V_{K_\pm}^2}. \tag{2.52}$$

We apply the Sobolev inequality, Proposition 1.13, (2.17) and Lemma 1.6, we have

$$\begin{aligned}\Big\|\sum_{N_2\ll N_1}Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{v}_{N_2}\Big\|_{L_t^2L_x^{d+1}} &\lesssim \Big\|\langle\nabla_x\rangle^{d(d-1)/2(d+1)}\sum_{N_2\ll N_1}Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{v}_{N_2}\Big\|_{L_{t,x}^2}\\ &\lesssim N_1^{-1/2}\Big\|\langle\nabla_x\rangle^{(d^2-3d-2)/2(d+1)}\sum_{N_2\ll N_1}v_{N_2}\Big\|_{V_{K_\pm}^2}\\ &\lesssim N_1^{-1/2}\|v\|_{Y_{K_\pm}^{s'}}.\end{aligned} \tag{2.53}$$

Collecting (2.50)–(2.53) and $N_1 \gg 1$, we obtain

$$|I_5| \lesssim T^{1/(d+1)}\|n_{N_3}\|_{V_{W_{\pm c}}^2}\|v\|_{Y_{K_\pm}^{s'}}\|u_{N_1}\|_{V_{K_\pm}^2}.$$

Finally, we prove it for spherically symmetric functions (u, v, n) and $d \geq 4$. By the Hölder inequality, Proposition 2.11, (2.17), the Sobolev inequality, $1 \ll N_1 \sim N_3$, Proposition 1.13 and Lemma 1.6, we have

$$\begin{aligned}|I_5| &\lesssim \|Q_3\tilde{n}_{N_3}\|_{L_t^4L_x^{2d/(d-1)}}\Big\|\sum_{N_2\ll N_1}Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{v}_{N_2}\Big\|_{L_t^2L_x^d}\|Q_1\tilde{u}_{N_1}\|_{L_t^4L_x^{2d/(d-1)}}\\ &\lesssim N_3^{1/4}\|n_{N_3}\|_{V_{W_{\pm c}}^2}\Big\|\langle\nabla_x\rangle^{(d-2)/2}\sum_{N_2\ll N_1}Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{v}_{N_2}\Big\|_{L_{t,x}^2}N_1^{1/4}\|u_{N_1}\|_{V_{K_\pm}^2}\\ &\lesssim N_1^{1/2}\|n_{N_3}\|_{V_{W_{\pm c}}^2}N_1^{-1/2}\Big\|\langle\nabla_x\rangle^{(d-2)/2}\sum_{N_2\ll N_1}\omega_1^{-1}\tilde{v}_{N_2}\Big\|_{V_{K_\pm}^2}\|u_{N_1}\|_{V_{K_\pm}^2}\\ &\lesssim \|n_{N_3}\|_{V_{W_{\pm c}}^2}\|v\|_{Y_{K_\pm}^{sc}}\|u_{N_1}\|_{V_{K_\pm}^2}.\end{aligned}$$

□

2.3. **Bilinear estimates.**

Proposition 2.14. *(i) Let* $(\theta, s) = (1/4, 1/4)$ *for* $d = 4$ *and* $(\theta, s) = (1/(d+1), (d^2 - 3d - 2)/2(d+1))$ *for* $d \geq 5$. *For any* $0 < T < 1$,

$$\|I_{T,K_\pm}(n, v)\|_{Z^s_{K_\pm}} \lesssim T^\theta \|n\|_{\dot{Y}^s_{W_{\pm c}}} \|v\|_{Y^s_{K_\pm}}, \tag{2.54}$$

$$\|I_{T,W_{\pm c}}(u, v)\|_{\dot{Z}^s_{W_{\pm c}}} \lesssim T^\theta \|u\|_{Y^s_{K_\pm}} \|v\|_{Y^s_{K_\pm}}. \tag{2.55}$$

(ii) We assume that (u, v, n) *are spherically symmetric functions. Then for* $d \geq 4$ *and for all* $0 < T < \infty$, (2.54), (2.55) *also holds with* $(\theta, s) = (0, d/2 - 2)$.

Proof. We denote $\tilde{u}_{N_1} := \mathbf{1}_{[0,T)} P_{N_1} u, \tilde{v}_{N_2} := \mathbf{1}_{[0,T)} P_{N_2} v, \tilde{n}_{N_3} := \mathbf{1}_{[0,T)} P_{N_3} n$. First, we prove (2.54).

$$\|I_{T,K_\pm}(n, v)\|^2_{Z^s_{K_\pm}} \lesssim \sum_{i=0}^{3} J_i$$

where

$$J_0 := \Big\| \int_0^t \mathbf{1}_{[0,T)}(t') K_\pm(t - t') P_{<1}(\tilde{n}(\omega_1^{-1}\tilde{v}))(t') dt' \Big\|^2_{U^2_{K_\pm}},$$

$$J_1 := \sum_{N_1 \geq 1} N_1^{2s} \Big\| \int_0^t \mathbf{1}_{[0,T)}(t') K_\pm(t - t') \sum_{N_2 \sim N_1} \sum_{N_3 \lesssim N_2} P_{N_1}(\tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2}))(t') dt' \Big\|^2_{U^2_{K_\pm}},$$

$$J_2 := \sum_{N_1 \geq 1} N_1^{2s} \Big\| \int_0^t \mathbf{1}_{[0,T)}(t') K_\pm(t - t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1}(\tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2}))(t') dt' \Big\|^2_{U^2_{K_\pm}},$$

$$J_3 := \sum_{N_1 \geq 1} N_1^{2s} \Big\| \int_0^t \mathbf{1}_{[0,T)}(t') K_\pm(t - t') \sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} P_{N_1}(\tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2}))(t') dt' \Big\|^2_{U^2_{K_\pm}}.$$

By Corollary 1.11, we have

$$J_0^{1/2} \lesssim \sup_{\|u\|_{V^2_{K_\pm}} = 1} \Big| \int_{\mathbb{R}^{1+d}} \tilde{n}(\omega_1^{-1}\tilde{v}) (\overline{P_{<1}\tilde{u}}) dx dt \Big|. \tag{2.56}$$

For $d = 4$ and $s = 1/4$, from (2.56), Lemma 2.13 (ii) and $\|P_{<1} u\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, we obtain

$$J_0^{1/2} \lesssim T \|n\|_{\dot{Y}^{1/4}_{W_{\pm c}}} \|v\|_{Y^{1/4}_{K_\pm}}.$$

We apply Corollary 1.11 to have

$$J_1 \lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{V^2_{K_\pm}} = 1} \Big| \sum_{N_2 \sim N_1} \sum_{N_3 \lesssim N_2} \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \Big|^2. \tag{2.57}$$

For $d=4$ and $s=1/4$, by (2.57), $N_1 \sim N_2$, Lemma 2.13 (iii) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, we have

$$J_1 \lesssim \sum_{N_2 \gtrsim 1} N_2^{1/2} T^{1/2} \|n\|^2_{\dot{Y}^{1/4}_{W_{\pm c}}} \|v_{N_2}\|^2_{V^2_{K_\pm}} \lesssim T^{1/2} \|n\|^2_{\dot{Y}^{1/4}_{W_{\pm c}}} \|v\|^2_{Y^{1/4}_{K_\pm}}.$$

For the estimate of J_2, we take $M = \varepsilon N_1$ for sufficiently small $\varepsilon > 0$. Then, from Lemma 3.4, we have

$$\begin{aligned} &Q^{K_\pm}_{<M}\big((Q^{W_{\pm c}}_{<M} \tilde{n}_{N_3})(Q^{K_\pm}_{<M} \omega_1^{-1} \tilde{v}_{N_2})\big) \\ &= Q^{K_\pm}_{<M}\Big[\mathcal{F}^{-1}\Big(\int_{\tau_1=\tau_2+\tau_3,\, \xi_1=\xi_2+\xi_3} (\widehat{Q^{W_{\pm c}}_{<M} \tilde{n}_{N_3}})(\tau_3,\xi_3)(\widehat{Q^{K_\pm}_{<M} \omega_1^{-1} \tilde{v}_{N_2}})(\tau_2,\xi_2)\Big)\Big] = 0 \end{aligned}$$

when $N_1 \gg \langle N_2 \rangle$. Therefore,

$$\tilde{n}_{N_3}(\omega_1^{-1} \tilde{v}_{N_2}) = \sum_{i=1}^{3} F_i,$$

where $Q_1, Q_2 \in \{Q^{K_\pm}_{<M}, Q^{K_\pm}_{\geq M}\}$, $Q_3 \in \{Q^{W_{\pm c}}_{<M}, Q^{W_{\pm c}}_{\geq M}\}$,

$$\begin{aligned} &F_1 := Q_1\big((Q^{W_{\pm c}}_{\geq M} \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2})\big), \qquad F_2 := Q_1\big((Q_3 \tilde{n}_{N_3})(Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2})\big), \\ &F_3 := Q^{K_\pm}_{\geq M}\big((Q_3 \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2})\big). \end{aligned}$$

For the estimate of F_1, we apply Corollary 1.11 and Lemma 1.16 to have

$$\begin{aligned} &\sum_{N_1 \geq 1} N_1^{2s} \Big\| \int_0^t \mathbf{1}_{[0,T)}(t') K_\pm(t-t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1} F_1(t') dt' \Big\|^2_{U^2_{K_\pm}} \\ &\lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{V^2_{K_\pm}}=1} \Big| \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} \int_{\mathbb{R}^{1+d}} (Q^{W_{\pm c}}_{\geq M} \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2})(\overline{Q_1 \tilde{u}_{N_1}}) dx dt \Big|^2. \end{aligned} \tag{2.58}$$

For $d=4$, $s=1/4$, from Lemma 2.13 (iv) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, the right-hand side of (2.58) is bounded by

$$T^{1/2} \sum_{N_3 \gtrsim 1} N_3^{1/2} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v\|^2_{Y^{1/4}_{K_\pm}} \lesssim T^{1/2} \|n\|^2_{\dot{Y}^{1/4}_{W_{\pm c}}} \|v\|^2_{Y^{1/4}_{K_\pm}}. \tag{2.59}$$

For the estimate of F_2, we apply Corollary 1.11 and Lemma 1.16 to have

$$\begin{aligned} &\sum_{N_1 \geq 1} N_1^{2s} \Big\| \int_0^t \mathbf{1}_{[0,T)}(t') K_\pm(t-t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1} F_2(t') dt' \Big\|^2_{U^2_{K_\pm}} \\ &\lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{V^2_{K_\pm}}=1} \Big| \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} \int_{\mathbb{R}^{1+d}} (Q_3 \tilde{n}_{N_3})(Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2})(\overline{Q_1 \tilde{u}_{N_1}}) dx dt \Big|^2. \end{aligned} \tag{2.60}$$

For $d=4, s=1/4$, we apply Lemma 2.13 (iv) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, then the right-hand side of (2.60) is bounded by

$$T^{1/2} \sum_{N_3 \gtrsim 1} N_3^{1/2} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v\|^2_{Y^{1/4}_{K_\pm}} \lesssim T^{1/2} \|n\|^2_{\dot{Y}^{1/4}_{W_{\pm c}}} \|v\|^2_{Y^{1/4}_{K_\pm}}. \tag{2.61}$$

For the estimate for F_3, we apply Corollary 1.11 and Lemma 1.16 to have

$$\begin{aligned} &\sum_{N_1 \geq 1} N_1^{2s} \Big\| \int_0^t \mathbf{1}_{[0,T)}(t') K_\pm(t-t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1} F_3(t') dt' \Big\|^2_{U^2_{K_\pm}} \\ &\lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{V^2_{K_\pm}}=1} \Big| \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} \int_{\mathbb{R}^{1+d}} (Q_3 \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2}) (\overline{Q^{K_\pm}_{\geq M} \tilde{u}_{N_1}}) dx dt \Big|^2. \end{aligned} \tag{2.62}$$

For $d=4, s=1/4$, we apply Lemma 2.13 (iv) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, then the right-hand side of (2.62) is bounded by

$$T^{1/2} \sum_{N_3 \gtrsim 1} N_3^{1/2} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v\|^2_{Y^{1/4}_{K_\pm}} \lesssim T^{1/2} \|n\|^2_{\dot{Y}^{1/4}_{W_{\pm c}}} \|v\|^2_{Y^{1/4}_{K_\pm}}. \tag{2.63}$$

Collecting (2.59), (2.61) and (2.63), we obtain $J_2 \lesssim T^{1/2} \|n\|^2_{\dot{Y}^{1/4}_{W_{\pm c}}} \|v\|^2_{Y^{1/4}_{K_\pm}}$. By Corollary 1.11 and the triangle inequality to have

$$\begin{aligned} J_3 &\lesssim \sum_{N_1 \geq 1} N_1^{2s} \sup_{\|u\|_{V^2_{K_\pm}}=1} \Big| \sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \Big|^2 \\ &\lesssim \sum_{N_1 \geq 1} N_1^{2s} \Big(\sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} \sup_{\|u\|_{V^2_{K_\pm}}=1} \Big| \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \Big| \Big)^2. \end{aligned} \tag{2.64}$$

In the same manner as (2.39), for $d=4, s=1/4$, we have

$$\Big| \int_{\mathbb{R}^{1+4}} \tilde{n}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dx dt \Big| \lesssim T^{1/4} N_3^{1/4} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}. \tag{2.65}$$

From (2.65), the right-hand side of (2.64) is bounded by

$$\sum_{N_1 \geq 1} \Big(\sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} N_1^{1/4} T^{1/4} N_3^{1/4} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \Big)^2.$$

Hence, $\|\cdot\|_{l^2 l^1} \lesssim \|\cdot\|_{l^1 l^2}$ and the Cauchy-Schwarz inequality to have

$$\begin{aligned} J_3^{1/2} &\lesssim \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} \Big(\sum_{N_1 \ll N_2} N_1^{1/2} T^{1/2} N_3^{1/2} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v_{N_2}\|^2_{V^2_{K_\pm}} \Big)^{1/2} \\ &\lesssim T^{1/4} \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} N_2^{1/4} N_3^{1/4} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \\ &\lesssim T^{1/4} \|n\|_{\dot{Y}^{1/4}_{W_{\pm c}}} \|v\|_{Y^{1/4}_{K_\pm}}. \end{aligned}$$

We prove (2.55). By Corollary 1.11, we only need to estimate K_i $(i=1,2,3)$:

$$K_1 := \sum_{N_3} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3} \sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}})dxdt\Big|^2,$$

$$K_2 := \sum_{N_3} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\ll N_3} \sum_{N_1\sim N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}})dxdt\Big|^2,$$

$$K_3 := \sum_{N_3} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \gtrsim N_3} \sum_{N_1\sim N_2} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}})dxdt\Big|^2.$$

First, we estimate K_1. Put $K_1 = K_{1,1} + K_{1,2}$ where

$$\begin{aligned} K_{1,1} := \sum_{N_3 \lesssim 1} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3} \sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}}) \\ \times (\overline{\omega\tilde{n}_{N_3}})dxdt\Big|^2, \end{aligned} \tag{2.66}$$

$$K_{1,2} := \sum_{N_3\gg 1} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3} \sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}})dxdt\Big|^2.$$

For $d=4, s=1/4$, by the same manner as the estimate for Lemma 2.13 (i) and $N_1 \ll N_3 \lesssim 1$, we find

$$\begin{aligned} &\Big|\int_{\mathbb{R}^{1+4}} \Big(\sum_{N_1\ll N_3} \omega_1^{-1}\tilde{u}_{N_1}\Big)(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}})dxdt\Big| \\ &\lesssim \|\mathbf{1}_{[0,T)}\|_{L^4_t} \Big\|\sum_{N_1\ll N_3} \omega_1^{-1}\tilde{u}_{N_1}\Big\|_{L^{20/3}_t L^{5/2}_x} \|\omega_1^{-1}\tilde{v}_{N_2}\|_{L^{20/3}_t L^{5/2}_x} \|\omega\tilde{n}_{N_3}\|_{L^{20/9}_t L^5_x} \\ &\lesssim T^{1/4}\Big\|\langle\nabla_x\rangle^{1/4} \sum_{N_1\ll N_3} \omega_1^{-1}\tilde{u}_{N_1}\Big\|_{V^2_{K_\pm}} \|\langle\nabla_x\rangle^{1/4}\omega_1^{-1}\tilde{v}_{N_2}\|_{V^2_{K_\pm}} \||\nabla_x|^{3/4}\omega\tilde{n}_{N_3}\|_{V^2_{W_{\pm c}}} \\ &\lesssim T^{1/4}\Big\|\langle\nabla_x\rangle^{1/4} \sum_{N_1\ll N_3} \tilde{u}_{N_1}\Big\|_{V^2_{K_\pm}} \langle N_2\rangle^{1/4-1}\|v_{N_2}\|_{V^2_{K_\pm}} N_3^{3/4+1}\|n_{N_3}\|_{V^2_{W_{\pm c}}} \\ &\lesssim T^{1/4}\|u\|_{Y^{1/4}_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}. \end{aligned}$$

Hence,

$$K_{1,1} \lesssim \sum_{N_2 \lesssim 1} N_2^{1/2}(T^{1/4}\|u\|_{Y^{1/4}_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim T^{1/2}\|u\|^2_{Y^{1/4}_{K_\pm}}\|v\|^2_{Y^{1/4}_{K_\pm}}.$$

We take $M = \varepsilon N_2$ for sufficiently small $\varepsilon > 0$. Then, from Lemma 3.4, we have

$$\begin{aligned} &Q^{K_\pm}_{<M}\omega_1^{-1}\big((Q^{K_\pm}_{<M}\omega_1^{-1}\tilde{v}_{N_2})(Q^{W_{\pm c}}_{<M}\omega\,\tilde{n}_{N_3})\big) \\ &= Q^{K_\pm}_{<M}\omega_1^{-1}\Big[\mathcal{F}^{-1}\Big(\int_{\tau_1=\tau_2+\tau_3,\,\xi_1=\xi_2+\xi_3} (\widehat{Q^{K_\pm}_{<M}\omega_1^{-1}}\tilde{v}_{N_2})(\tau_2,\xi_2)(\widehat{Q^{W_{\pm c}}_{<M}\omega}\,\tilde{n}_{N_3})(\tau_3,\xi_3)\Big)\Big] \\ &= 0 \end{aligned}$$

when $N_2 \gg \langle N_1\rangle$. Therefore,

$$(\omega_1^{-1}\tilde{v}_{N_2})(\omega\,\tilde{n}_{N_3}) = \sum_{i=1}^{3} G_i,$$

where $Q_1, Q_2 \in \{Q_{<M}^{K_\pm}, Q_{\geq M}^{K_\pm}\}, Q_3 \in \{Q_{<M}^{W_{\pm c}}, Q_{\geq M}^{W_{\pm c}}\}$,

$$G_1 := Q_{\geq M}^{K_\pm}\big((Q_2\omega_1^{-1}\tilde{v}_{N_2})(Q_3\omega\,\tilde{n}_{N_3})\big), \qquad G_2 := Q_1\big((Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{v}_{N_2})(Q_3\omega\,\tilde{n}_{N_3})\big),$$
$$G_3 := Q_1\big((Q_2\omega_1^{-1}\tilde{v}_{N_2})(Q_{\geq M}^{W_{\pm c}}\omega\,\tilde{n}_{N_3})\big).$$

Hence, it follows that

$$K_{1,2} \leq K_{1,2,1} + K_{1,2,2} + K_{1,2,3}$$

where

$$K_{1,2,1} := \sum_{N_3\gg 1} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3}\sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})\overline{G_1}dxdt\Big|^2, \tag{2.67}$$

$$K_{1,2,2} := \sum_{N_3\gg 1} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3}\sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})\overline{G_2}dxdt\Big|^2, \tag{2.68}$$

$$K_{1,2,3} := \sum_{N_3\gg 1} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3}\sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})\overline{G_3}dxdt\Big|^2. \tag{2.69}$$

