On modulation spaces and their applications to dispersive equations

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Abstract

This thesis is mainly written on the basis of papers [42, 43, 44, 45] and is organized by the following six chapters. Chapter 1 describes the background and previous results of importance on modulation spaces. Some results got by the author are also described there. In Chapter 2, all the basic notations and the definition of function spaces, which will be used throughout this thesis, are introduced. Elementary properties on the function spaces can be also found. Chapters 3, 4, 5 and 6 will present the author's works in [42], [43], [45] and [44], respectively. Each work is briefly explained in the next paragraphs.

In Chapter 3, we discuss decay estimates and Strichartz estimates for dispersive equations with non-homogeneous symbols on modulation spaces. These are some of tools to obtain the global well-posedness of the Cauchy problems for nonlinear dispersive equations. As a result, we are able to deal with phase functions of linear solution in wide class by establishing these estimates in the frame of modulation spaces. Moreover, we have a generalization of the result in [97] which treated the Schrödinger equations with a non-linearity of wider class.

In Chapter 4, we consider the Cauchy problem for the nonlinear higher order Schrödinger equations on modulation spaces and show the existence of a unique global solution by using integrability of time decay factors of time decay estimates. As a result, we are able to deal with wider classes of a non-linearity and a solution space. Moreover, we study time decay estimates of a semi–group $e^{it\phi(\sqrt{-\Delta})}$ with a polynomial type of phase ϕ. Considering multiplicities of critical points and inflection points of ϕ carefully, we have time decay estimates with better time decay rate.

In Chapter 5, we mainly consider the Cauchy problem for the two dimensional generalized Zakharov–Kuznetzov equations: $\partial_t u + \partial_x \Delta u = \partial_x(u^{p+1})$, where $(x, y) \in \mathbf{R}^2$, $t > 0$, and integers $p \geq 4$. Considering the maximal function estimate (for $L^4_x L^\infty_{y,t}$) on modulation spaces, we see that it is bounded on $M^{1/4}_{2,1}$ globally in time. As a result, we have the global well–posedness in $M^{1/p}_{2,1}$ with small initial data for $p \geq 4$.

In Chapter 6, we mention the structure of α–modulation spaces, which are function spaces like generalized modulation spaces. We first discuss equivalent norms to α–modulation spaces norms, which are composed from decomposition with non–compact support. Then, we determine sharp inclusion relations between α–modulation spaces and L^p–Sobolev spaces, and between α–modulation spaces and local Hardy spaces.

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Contents

1 Introduction

1.1 Modulation spaces

1.1.1 Backgrounds

The modulation spaces were introduced by Feichtinger in his paper [22] in 1983. He originally defined modulation spaced by using the short–time Fourier transform, which is one kind of technique to analyze the information of time and frequency like sounds, voice, and so on. The short–time Fourier transform $V_g[f](x.\omega)$ is defined by

$$V_g[f](x,\omega) := \int_{\mathbf{R}^n} e^{-it\cdot\omega}\overline{g(t-x)}f(t)dt,$$

where $g \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$ called a "window function". The Fourier transform $\mathcal{F}[f](\omega)$;

$$\mathcal{F}[f](\omega) := \int_{\mathbf{R}^n} e^{-it\cdot\omega}f(t)dt,$$

are used to analyze the information of frequency for a long time. However, as we know from the definition, although frequency of sounds and so on changes their properties as time goes on, the Fourier transform does not give us the relations between time and frequency. On the other hand, for the short–time Fourier transform $V_g[f](x,\omega)$, we set the window function g has a compact support with its support centered at the origin and slide the window function to different positions along x–axis (time–axis). Then, the short–time Fourier transform informs us how the frequency changes as time passes. Thus, the short–time Fourier transform is quite useful method for time–frequency analysis.

For $0 < p,q \le \infty$ and $s \in \mathbf{R}$, the (original) modulation spaces $M^s_{p,q}$ are denoted as the spaces of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the norms

$$\|f\|^{original}_{M^s_{p,q}} = \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n}|V_g[f](x,\omega)|^p\,dx\right)^{q/p}\langle\omega\rangle^s q d\omega\right)^{1/q},$$

is finite, where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. In the paper [22], Feichtinger invented these function spaces, and then showed that these norms are equivalent to the Besov–type spaces with the frequency–uniform decomposition $\{\sigma_k\}_{k\in\mathbf{Z}^n}$ as follows;

$$\|f\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq}\|\mathcal{F}^{-1}\sigma_k\mathcal{F}f\|^q_p\right)^{1/q}.$$

Here, $\mathcal{F}^{-1}$ is the inverse Fourier transform, and $\sigma_k = \sigma(\cdot - k)$ satisfies that

$$\text{supp}\,\sigma \subset [-1,1]^n \text{ and } \sum_{k\in\mathbf{Z}^n}\sigma_k(\xi) \equiv 1$$

for any $\xi \in \mathbf{R}^n$. Since σ_k is a translation of σ and all σ_k have the same structures, σ_k is called the frequency–uniform decomposition. All statements written in this thesis are considered on this equivalent norm. Feichtinger showed the equivalence for the case $1 \le p, q \le \infty$ and Wang–Hudzik proved it for the case $0 < p, q < 1$ in [97, Appendix A]. Some elementary properties are written in Subsection 2.2.3.