By Lemma 1.16,

$$\begin{aligned} K_{1,2,1} \lesssim{}& \sum_{N_3\gg 1} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3}\sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+d}} (Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{u}_{N_1})(\overline{Q_2\omega_1^{-1}\tilde{v}_{N_2}}) \\ &\times (\overline{Q_3\omega\tilde{n}_{N_3}})dxdt\Big|^2. \end{aligned} \tag{2.70}$$

By the same manner as the estimate for Lemma 2.13 $(iv), i = 5$, for $d = 4, s = 1/4$, we find

$$\begin{aligned} &\Big|\int_{\mathbb{R}^{1+4}} \Big(\sum_{N_1\ll N_3} Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{u}_{N_1}\Big)(\overline{Q_2\omega_1^{-1}\tilde{v}_{N_2}})(\overline{Q_3\omega\tilde{n}_{N_3}})dxdt\Big| \\ &\lesssim T^{1/4}\|u\|_{Y^{1/4}_{K_\pm}}\|\omega_1^{-1}v_{N_2}\|_{V^2_{K_\pm}}\|\omega n_{N_3}\|_{V^2_{W_{\pm c}}}. \end{aligned} \tag{2.71}$$

Hence, from (2.70) and (2.71), we have

$$\begin{aligned} K_{1,2,1} \lesssim{}& \sum_{N_3\gg 1} N_3^{1/2} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3}\sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+4}} (Q_{\geq M}^{K_\pm}\omega_1^{-1}\tilde{u}_{N_1})(\overline{Q_2\omega_1^{-1}\tilde{v}_{N_2}}) \\ &\times (\overline{Q_3\omega\tilde{n}_{N_3}})dxdt\Big|^2 \\ \lesssim{}& \sum_{N_2\gg 1} N_2^{1/2}(T^{1/4}\|u\|_{Y^{1/4}_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim T^{1/2}\|u\|^2_{Y^{1/4}_{K_\pm}}\|v\|^2_{Y^{1/4}_{K_\pm}}. \end{aligned}$$

By Lemma 1.16,

$$K_{1,2,2} \lesssim \sum_{N_3 \gg 1} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2}}) \times (\overline{Q_3 \omega \tilde{n}_{N_3}}) dxdt \Big|^2. \quad (2.72)$$

By the same manner as the estimate for Lemma 2.13 $(iv), i = 6$, for $d = 4, s = 1/4$, we find

$$\Big| \int_{\mathbb{R}^{1+4}} \Big(\sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \tilde{u}_{N_1} \Big) (\overline{Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q_3 \omega \tilde{n}_{N_3}}) dxdt \Big| \lesssim T^{1/4} \|u\|_{Y^{1/4}_{K_\pm}} \|\omega_1^{-1} v_{N_2}\|_{V^2_{K_\pm}} \|\omega n_{N_3}\|_{V^2_{W_{\pm c}}}. \quad (2.73)$$

Hence, from (2.72) and (2.73), we have

$$\begin{aligned} K_{1,2,2} &\lesssim \sum_{N_3 \gg 1} N_3^{1/2} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+4}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2}}) \times (\overline{Q_3 \omega \tilde{n}_{N_3}}) dxdt \Big|^2 \\ &\lesssim \sum_{N_2 \gg 1} N_2^{1/2} (T^{1/4} \|u\|_{Y^{1/4}_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim T^{1/2} \|u\|^2_{Y^{1/4}_{K_\pm}} \|v\|^2_{Y^{1/4}_{K_\pm}}. \end{aligned}$$

By Lemma 1.16,

$$K_{1,2,3} \lesssim \sum_{N_3 \gg 1} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) \times (\overline{Q^{W_{\pm c}}_{\geq M} \omega \tilde{n}_{N_3}}) dxdt \Big|^2. \quad (2.74)$$

By the same manner as the estimate for Lemma 2.13 $(iv), i = 4$, for $d = 4, s = 1/4$, we find

$$\Big| \int_{\mathbb{R}^{1+4}} \Big(\sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \tilde{u}_{N_1} \Big) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q^{W_{\pm c}}_{\geq M} \omega \tilde{n}_{N_3}}) dxdt \Big| \lesssim T^{1/4} \|u\|_{Y^{1/4}_{K_\pm}} \|\omega_1^{-1} v_{N_2}\|_{V^2_{K_\pm}} \|\omega n_{N_3}\|_{V^2_{W_{\pm c}}}. \quad (2.75)$$

Hence, from (2.74) and (2.75), we have

$$\begin{aligned} K_{1,2,3} &\lesssim \sum_{N_3 \gg 1} N_3^{1/2} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+4}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) \times (\overline{Q^{W_{\pm c}}_{\geq M} \omega \tilde{n}_{N_3}}) dxdt \Big|^2 \\ &\lesssim \sum_{N_2 \gg 1} N_2^{1/2} (T^{1/4} \|u\|_{Y^{1/4}_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim T^{1/2} \|u\|^2_{Y^{1/4}_{K_\pm}} \|v\|^2_{Y^{1/4}_{K_\pm}}. \end{aligned}$$

By symmetry, the estimate for K_2 is obtained by the same manner as the estimate for K_1. We estimate K_3. By the triangle inequality, we have

$$K_3^{1/2} \lesssim \sum_{N_2} \sum_{N_1 \sim N_2} \Big\{ \sum_{N_3 \lesssim N_2} N_3^{2s} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega \tilde{n}_{N_3}}) dxdt \Big|^2 \Big\}^{1/2}. \tag{2.76}$$

For $d=4, s=1/4$, we apply Lemma 2.13 (i) and the Cauchy-Schwarz inequality, the right-hand side of (2.76) is bounded by

$$\begin{aligned} &\sum_{N_2} \sum_{N_1 \sim N_2} \Big\{ \sum_{N_3 \lesssim N_2} N_3^{1/2} (T^{1/4} N_3^{1/4} \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \Big\}^{1/2} \\ &\lesssim T^{1/4} \sum_{N_2} \sum_{N_1 \sim N_2} (N_2 \|u_{N_1}\|^2_{V^2_{K_\pm}} \|v_{N_2}\|^2_{V^2_{K_\pm}})^{1/2} \\ &\lesssim T^{1/4} \Big(\sum_N N^{1/2} \|u_N\|^2_{V^2_{K_\pm}} \Big)^{1/2} \Big(\sum_N N^{1/2} \|v_N\|^2_{V^2_{K_\pm}} \Big)^{1/2}. \end{aligned}$$

Since

$$\sum_{N<1} N^{1/2} \|u_N\|^2_{V^2_{K_\pm}} \lesssim \sum_{N<1} N^{1/2} \|P_{<1} u\|^2_{V^2_{K_\pm}} \lesssim \|P_{<1} u\|^2_{V^2_{K_\pm}},$$

we obtain $K_3^{1/2} \lesssim T^{1/4} \|u\|_{Y^{1/4}_{K_\pm}} \|v\|_{Y^{1/4}_{K_\pm}}$.

Next, we prove (2.54) for $d \geq 5$ and $s = s' = (d^2 - 3d - 2)/2(d+1)$ by the same manner as the proof for $d=4, s=1/4$. From (2.56) and Lemma 2.13 (ii), we have

$$J_0^{1/2} \lesssim T^{1/(d+1)} \|n\|_{\dot{Y}^{s'}_{W_{\pm c}}} \|v\|_{Y^{s'}_{K_\pm}}.$$

By (2.57), $N_1 \sim N_2$, Lemma 2.13 (iii) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, we have

$$J_1 \lesssim \sum_{N_2 \gtrsim 1} N_2^{2s'} T^{2/(d+1)} \|n\|^2_{\dot{Y}^{s'}_{W_{\pm c}}} \|v_{N_2}\|^2_{V^2_{K_\pm}} \lesssim T^{2/(d+1)} \|n\|^2_{\dot{Y}^{s'}_{W_{\pm c}}} \|v\|^2_{Y^{s'}_{K_\pm}}.$$

From Lemma 2.13 (iv) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, the right-hand side of (2.58) is bounded by

$$T^{2/(d+1)} \sum_{N_3 \gtrsim 1} N_3^{2s'} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v\|^2_{Y^{s'}_{K_\pm}} \lesssim T^{2/(d+1)} \|n\|^2_{\dot{Y}^{s'}_{W_{\pm c}}} \|v\|^2_{Y^{s'}_{K_\pm}}. \tag{2.77}$$

From Lemma 2.13 (iv) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, the right-hand side of (2.60) is bounded by

$$T^{2/(d+1)} \sum_{N_3 \gtrsim 1} N_3^{2s'} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v\|^2_{Y^{s'}_{K_\pm}} \lesssim T^{2/(d+1)} \|n\|^2_{\dot{Y}^{s'}_{W_{\pm c}}} \|v\|^2_{Y^{s'}_{K_\pm}}. \tag{2.78}$$

From Lemma 2.13 (iv) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, the right-hand side of (2.62) is bounded by

$$T^{2/(d+1)} \sum_{N_3 \gtrsim 1} N_3^{2s'} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v\|^2_{Y^{s'}_{K_\pm}} \lesssim T^{2/(d+1)} \|n\|^2_{\dot{Y}^{s'}_{W_{\pm c}}} \|v\|^2_{Y^{s'}_{K_\pm}}. \tag{2.79}$$

Collecting (2.77)–(2.79), we obtain $J_2 \lesssim T^{2/(d+1)} \|n\|^2_{\dot{Y}^{s'}_{W_{\pm c}}} \|v\|^2_{Y^{s'}_{K_\pm}}$. By the same manner as the estimate for Lemma 2.13 (iii), we obtain

$$\Big| \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dxdt \Big| \lesssim T^{1/(d+1)} N_3^{s'} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}. \tag{2.80}$$

From (2.80), the right-hand side of (2.64) is bounded by

$$\sum_{N_1 \geq 1} \Big(\sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} N_1^{s'} T^{1/(d+1)} N_3^{s'} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \Big)^2.$$

Hence, $\|\cdot\|_{l^2 l^1} \lesssim \|\cdot\|_{l^1 l^2}$ and the Cauchy-Schwarz inequality to have

$$\begin{aligned} J_3^{1/2} &\lesssim \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} \Big(\sum_{N_1 \ll N_2} N_1^{2s'} T^{2/(d+1)} N_3^{2s'} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v_{N_2}\|^2_{V^2_{K_\pm}} \Big)^{1/2} \\ &\lesssim T^{1/(d+1)} \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} N_2^{s'} N_3^{s'} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \\ &\lesssim T^{1/(d+1)} \|n\|_{\dot{Y}^{s'}_{W_{\pm c}}} \|v\|_{Y^{s'}_{K_\pm}}. \end{aligned}$$

We prove (2.55) for $d \geq 5, s = s' = (d^2 - 3d - 2)/2(d+1)$ by the same manner as the proof for $d = 4, s = 1/4$. By the Hölder inequality to have

$$\begin{aligned} &\Big| \int_{\mathbb{R}^{1+d}} \Big(\sum_{N_1 \ll N_3} \omega_1^{-1} \tilde{u}_{N_1} \Big) (\overline{\omega_1^{-1} \tilde{v}_{N_2}}) (\overline{\omega \tilde{n}_{N_3}}) dxdt \Big| \\ &\lesssim \Big\| \sum_{N_1 \ll N_3} \omega_1^{-1} \tilde{u}_{N_1} \Big\|_{L^{2(d+1)/(d-1)}_{t,x}} \|\omega_1^{-1} \tilde{v}_{N_2}\|_{L^{2(d+1)/(d-1)}_{t,x}} \|\omega \tilde{n}_{N_3}\|_{L^{(d+1)/2}_{t,x}}. \end{aligned} \tag{2.81}$$

By Proposition 2.7, $N_1 \ll N_3 \lesssim 1$ and discarding ω_1^{-1} to have

$$\begin{aligned} \Big\| \sum_{N_1 \ll N_3} \omega_1^{-1} \tilde{u}_{N_1} \Big\|_{L^{2(d+1)/(d-1)}_{t,x}} &\lesssim \Big\| \langle \nabla_x \rangle^{1/2} \sum_{N_1 \ll N_3} \omega_1^{-1} \tilde{u}_{N_1} \Big\|_{V^2_{K_\pm}} \\ &\lesssim \Big\| \sum_{N_1 \ll N_3} \tilde{u}_{N_1} \Big\|_{V^2_{K_\pm}} \\ &\lesssim \|P_{<1} u\|_{V^2_{K_\pm}}. \end{aligned} \tag{2.82}$$

From (2.66), (2.81), (2.82), (2.19), (2.23) and $N_2 \sim N_3 \lesssim 1$, we obtain

$$\begin{aligned} K_{1,1} &\lesssim \sum_{N_2 \lesssim 1} N_2^{2s'} (T^{1/(d+1)} \langle N_2 \rangle^{-1/2} N_2^{s'+1} \| P_{<1} u \|_{V^2_{K_\pm}} \| v_{N_2} \|_{V^2_{K_\pm}})^2 \\ &\lesssim T^{2/(d+2)} \| P_{<1} u \|^2_{V^2_{K_\pm}} \sum_{N_2 \lesssim 1} N_2^{2s'} \| v_{N_2} \|^2_{V^2_{K_\pm}} \\ &\lesssim T^{2/(d+2)} \| u \|^2_{Y^{s'}_{K_\pm}} \| v \|^2_{Y^{s'}_{K_\pm}}. \end{aligned}$$

By the same manner as the estimate for Lemma 2.13 $(iv), i = 5$, we see

$$\begin{aligned} &\Big| \int_{\mathbb{R}^{1+d}} \Big(\sum_{N_1 \ll N_3} Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{u}_{N_1} \Big) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt \Big| \\ &\lesssim T^{1/(d+1)} \| u \|_{Y^{s'}_{K_\pm}} \| \omega_1^{-1} v_{N_2} \|_{V^2_{K_\pm}} \| \omega n_{N_3} \|_{V^2_{W_{\pm c}}}. \end{aligned} \tag{2.83}$$

From (2.70) and (2.83), we have

$$\begin{aligned} K_{1,2,1} &\lesssim \sum_{N_3 \gg 1} N_3^{2s'} \sup_{\|n\|_{V^2_{W_{\pm c}}} = 1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) \\ &\qquad \times (\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt \Big|^2 \\ &\lesssim \sum_{N_2 \gg 1} N_2^{2s'} (T^{1/(d+1)} \| u \|_{Y^{s'}_{K_\pm}} \| v_{N_2} \|_{V^2_{K_\pm}})^2 \lesssim T^{2/(d+1)} \| u \|^2_{Y^{s'}_{K_\pm}} \| v \|^2_{Y^{s'}_{K_\pm}}. \end{aligned}$$

By the same manner as the estimate for Lemma 2.13 $(iv), i = 6$, we see

$$\begin{aligned} &\Big| \int_{\mathbb{R}^{1+d}} \Big(\sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \tilde{u}_{N_1} \Big) (\overline{Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt \Big| \\ &\lesssim T^{1/(d+1)} \| u \|_{Y^{s'}_{K_\pm}} \| \omega_1^{-1} v_{N_2} \|_{V^2_{K_\pm}} \| \omega n_{N_3} \|_{V^2_{W_{\pm c}}}. \end{aligned} \tag{2.84}$$

From (2.72) and (2.84), we have

$$\begin{aligned} K_{1,2,2} &\lesssim \sum_{N_3 \gg 1} N_3^{2s'} \sup_{\|n\|_{V^2_{W_{\pm c}}} = 1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2}}) \\ &\qquad \times (\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt \Big|^2 \\ &\lesssim \sum_{N_2 \gg 1} N_2^{2s'} (T^{1/(d+1)} \| u \|_{Y^{s'}_{K_\pm}} \| v_{N_2} \|_{V^2_{K_\pm}})^2 \lesssim T^{2/(d+1)} \| u \|^2_{Y^{s'}_{K_\pm}} \| v \|^2_{Y^{s'}_{K_\pm}}. \end{aligned}$$

By the same manner as the estimate for Lemma 2.13 $(iv), i = 4$, we see

$$\begin{aligned} &\Big| \int_{\mathbb{R}^{1+d}} \Big(\sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \tilde{u}_{N_1} \Big) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q^{W_{\pm c}}_{\geq M} \omega \tilde{n}_{N_3}}) dx dt \Big| \\ &\lesssim T^{1/(d+1)} \| u \|_{Y^{s'}_{K_\pm}} \| \omega_1^{-1} v_{N_2} \|_{V^2_{K_\pm}} \| \omega n_{N_3} \|_{V^2_{W_{\pm c}}}. \end{aligned} \tag{2.85}$$

From (2.74) and (2.85), we have

$$K_{1,2,3} \lesssim \sum_{N_3 \gg 1} N_3^{2s'} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}})$$
$$\times (\overline{Q^{W_{\pm c}}_{\geq M} \omega \tilde{n}_{N_3}}) dxdt \Big|^2$$
$$\lesssim \sum_{N_2 \gg 1} N_2^{2s'} (T^{1/(d+1)} \|u\|_{Y^{s'}_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim T^{2/(d+1)} \|u\|^2_{Y^{s'}_{K_\pm}} \|v\|^2_{Y^{s'}_{K_\pm}}.$$

By symmetry, the estimate for K_2 is obtained by the same manner as the estimate for K_1. We apply Lemma 2.13 (i) and the Cauchy-Schwarz inequality, the right-hand side of (2.76) is bounded by

$$\sum_{N_2} \sum_{N_1 \sim N_2} \Big\{ \sum_{N_3 \lesssim N_2} N_3^{2s'} (T^{1/(d+1)} N_3^{s'} \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \Big\}^{1/2}$$
$$\lesssim T^{1/(d+1)} \sum_{N_2} \sum_{N_1 \sim N_2} (N_2^{4s'} \|u_{N_1}\|^2_{V^2_{K_\pm}} \|v_{N_2}\|^2_{V^2_{K_\pm}})^{1/2}$$
$$\lesssim T^{1/(d+1)} \Big(\sum_N N^{2s'} \|u_N\|^2_{V^2_{K_\pm}} \Big)^{1/2} \Big(\sum_N N^{2s'} \|v_N\|^2_{V^2_{K_\pm}} \Big)^{1/2}.$$

Since $s' > 0$, we have

$$\sum_{N<1} N^{2s'} \|u_N\|^2_{V^2_{K_\pm}} \lesssim \sum_{N<1} N^{2s'} \|P_{<1} u\|^2_{V^2_{K_\pm}} \lesssim \|P_{<1} u\|^2_{V^2_{K_\pm}}.$$

Thus, we obtain $K_3^{1/2} \lesssim T^{1/(d+1)} \|u\|_{Y^{s'}_{K_\pm}} \|v\|_{Y^{s'}_{K_\pm}}$.

Finally, we prove (2.54) for $d \geq 4, s = s_c = d/2 - 2$ and spherically symmetric functions (u, v, n) by the same manner as the proof of $d = 4, s = 1/4$. From (2.56) and Lemma 2.13 (ii), we obtain

$$J_0^{1/2} \lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|_{Y^{s_c}_{K_\pm}}.$$

By (2.57), $N_1 \sim N_2$, Lemma 2.13 (iii) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, we have

$$J_1 \lesssim \sum_{N_2 \gtrsim 1} N_2^{2s_c} \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v_{N_2}\|^2_{V^2_{K_\pm}} \lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|^2_{Y^{s_c}_{K_\pm}}.$$

From Lemma 2.13 (iv) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, the right-hand side of (2.58) is bounded by

$$\sum_{N_3 \gtrsim 1} N_3^{2s_c} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v\|^2_{Y^{s_c}_{K_\pm}} \lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|^2_{Y^{s_c}_{K_\pm}}. \tag{2.86}$$

From Lemma 2.13 (iv) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, the right-hand side of (2.60) is bounded by

$$\sum_{N_3 \gtrsim 1} N_3^{2s_c}\|n_{N_3}\|^2_{V^2_{W_{\pm c}}}\|v\|^2_{Y^{s_c}_{K_\pm}} \lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}}\|v\|^2_{Y^{s_c}_{K_\pm}}. \tag{2.87}$$

From Lemma 2.13 (iv) and $\|u_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, the right-hand side of (2.62) is bounded by

$$\sum_{N_3 \gtrsim 1} N_3^{2s_c}\|n_{N_3}\|^2_{V^2_{W_{\pm c}}}\|v\|^2_{Y^{s_c}_{K_\pm}} \lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}}\|v\|^2_{Y^{s_c}_{K_\pm}}. \tag{2.88}$$

Collecting (2.86)–(2.88), we have $J_2 \lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}}\|v\|^2_{Y^{s_c}_{K_\pm}}$. By the same manner as the estimate for Lemma 2.13 (iii), we obtain

$$\left|\int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2})\overline{\tilde{u}_{N_1}}dxdt\right| \lesssim N_3^{s_c}\|n_{N_3}\|_{V^2_{W_{\pm c}}}\|v_{N_2}\|_{V^2_{K_\pm}}\|u_{N_1}\|_{V^2_{K_\pm}}. \tag{2.89}$$

From (2.89), the right-hand side of (2.64) is bounded by

$$\sum_{N_1\geq 1}\Big(\sum_{N_2\gg N_1}\sum_{N_3\sim N_2} N_1^{s_c}N_3^{s_c}\|n_{N_3}\|_{V^2_{W_{\pm c}}}\|v_{N_2}\|_{V^2_{K_\pm}}\Big)^2.$$

Hence, when $d > 4$, by $s_c > 0$, $\|\cdot\|_{l^2l^1} \lesssim \|\cdot\|_{l^1l^2}$ and the Cauchy-Schwarz inequality to have

$$\begin{aligned} J_3^{1/2} &\lesssim \sum_{N_2 \gtrsim 1}\sum_{N_3\sim N_2}\Big(\sum_{N_1\ll N_2} N_1^{2s_c}N_3^{2s_c}\|n_{N_3}\|^2_{V^2_{W_{\pm c}}}\|v_{N_2}\|^2_{V^2_{K_\pm}}\Big)^{1/2} \\ &\lesssim \sum_{N_2 \gtrsim 1}\sum_{N_3\sim N_2} N_2^{s_c}N_3^{s_c}\|n_{N_3}\|_{V^2_{W_{\pm c}}}\|v_{N_2}\|_{V^2_{K_\pm}} \\ &\lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}}\|v\|_{Y^{s_c}_{K_\pm}}. \end{aligned}$$

If $d = 4$, then we apply (2.25)–(2.27) and $1 \leq N_1 \ll N_2 \sim N_3$ to have

$$\left|\int_{\mathbb{R}^{1+4}} \tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2})\overline{\tilde{u}_{N_1}}dxdt\right| \lesssim N_1^{1/3}N_2^{-1/3}\|n_{N_3}\|_{V^2_{W_{\pm c}}}\|v_{N_2}\|_{V^2_{K_\pm}}\|u_{N_1}\|_{V^2_{K_\pm}}. \tag{2.90}$$

From (2.90), (2.64) and the Cauchy-Schwarz inequality to have

$$\begin{aligned} J_3^{1/2} &\lesssim \sum_{N_2 \gtrsim 1}\sum_{N_3\sim N_2}\Big(\sum_{N_1\ll N_2} N_1^{2/3}N_2^{-2/3}\|n_{N_3}\|^2_{V^2_{W_{\pm c}}}\|v_{N_2}\|^2_{V^2_{K_\pm}}\Big)^{1/2} \\ &\lesssim \sum_{N_2 \gtrsim 1}\sum_{N_3\sim N_2} \|n_{N_3}\|_{V^2_{W_{\pm c}}}\|v_{N_2}\|_{V^2_{K_\pm}} \\ &\lesssim \|n\|_{\dot{Y}^{0}_{W_{\pm c}}}\|v\|_{Y^{0}_{K_\pm}}. \end{aligned}$$

We prove (2.55) for $d \geq 4, s = s_c = d/2 - 2$ and spherically symmetric functions (u, v, n) by the same manner as the proof of $d = 4, s = 1/4$. By the Hölder inequality to have

$$\begin{aligned} &\Big|\int_{\mathbb{R}^{1+d}} \Big(\sum_{N_1 \ll N_3} \omega_1^{-1} \tilde{u}_{N_1}\Big)(\overline{\omega_1^{-1} \tilde{v}_{N_2}})(\overline{\omega \tilde{n}_{N_3}}) dx dt\Big| \\ &\lesssim \Big\|\sum_{N_1 \ll N_3} \omega_1^{-1} \tilde{u}_{N_1}\Big\|_{L^3_{t,x}} \|\omega_1^{-1} \tilde{v}_{N_2}\|_{L^3_{t,x}} \|\omega \tilde{n}_{N_3}\|_{L^3_{t,x}}. \end{aligned} \tag{2.91}$$

Discarding ω_1^{-1}, then $N_1 \ll N_3 \lesssim 1$ and the same manner as (2.25), we find

$$\Big\|\sum_{N_1 \ll N_3} \omega_1^{-1} \tilde{u}_{N_1}\Big\|_{L^3_{t,x}} \lesssim \Big\|\langle \nabla_x \rangle^{(d-8)/6} \sum_{N_1 \ll N_3} \tilde{u}_{N_1}\Big\|_{V^2_{K_\pm}} \lesssim \|P_{<1} u\|_{V^2_{K_\pm}}. \tag{2.92}$$

Collecting (2.66), (2.91), (2.92), (2.26), (2.27) and $N_2 \sim N_3 \lesssim 1$, we obtain

$$\begin{aligned} K_{1,1} &\lesssim \sum_{N_2 \lesssim 1} N_2^{2s_c} (\|P_{<1} u\|_{V^2_{K_\pm}} \langle N_2 \rangle^{(d-8)/6} \|v_{N_2}\|_{V^2_{K_\pm}} N_2^{(d+4)/6})^2 \\ &\lesssim \|P_{<1} u\|^2_{V^2_{K_\pm}} \sum_{N_2 \lesssim 1} N_2^{2s_c} \|v_{N_2}\|^2_{V^2_{K_\pm}} \\ &\lesssim \|u\|^2_{Y^{s_c}_{K_\pm}} \|v\|^2_{Y^{s_c}_{K_\pm}}. \end{aligned}$$

By the same manner as the estimate for Lemma 2.13 $(iv), i = 5$, we obtain

$$\begin{aligned} &\Big|\int_{\mathbb{R}^{1+d}} \Big(\sum_{N_1 \ll N_3} Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{u}_{N_1}\Big)(\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}})(\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt\Big| \\ &\lesssim \|u\|_{Y^{s_c}_{K_\pm}} \|\omega_1^{-1} v_{N_2}\|_{V^2_{K_\pm}} \|\omega n_{N_3}\|_{V^2_{W_{\pm c}}}. \end{aligned} \tag{2.93}$$

From (2.70) and (2.93), we have

$$\begin{aligned} K_{1,2,1} \lesssim &\sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}} = 1} \Big|\sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{u}_{N_1})(\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) \\ &\qquad \times (\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt\Big|^2 \\ \lesssim &\sum_{N_2 \gg 1} N_2^{2s_c} (\|u\|_{Y^{s_c}_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim \|u\|^2_{Y^{s_c}_{K_\pm}} \|v\|^2_{Y^{s_c}_{K_\pm}}. \end{aligned}$$

By the same manner as the estimate for Lemma 2.13 $(iv), i = 6$, we obtain

$$\begin{aligned} &\Big|\int_{\mathbb{R}^{1+d}} \Big(\sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \tilde{u}_{N_1}\Big)(\overline{Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2}})(\overline{Q_3 \omega \tilde{n}_{N_3}}) dx dt\Big| \\ &\lesssim \|u\|_{Y^{s_c}_{K_\pm}} \|\omega_1^{-1} v_{N_2}\|_{V^2_{K_\pm}} \|\omega n_{N_3}\|_{V^2_{W_{\pm c}}}. \end{aligned} \tag{2.94}$$

From (2.72) and (2.94), we have

$$K_{1,2,2} \lesssim \sum_{N_3\gg 1} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3} \sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1\omega_1^{-1}\tilde{u}_{N_1})(\overline{Q^{K_\pm}_{\geq M}\omega_1^{-1}\tilde{v}_{N_2}}) \times (\overline{Q_3\omega\tilde{n}_{N_3}}) dxdt \Big|^2$$
$$\lesssim \sum_{N_2\gg 1} N_2^{2s_c} (\|u\|_{Y^{s_c}_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim \|u\|^2_{Y^{s_c}_{K_\pm}} \|v\|^2_{Y^{s_c}_{K_\pm}}.$$

By the same manner as the estimate for Lemma 2.13 $(iv), i=4$, we obtain

$$\Big|\int_{\mathbb{R}^{1+d}} \Big(\sum_{N_1\ll N_3} Q_1\omega_1^{-1}\tilde{u}_{N_1}\Big)(\overline{Q_2\omega_1^{-1}\tilde{v}_{N_2}})(\overline{Q^{W_{\pm c}}_{\geq M}\omega\tilde{n}_{N_3}}) dxdt\Big|$$
$$\lesssim \|u\|_{Y^{s_c}_{K_\pm}} \|\omega_1^{-1}v_{N_2}\|_{V^2_{K_\pm}} \|\omega n_{N_3}\|_{V^2_{W_{\pm c}}}. \tag{2.95}$$

From (2.74) and (2.95), we have

$$K_{1,2,3} \lesssim \sum_{N_3\gg 1} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2\sim N_3} \sum_{N_1\ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1\omega_1^{-1}\tilde{u}_{N_1})(\overline{Q_2\omega_1^{-1}\tilde{v}_{N_2}}) \times (\overline{Q^{W_{\pm c}}_{\geq M}\omega\tilde{n}_{N_3}}) dxdt \Big|^2$$
$$\lesssim \sum_{N_2\gg 1} N_2^{2s_c} (\|u\|_{Y^{s_c}_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim \|u\|^2_{Y^{s_c}_{K_\pm}} \|v\|^2_{Y^{s_c}_{K_\pm}}.$$

By symmetry, the estimate for K_2 is obtained by the same manner as the estimate for K_1. For $d=4$, from (2.76), Lemma 2.13 (i) and the Cauchy-Schwarz inequality to have

$$K_3^{1/2} \lesssim \sum_{N_2} \sum_{N_1\sim N_2} \Big\{ \sum_{N_3\lesssim N_2} (\langle N_2\rangle^{-4/3} N_3^{4/3} \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \Big\}^{1/2}$$
$$\lesssim \sum_{N_2} \sum_{N_1\sim N_2} \big(\langle N_2\rangle^{-8/3} N_2^{8/3} \|u_{N_1}\|^2_{V^2_{K_\pm}} \|v_{N_2}\|^2_{V^2_{K_\pm}}\big)^{1/2}$$
$$\lesssim \Big(\sum_N \langle N\rangle^{-4/3} N^{4/3} \|u_N\|^2_{V^2_{K_\pm}}\Big)^{1/2} \Big(\sum_N \langle N\rangle^{-4/3} N^{4/3} \|v_N\|^2_{V^2_{K_\pm}}\Big)^{1/2}.$$

By $\langle N\rangle^{-4/3} \leq 1$, we have

$$\sum_{N<1} \langle N\rangle^{-4/3} N^{4/3} \|u_N\|^2_{V^2_{K_\pm}} \lesssim \sum_{N<1} N^{4/3} \|P_{<1}u\|^2_{V^2_{K_\pm}} \lesssim \|P_{<1}u\|^2_{V^2_{K_\pm}}.$$

Hence, for $d=4$, we obtain

$$K_3^{1/2} \lesssim \|u\|_{Y^0_{K_\pm}} \|v\|_{Y^0_{K_\pm}}. \tag{2.96}$$

For $d > 4$, from (2.76) and Lemma 2.13 (i), we have

$$K_3^{1/2} \lesssim \sum_{N_2} \sum_{N_1 \sim N_2} \Big\{ \sum_{N_3 \lesssim N_2} N_3^{2s_c} (\langle N_2\rangle^{(d-8)/3} N_3^{(d+4)/6} \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \Big\}^{1/2}$$
$$\lesssim \sum_{N_2} \sum_{N_1 \sim N_2} \Big(\sum_{N_3 \lesssim N_2} N_3^{4(d-2)/3} \langle N_2\rangle^{2(d-8)/3} \|u_{N_1}\|^2_{V^2_{K_\pm}} \|v_{N_2}\|^2_{V^2_{K_\pm}} \Big)^{1/2}$$
$$\lesssim \sum_{N_2} \sum_{N_1 \sim N_2} N_2^{2(d-2)/3} \langle N_2\rangle^{(d-8)/3} \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}}. \tag{2.97}$$

For $d \le 8$, then $\langle N_2\rangle^{(d-8)/3} \le N_2^{(d-8)/3}$. Hence by (2.97) and the Cauchy-Schwarz inequality, for $4 < d \le 8$, we have

$$K_3^{1/2} \lesssim \sum_{N_2} \sum_{N_1 \sim N_2} N_2^{d-4} \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}}$$
$$\lesssim \|u\|_{Y^{s_c}_{K_\pm}} \|v\|_{Y^{s_c}_{K_\pm}}. \tag{2.98}$$

For $d > 8$ and $N_2 < 1$, it holds that $\langle N_2\rangle \lesssim 1$. Hence, by (2.97) to have

$$K_3^{1/2} \lesssim \sum_{N_2<1} \sum_{N_1 \sim N_2} N_2^{2(d-2)/3} \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}}$$
$$\lesssim \|u\|_{Y^{s_c}_{K_\pm}} \|P_{<1} v\|_{V^2_{K_\pm}}. \tag{2.99}$$

For $d > 8$ and $N_2 \ge 1$, it holds that $\langle N_2\rangle^{(d-8)/3} \sim N_2^{(d-8)/3}$. Thus by (2.97) and the Cauchy-Schwarz inequality to have

$$K_3^{1/2} \lesssim \sum_{N_2\ge 1} \sum_{N_1 \sim N_2} N_2^{d-4} \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}}$$
$$\lesssim \|u\|_{Y^{s_c}_{K_\pm}} \|v\|_{Y^{s_c}_{K_\pm}}. \tag{2.100}$$

Collecting (2.96), (2.98)–(2.100), we obtain $K_3^{1/2} \lesssim \|u\|_{Y^{s_c}_{K_\pm}} \|v\|_{Y^{s_c}_{K_\pm}}$ for $d \ge 4$.

□

2.4. **The proof of the main theorem.** By the Duhamel principle, we consider the following integral equation corresponding to (2.2) on the time interval $[0, T]$ with $0 < T < \infty$:

$$u_\pm = \Phi_1(u_\pm, n_+, n_-), \quad n_\pm = \Phi_2(n_\pm, u_+, u_-), \tag{2.101}$$

where

$$\Phi_1(u_\pm, n_+, n_-) := K_\pm(t)u_{\pm 0} \pm (1/4)\{I_{T,K_\pm}(n_+, u_+)(t) + I_{T,K_\pm}(n_+, u_-)(t) + I_{T,K_\pm}(n_-, u_+)(t) + I_{T,K_\pm}(n_-, u_-)(t)\},$$

$$\Phi_2(n_\pm, u_+, u_-) := W_{\pm c}(t)n_{\pm 0} \pm (4c)^{-1}\{I_{T,W_{\pm c}}(u_+, u_+)(t) + I_{T,W_{\pm c}}(u_+, u_-)(t) + I_{T,W_{\pm c}}(u_-, u_+)(t) + I_{T,W_{\pm c}}(u_-, u_-)(t)\}.$$

Proposition 2.15. *(i) Let $s = 1/4$ for $d = 4$ or $s = (d^2-3d-2)/2(d+1)$ for $d \geq 5$. Let $\delta > 0$ be arbitrary. Then, for any initial data $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$, there exists $T > 0$ and a unique solution of (2.101) on $[0, T]$ such that*

$$(u_\pm, n_\pm) \in Y^s_{K_\pm}([0, T]) \times \dot{Y}^s_{W_{\pm c}}([0, T]) \subset C([0, T]; H^s(\mathbb{R}^d)) \times C([0, T]; \dot{H}^s(\mathbb{R}^d)).$$

Moreover, let $d \geq 4$, $s = s_c = d/2 - 2$ and $\delta > 0$ be sufficiently small. If $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$ be radial, then for all $0 < T < \infty$, there exists a unique spherically symmetric solution of (2.101) on $[0, T]$ such that

$$(u_\pm, n_\pm) \in Y^s_{K_\pm}([0, T]) \times \dot{Y}^s_{W_{\pm c}}([0, T]) \subset C([0, T]; H^s(\mathbb{R}^d)) \times C([0, T]; \dot{H}^s(\mathbb{R}^d)).$$

(ii) The flow map obtained by (i):
$B_\delta(H^s(\mathbb{R}^d)) \times B_\delta(\dot{H}^s(\mathbb{R}^d)) \ni (u_{\pm 0}, n_{\pm 0}) \mapsto (u_\pm, n_\pm) \in Y^s_{K_\pm}([0, T]) \times \dot{Y}^s_{W_{\pm c}}([0, T])$ is Lipschitz continuous.

Remark 2.1. Due to the time reversibility of the Klein-Gordon-Zakharov equation, Porpositions 2.15 also holds in corresponding time interval $[-T, 0]$

Remark 2.2. By (i) in Proposition 2.15 and Remark 2.1, for any $T > 0$, we have solutions to (2.101) $(u_\pm(t), n_\pm(t))$ on $[0, T]$ and $[-T, 0]$. If radial initial data $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$, then we can take T arbitrary large and by uniqueness, spherically symmetric function $(u_\pm(t), n_\pm(t)) \in C((-\infty, \infty); H^s(\mathbb{R}^d)) \times C((-\infty, \infty); \dot{H}^s(\mathbb{R}^d))$ can be defined uniquely.