The modulation spaces had not been studied so studiously for the past twenty years after the first introduction. However, since modulation spaces were first applied to the theory of partial differential equations in the early twenty–first century, studies for modulation spaces have been developing rapidly. In the next subsection, we will state some important results on modulation spaces.

1.1.2 Previous results for modulation spaces

In this subsection, we give three results and applications of special importance about modulation spaces.

The first one is about a Gabor frame, which are quite important theory of time–frequency analysis. Let $g \in L^2(\mathbf{R}^n)$ and $\alpha, \beta > 0$. We set the time–frequency shift $g_{k,\ell}$;

$$g_{k,\ell}(x) := e^{i\alpha\ell\cdot x} g(x - \beta k),$$

and the set $\mathcal{G}(g, \alpha, \beta)$ called a Gabor system;

$$\mathcal{G}(g, \alpha, \beta) := \{g_{k,\ell}; (k, \ell) \in \mathbf{Z}^n \times \mathbf{Z}^n\}.$$

A Gabor system $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbf{R}^n)$ if the frame operator S_g;

$$S_g f = S_g^{\alpha,\beta} f = \sum_{(k,\ell)\in\mathbf{Z}^{2n}} (f, g_{k,\ell}) g_{k,\ell}$$

is bounded and invertible on L^2. This is equivalent to satisfy that

$$\left(\sum_{k,\ell} |(f, g_{k,\ell})|^2 \right)^{1/2} \sim \|f\|_{L^2}.$$

Here, $(f, g) = \int_{\mathbf{R}^n} f(x)\overline{g(x)}dx$. It is known that if $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for L^2, then every $f \in L^2$ is expanded to

$$f = \sum_{k,\ell} (f, S_g^{-1} g_{k,\ell}) g_{k,\ell} \tag{1.1}$$

(see [31, Section 5.2]). It was shown that the similar property is given for modulation spaces ($p, q \ne \infty$), that is, every functions in modulation spaces has the expansion (1.1) if $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for L^2. This statement can be found in [24, 93]. This property on modulation spaces has been applied to the theory of partial differential equations. For instance, setting $g(x) = e^{-|x|^2/2}$ (the Gauss function), which is one of the example of the Gabor frame, then the solution to the linear Schrödinger equations;

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

with $t \in \mathbf{R}$ and $x \in \mathbf{R}^n$, is given by a concrete expression (see for example [17, 18, 94]).

The second one is the application to the Cauchy problem for the nonlinear Schrödinger equations;

$$\begin{cases} i\partial_t u + \Delta u = F(u), \\ u(0) = u_0, \end{cases}$$

with $t \in \mathbf{R}$ and $x \in \mathbf{R}^n$. In order to solve this problem by fixed point argument, the following two estimates for the Schrödinger semi–group $e^{it\Delta}$ play a crucial roles, where $e^{it\Delta} = \mathcal{F}^{-1} e^{-it|\xi|^2} \mathcal{F}$ (see [13]). The first one is the time decay estimates; for $2 \le p \le \infty$ and $1/p + 1/p' = 1$

$$\left\| e^{it\Delta} f \right\|_{L^p} \lesssim |t|^{-n(1/2-1/p)} \|f\|_{L^{p'}}, \tag{1.3}$$

and the second one is the Strichartz estimates; for $2 \le p \le \infty$ ($p \neq \infty$ if $n = 2$),

$$\left\| e^{it\Delta} f \right\|_{L^\gamma(\mathbf{R}, L^p)} \lesssim \|f\|_2, \tag{1.4}$$

where $\gamma = \gamma(p)$, $2/\gamma(p) = n(1/2 - 1/p)$ and $\|f\|_{L^\gamma(\mathbf{R},L^p)} = (\int_{\mathbf{R}} \|f(t)\|_{L^p}^\gamma dt)^{1/\gamma}$. Keel–Tao [46] proved that if the time decay estimates (1.3) have the truncated time decay, that is,

$$\left\| e^{it\Delta} f \right\|_{L^p} \lesssim (1 + |t|)^{-n(1/2-1/p)} \|f\|_{L^{p'}} \tag{1.5}$$