Proposition 2.16. *Let the spherically symmetric solution $(u_\pm(t), n_\pm(t))$ to (2.101) on $(-\infty, \infty)$ obtained by Proposirion 2.15, Remark 2.1 and Remark 2.2 with radial initial data $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$. Then, there exist $(u_{\pm,+\infty}, n_{\pm,+\infty})$ and $(u_{\pm,-\infty}, n_{\pm,-\infty})$ in $H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$ such that*

$$\|u_\pm(t) - K_\pm(t)u_{\pm,+\infty}\|_{H^s_x(\mathbb{R}^d)} + \|n_\pm(t) - W_{\pm c}(t)n_{\pm,+\infty}\|_{\dot{H}^s_x(\mathbb{R}^d)} \to 0$$

as $t \to +\infty$ and

$$\|u_\pm(t) - K_\pm(t)u_{\pm,-\infty}\|_{H^s_x(\mathbb{R}^d)} + \|n_\pm(t) - W_{\pm c}(t)n_{\pm,-\infty}\|_{\dot{H}^s_x(\mathbb{R}^d)} \to 0$$

as $t \to -\infty$.

proof of Proposition 2.15. First, we prove (i). By Proposition 2.7, there exists $C > 0$ such that

$$\|K_\pm(t)u_{\pm 0}\|_{Y^s_{K_\pm}} \le C\|u_{\pm 0}\|_{H^s}, \qquad \|W_{\pm c}(t)n_{\pm 0}\|_{\dot{Y}^s_{W_{\pm c}}} \le C\|n_{\pm 0}\|_{\dot{H}^s}.$$

We denote time interval $I := [0,T]$. If $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)), (u_\pm, n_\pm) \in B_r(Y^s_{K_\pm}(I) \times \dot{Y}^s_{W_{\pm c}}(I))$, then by Proposition 2.14, for $(\theta, s) = (1/4, 1/4), d = 4$ or for $(\theta, s) = (1/(d+1), (d^2 - 3d - 2)/2(d+1)), d \ge 5$, it holds that

$$\begin{aligned}
&\|\Phi_1(u_\pm, n_+, n_-)\|_{Y^s_{K_\pm}(I)} \\
&\le C\|u_{\pm 0}\|_{H^s} + (1/4)CT^\theta(\|n_+\|_{\dot{Y}^s_{W_{+c}}(I)}\|u_+\|_{Y^s_{K_+}(I)} + \|n_+\|_{\dot{Y}^s_{W_{+c}}(I)}\|u_-\|_{Y^s_{K_-}(I)} \\
&\qquad + \|n_-\|_{\dot{Y}^s_{W_{-c}}(I)}\|u_+\|_{Y^s_{K_+}(I)} + \|n_-\|_{\dot{Y}^s_{W_{-c}}(I)}\|u_-\|_{Y^s_{K_-}(I)}) \\
&\le C\delta + CT^\theta r^2, \\
&\|\Phi_2(n_\pm, u_+, u_-)\|_{\dot{Y}^s_{W_{\pm c}}(I)} \\
&\le C\|n_{\pm 0}\|_{H^s} + (CT^\theta/4c)(\|u_+\|^2_{Y^s_{K_+}(I)} + 2\|u_+\|_{Y^s_{K_+}(I)}\|u_-\|_{Y^s_{K_-}(I)} + \|u_-\|^2_{Y^s_{K_-}(I)}) \\
&\le C\delta + CT^\theta r^2/c.
\end{aligned}$$

We take $r = 2C\delta$ and $T > 0$ satisfying

$$4CT^\theta r \le \min\{1, c\}. \tag{2.102}$$

Then we have

$$\|\Phi_1(u_\pm, n_+, n_-)\|_{Y^s_{K_\pm}(I)} \le r, \qquad \|\Phi_2(n_\pm, u_+, u_-)\|_{\dot{Y}^s_{W_{\pm c}}(I)} \le r.$$

Hence, (Φ_1, Φ_2) is a map from $B_r(Y^s_{K_\pm}([0,T]) \times \dot{Y}^s_{W_{\pm c}}([0,T]))$ into itself. Similarly, we assume $(v_{\pm 0}, m_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)), (v_\pm, m_\pm) \in B_r(Y^s_{K_\pm}(I) \times \dot{Y}^s_{W_{\pm c}}(I))$, then it holds that

$$\begin{aligned}
&\|\Phi_1(u_\pm, n_+, n_-) - \Phi_1(v_\pm, m_+, m_-)\|_{Y^s_{K_\pm}(I)} \\
&\le (1/4)(\|I_{T,K_\pm}(n_+, u_+)(t) - I_{T,K_\pm}(m_+, v_+)(t)\|_{Y^s_{K_\pm}(I)} \\
&\qquad + \|I_{T,K_\pm}(n_+, u_-)(t) - I_{T,K_\pm}(m_+, v_-)(t)\|_{Y^s_{K_\pm}(I)} \\
&\qquad + \|I_{T,K_\pm}(n_-, u_+)(t) - I_{T,K_\pm}(m_-, v_+)(t)\|_{Y^s_{K_\pm}(I)} \\
&\qquad + \|I_{T,K_\pm}(n_-, u_-)(t) - I_{T,K_\pm}(m_-, v_-)(t)\|_{Y^s_{K_\pm}(I)}).
\end{aligned} \tag{2.103}$$

By Proposition 2.14, we have

$$\begin{aligned}&\|I_{T,K_\pm}(n_+,u_+)(t)-I_{T,K_\pm}(m_+,v_+)(t)\|_{Y^s_{K_\pm}(I)}\\&\le CT^\theta(\|n_+-m_+\|_{\dot Y^s_{W_{+c}}(I)}\|u_+\|_{Y^s_{K_+}(I)}+\|m_+\|_{Y^s_{W_{+c}}(I)}\|u_+-v_+\|_{Y^s_{K_+}(I)}).\end{aligned}\tag{2.104}$$

Similarly, we have

$$\begin{aligned}&\|I_{T,K_\pm}(n_+,u_-)(t)-I_{T,K_\pm}(m_+,v_-)(t)\|_{Y^s_{K_\pm}(I)}\\&\le CT^\theta(\|n_+-m_+\|_{\dot Y^s_{W_{+c}}(I)}\|u_-\|_{Y^s_{K_-}(I)}+\|m_+\|_{\dot Y^s_{W_{+c}}(I)}\|u_--v_-\|_{Y^s_{K_-}(I)}),\end{aligned}\tag{2.105}$$

$$\begin{aligned}&\|I_{T,K_\pm}(n_-,u_+)(t)-I_{T,K_\pm}(m_-,v_+)(t)\|_{Y^s_{K_\pm}(I)}\\&\le CT^\theta(\|n_--m_-\|_{\dot Y^s_{W_{-c}}(I)}\|u_+\|_{Y^s_{K_+}(I)}+\|m_-\|_{\dot Y^s_{W_{-c}}(I)}\|u_+-v_+\|_{Y^s_{K_+}(I)}),\end{aligned}\tag{2.106}$$

$$\begin{aligned}&\|I_{T,K_\pm}(n_-,u_-)(t)-I_{T,K_\pm}(m_-,v_-)(t)\|_{Y^s_{K_\pm}(I)}\\&\le CT^\theta(\|n_--m_-\|_{\dot Y^s_{W_{-c}}(I)}\|u_-\|_{Y^s_{K_-}(I)}+\|m_-\|_{\dot Y^s_{W_{-c}}(I)}\|u_--v_-\|_{Y^s_{K_-}(I)}).\end{aligned}\tag{2.107}$$

Hence from $\|u_\pm\|_{Y^s_{K_\pm}(I)}\le r$, $\|m_\pm\|_{\dot Y^s_{W_{\pm c}}(I)}\le r$, (2.103)–(2.107) and (2.102), we have

$$\begin{aligned}&\|\Phi_1(u_\pm,n_+,n_-)-\Phi_1(v_\pm,m_+,m_-)\|_{Y^s_{K_\pm}(I)}\\&\le(1/8)(\|u_+-v_+\|_{Y^s_{K_+}(I)}+\|u_--v_-\|_{Y^s_{K_-}(I)}\\&\qquad+\|n_+-m_+\|_{\dot Y^s_{W_{+c}}(I)}+\|n_--m_-\|_{\dot Y^s_{W_{-c}}(I)}).\end{aligned}\tag{2.108}$$

Similarly, we have

$$\begin{aligned}&\|\Phi_2(n_\pm,u_+,u-)-\Phi_2(m_\pm,v_+,v_-)\|_{\dot Y^s_{W_{\pm c}}(I)}\\&=(4c)^{-1}(\|I_{T,W_{\pm c}}(u_+,u_+)(t)-I_{T,W_{\pm c}}(v_+,v_+)(t)\|_{\dot Y^s_{W_{\pm c}}(I)}\\&\qquad+\|I_{T,W_{\pm c}}(u_+,u_-)(t)-I_{T,W_{\pm c}}(v_+,v_-)(t)\|_{\dot Y^s_{W_{\pm c}}(I)}\\&\qquad+\|I_{T,W_{\pm c}}(u_-,u_+)(t)-I_{T,W_{\pm c}}(v_-,v_+)(t)\|_{\dot Y^s_{W_{\pm c}}(I)}\\&\qquad+\|I_{T,W_{\pm c}}(u_-,u_-)(t)-I_{T,W_{\pm c}}(v_-,v_-)(t)\|_{\dot Y^s_{W_{\pm c}}(I)}).\end{aligned}\tag{2.109}$$

By Proposition 2.14, we have

$$\begin{aligned}&\|I_{T,W_{\pm c}}(u_+,u_+)(t)-I_{T,W_{\pm c}}(v_+,v_+)(t)\|_{\dot Y^s_{W_{\pm c}}(I)}\\&\le CT^\theta(\|u_+\|_{Y^s_{K_+}(I)}+\|v_+\|_{Y^s_{K_+}(I)})\|u_+-v_+\|_{Y^s_{K_+}(I)}.\end{aligned}\tag{2.110}$$

Similarly, we have

$$\begin{aligned}
&\|I_{T,W_{\pm c}}(u_+,u_-)(t)-I_{T,W_{\pm c}}(v_+,v_-)(t)\|_{\dot Y^s_{W_{\pm c}}(I)}\\
&\le CT^\theta(\|u_+-v_+\|_{Y^s_{K_+}(I)}\|u_-\|_{Y^s_{K_-}(I)}+\|v_+\|_{Y^s_{K_+}(I)}\|u_--v_-\|_{Y^s_{K_-}(I)}), \qquad (2.111)\\
&\|I_{T,W_{\pm c}}(u_-,u_+)(t)-I_{T,W_{\pm c}}(v_-,v_+)(t)\|_{\dot Y^s_{W_{\pm c}}(I)}\\
&\le CT^\theta(\|u_+-v_+\|_{Y^s_{K_+}(I)}\|u_+\|_{Y^s_{K_+}(I)}+\|v_-\|_{Y^s_{K_-}(I)}\|u_--v_-\|_{Y^s_{K_-}(I)}), \qquad (2.112)\\
&\|I_{T,W_{\pm c}}(u_-,u_-)(t)-I_{T,W_{\pm c}}(v_-,v_-)(t)\|_{\dot Y^s_{W_{\pm c}}(I)})\\
&\le CT^\theta(\|u_-\|_{Y^s_{K_-}(I)}+\|v_-\|_{Y^s_{K_-}(I)})\|u_--v_-\|_{Y^s_{K_+}(I)}. \qquad (2.113)
\end{aligned}$$

From $\|u_\pm\|_{Y^s_{K_\pm}(I)}\le r, \|v_\pm\|_{Y^s_{K_\pm}(I)}\le r$, (2.109)–(2.113) and (2.102), we obtain

$$\begin{aligned}
&\|\Phi_2(n_\pm,u_+,u-)-\Phi_2(m_\pm,v_+,v_-)\|_{\dot Y^s_{W_{\pm c}}(I)}\\
&\le (1/4)(\|u_+-v_+\|_{Y^s_{K_+}(I)}+\|u_--v_-\|_{Y^s_{K_-}(I)}). \qquad (2.114)
\end{aligned}$$

Therefore, (Φ_1,Φ_2) is a contraction mapping on $B_r(Y^s_{K_\pm}([0,T])\times \dot Y^s_{W_{\pm c}}([0,T]))$. Hence, by the Banach fixed point theorem, we have a solution to (2.101) in it.

Next, we prove uniqueness. Let $(u_\pm,n_\pm),(v_\pm,m_\pm)\in Y^s_{K_\pm}([0,T])\times \dot Y^s_{W_{\pm c}}([0,T])$ are two solutions satisfying $(u_\pm(0),n_\pm(0))=(v_\pm(0),m_\pm(0))$. Moreover,

$$T':=\sup\{0\le t\le T\,;u_\pm(t)=v_\pm(t),n_\pm(t)=m_\pm(t)\}<T.$$

By a translation in t, it suffices to consider $T'=0$. Fix $0<\tau\le T$ sufficiently small. From (2.103)–(2.107) and Proposition 2.12, we obtain

$$\begin{aligned}
&\|u_+-v_+\|_{Y^s_{K_+}([0,\tau])}\\
&\le (1/4)CT^\theta\big\{(\|u_+\|_{Y^s_{K_+}([0,\tau])}+\|u_-\|_{Y^s_{K_-}([0,\tau])})\\
&\qquad \times(\|n_+-m_+\|_{\dot Y^s_{W_{+c}}([0,\tau])}+\|n_--m_-\|_{\dot Y^s_{W_{-c}}([0,\tau])})\\
&\qquad +(\|m_+\|_{\dot Y^s_{K_+}([0,\tau])}+\|m_-\|_{\dot Y^s_{K_-}([0,\tau])})(\|u_+-v_+\|_{Y^s_{K_+}([0,\tau])}+\|u_--v_-\|_{Y^s_{K_-}([0,\tau])})\big\}\\
&\le (1/8)(\|n_+-m_+\|_{\dot Y^s_{W_{+c}}([0,\tau])}+\|n_--m_-\|_{\dot Y^s_{W_{-c}}([0,\tau])}\\
&\qquad +\|u_+-v_+\|_{Y^s_{K_+}([0,\tau])}+\|u_--v_-\|_{Y^s_{K_-}([0,\tau])}). \qquad (2.115)
\end{aligned}$$

From (2.115), we obtain

$$\begin{aligned}
&\|u_+-v_+\|_{Y^s_{K_+}([0,\tau])}\\
&\le (1/7)(\|n_+-m_+\|_{\dot Y^s_{W_{+c}}([0,\tau])}+\|n_--m_-\|_{\dot Y^s_{W_{-c}}([0,\tau])}+\|u_--v_-\|_{Y^s_{K_-}([0,\tau])}). \qquad (2.116)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\|u_- - v_-\|_{Y^s_{K_-}([0,\tau])}\\&\le (1/7)(\|n_+ - m_+\|_{\dot Y^s_{W_{+c}}([0,\tau])} + \|n_- - m_-\|_{\dot Y^s_{W_{-c}}([0,\tau])} + \|u_+ - v_+\|_{Y^s_{K_+}([0,\tau])}).\end{aligned} \tag{2.117}$$

From (2.109)–(2.113) and Proposition 2.12, we have

$$\|n_\pm - m_\pm\|_{\dot Y^s_{W_{\pm c}}([0,\tau])} \le (1/4)(\|u_+ - v_+\|_{Y^s_{K_+}([0,\tau])} + \|u_- - v_-\|_{Y^s_{K_-}([0,\tau])}). \tag{2.118}$$

Hence, collecting (2.115)–(2.118), we obtain

$$u_\pm = v_\pm, \quad n_\pm = m_\pm$$

on $[0,\tau]$. This contradicts the definition of T'.

If $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot H^s(\mathbb{R}^d))$ is radial, $s = s_c = d/2 - 2$ with $d \ge 4$ and $(u_\pm, n_\pm) \in B_r(Y^s_{K_\pm}(I) \times \dot Y^s_{W_{\pm c}}(I))$ is spherically symmetric, then by Proposition 2.14, we have

$$\begin{aligned}&\|\Phi_1(u_\pm, n_+, n_-)\|_{Y^s_{K_\pm}(I)}\\&\le C\delta + (1/4)C(\|n_+\|_{\dot Y^s_{W_{+c}}(I)}\|u_+\|_{Y^s_{K_+}(I)} + \|n_+\|_{\dot Y^s_{W_{+c}}(I)}\|u_-\|_{Y^s_{K_-}(I)}\\&\qquad + \|n_-\|_{\dot Y^s_{W_{-c}}(I)}\|u_+\|_{Y^s_{K_+}(I)} + \|n_-\|_{\dot Y^s_{W_{-c}}(I)}\|u_-\|_{Y^s_{K_-}(I)}),\\&\|\Phi_2(n_\pm, u_+, u_-)\|_{\dot Y^s_{W_{\pm c}}(I)}\\&\le C\delta + (C/4c)(\|u_+\|^2_{Y^s_{K_+}(I)} + 2\|u_+\|_{Y^s_{K_+}(I)}\|u_-\|_{Y^s_{K_-}(I)} + \|u_-\|^2_{Y^s_{K_-}(I)}).\end{aligned}$$

Taking $\delta = r^2$ and $r = \min\{1, c\}/(4C)$, then we have

$$\|\Phi_1(u_\pm, n_+, n_-)\|_{Y^s_{K_\pm}(I)} \le r, \qquad \|\Phi_2(n_\pm, u_+, u_-)\|_{\dot Y^s_{W_{\pm c}}(I)} \le r.$$

Hence, (Φ_1, Φ_2) is a map from $B_r(Y^s_{K_\pm}([0,T]) \times \dot Y^s_{W_{\pm c}}([0,T]))$ into itself. If we also assume $(v_{\pm 0}, m_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d)) \times \dot H^s(\mathbb{R}^d))$ is radial and $(v_\pm, m_\pm) \in B_r(Y^s_{K_\pm}(I) \times \dot Y^s_{W_{\pm c}}(I))$ is spherically symmetric, then by the same manner as the estimate for (2.108) and (2.114), we have

$$\begin{aligned}&\|\Phi_1(u_\pm, n_+, n_-) - \Phi_1(v_\pm, m_+, m_-)\|_{Y^s_{K_\pm}(I)}\\&\le (1/8)(\|u_+ - v_+\|_{Y^s_{K_+}(I)} + \|u_- - v_-\|_{Y^s_{K_-}(I)}\\&\qquad + \|n_+ - m_+\|_{\dot Y^s_{W_{+c}}(I)} + \|n_- - m_-\|_{\dot Y^s_{W_{-c}}(I)}),\\&\|\Phi_2(n_\pm, u_+, u-) - \Phi_2(m_\pm, v_+, v_-)\|_{\dot Y^s_{W_{\pm c}}(I)}\\&\le (1/4)(\|u_+ - v_+\|_{Y^s_{K_+}(I)} + \|u_- - v_-\|_{Y^s_{K_-}(I)}).\end{aligned}$$

Thus, (Φ_1, Φ_2) is a contraction mapping on $B_r(Y^s_{K\pm}([0,T]) \times \dot{Y}^s_{W\pm c}([0,T]))$. Hence, by the Banach fixed point theorem, we have a solution to (2.101) in it. We assume that $(u_\pm(0), n_\pm(0)), (v_\pm(0), m_\pm(0))$ are both radial and $s = s_c = d/2 - 2$ with $d \geq 4$. Let $(u_\pm, n_\pm), (v_\pm, m_\pm) \in Y^s_{K\pm}([0,T]) \times \dot{Y}^s_{W\pm c}([0,T])$ are two spherically symmetric solutions satisfying $(u_\pm(0), n_\pm(0)) = (v_\pm(0), m_\pm(0))$. Then by the same manner as the proof for non-radial initial data, the uniqueness of the solution $(u_\pm, n_\pm)$ is showed. (ii) follows from the standard argument, so we omit the proof. □

Finally, we prove Proposition 2.16. The proof is the same manner as the proof for Proposition 4.2 in [24].

Proof. There exists $M > 0$ such that for all $0 < T < \infty$,

$$\|u_\pm\|_{Y^s_{K\pm}([0,T])} + \|n_\pm\|_{\dot{Y}^s_{W\pm c}([0,T])} < M,$$
$$\|u_\pm\|_{Y^s_{K\pm}([-T,0])} + \|n_\pm\|_{\dot{Y}^s_{W\pm c}([-T,0])} < M$$

holds since r in the proof of Proposition 2.15 does not depend on T. Take $\{t_k\}_{k=0}^K \in \mathcal{Z}_0$ and $0 < T < \infty$ such that $-T < t_0, t_K < T$. By L^2_x orthogonality,

$$\begin{aligned}
&\Big(\sum_{k=1}^{K} \|\langle\nabla_x\rangle^s\big(K_\pm(-t_k)u_\pm(t_k) - K_\pm(-t_{k-1})u_\pm(t_{k-1})\big)\|^2_{L^2_x}\Big)^{1/2} \\
&\lesssim \|\langle\nabla_x\rangle^s u_\pm\|_{V^2_{K\pm}([0,T])} + \|\langle\nabla_x\rangle^s u_\pm\|_{V^2_{K\pm}([-T,0])} \\
&\lesssim \|u_\pm\|_{Y^s_{K\pm}([0,T])} + \|u_\pm\|_{Y^s_{K\pm}([-T,0])} \\
&< 2M.
\end{aligned}$$

Thus,

$$\sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}_0} \Big(\sum_{k=1}^{K} \|\langle\nabla_x\rangle^s K_\pm(-t_k)u_\pm(t_k) - \langle\nabla_x\rangle^s K_\pm(-t_{k-1})u_\pm(t_{k-1})\|^2_{L^2_x}\Big)^{1/2} < 2M.$$

Hence, there exists $f_\pm := \lim_{t\to\pm\infty} \langle\nabla_x\rangle^s K_\pm(-t)u_\pm(t)$ in $L^2_x(\mathbb{R}^d)$. Then put $u_{\pm\infty} := \langle\nabla_x\rangle^{-s} f_\pm$, we obtain

$$\|\langle\nabla_x\rangle^s K_\pm(-t)u_\pm(t) - f_\pm\|_{L^2_x} = \|u_\pm(t) - K_\pm(t)u_{\pm\infty}\|_{H^s_x} \to 0$$

as $t \to \pm\infty$. The scattering result for the wave equation is obtained similarly.

□

3. Scattering and well-posedness for the Zakharov system at a critical space in four and more spatial dimensions

3.1. **Introduction.** We consider the Cauchy problem for the Zakharov system:

$$\begin{cases} i\partial_t u + \Delta u = nu, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ \partial_t^2 n - \Delta n = \Delta |u|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ \big(u(0), n(0), \partial_t n(0)\big) = (u_0, n_0, n_1) \in H^k(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d) \times \dot{H}^{l-1}(\mathbb{R}^d), \end{cases} \tag{3.1}$$

where $u = u(t,x)$ is complex valued, the slowly varying envelope of electric field and $n = n(t,x)$ is real valued, the deviation of ion density from its mean background density. (3.1) describes the Langmuir turbulence in a plasma. We consider well-posedness for (3.1) in spatial dimension $d \geq 4$. (3.1) does not have scaling invariant transformation because of the difference of dilation transformations for the linear wave equation and the Schrödinger equation. However, in [10], Ginibre, Tsutsumi and Velo introduced a critical exponent for (3.1) which corresponds to the scaling criticality in the following sense. We transform n into $n_\pm$ as $n_\pm := n \pm i\omega^{-1}\partial_t n$, $\omega := \sqrt{-\Delta}$. Then (3.1) is rewritten into

$$\begin{cases} i\partial_t u + \Delta u = u(n_+ + n_-)/2, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (i\partial_t \mp \omega) n_\pm = \pm \omega |u|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ \big(u(0), n_+(0), n_-(0)\big) = (u_0, n_{+0}, n_{-0}). \end{cases} \tag{3.2}$$

In the second equation of (3.2), if we disregard the second term of the left-hand side, then (3.2) is invariant under the dilation

$$u \to u_\lambda = \lambda^{3/2} u(\lambda x, \lambda^2 t), \qquad n \to n_\lambda = \lambda^2 n(\lambda x, \lambda^2 t),$$

and the the scaling critical exponent is $(k,l) = \big((d-3)/2, (d-4)/2\big)$. Our main result is the scattering and the small data global well-posedness for (3.2) at the critical exponent in spatial dimension $d \geq 4$.

Theorem 3.1. *Let $d \geq 4, k = (d-3)/2, l = (d-4)/2$. Then (3.2) is globally well-posed for small data in $H^k(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d)$ (resp. $H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^l(\mathbb{R}^d)$). Moreover, the solution scatters in $H^k(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d)$ (resp. $H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^l(\mathbb{R}^d)$).*

Remark 3.1. Note that $(n_+, n_-) \in \dot{H}^l(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d)$ (resp. $H^l(\mathbb{R}^d) \times H^l(\mathbb{R}^d)$) is equivalent to $(n, \partial_t n) \in \dot{H}^l(\mathbb{R}^d) \times \dot{H}^{l-1}(\mathbb{R}^d)$ (resp. $(n, \omega^{-1}\partial_t n) \in H^l(\mathbb{R}^d) \times H^l(\mathbb{R}^d)$).

For more precise statement of Theorem 3.1, see Propositions 3.11, 3.12. Here, we briefly mention the known results for the Cauchy problem for (3.1). There are many results for $3 \geq d \geq 1$. For the well-posedness and the scattering on $\mathbb{R}^d$, see [2], [3], [6], [7], [9], [10], [13], [14], [16], [21], [22], [27], [34], [38], [39], [42], [43]. For the case on $\mathbb{T}^d$, see [5], [28], [29], [45]. All these results are for the sub critical case. For $d \geq 4$, Ginibre-Tsutsumi-Velo [10] proved the local well-posedness of (3.1) when the initial data is in $H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^{l-1}(\mathbb{R}^d)$ with $2k > l + (d-2)/2, l > (d-4)/2, l+1 \geq k \geq l$, which is the sub critical case. Recently, Bejenaru, Guo, Herr and Nakanishi [1] have proved the small data global well-posedness and the scattering in a range of (k,l) for $d = 4$, which includes the critical case $(k,l) = (1/2, 0)$ and the energy space $(k,l) = (1,0)$.

The main difficulty in the study of the well-posedness of the Zakharov system arise from so called "derivative loss". The both nonlinear terms of (3.2) have a half derivative loss when $k = l + 1/2$. To recover the derivative loss, Ginibre-Tsutsumi-Velo [10] applied the Forier restriction norm method, which is introduced by Bourgain [4]. However, it seems difficult to apply the method to the critical case. Bejenaru, Guo, Herr and Nakanishi [1] used the normal form reduction and transformed (3.2) into a system which does not have derivative loss. Our proof is more direct than their proof. We use the U^2, V^2 type spaces, which are introduced by Hadac-Herr-Koch [19] to study the small data global well-posedness and the scattering for the KP-II equation at the scale critical space. There are two merits for using these function spaces. One is that we can recover the derivative loss, by combining Lemma 3.4 and (1.7) in Proposition 1.13. The other is that we can employ the Strichartz estimate (see Proposition 3.5) by Colollary 1.15 and we gain some integrability. Actually, the L^4 Strichartz estimate was used in [19]. On the other hand, by the difference of the dilation scale of the Schödinger equation and the wave equation, the effect by oscillatory integral for the Schrödinger equation works more effective than that of the wave equation. Therefore, in our problem we have to use the endpoint Strichartz estimate for the Schrödinger equation, that is to say the case of $(p_1, q_1) = (2, 2d/(d-2))$ in Proposition 3.5. This causes the following problem: if we use the U^2 type function space and follow the argument by Hadac-Herr-Koch [19], then by duality argument (see Proposition 1.10) we need to estimate $L_t^2 L_x^{2d/(d-2)}$ norm by the V^2 type norm. However, we can not get such estimate by Corollary 1.15 because the V^2 type norm is slightly weaker than U^2 type norm. For this reason, we need the function space weaker than the U^2 type and stronger than the V^2 type. For that purpose, we use an intersection space of V^2

type space and $E := L_t^2 L_x^{2d/(d-2)}$. See the definition of $\|u\|_{X_S^k}$ in Definition 4, which is the main idea in the present paper.

Finally, we refer to the plan of the rest of the paper. We introduce function spaces, their properties and some lemmas in Section 2. In section 3, we derive the key bilinear estimate for the homogeneous case, Proposition 3.9. As a corollary, we also prove the bilinear estimate for the inhomogeneous case, Corollary 3.10. In section 4, we mention the detail of main theorem and its proof.

3.2. **Notations and preliminary lemmas.**

Proposition 3.2. *Let $1 < p < \infty, v \in V_-^1$ be absolutely continuous on compact intervals and u be a $U^{p'}$-atom. Then,*

$$B(u,v) = \int_{-\infty}^{\infty} \langle u(t), v'(t)\rangle_{L_x^2} dt - \lim_{t\to\infty} \langle u(t), v(t)\rangle_{L_x^2}. \tag{3.3}$$

Proof. By Corollary 2.6 in [19], we have $v \in V^p$. Therefore, the left-hand side of (3.3) make sense. From our assumption, it follows that $v' \in L^1(\mathbb{R}; L_x^2)$ with $\|v'\|_{L^1(\mathbb{R};L_x^2)} \le \|v\|_{V^1} < \infty$ and

$$u = \sum_{k=1}^{K} \mathbf{1}_{[t_{k-1},t_k)}\phi_{k-1}$$

with $\{t_k\}_{k=0}^K \in \mathcal{Z}_0$, $\{\phi_k\}_{k=0}^{K-1} \subset L_x^2$ and $\sum_{k=0}^{K-1} \|\phi_k\|_{L_x^2}^{p'} = 1$. By the definition of B, for any $\varepsilon > 0$, there exists $\widetilde{t} = \{\widetilde{t}_k\}_{k=0}^N \in \mathcal{Z}_0$ such that for any $\mathcal{Z}_0 \ni t' = \{t'_k\}_{k=0}^M \supset \widetilde{t}$ the estimate

$$|B_{t'}(u,v) - B(u,v)| < \varepsilon$$

holds where

$$B_{t'}(u,v) = \sum_{k=0}^{M} \langle u(t'_{k-1}), v(t'_k) - v(t'_{k-1})\rangle_{L_x^2}.$$

Put $t' = \{t_k\}_{k=0}^K \cup \{\widetilde{t}_k\}_{k=0}^N$. Since $u(s) = u(t'_{n-1})$ on $s \in [t'_{n-1}, t'_n)$, we have

$$\langle u(t'_{n-1}), v(t'_n) - v(t'_{n-1})\rangle_{L_x^2} = \int_{t'_{n-1}}^{t'_n} \langle u(s), v'(s)\rangle_{L_x^2} ds$$

when $t'_n \neq \infty$ and

$$\begin{aligned}\langle u(t'_{n-1}), v(t'_n) - v(t'_{n-1})\rangle_{L_x^2} &= \lim_{t\to\infty} \langle u(t'_{n-1}), v(t) - v(t'_{n-1})\rangle_{L_x^2} - \lim_{t\to\infty} \langle u(t'_{n-1}), v(t)\rangle_{L_x^2} \\ &= \int_{t'_{n-1}}^{t'_n} \langle u(s), v'(s)\rangle_{L_x^2} ds - \lim_{t\to\infty} \langle u(t), v(t)\rangle_{L_x^2}\end{aligned}$$

when $t_n' = \infty$. Thus, we conclude

$$\Big|\int_{-\infty}^{\infty} \langle u(s), v'(s)\rangle_{L^2_x} ds - \lim_{t\to\infty} \langle u(t), v(t)\rangle_{L^2_x} - B(u,v)\Big| < \varepsilon.$$

□

Combining Remark 1.3 and Proposition 3.2, we have the following corollary.

Corollary 3.3. *Let $A = S$ or $W_\pm$ and $v \in V^1_{-,A} \subset V^2_{-,A}$ be absolutely continuous on compact intervals. Then,*

$$\|v\|_{V^2_A} \le \sup_{u\in U^2_A,\, \|u\|_{U^2_A}=1} \Big|\int_{-\infty}^{\infty} \langle u(t), A(t)\big(A(-\cdot)v\big)'(t)\rangle_{L^2_x} dt - \lim_{t\to\infty} \langle u(t), v(t)\rangle_{L^2_x}\Big|.$$

Lemma 3.4. *Let $\tau_3 = \tau_1 - \tau_2,\ \xi_3 = \xi_1 - \xi_2$. If $|\xi_1| \gg \langle \xi_2\rangle$ or $\langle \xi_1\rangle \ll |\xi_2|$, then it holds that*

$$\max\Big\{\big|\tau_1 + |\xi_1|^2\big|, \big|\tau_2 + |\xi_2|^2\big|, \big|\tau_3 \pm |\xi_3|\big|\Big\} \gtrsim \max\{|\xi_1|^2, |\xi_2|^2\}. \tag{3.4}$$

Proof. We only prove the case of $|\xi_1| \gg \langle\xi_2\rangle$. By triangle inequality, $\tau_3 = \tau_1 - \tau_2$ and $\xi_3 = \xi_1 - \xi_2$, we have

$$\begin{aligned}(LHS\ of\ (3.4)) &\gtrsim \big|\tau_1 + |\xi_1|^2\big| + \big|\tau_2 + |\xi_2|^2\big| + \big|\tau_3 \pm |\xi_3|\big| \\ &\ge \big|\tau_1 + |\xi_1|^2 - (\tau_2 + |\xi_2|^2) - (\tau_3 \pm |\xi_3|)\big| \\ &= \big||\xi_1|^2 - |\xi_2|^2 \mp |\xi_1 - \xi_2|\big|. \end{aligned}\tag{3.5}$$

Since $|\xi_1| \gg \langle\xi_2\rangle$, we see that $|\xi_1 - \xi_2| \sim |\xi_1|$. Hence

$$(3.5) \gtrsim |\xi_1|^2.$$

□

We define the Duhamel terms as follows.

Definition 6.

$$I_{T,S}(n,v)(t) := -i/2 \int_0^t \mathbf{1}_{[0,T]}(t') S(t-t') n(t') v(t')\, dt', \tag{3.6}$$

$$I_{T,W_\pm}(u,v)(t) := \pm \int_0^t \mathbf{1}_{[0,T]}(t') W_\pm(t-t')\, \omega\big(u(t')\bar v(t')\big)\, dt' \tag{3.7}$$

where $\omega = (-\Delta)^{1/2}$.

The following statement is the Strichartz estimate for the Schrödinger equation.

Proposition 3.5. *Let $d \geq 3$ and (p_1, q_1), (p_2, q_2) satisfy $2 \leq q_i \leq 2d/(d-2)$ and $2/p_i = d(1/2 - 1/q_i)$ for $i = 1, 2$. p_2', q_2' satisfy $1/p_2 + 1/p_2' = 1$, $1/q_2 + 1/q_2' = 1$. Then, it holds that*

$$\|S(t)f\|_{L_t^{p_i} L_x^{q_i}} \lesssim \|f\|_{L_x^2}, \quad i = 1, 2, \tag{3.8}$$

$$\Big\| \int_{-\infty}^{t} S(t-t')g(t')dt' \Big\|_{L_t^{p_1} L_x^{q_1}} \lesssim \|g\|_{L_t^{p_2'} L_x^{q_2'}}. \tag{3.9}$$

Moreover, by duality, we have

$$\|I_{T,S}(n, v)(t)\|_{L_t^{p_1} L_x^{q_1}} \lesssim \sup_{\|u\|_{L_t^{p_2} L_x^{q_2}} = 1} \Big| \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} n v \bar{u} dx dt \Big|.$$

For the proofs of (3.8) and (3.9), see [47], [12] and [26].

Proposition 3.6. *Let $d \geq 4$, $k = (d-3)/2$ and $l = (d-4)/2$.*
(i) Let $T > 0$ and $u \in X_S^k([0, T])$, $u(0) = 0$. Then, for any $\varepsilon > 0$, there exists $0 \leq T' \leq T$ such that $\|u\|_{X_S^k([0,T'])} < \varepsilon$.
(ii) Let $T > 0$ and $u \in Y_S^k([0, T])$, $u(0) = 0$. Then, for any $\varepsilon > 0$, there exists $0 \leq T' \leq T$ such that $\|u\|_{Y_S^k([0,T'])} < \varepsilon$.
(iii) Let $T > 0$ and $n \in \dot{Y}_{W_\pm}^l([0, T])$, (resp. $Y_{W_\pm}^l([0, T])$), $n(0) = 0$. Then, for any $\varepsilon > 0$, there exists $0 \leq T' \leq T$ such that $\|n\|_{\dot{Y}_{W_\pm}^l([0,T'])}$ (resp. $\|n\|_{Y_{W_\pm}^l([0,T'])}$) $< \varepsilon$.

Proof. For the proofs of (ii) and (iii), see Proposition 2.24 in [19]. For the proof of (i), we only see that $\|u\|_{E^k([0,T'])} < \varepsilon$, which follows from $\|u\|_{E^k([0,T])} < \infty$. □

Lemma 3.7. *If f, g are measurable functions, then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} f(t, x) \overline{Q_{\geq M}^S g(t, x)} dx dt = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Big(Q_{\geq M}^S f(t, x) \Big) \overline{g(t, x)} dx dt. \tag{3.10}$$

Proof. From the definition of $Q_{\geq M}^S$, we obtain

$$\begin{aligned} \mathcal{F}_x[Q_{\geq M}^S g](t, \xi) &= \sum_{N \geq M} \mathcal{F}_x[S(\cdot) Q_N S(-\cdot) g](t, \xi) \\ &= \sum_{2^n \geq M} e^{-it|\xi|^2} \mathcal{F}_x \big[\mathcal{F}_\tau^{-1} [\phi_n(\tau) \mathcal{F}_t[S(-\cdot)g](\tau)] \big](t, \xi) \\ &= \sum_{2^n \geq M} e^{-it|\xi|^2} \mathcal{F}_\tau^{-1} \big[\phi_n(\tau) \mathcal{F}_t \big[e^{i \cdot |\xi|^2} \mathcal{F}_x[g] \big](\tau) \big](t, \xi) \\ &= \sum_{2^n \geq M} e^{-it|\xi|^2} (\mathcal{F}_\tau^{-1}[\phi_n] *_{(t)} e^{i \cdot |\xi|^2} \mathcal{F}_x[g])(t, \xi) \end{aligned} \tag{3.11}$$

Applying the Plancherel theorem and (3.11), we obtain that the left-hand side of (3.10) is equal to

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathcal{F}_x[f](t,\xi)\overline{\mathcal{F}_x[Q^S_{\geq M}g](t,\xi)}d\xi dt\\
&=\sum_{2^n\geq M}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{i(t-t')|\xi|^2}\mathcal{F}_x[f](t,\xi)\overline{\mathcal{F}_\tau^{-1}[\phi_n](t-t')\mathcal{F}_x[g](t',\xi)}dt'd\xi dt\\
&=\sum_{2^n\geq M}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{i(t-t')|\xi|^2}\mathcal{F}_x[f](t,\xi)\mathcal{F}_\tau^{-1}[\phi_n](t'-t)\overline{\mathcal{F}_x[g](t',\xi)}dt'd\xi dt.
\end{aligned}
$$

In the last line, we used $\overline{\mathcal{F}_\tau^{-1}[\phi_n](t-t')}=\mathcal{F}_\tau^{-1}[\phi_n](t'-t)$, which holds because ϕ_n is real valued. Applying the Plancherel theorem and (3.11), we obtain that the right-hand side of (3.10) is equal to

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathcal{F}_x[Q^S_{\geq M}f](t,\xi)\overline{\mathcal{F}_x[g](t,\xi)}d\xi dt\\
&=\sum_{2^n\geq M}\int_{\mathbb{R}}\int_{\mathbb{R}^d} e^{-it|\xi|^2}(\mathcal{F}_\tau^{-1}[\phi_n]*_{(t)} e^{i\cdot|\xi|^2}\mathcal{F}_x[f])(t,\xi)\overline{\mathcal{F}_x[g](t,\xi)}d\xi dt\\
&=\sum_{2^n\geq M}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}} e^{i(t'-t)|\xi|^2}\mathcal{F}_x[f](t',\xi)\mathcal{F}_\tau^{-1}[\phi_n](t-t')\overline{\mathcal{F}_x[g](t,\xi)}dt'd\xi dt.
\end{aligned}
$$

Thus, we conclude (3.10). □

Lemma 3.8. *Let $d\geq 4$, $k=(d-3)/2$, $l=(d-4)/2$, $f_{N_1}:=P_{N_1}f$, $g_{N_2}:=P_{N_2}g$, $h_{N_3}:=P_{N_3}h$, $Q_1^{W_\pm}\in\{Q^{W_\pm}_{<M},Q^{W_\pm}_{\geq M}\}$, Q_2^S, $Q_3^S\in\{Q^S_{<M},Q^S_{\geq M}\}$. Then, the following estimates hold:*

(i)
$$\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d} f_{N_1}g_{N_2}\overline{h_{N_3}}dxdt\Big|\lesssim N_1^l\|f_{N_1}\|_{V^2_{W_\pm}}\|g_{N_2}\|_E\|h_{N_3}\|_E,$$

(ii)
$$\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Big(\sum_{N_1\ll N_2} f_{N_1}\Big)g_{N_2}\overline{h_{N_3}}dxdt\Big|\lesssim \|f\|_{\dot{Y}^l_{W_\pm}}\|g_{N_2}\|_E\|h_{N_3}\|_E,$$

(iii) If $N_2\ll N_1\sim N_3, N_1>2^2$, $M=\varepsilon N_1^2$ and $\varepsilon>0$ is small, then
$$\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q^{W_\pm}_{\geq M}f_{N_1})\Big(\sum_{N_2\ll N_1} g_{N_2}\Big)\overline{h_{N_3}}dxdt\Big|\lesssim N_1^{-1/2}\|f_{N_1}\|_{V^2_{W_\pm}}\|g\|_{Y^k_S}\|h_{N_3}\|_E,$$

(iv) If $N_2\ll N_1\sim N_3, N_1>2^2$, $M=\varepsilon N_1^2$ and $\varepsilon>0$ is small, then
$$\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q^{W_\pm}_{<M}f_{N_1})\Big(\sum_{N_2\ll N_1} g_{N_2}\Big)\overline{(Q^S_{\geq M}h_{N_3})}dxdt\Big|\lesssim N_1^{-1/2}\|f_{N_1}\|_{V^2_{W_\pm}}\|g\|_{E^k}\|h_{N_3}\|_{V^2_S},$$

(v) If $N_2\ll N_1\sim N_3, N_1>2^2$, $M=\varepsilon N_1^2$ and $\varepsilon>0$ is small, then
$$\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q^{W_\pm}_{<M}f_{N_1})\Big(\sum_{N_2\ll N_1} Q^S_{\geq M}g_{N_2}\Big)\overline{h_{N_3}}dxdt\Big|\lesssim N_1^{-1/2}\|f_{N_1}\|_{V^2_{W_\pm}}\|g\|_{Y^k_S}\|h_{N_3}\|_E,$$

(vi) If $N_2 \ll N_1 \sim N_3, N_1 > 2^2,\ M = \varepsilon N_1^2$ and $\varepsilon > 0$ is small, then

$$\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}(Q^{W_\pm}_{<M} f_{N_1})\Big(\sum_{N_2 \ll N_1} Q^S_{\geq M} g_{N_2}\Big)\overline{(Q^S_{\geq M} h_{N_3})}dxdt\Big| \lesssim N_1^{-1/2}\|f_{N_1}\|_{V^2_{W_\pm}}\|g\|_{Y^k_S}\|h_{N_3}\|_{V^2_S}.$$

Here, the implicit constants may depend on ε.

Proof. First, we show (i). By the Hölder inequality, we have

$$(\text{LHS of } (i)) \lesssim \|f_{N_1}\|_{L^\infty_t L^{d/2}_x}\|g_{N_2}\|_{L^2_t L^{2d/(d-2)}_x}\|h_{N_3}\|_{L^2_t L^{2d/(d-2)}_x}. \tag{3.12}$$

The Sobolev inequality and Remark 1.1 gives

$$\|f_{N_1}\|_{L^\infty_t L^{d/2}_x} \lesssim \big\||\nabla_x|^{(d-4)/2} f_{N_1}\big\|_{L^\infty_t L^2_x} \lesssim N_1^{(d-4)/2}\|f_{N_1}\|_{V^2_{W_\pm}}. \tag{3.13}$$

Hence, from (3.12) and (3.13), we obtain (i). By L^2_x orthogonality,

$$\begin{aligned}\Big\||\nabla_x|^{(d-4)/2}\sum_{N_1 \ll N_2} f_{N_1}\Big\|_{L^\infty_t L^2_x} &\lesssim \Big\|\Big(\sum_{N_1 \ll N_2} N_1^{d-4}\|f_{N_1}\|^2_{L^2_x}\Big)^{1/2}\Big\|_{L^\infty_t} \\ &\lesssim \Big(\sum_{N_1 \ll N_2} N_1^{d-4}\|f_{N_1}\|^2_{L^\infty_t L^2_x}\Big)^{1/2}.\end{aligned}$$

Thus, we obtain (ii) in the same manner as (i). Next, we show (iii). By the Hölder inequality, the Sobolev inequality and Proposition 1.13, we have

$$\begin{aligned}(\text{LHS of } (iii)) &\lesssim \|Q^{W_\pm}_{\geq M} f_{N_1}\|_{L^2_{t,x}}\Big\|\sum_{N_2 \ll N_1} g_{N_2}\Big\|_{L^\infty_t L^d_x}\|h_{N_3}\|_{L^2_t L^{2d/(d-2)}_x} \\ &\lesssim N_1^{-1}\|f_{N_1}\|_{V^2_{W_\pm}}\Big\||\nabla_x|^{(d-2)/2}\sum_{N_2 \ll N_1} g_{N_2}\Big\|_{L^\infty_t L^2_x}\|h_{N_3}\|_E.\end{aligned} \tag{3.14}$$

By Remark 1.1, we have

$$\Big\||\nabla_x|^{(d-2)/2}\sum_{N_2 < 1} g_{N_2}\Big\|_{L^\infty_t L^2_x} \lesssim \|P_{<1} g\|_{L^\infty_t L^2_x} \lesssim \|P_{<1} g\|_{V^2_S} \lesssim \|g\|_{Y^k_S}. \tag{3.15}$$

By L^2_x orthogonality and Remark 1.1, we have

$$\Big\||\nabla_x|^{(d-2)/2}\sum_{1 \leq N_2 \ll N_1} g_{N_2}\Big\|_{L^\infty_t L^2_x} \lesssim \Big(\sum_{1 \leq N_2 \ll N_1} N_2^{d-2}\|g_{N_2}\|^2_{V^2_S}\Big)^{1/2} \lesssim N_1^{1/2}\|g\|_{Y^k_S}. \tag{3.16}$$

Collecting (3.14)–(3.16), we obtain (iii). Next, we show (iv). Applying the Hölder inequality, we have

$$(\text{LHS of } (iv)) \lesssim \|Q^{W_\pm}_{<M} f_{N_1}\|_{L^\infty_t L^2_x}\Big\|\sum_{N_2 \ll N_1} g_{N_2}\Big\|_{L^2_t L^\infty_x}\|Q^S_{\geq M} h_{N_3}\|_{L^2_{t,x}}. \tag{3.17}$$

By Remark 1.1 and Proposition 1.13, we have

$$\|Q^{W_\pm}_{<M} f_{N_1}\|_{L^\infty_t L^2_x} \lesssim \|Q^{W_\pm}_{<M} f_{N_1}\|_{V^2_{W_\pm}} \lesssim \|f_{N_1}\|_{V^2_{W_\pm}}, \tag{3.18}$$

$$\|Q^S_{\geq M} h_{N_3}\|_{L^2_{t,x}} \lesssim N_3^{-1}\|h_{N_3}\|_{V^2_S}. \tag{3.19}$$

By the Bernstein inequality (see e.g. (A.6) on page 333 in [46]), we have

$$\Big\| \sum_{N_2 \ll N_1} g_{N_2} \Big\|_{L_t^2 L_x^\infty} \lesssim \Big\| |\nabla_x|^{(d-2)/2} \sum_{N_2 \ll N_1} g_{N_2} \Big\|_{L_t^2 L_x^{2d/(d-2)}}. \tag{3.20}$$

By Mihlin-Hörmander's multiplier theorem, we have

$$\Big\| |\nabla_x|^{(d-2)/2} \sum_{N_2 < 1} g_{N_2} \Big\|_E \lesssim \|P_{<1} g\|_E. \tag{3.21}$$

By Mihlin-Hörmander's multiplier theorem and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \Big\| |\nabla_x|^{(d-2)/2} \sum_{1 \le N_2 \ll N_1} g_{N_2} \Big\|_E &\lesssim \sum_{1 \le N_2 \ll N_1} N_2^{(d-2)/2} \|g_{N_2}\|_E \\ &\lesssim \Big(\sum_{1 \le N_2 \ll N_1} N_2 \Big)^{1/2} \Big(\sum_{1 \le N_2 \ll N_1} N_2^{d-3} \|g_{N_2}\|_E^2 \Big)^{1/2} \\ &\lesssim N_1^{1/2} \|g\|_{E^k}. \end{aligned} \tag{3.22}$$

Collecting (3.17)–(3.22) and $N_1 \sim N_3$, we obtain (iv). Next, we show (v). Applying the Hölder inequality, the Sobolev inequality and (3.18), we have

$$\begin{aligned} (\text{LHS of } (v)) &\lesssim \|Q_{<M}^{W_\pm} f_{N_1}\|_{L_t^\infty L_x^2} \Big\| \sum_{N_2 \ll N_1} Q_{\ge M}^S g_{N_2} \Big\|_{L_t^2 L_x^d} \|h_{N_3}\|_{L_t^2 L_x^{2d/(d-2)}} \\ &\lesssim \|f_{N_1}\|_{V_{W_\pm}^2} \Big\| |\nabla_x|^{(d-2)/2} \sum_{N_2 \ll N_1} Q_{\ge M}^S g_{N_2} \Big\|_{L_{t,x}^2} \|h_{N_3}\|_E. \end{aligned} \tag{3.23}$$

By Proposition 1.13, we have

$$\Big\| |\nabla_x|^{(d-2)/2} \sum_{N_2 < 1} Q_{\ge M}^S g_{N_2} \Big\|_{L_{t,x}^2} \lesssim \|Q_{\ge M}^S P_{<1} g\|_{L_{t,x}^2} \lesssim N_1^{-1} \|P_{<1} g\|_{V_S^2}. \tag{3.24}$$

By L_x^2 orthogonality and Proposition 1.13, we have

$$\begin{aligned} \Big\| |\nabla_x|^{(d-2)/2} \sum_{1 \le N_2 \ll N_1} Q_{\ge M}^S g_{N_2} \Big\|_{L_{t,x}^2} &\lesssim \Big(\sum_{1 \le N_2 \ll N_1} \big\| |\nabla_x|^{(d-2)/2} Q_{\ge M}^S g_{N_2} \big\|_{L_{t,x}^2}^2 \Big)^{1/2} \\ &\lesssim \Big(\sum_{1 \le N_2 \ll N_1} N_2^{d-2} N_1^{-2} \|g_{N_2}\|_{V_S^2}^2 \Big)^{1/2} \\ &\lesssim N_1^{-1/2} \|g\|_{Y_S^k}. \end{aligned} \tag{3.25}$$

From (3.23)–(3.25), we obtain (v). Finally, we show (vi). By the Hölder inequality, the Bernstein inequality, (3.18) and (3.19), we have

$$\begin{aligned} (\text{LHS of } (vi)) &\lesssim \|Q_{<M}^{W_\pm} f_{N_1}\|_{L_t^\infty L_x^2} \Big\| \sum_{N_2 \ll N_1} Q_{\ge M}^S g_{N_2} \Big\|_{L_t^2 L_x^\infty} \|Q_{\ge M}^S h_{N_3}\|_{L_{t,x}^2} \\ &\lesssim \|f_{N_1}\|_{V_{W_\pm}^2} \Big\| |\nabla_x|^{d/2} \sum_{N_2 \ll N_1} Q_{\ge M}^S g_{N_2} \Big\|_{L_{t,x}^2} N_3^{-1} \|h_{N_3}\|_{V_S^2}. \end{aligned} \tag{3.26}$$

By Proposition 1.13, we have

$$\Big\||\nabla_x|^{d/2}\sum_{N_2<1}Q^S_{\geq M}g_{N_2}\Big\|_{L^2_{t,x}} \lesssim \Big\|Q^S_{\geq M}P_{<1}g\Big\|_{L^2_{t,x}} \lesssim N_1^{-1}\|P_{<1}g\|_{V^2_S}. \tag{3.27}$$

By L^2_x orthogonality and Proposition 1.13, we obtain

$$\begin{aligned}\Big\||\nabla_x|^{d/2}\sum_{1\leq N_2\ll N_1}Q^S_{\geq M}g_{N_2}\Big\|_{L^2_{t,x}} &\lesssim \Big(\sum_{1\leq N_2\ll N_1}\big\||\nabla_x|^{d/2}Q^S_{\geq M}g_{N_2}\big\|^2_{L^2_{t,x}}\Big)^{1/2}\\ &\lesssim \Big(\sum_{1\leq N_2\ll N_1}N_2^dN_1^{-2}\|g_{N_2}\|^2_{V^2_S}\Big)^{1/2}\\ &\lesssim N_1^{1/2}\|g\|_{Y^k_S}.\end{aligned} \tag{3.28}$$

From (3.26)–(3.28) and $N_1\sim N_3$, we obtain (vi). □

3.3. **Bilinear estimates.** In this section, we give bilinear estimates for the Duhamel terms (3.6) and (3.7).

Proposition 3.9. *Let $d\geq 4$, $k=(d-3)/2$ and $l=(d-4)/2$. Then for all $0<T<\infty$, it holds that*

$$\|I_{T,S}(n,v)\|_{X^k_S} \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}\|v\|_{X^k_S}, \tag{3.29}$$

$$\|I_{T,W_\pm}(u,v)\|_{\dot{Z}^l_{W_\pm}} \lesssim \|u\|_{X^k_S}\|v\|_{X^k_S}. \tag{3.30}$$

Proof. Let $u_{N_1}=P_{N_1}u,\ v_{N_2}=P_{N_2}v,\ n_{N_3}=P_{N_3}n$. First, we prove (3.29). Since $\|\cdot\|_{X^k_S}=\|\cdot\|_{Y^k_S}+\|\cdot\|_{E^k}$, we need to show

$$\|I_{T,S}(n,v)\|_{E^k} \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}\|v\|_{X^k_S}, \tag{3.31}$$

$$\|I_{T,S}(n,v)\|_{Y^k_S} \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}\|v\|_{X^k_S}. \tag{3.32}$$

By the definition of E^k norm, we have

$$(\text{LHS of (3.31)})^2 \lesssim \|P_{<1}I_{T,S}(n,v)\|^2_E+\sum_{N_1\geq 1}N_1^{d-3}\|P_{N_1}I_{T,S}(n,v)\|^2_E. \tag{3.33}$$

Put

$$J_{1,E}:=\sum_{N_1\geq 1}N_1^{d-3}\Big\|\int_0^t \mathbf{1}_{[0,T]}(t')S(t-t')\sum_{N_2\sim N_1}\sum_{N_3\ll N_2}P_{N_1}\big(n_{N_3}(t')v_{N_2}(t')\big)dt'\Big\|^2_E,$$

$$J_{2,E}:=\sum_{N_1\geq 1}N_1^{d-3}\Big\|\int_0^t \mathbf{1}_{[0,T]}(t')S(t-t')\sum_{N_2\gtrsim N_1}\sum_{N_3\sim N_2}P_{N_1}\big(n_{N_3}(t')v_{N_2}(t')\big)dt'\Big\|^2_E,$$

$$J_{3,E}:=\sum_{N_1\geq 1}N_1^{d-3}\Big\|\int_0^t \mathbf{1}_{[0,T]}(t')S(t-t')\sum_{N_2\ll N_1}\sum_{N_3\sim N_1}P_{N_1}\big(n_{N_3}(t')v_{N_2}(t')\big)dt'\Big\|^2_E.$$

We will prove $J_{i,E} \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}^2 \|v\|_{X^k_S}^2$ for $i = 1,2,3$ below. By Proposition 3.5 and Lemma 3.8 (ii), we have

$$\begin{aligned} J_{1,E} &\lesssim \sum_{N_1 \geq 1} N_1^{d-3} \sup_{\|u\|_E=1} \Big| \sum_{N_2 \sim N_1} \sum_{N_3 \ll N_2} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} n_{N_3} v_{N_2} \overline{u_{N_1}} dx dt \Big|^2 \\ &\lesssim \|n\|_{\dot{Y}^l_{W_\pm}}^2 \sum_{N_1 \geq 1} \sum_{N_2 \sim N_1} N_1^{d-3} \|v_{N_2}\|_E^2 \sup_{\|u\|_E=1} \|u_{N_1}\|_E^2. \end{aligned}$$

Since $\sup_{\|u\|_E=1} \|u_{N_1}\|_E \lesssim 1$, we obtain

$$J_{1,E} \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}^2 \sum_{N_2 \gtrsim 1} N_2^{d-3} \|v_{N_2}\|_E^2 \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}^2 \|v\|_{X^k_S}^2.$$

By the triangle inequality, Proposition 3.5 and Lemma 3.8 (i), we have

$$\begin{aligned} J_{2,E} &\lesssim \sum_{N_1 \geq 1} N_1^{d-3} \Big(\sum_{N_2 \gtrsim N_1} \sum_{N_3 \sim N_2} \Big\| \int_0^t \mathbf{1}_{[0,T]}(t') S(t-t') P_{N_1}\big(n_{N_3}(t') v_{N_2}(t')\big) dt' \Big\|_E \Big)^2 \\ &\lesssim \sum_{N_1 \geq 1} N_1^{d-3} \Big(\sum_{N_2 \gtrsim N_1} \sum_{N_3 \sim N_2} \sup_{\|u\|_E=1} \Big| \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} n_{N_3} v_{N_2} \overline{u_{N_1}} dx dt \Big| \Big)^2 \\ &\lesssim \sum_{N_1 \geq 1} \Big(\sum_{N_2 \gtrsim N_1} \sum_{N_3 \sim N_2} N_1^{(d-3)/2} N_3^{(d-4)/2} \|v_{N_2}\|_E \|n_{N_3}\|_{V^2_{W_\pm}} \Big)^2. \end{aligned}$$

Since $\|\cdot\|_{\ell^2 \ell^1} \leq \|\cdot\|_{\ell^1 \ell^2}$, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} J_{2,E}^{1/2} &\lesssim \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} \Big(\sum_{N_1 \lesssim N_2} N_1^{d-3} N_3^{d-4} \|v_{N_2}\|_E^2 \|n_{N_3}\|_{V^2_{W_\pm}}^2 \Big)^{1/2} \\ &\lesssim \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} N_2^{(d-3)/2} N_3^{(d-4)/2} \|v_{N_2}\|_E \|n_{N_3}\|_{V^2_{W_\pm}} \\ &\lesssim \|n\|_{\dot{Y}^l_{W_\pm}} \|v\|_{X^k_S}. \end{aligned}$$

Next, we consider the estimate of $J_{3,E}$. We take $M = \varepsilon N_1^2$ for sufficiently small $\varepsilon > 0$. Then, from Lemma 3.4, we have

$$\begin{aligned} &Q^S_{<M}\big((Q^{W_\pm}_{<M} n_{N_3})(Q^S_{<M} v_{N_2})\big) \\ &= Q^S_{<M}\Big[\mathcal{F}^{-1}\Big(\int_{\tau_1=\tau_2+\tau_3,\, \xi_1=\xi_2+\xi_3} (\widehat{Q^{W_\pm}_{<M} n_{N_3}})(\tau_3,\xi_3)(\widehat{Q^S_{<M} v_{N_2}})(\tau_2,\xi_2)\Big)\Big] = 0 \end{aligned}$$

when $N_1 \gg \langle N_2 \rangle$. Therefore,

$$n_{N_3} v_{N_2} = \sum_{i=1}^4 F_i,$$

where

$$F_1 := (Q^{W_\pm}_{\geq M} n_{N_3}) v_{N_2}, \qquad F_2 := Q^S_{\geq M}\big((Q^{W_\pm}_{<M} n_{N_3}) v_{N_2}\big),$$
$$F_3 := (Q^{W_\pm}_{<M} n_{N_3})(Q^S_{\geq M} v_{N_2}), \qquad F_4 := -Q^S_{\geq M}\big((Q^{W_\pm}_{<M} n_{N_3})(Q^S_{\geq M} v_{N_2})\big).$$

For the estimate of F_1, we apply Proposition 3.5 and Lemma 3.8 (iii) to have

$$\begin{aligned}
&\sum_{N_1\geq 1} N_1^{d-3}\Big\|\int_0^t \mathbf{1}_{[0,T]}(t')S(t-t')\sum_{N_2\ll N_1}\sum_{N_3\sim N_1} P_{N_1}F_1 dt'\Big\|_E^2\\
&\lesssim \sum_{N_1\geq 1} N_1^{d-3}\sup_{\|u\|_E=1}\Big|\sum_{N_2\ll N_1}\sum_{N_3\sim N_1}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}(Q^{W_\pm}_{\geq M} n_{N_3}) v_{N_2}\overline{u_{N_1}}dxdt\Big|^2\\
&\lesssim \sum_{N_3\ \gtrsim\ 1} N_3^{d-3}\big(N_3^{-1/2}\|n_{N_3}\|_{V^2_{W_\pm}}\|v\|_{Y^k_S}\big)^2\\
&\lesssim \|n\|^2_{\dot{Y}^l_{W_\pm}}\|v\|^2_{X^k_S}.
\end{aligned}$$

For the estimate of F_2, we apply Corollary 1.15, Corollary 1.11, Lemma 1.16, Lemma 3.8 (iv) and

$$\|\mathbf{1}_{[0,T]}u_{N_1}\|_{V^2_S} \lesssim \|u_{N_1}\|_{V^2_S} \lesssim \|u\|_{V^2_S} \tag{3.34}$$

to have

$$\begin{aligned}
&\sum_{N_1\geq 1} N_1^{d-3}\Big\|\int_0^t \mathbf{1}_{[0,T]}(t')S(t-t')\sum_{N_2\ll N_1}\sum_{N_3\sim N_1} P_{N_1}F_2 dt'\Big\|_E^2\\
&\lesssim \sum_{N_1\geq 1} N_1^{d-3}\Big\|\int_0^t \mathbf{1}_{[0,T]}(t')S(t-t')\sum_{N_2\ll N_1}\sum_{N_3\sim N_1} P_{N_1}F_2 dt'\Big\|_{U^2_S}^2\\
&\lesssim \sum_{N_1\geq 1} N_1^{d-3}\sup_{\|u\|_{V^2_S}=1}\Big|\sum_{N_2\ll N_1}\sum_{N_3\sim N_1}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}\Big(Q^S_{\geq M}\big((Q^{W_\pm}_{<M} n_{N_3}) v_{N_2}\big)\Big)\overline{u_{N_1}}dxdt\Big|^2\\
&\lesssim \sum_{N_3\ \gtrsim\ 1} N_3^{d-3}\big(N_3^{-1/2}\|n_{N_3}\|_{V^2_{W_\pm}}\|v\|_{E^k}\big)^2\\
&\lesssim \|n\|^2_{\dot{Y}^l_{W_\pm}}\|v\|^2_{X^k_S}.
\end{aligned}$$

For the estimate of F_3, we apply Proposition 3.5 and Lemma 3.8 (v) to have

$$\begin{aligned}
&\sum_{N_1\geq 1} N_1^{d-3}\Big\|\int_0^t \mathbf{1}_{[0,T]}(t')S(t-t')\sum_{N_2\ll N_1}\sum_{N_3\sim N_1} P_{N_1}F_3 dt'\Big\|_E^2\\
&\lesssim \sum_{N_1\geq 1} N_1^{d-3}\sup_{\|u\|_E=1}\Big|\sum_{N_2\ll N_1}\sum_{N_3\sim N_1}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}(Q^{W_\pm}_{<M}n_{N_3})(Q^S_{\geq M}v_{N_2})\overline{u_{N_1}}dxdt\Big|^2\\
&\lesssim \sum_{N_3\ \gtrsim\ 1} N_3^{d-3}\big(N_3^{-1/2}\|n_{N_3}\|_{V^2_{W_\pm}}\|v\|_{Y^k_S}\big)^2\\
&\lesssim \|n\|^2_{\dot{Y}^l_{W_\pm}}\|v\|^2_{X^k_S}.
\end{aligned}$$

For the estimate of F_4, we apply Corollary 1.15, Corollary 1.11, Lemma 1.16, Lemma 3.8 (vi) and (3.34) to have

$$\begin{aligned}
&\sum_{N_1\geq 1} N_1^{d-3}\Big\|\int_0^t \mathbf{1}_{[0,T]}(t')S(t-t')\sum_{N_2\ll N_1}\sum_{N_3\sim N_1} P_{N_1}F_4 dt'\Big\|_E^2\\
&\lesssim \sum_{N_1\geq 1} N_1^{d-3}\Big\|\int_0^t \mathbf{1}_{[0,T]}(t')S(t-t')\sum_{N_2\ll N_1}\sum_{N_3\sim N_1} P_{N_1}F_4 dt'\Big\|_{U^2_S}^2\\
&\lesssim \sum_{N_1\geq 1} N_1^{d-3}\sup_{\|u\|_{V^2_S}=1}\Big|\sum_{N_2\ll N_1}\sum_{N_3\sim N_1}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}\Big(Q^S_{\geq M}\big((Q^{W_\pm}_{<M}n_{N_3})(Q^S_{\geq M}v_{N_2})\big)\Big)\overline{u_{N_1}}dxdt\Big|^2\\
&\lesssim \sum_{N_3\ \gtrsim\ 1} N_3^{d-3}\big(N_3^{-1/2}\|n_{N_3}\|_{V^2_{W_\pm}}\|v\|_{Y^k_S}\big)^2\\
&\lesssim \|n\|^2_{\dot{Y}^l_{W_\pm}}\|v\|^2_{X^k_S}.
\end{aligned}$$

Collecting the estimates of F_1, F_2, F_3 and F_4, we obtain $J_{3,E}\lesssim \|n\|^2_{\dot{Y}^l_{W_\pm}}\|v\|^2_{X^k_S}$. Thus,

$$\sum_{N_1\geq 1} N_1^{d-3}\|P_{N_1}I_{T,S}(n,v)\|_E^2 \lesssim \|n\|^2_{\dot{Y}^l_{W_\pm}}\|v\|^2_{X^k_S}. \tag{3.35}$$

Note that we also have

$$\sum_{N_1\geq 1} N_1^{d-3}\|P_{N_1}I_{T,S}(n,v)\|^2_{L^\infty_t L^2_x} \lesssim \|n\|^2_{\dot{Y}^l_{W_\pm}}\|v\|^2_{X^k_S} \tag{3.36}$$

in the same manner as the proof of (3.35) since $(p_1,q_1)=(\infty,2)$ also satisfies the assumption of Proposition 3.5. Next, we show

$$\|P_{<1}I_{T,S}(n,v)\|_E \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}\|v\|_{X^k_S}. \tag{3.37}$$

In the same manner as the proof of Lemma 3.8 (ii), we have

$$\|n\|_{L^\infty_t L^d_x} \lesssim \Big\||\nabla_x|^{(d-4)/2}\sum_N P_N n\Big\|_{L^\infty_t L^2_x} \lesssim \Big(\sum_N N^{2l}\|P_N n\|^2_{V^2_{W_\pm}}\Big)^{1/2} = \|n\|_{\dot{Y}^l_{W_\pm}}.$$

Thus, by Proposition 3.5 and the Hölder inequality, the left-hand side of (3.37) is bounded by

$$\begin{aligned} &\sup_{\|u\|_E=1}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}nv\overline{P_{<1}u}dxdt\Big| \\ &\lesssim \|n\|_{L_t^\infty L_x^d}\|v\|_E\sup_{\|u\|_E=1}\|P_{<1}u\|_E \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}\|v\|_{E^k}. \end{aligned} \tag{3.38}$$

Thus, we obtain (3.37). From (3.33), (3.35) and (3.37), we conclude (3.31).

Next, we prove (3.32). By the definition of $\|\cdot\|_{Y_S^k}$, we only need to show

$$\sum_{N_1\geq 1}N_1^{d-3}\|P_{N_1}I_{T,S}(n,v)\|_{V_S^2}^2 \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}^2\|v\|_{X_S^k}^2, \tag{3.39}$$

$$\|P_{<1}I_{T,S}(n,v)\|_{V_S^2}^2 \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}^2\|v\|_{X_S^k}^2. \tag{3.40}$$

By Corollary 3.3 and Remark 1.1, the left-hand side of (3.39) is bounded by

$$\begin{aligned} &\sum_{N_1\geq 1}N_1^{d-3}\sup_{\|u\|_{U_S^2}=1}\Big|\int_{-\infty}^{\infty}\langle u(t),S(t)\big(S(-\cdot)P_{N_1}I_{T,S}(n,v)\big)'(t)\rangle_{L_x^2}dt \\ &\qquad\qquad - \lim_{t\to\infty}\langle u(t),P_{N_1}I_{T,S}(n,v)\rangle_{L_x^2}\Big|^2, \\ &\lesssim \sum_{N_1\geq 1}N_1^{d-3}\sup_{\|u\|_{U_S^2}=1}\Big(\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}nv\,\overline{u_{N_1}}dxdt\Big|^2+\|u\|_{L_t^\infty L_x^2}^2\|P_{N_1}I_{T,S}(n,v)\|_{L_t^\infty L_x^2}^2\Big) \\ &\lesssim \sum_{N_1\geq 1}N_1^{d-3}\sup_{\|u\|_{U_S^2}=1}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}nv\,\overline{u_{N_1}}dxdt\Big|^2+\sum_{N_1\geq 1}N_1^{d-3}\|P_{N_1}I_{T,S}(n,v)\|_{L_t^\infty L_x^2}^2 \\ &\lesssim \sum_{i=1}^{3}J_{i,Y}+\sum_{N_1\geq 1}N_1^{d-3}\|P_{N_1}I_{T,S}(n,v)\|_{L_t^\infty L_x^2}^2 \end{aligned}$$

where

$$\begin{aligned} J_{1,Y} &:= \sum_{N_1\geq 1}N_1^{d-3}\sup_{\|u\|_{U_S^2}=1}\Big|\sum_{N_2\sim N_1}\sum_{N_3\ll N_2}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}n_{N_3}v_{N_2}\overline{u_{N_1}}dxdt\Big|^2, \\ J_{2,Y} &:= \sum_{N_1\geq 1}N_1^{d-3}\sup_{\|u\|_{U_S^2}=1}\Big|\sum_{N_2\gtrsim N_1}\sum_{N_3\sim N_2}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}n_{N_3}v_{N_2}\overline{u_{N_1}}dxdt\Big|^2, \\ J_{3,Y} &:= \sum_{N_1\geq 1}N_1^{d-3}\sup_{\|u\|_{U_S^2}=1}\Big|\sum_{N_2\ll N_1}\sum_{N_3\sim N_1}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}n_{N_3}v_{N_2}\overline{u_{N_1}}dxdt\Big|^2. \end{aligned}$$

By Corollary 1.15 and Remark 1.1, it follows that

$$\|u\|_E \lesssim \|u\|_{U_S^2}, \quad \|u\|_{V_S^2} \lesssim \|u\|_{U_S^2}. \tag{3.41}$$

We obtain $J_{i,Y} \lesssim \|n\|_{\dot{Y}^l_{W_\pm}}^2\|v\|_{X_S^k}^2$ in the same manner as the estimates for $J_{i,E}$ with $i=1,2,3$ if we use (3.41). Collecting (3.36) and the estimates above, we conclude

(3.39). Next, we show (3.40). By Corollary 3.3 and Remark 1.1, we have

$$\begin{aligned}
&\|P_{<1}I_{T,S}(n,v)\|_{V_S^2}\\
&= \sup_{\|u\|_{U_S^2}=1}\Big|\int_{-\infty}^{\infty}\langle u(t), S(t)\big(S(-\cdot)P_{<1}I_{T,S}(n,v)\big)'(t)\rangle_{L_x^2}dt\\
&\qquad\qquad - \lim_{t\to\infty}\langle u(t), \big(P_{<1}I_{T,S}(n,v)\big)(t)\rangle_{L_x^2}\Big|\\
&\lesssim \sup_{\|u\|_{U_S^2}=1}\Big(\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}nv\overline{P_{<1}u}dxdt\Big| + \|u\|_{L_t^\infty L_x^2}\|P_{<1}I_{T,S}(n,v)\|_{L_t^\infty L_x^2}\Big)\\
&\lesssim \sup_{\|u\|_E=1}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T)}nv\overline{P_{<1}u}dxdt\Big| + \|P_{<1}I_{T,S}(n,v)\|_{L_t^\infty L_x^2}. \qquad (3.42)
\end{aligned}$$

By Proposition 3.5, we have

$$\|P_{<1}I_{T,S}(n,v)\|_{L_t^\infty L_x^2} \lesssim \sup_{\|u\|_E=1}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}nv\overline{P_{<1}u}dxdt\Big|. \qquad (3.43)$$

Collecting (3.42), (3.43) and (3.38), we obtain (3.40). From (3.39) and (3.40), we obtain (3.32). From (3.31) and (3.32), we conclude (3.29).

Finally, we prove (3.30). By Corollary 1.11, we only need to estimate $K_i \lesssim \|u\|_{X_S^k}^2\|v\|_{X_S^k}^2$ for $i=1,2,3$, where

$$\begin{aligned}
K_1 &:= \sum_{N_3} N_3^{d-4}\sup_{\|n\|_{V_{W_\pm}^2}=1}\Big|\sum_{N_2\gtrsim N_3}\sum_{N_1\sim N_2}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}u_{N_1}\overline{v_{N_2}\omega\, n_{N_3}}dxdt\Big|^2,\\
K_2 &:= \sum_{N_3} N_3^{d-4}\sup_{\|n\|_{V_{W_\pm}^2}=1}\Big|\sum_{N_2\sim N_3}\sum_{N_1\ll N_2}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}u_{N_1}\overline{v_{N_2}\omega\, n_{N_3}}dxdt\Big|^2,\\
K_3 &:= \sum_{N_3} N_3^{d-4}\sup_{\|n\|_{V_{W_\pm}^2}=1}\Big|\sum_{N_2\ll N_3}\sum_{N_1\sim N_3}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}u_{N_1}\overline{v_{N_2}\omega\, n_{N_3}}dxdt\Big|^2.
\end{aligned}$$

By the triangle inequality, Lemma 3.8 (i) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
K_1^{1/2} &\lesssim \sum_{N_2}\sum_{N_1\sim N_2}\Big\{\sum_{N_3\lesssim N_2}N_3^{d-4}\sup_{\|n\|_{V_{W_\pm}^2}=1}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbf{1}_{[0,T]}u_{N_1}\overline{v_{N_2}\omega\, n_{N_3}}dxdt\Big|^2\Big\}^{1/2}\\
&\lesssim \sum_{N_2}\sum_{N_1\sim N_2}\Big\{\sum_{N_3\lesssim N_2}N_3^{d-4}(N_3^{(d-4)/2}N_3\|u_{N_1}\|_E\|v_{N_2}\|_E)^2\Big\}^{1/2}\\
&\lesssim \sum_{N_2}\sum_{N_1\sim N_2}\big(N_2^{2d-6}\|u_{N_1}\|_E^2\|v_{N_2}\|_E^2\big)^{1/2}\\
&\lesssim \Big(\sum_N N^{d-3}\|u_N\|_E^2\Big)^{1/2}\Big(\sum_N N^{d-3}\|v_N\|_E^2\Big)^{1/2}.
\end{aligned}$$

By Mihlin-Hörmander's multiplier theorem, it follows that

$$\sum_{N<1} N^{d-3}\|u_N\|_E^2 \lesssim \sum_{N<1} N^{d-3}\|P_{<1}u\|_E^2 \lesssim \|P_{<1}u\|_E^2. \tag{3.44}$$

Thus, we conclude $K_1 \lesssim \|u\|_{X_S^k}^2\|v\|_{X_S^k}^2$. Next, we estimate K_2. Put $K_2 = K_{2,1} + K_{2,2}$ where

$$K_{2,1} := \sum_{N_3 \lesssim 1} N_3^{d-4} \sup_{\|n\|_{V^2_{W_\pm}}=1} \Big| \sum_{N_2\sim N_3} \sum_{N_1\ll N_2} \int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} u_{N_1} \overline{v_{N_2}\omega\, n_{N_3}} dxdt \Big|^2,$$

$$K_{2,2} := \sum_{N_3\gg 1} N_3^{d-4} \sup_{\|n\|_{V^2_{W_\pm}}=1} \Big| \sum_{N_2\sim N_3} \sum_{N_1\ll N_2} \int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} u_{N_1} \overline{v_{N_2}\omega\, n_{N_3}} dxdt \Big|^2.$$

By Lemma 3.8 (i), we have

$$\begin{aligned} K_{2,1} &\lesssim \sum_{N_2 \lesssim 1} N_2^{d-4}\Big(N_2^{(d-4)/2} N_2 \Big\| \sum_{N_1\ll N_2} u_{N_1}\Big\|_E \|v_{N_2}\|_E\Big)^2 \\ &\lesssim \|P_{<1}u\|_E^2 \sum_{N_2 \lesssim 1} N_2^{2d-6}\|v_{N_2}\|_E^2 \\ &\lesssim \|u\|_{X_S^k}^2\|v\|_{X_S^k}^2. \end{aligned} \tag{3.45}$$

For the estimate of $K_{2,2}$, we take $M = \varepsilon N_2^2$ for sufficiently small $\varepsilon > 0$. Then, from Lemma 3.4, we have

$$\begin{aligned} &Q^S_{<M}\big((Q^S_{<M}v_{N_2})(Q^{W_\pm}_{<M}\omega\, n_{N_3})\big) \\ &= Q^S_{<M}\Big[\mathcal{F}^{-1}\Big(\int_{\tau_1=\tau_2+\tau_3,\, \xi_1=\xi_2+\xi_3} (\widehat{Q^S_{<M}v_{N_2}})(\tau_2,\xi_2)(\widehat{Q^{W_\pm}_{<M}\omega\, n_{N_3}})(\tau_3,\xi_3)\Big)\Big] = 0, \end{aligned}$$

when $N_2 \gg \langle N_1\rangle$. Therefore,

$$v_{N_2}\omega\, n_{N_3} = \sum_{i=1}^4 G_i,$$

where

$$G_1 := v_{N_2}(Q^{W_\pm}_{\geq M}\omega\, n_{N_3}), \qquad G_2 := Q^S_{\geq M}\big(v_{N_2}(Q^{W_\pm}_{<M}\omega\, n_{N_3})\big),$$
$$G_3 := (Q^S_{\geq M}v_{N_2})(Q^{W_\pm}_{<M}\omega\, n_{N_3}), \qquad G_4 := -Q^S_{\geq M}\big((Q^S_{\geq M}v_{N_2})(Q^{W_\pm}_{<M}\omega\, n_{N_3})\big).$$

Therefore, it follows that

$$K_{2,2} \leq K_{2,2}^{(1)} + K_{2,2}^{(2)} + K_{2,2}^{(3)} + K_{2,2}^{(4)}$$

where

$$K_{2,2}^{(1)} := \sum_{N_3 \gg 1} N_3^{d-4} \sup_{\|n\|_{V^2_{W_\pm}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_2} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} u_{N_1} \overline{G_1} dx dt \Big|^2,$$

$$K_{2,2}^{(2)} := \sum_{N_3 \gg 1} N_3^{d-4} \sup_{\|n\|_{V^2_{W_\pm}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_2} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} u_{N_1} \overline{G_2} dx dt \Big|^2,$$

$$K_{2,2}^{(3)} := \sum_{N_3 \gg 1} N_3^{d-4} \sup_{\|n\|_{V^2_{W_\pm}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_2} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} u_{N_1} \overline{G_3} dx dt \Big|^2,$$

$$K_{2,2}^{(4)} := \sum_{N_3 \gg 1} N_3^{d-4} \sup_{\|n\|_{V^2_{W_\pm}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_2} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} u_{N_1} \overline{G_4} dx dt \Big|^2.$$

Note that $N_3 \gg 1$ and $N_2 \sim N_3$ implies $N_2 > 2^2$. By Lemma 3.8 (iii), we have

$$\begin{aligned} K_{2,2}^{(1)} &\lesssim \sum_{N_2 > 2^2} N_2^{d-4} \big(N_2^{-1/2} N_2 \|u\|_{Y_S^k} \|v_{N_2}\|_E \big)^2 \\ &\lesssim \sum_{N_2 > 2^2} N_2^{d-3} \|u\|_{Y_S^k}^2 \|v_{N_2}\|_E^2 \lesssim \|u\|_{Y_S^k}^2 \|v\|_{E^k}^2. \end{aligned} \tag{3.46}$$

We apply Lemma 1.16, Lemma 3.8 (v) and (3.34), then we have

$$K_{2,2}^{(2)} \lesssim \sum_{N_2 > 2^2} N_2^{d-4} \big(N_2^{-1/2} N_2 \|u\|_{Y_S^k} \|v_{N_2}\|_E \big)^2 \lesssim \|u\|_{Y_S^k}^2 \|v\|_{E^k}^2. \tag{3.47}$$

By Lemma 3.8 (iv), we have

$$K_{2,2}^{(3)} \lesssim \sum_{N_2 > 2^2} N_2^{d-4} \big(N_2^{-1/2} N_2 \|u\|_{E^k} \|v_{N_2}\|_{V_S^2} \big)^2 \lesssim \|u\|_{E^k}^2 \|v\|_{Y_S^k}^2. \tag{3.48}$$

Applying Lemma 1.16, Lemma 3.8 (vi) and (3.34), we obtain

$$K_{2,2}^{(4)} \lesssim \sum_{N_2 > 2^2} N_2^{d-4} \big(N_2^{-1/2} N_2 \|u\|_{Y_S^k} \|v_{N_2}\|_{V_S^2} \big)^2 \lesssim \|u\|_{Y_S^k}^2 \|v\|_{Y_S^k}^2. \tag{3.49}$$

Hence, collecting (3.45), (3.46), (3.47), (3.48) and (3.49), we have $K_2 \lesssim \|u\|_{X_S^k}^2 \|v\|_{X_S^k}^2$. By symmetry, we also obtain $K_3 \lesssim \|u\|_{X_S^k}^2 \|v\|_{X_S^k}^2$ in the same manner as the estimate of K_2. □

Next, we consider the inhomogeneous case for the wave equation.

Corollary 3.10. *Let* $d \geq 4,\ k = (d-3)/2$ *and* $l = (d-4)/2$. *Then for all* $0 < T < \infty$, *it holds that*

$$\|I_{T,W_\pm}(u,v)\|_{Z^l_{W_\pm}} \lesssim \|u\|_{X_S^k} \|v\|_{X_S^k}.$$

Proof. From Proposition 3.9, we have

$$\Big\|\sum_{N\geq 1} P_N I_{T,W_\pm}(u,v)\Big\|_{Z^l_{W_\pm}} \sim \Big\|\sum_{N\geq 1} P_N I_{T,W_\pm}(u,v)\Big\|_{\dot Z^l_{W_\pm}} \lesssim \|u\|_{X^k_S}\|v\|_{X^k_S}.$$

Hence, we only need to show the following.

$$\|P_{<1} I_{T,W_\pm}(u,v)\|_{U^2_{W_\pm}} \lesssim \|u\|_{X^k_S}\|v\|_{X^k_S}. \tag{3.50}$$

By Corollary 1.11 and Hölder's inequality, we have

$$\begin{aligned}(\text{LHS of (3.50)}) &= \sup_{\|n\|_{V^2_{W_\pm}}=1}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathbf{1}_{[0,T]} u\,\bar v\,\overline{P_{<1}\omega\, n}\,dxdt\Big| \\ &\lesssim \sup_{\|n\|_{V^2_{W_\pm}}=1} \|u\|_E\|v\|_E\|P_{<1}\omega\, n\|_{L^\infty_t L^{d/2}_x}. \end{aligned}\tag{3.51}$$

Since

$$\|u\|_E \leq \|P_{<1}u\|_E + \Big(\sum_{N\geq 1} N^{-2k}\Big)^{1/2}\Big(\sum_{N\geq 1} N^{2k}\|P_N u\|_E^2\Big)^{1/2}$$

by the Cauchy-Schwarz inequality, we have

$$\|u\|_E \lesssim \|u\|_{X^k_S}, \qquad \|v\|_E \lesssim \|v\|_{X^k_S}. \tag{3.52}$$

By the Sobolev inequality and Remark 1.1, we have

$$\|P_{<1}\omega\, n\|_{L^\infty_t L^{d/2}_x} \lesssim \big\||\nabla_x|^{(d-4)/2} P_{<1}\omega\, n\big\|_{L^\infty_t L^2_x} \lesssim \|P_{<1} n\|_{L^\infty_t L^2_x} \lesssim \|n\|_{V^2_{W_\pm}}. \tag{3.53}$$

Hence, collecting (3.51), (3.52) and (3.53), we obtain (3.50). □

3.4. **The proof of the main theorem.** By the Duhamel principle, we consider the following integral equation corresponding to (3.2) on the time interval $[0,T]$ with $0 < T < \infty$:

$$(u, n_\pm) = (\Phi_1(u,n_\pm), \Phi_{2\pm}(u)), \tag{3.54}$$

where

$$\begin{aligned}\Phi_1(u,n_\pm) &:= S(t)u_0 + I_{T,S}(n_+,u)(t) + I_{T,S}(n_-,u)(t),\\ \Phi_{2\pm}(u) &:= W_\pm(t)n_{\pm 0} + I_{T,W_\pm}(u,u)(t).\end{aligned}$$

Proposition 3.11. *Let $d \geq 4, k = (d-3)/2$ and $l = (d-4)/2$.*
(i) (existence) Let $\delta > 0$ be sufficiently small. Then, for any $0 < T < \infty$ and any initial data

$$(u_0, n_{\pm 0}) \in B_\delta(H^k(\mathbb{R}^d)\times \dot H^l(\mathbb{R}^d)) \ (\textit{resp.}\ B_\delta(H^k(\mathbb{R}^d)\times H^l(\mathbb{R}^d))),$$

there exists a solution to (3.54) *on* $[0,T]$ *satisfying*

$$(u,n_\pm)\in X_S^k([0,T])\times \dot{Y}_{W_\pm}^l([0,T])\subset C([0,T];H^k(\mathbb{R}^d))\times C([0,T];\dot{H}^l(\mathbb{R}^d))$$

$$\textit{(resp. } (u,n_\pm)\in X_S^k([0,T])\times Y_{W_\pm}^l([0,T])\subset C([0,T];H^k(\mathbb{R}^d))\times C([0,T];H^l(\mathbb{R}^d))\textit{).}$$

(ii) (uniqueness) Let

$$(u,n_\pm),(v,m_\pm)\in X_S^k([0,T])\times \dot{Y}_{W_\pm}^l([0,T])\ (\textit{resp.}\ \in X_S^k([0,T])\times Y_{W_\pm}^l([0,T]))$$

be solutions to (3.54) *on* $[0,T]$ *for some* $T>0$ *with the same initial data. Then* $(u(t),n_\pm(t))=(v(t),m_\pm(t))$ *on* $t\in[0,T]$.

(iii) (continuous dependence of the solution on the initial data) The flow map obtained by (i):

$$B_\delta(H^k(\mathbb{R}^d)\times \dot{H}^l(\mathbb{R}^d))\ni(u_0,n_{\pm0})\mapsto(u,n_\pm)\in X_S^k([0,T])\times \dot{Y}_{W_\pm}^l([0,T])$$

$$(\textit{resp.}\ B_\delta(H^k(\mathbb{R}^d)\times H^l(\mathbb{R}^d))\ni(u_0,n_{\pm0})\mapsto(u,n_\pm)\in X_S^k([0,T])\times Y_{W_\pm}^l([0,T]))$$

is Lipschitz continuous.

(iv) (persistence) Let $\delta>0$ *be sufficiently small and*

$$(u_0,n_{\pm0})\in B_\delta(H^k(\mathbb{R}^d)\times H^l(\mathbb{R}^d))\cap H^{k+a}(\mathbb{R}^d)\times H^{l+a}(\mathbb{R}^d)$$

for some $a>0$. *Then, the solution* $(u,n_\pm)$ *obtained by (i) is in*

$$X_S^{k+a}([0,T])\times Y_{W_\pm}^{l+a}([0,T])\subset C([0,T];H^{k+a}(\mathbb{R}^d))\times C([0,T];H^{l+a}(\mathbb{R}^d))$$

for any $0<T<\infty$.

Remark 3.2. Due to the time reversibility of the Zakharov equation, Porposition 3.11 holds on corresponding time interval $[-T,0]$.

Remark 3.3. By (i) in Proposition 3.11 and Remark 3.2, we have solutions to (3.54) on $[0,T]$ and $[-T,0]$ for any $T>0$. Since we can take any large T and have the uniqueness, the solution $(u(t),n_\pm(t))\in C((-\infty,\infty);H^k(\mathbb{R}^d))\times C((-\infty,\infty);\dot{H}^l(\mathbb{R}^d))$ (resp. $C((-\infty,\infty);H^k(\mathbb{R}^d))\times C((-\infty,\infty);H^l(\mathbb{R}^d))$) can be defined uniquely when $(u_0,n_{\pm0})\in B_\delta(H^k(\mathbb{R}^d)\times \dot{H}^l(\mathbb{R}^d))$ (resp. $B_\delta(H^k(\mathbb{R}^d)\times H^l(\mathbb{R}^d))$).

Proposition 3.12. *(scattering) Let* $(u(t),n_\pm(t))$ *be the solution to* (3.54) *with* $(u_0,n_{\pm0})\in B_\delta(H^k(\mathbb{R}^d)\times \dot{H}^l(\mathbb{R}^d))$ *on* $(-\infty,\infty)$ *obtained by Proposition 3.11, Remark 3.2 and Remark 3.3. Then, there exist* $(u_{+\infty},n_{\pm,+\infty})$ *and* $(u_{-\infty},n_{\pm,-\infty})$ *in* $H^k(\mathbb{R}^d)\times \dot{H}^l(\mathbb{R}^d)$ *such that*

$$\|u(t)-S(t)u_{+\infty}\|_{H^k}+\|n_\pm(t)-W_\pm(t)n_{\pm,+\infty}\|_{\dot{H}^l}\to 0$$

as $t \to \infty$ and

$$\|u(t) - S(t)u_{-\infty}\|_{H^k} + \|n_\pm(t) - W_\pm(t)n_{\pm,-\infty}\|_{\dot{H}^l} \to 0$$

as $t \to -\infty$. The similar result holds for the inhomogeneous case.

Proof of Proposition 3.11. We will show only the case $(u_0, n_{\pm 0}) \in B_\delta(H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d))$ because the proof of the case $(u_0, n_{\pm 0}) \in B_\delta(H^k(\mathbb{R}^d) \times \dot{H}^l(\mathbb{R}^d))$ follows from the same argument if we use (3.30) instead of Corollary 3.10.

First, we prove (i). We denote $I := [0, T]$. By Proposition 3.5 and the definition of $X^k_S, Y^l_{W_\pm}$, there exists $C > 0$ such that

$$\|S(t)u_0\|_{X^k_S(I)} \leq C\|u_0\|_{H^k},\ \|W_\pm(t)n_{\pm 0}\|_{Y^l_{W_\pm}(I)} \leq C\|n_{\pm 0}\|_{H^l}.$$

Assume that $(u_0, n_{\pm 0}) \in B_\delta(H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d)),\ (u, n_\pm) \in B_r(X^k_S(I) \times Y^l_{W_\pm}(I))$. Then, by Proposition 3.9, Corollary 3.10 and $\|\cdot\|_{Y^l_{W_\pm}} \lesssim \|\cdot\|_{Z^l_{W_\pm}}$, we have

$$\begin{aligned}
&\|\Phi_1(u, n_\pm)\|_{X^k_S(I)} \leq C\|u_0\|_{H^k} + C\|n_\pm\|_{Y^l_{W_\pm}(I)}\|u\|_{X^k_S(I)} \leq C\delta + Cr^2,\\
&\|\Phi_{2\pm}(u)\|_{Y^l_{W_\pm}(I)} \leq C\|n_{\pm 0}\|_{H^l} + C\|u\|^2_{X^k_S(I)} \leq C\delta + Cr^2.
\end{aligned}$$

We choose $\delta = r^2,\ r = 1/4C$, then we have

$$\|\Phi_1(u, n_\pm)\|_{X^k_S(I)} + \|\Phi_{2\pm}(u)\|_{Y^l_{W_\pm}(I)} \leq r.$$

Hence, $(\Phi_1, \Phi_{2\pm})$ is a map from $B_r(X^k_S([0, T]) \times Y^l_{W_\pm}([0, T]))$ into itself. Note that r does not depend on T. Moreover, we assume $(v_0, m_{\pm 0}) \in B_\delta(H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d)),\ (v, m_\pm) \in B_r(X^k_S(I) \times Y^l_{W_\pm}(I))$, then

$$\begin{aligned}
&\|\Phi_1(u, n_\pm) - \Phi_1(v, m_\pm)\|_{X^k_S(I)}\\
&= \|I_{T,S}(n_\pm, u)(t) - I_{T,S}(m_\pm, v)(t)\|_{X^k_S(I)}\\
&\leq \|I_{T,S}(n_\pm, u - v)\|_{X^k_S(I)} + \|I_{T,S}(n_\pm - m_\pm, v)\|_{X^k_S(I)}\\
&\leq C(\|n_\pm\|_{Y^l_{W_\pm}(I)}\|u - v\|_{X^k_S(I)} + \|n_\pm - m_\pm\|_{Y^l_{W_\pm}(I)}\|v\|_{X^k_S(I)}) \qquad (3.55)\\
&\leq (1/4)(\|u - v\|_{X^k_S(I)} + \|n_\pm - m_\pm\|_{Y^l_{W_\pm}(I)}),\\
&\|\Phi_2(u) - \Phi_2(v)\|_{Y^l_{W_\pm}(I)}\\
&= \|I_{T,W_\pm}(u, u)(t) - I_{T,W_\pm}(v, v)(t)\|_{Y^l_{W_\pm}(I)}\\
&\leq C(\|u\|_{X^k_S(I)} + \|v\|_{X^k_S(I)})\|u - v\|_{X^k_S(I)} \qquad (3.56)\\
&\leq (1/2)\|u - v\|_{X^k_S(I)}.
\end{aligned}$$

Therefore, $(\Phi_1, \Phi_{2\pm})$ is a contraction mapping on $B_r(X^k_S([0,T])\times Y^l_{W_\pm}([0,T]))$. Thus, by the Banach fixed point theorem, we have a solution to (3.54) in it.

Next, we prove (ii) by contradiction. Let $(u,n_\pm)$, $(v,m_\pm) \in X^k_S([0,T])\times Y^l_{W_\pm}([0,T])$ are two solutions satisfying $(u(0),n_\pm(0)) = (v(0),m_\pm(0))$. Assume that

$$T' := \sup\{0 \le t < T\,; u(t) = v(t), n_\pm(t) = m_\pm(t)\} < T.$$

By a translation in t, it suffices to consider the case $T' = 0$. Fix $0 < \tau \le T$ sufficiently small. From (3.55) and Proposition 3.6, we obtain

$$\begin{aligned}\|u-v\|_{X^k_S([0,\tau])} &\le C(\|n_\pm\|_{Y^l_{W_\pm}([0,\tau])}\|u-v\|_{X^k_S([0,\tau])} + \|n_\pm - m_\pm\|_{Y^l_{W_\pm}([0,\tau])}\|v\|_{X^k_S([0,\tau])})\\ &\le (1/4)\big(\|u-v\|_{X^k_S([0,\tau])} + \|n_\pm - m_\pm\|_{Y^l_{W_\pm}([0,\tau])}\big).\end{aligned}$$

Hence, we obtain

$$\|u-v\|_{X^k_S([0,\tau])} \le (1/3)\|n_\pm - m_\pm\|_{Y^l_{W_\pm}([0,\tau])}. \tag{3.57}$$

Similarly, by (3.56) and Proposition 3.6, we obtain

$$\begin{aligned}\|n_\pm - m_\pm\|_{Y^l_{W_\pm}([0,\tau])} &\le C(\|u\|_{X^k_S([0,\tau])} + \|v\|_{X^k_S([0,\tau])})\|u-v\|_{X^k_S([0,\tau])}\\ &\le (1/2)\|u-v\|_{X^k_S([0,\tau])}. \end{aligned}\tag{3.58}$$

Hence, from (3.57) and (3.58), we obtain $u(t) = v(t), n_\pm(t) = m_\pm(t)$ on $[0,\tau]$, which contradicts to the definition of T'.

We omit the proof of (iii) because it follows from the standard argument. Finally, we prove (iv). Fix $0 < T < \infty$. Since $\langle\xi\rangle^a \le \langle\xi-\xi_1\rangle^a + \langle\xi_1\rangle^a$, we easily have

$$\|I_{T,S}(n_\pm,u)\|_{X^{k+a}_S} \lesssim \|n_\pm\|_{Y^{l+a}_{W_\pm}}\|u\|_{X^k_S} + \|n_\pm\|_{Y^l_{W_\pm}}\|u\|_{X^{k+a}_S}, \tag{3.59}$$

$$\|I_{T,W_\pm}(u,u)\|_{Z^{l+a}_{W_\pm}} \lesssim \|u\|_{X^{k+a}_S}\|u\|_{X^k_S}, \tag{3.60}$$

from Proposition 3.9 and Corollary 3.10. Thus, by a similar argument as (i), we obtain

$$\begin{aligned}&\|u\|_{X^{k+a}_S(I)} \le C\|u_0\|_{H^{k+a}} + Cr(\|u\|_{X^{k+a}_S(I)} + \|n_+\|_{Y^{l+a}_{W_+}(I)} + \|n_-\|_{Y^{l+a}_{W_-}(I)}),\\ &\|n_\pm\|_{Y^{l+a}_{W_\pm}(I)} \le C\|n_{\pm0}\|_{H^{l+a}} + Cr\|u\|_{X^{k+a}_S}\end{aligned}$$

for the solution to (3.54) such that $(u,n_\pm) \in B_r(X^k_S(I)\times Y^l_{W_\pm}(I))$. Since $4Cr = 1$, we conclude

$$\|u\|_{X^{k+a}_S(I)} + \|n_\pm\|_{Y^{l+a}_{W_\pm}(I)} \le C(\|u_0\|_{H^{k+a}} + \|n_{\pm0}\|_{H^{l+a}}).$$

□

Finally, we prove Proposition 4.2.

Proof. Since r in the proof of Proposition 3.11 does not depend on T, it follows that

$$\|u\|_{X_S^k([0,T])} + \|n_\pm\|_{Y^l_{W_\pm}([0,T])} < M,$$
$$\|u\|_{X_S^k([-T,0])} + \|n_\pm\|_{Y^l_{W_\pm}([-T,0])} < M$$

for any $T > 0$, where the constant M does not depends on T. For any $\{t_k\}_{k=0}^K \in \mathcal{Z}_0$ we can take $0 < T < \infty$ such that $-T < t_0$ and $t_K < T$. Then, by Lemma 1.6, we have

$$\begin{aligned}&\Big(\sum_{k=1}^{K} \|\langle\nabla_x\rangle^k (S(-t_k)u(t_k) - S(-t_{k-1})u(t_{k-1}))\|_{L^2}^2\Big)^{1/2}\\ &\lesssim \|\langle\nabla_x\rangle^k u\|_{V_S^2([0,T])} + \|\langle\nabla_x\rangle^k u\|_{V_S^2([-T,0])}\\ &\lesssim \|u\|_{X_S^k([0,T])} + \|u\|_{X_S^k([-T,0])} < 2M.\end{aligned}$$

Therefore, we have

$$\sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}_0} \Big(\sum_{k=1}^{K} \|\langle\nabla_x\rangle^k S(-t_k)u(t_k) - \langle\nabla_x\rangle^k S(-t_{k-1})u(t_{k-1})\|_{L^2}^2\Big)^{1/2} < 2M.$$

By Proposition 1.5, $f_\pm := \lim_{t\to\pm\infty} \langle\nabla_x\rangle^k S(-t)u(t)$ exists in L^2. Put $u_{\pm\infty} := \langle\nabla_x\rangle^{-k} f_\pm$. Then, we conclude

$$\|\langle\nabla_x\rangle^k S(-t)u(t) - f_\pm\|_{L^2} = \|u(t) - S(t)u_{\pm\infty}\|_{H^k} \to 0 \qquad \text{as } \ t \to \pm\infty.$$

Similarly, we obtain the scattering result for the wave equation. □

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