holds true, then the Strichartz estimates (1.4) hold for all $\gamma \ge \max(\gamma(p), 2)$. Since, by the property of modulation spaces, we have

$$\|e^{it\Delta} f\|_{M^s_{p,q}} \lesssim \|f\|_{M^s_{p',q}},$$

we see easily that

$$\|e^{it\Delta} f\|_{M^s_{p,q}} \lesssim (1 + |t|)^{-n(1/2-1/p)} \|f\|_{M^s_{p',q}} \tag{1.6}$$

hold true. The time decay estimates on modulation spaces (1.6) have the quite similar structure to the estimates (1.5). Thus, optimizing statements by Keel–Tao on modulation spaces, we can get the Strichartz estimates on modulation spaces with parameter $\gamma \ge \max(\gamma(p), 2)$. These estimates were first unveiled by Wang–Hudzik [97]. Moreover, by using the Strichartz estimates on modulation spaces, Wang–Hudzik also proved the global well–poseness for the nonlinear Schrödinger equations with small rough data in $M_{2,1}$. From a point of view of the scaling, the critical regularity is $s_c = n/2 - 2/\kappa$ as well known when the non-linearity is $F(u) = u^{\kappa+1}$. Cazenave–Weissler [11] proved the local well-posedness in the scaling critical space H^{s_c} with $s_c \ge 0$ and, moreover, there a global solution in $C(\mathbf{R}, H^{s_c})$ if the initial data in $\dot{H}^{s_c}$ is sufficiently small. Here, H^s and $\dot{H}^s$ are the inhomogeneous and homogeneous Sobolev spaces equipped with the norms $\|f\|_{H^s} := \|(1+|\xi|^2)^{s/2}\hat{f}\|_{L^2}$ and $\|f\|_{\dot{H}^s} := \||\xi|^s \hat{f}\|_{L^2}$, respectively. From the inclusion $M_{2,1} \not\subset H^\varepsilon$ for any $\varepsilon > 0$, we see that Wang–Hudzik showed the global well–posedness in the new class. Moreover, it is also surprising that we don't need any regularity if we construct a global solution in modulation spaces.

The final one is the application to uniform boundedness of unimodular Fourier multipliers $e^{i(-\Delta)^{m/2}}$;

$$e^{i(-\Delta)^{m/2}} := \mathcal{F}^{-1} e^{i|\xi|^m} \mathcal{F}$$

for $m > 0$. This operator arises in the solution of linear partial differential equations as we stated above (for the free Schrödinger equation). Thus, this boundedness property is also the application to the theory of partial differential equations. Miyachi [67] proved that for $1 < p < \infty$ and $m > 1$ the operator $e^{i(-\Delta)^{m/2}}$ is bounded from L^p_s into L^p, that is,

$$\|e^{i(-\Delta)^{m/2}} f\|_{L^p} \le C \|f\|_{L^p_s} \tag{1.7}$$

holds, if and only if $s \ge mn|1/p - 1/2|$. Here L^p_s is the L^p–Sobolev function space equipped with the norm $\|f\|_{L^p_s} := \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}f\|_{L^p}$. Moreover, it is well known that for $m > 1$, $e^{i(-\Delta)^{m/2}}$ is bounded from L^p into L^p if and only if $p = 2$ (see also [58]). On the other hand, let us consider this boundedness on modulation spaces. Bényi et al. [3] first apply modulation spaces to this problem, and get the fact that if $0 \le m \le 2$, then $e^{i(-\Delta)^{m/2}}$ is bounded from $M_{p,q}$ into $M_{p,q}$ for all $1 \le p, q \le \infty$. In contrast with the Lebesgue space, the unimodular Fourier multiplier is bounded on all modulation spaces. Thus, we can say that modulation spaces are a good class to consider this kind of problem. Especially, This fact by Bényi et al. means that the solution to the linear Schrödinger equation (1.2) belongs to $M_{p,q}$ for all $1 \le p, q \le \infty$, if the initial data u_0 is in $M_{p,q}$. Then, Miyachi et al. [70] extended the result to the case $m \ge 2$. They studied that for $m \ge 2$, $e^{i(-\Delta)^{m/2}}$ is bounded from $M^s_{p,q}$ into $M_{p,q}$, that is,

$$\|e^{i(-\Delta)^{m/2}} f\|_{M_{p,q}} \le C \|f\|_{M^s_{p,q}} \tag{1.8}$$

holds, if and only if $s \ge (m-2)n|1/p-1/2|$. Comparing their two boundedness (1.7) and (1.8), we see that the loss of regularity is smaller if we consider the unimodular Fourier transform on modulation spaces. Moreover, Chen–Fan–Sun [14] refined the results for the cases $m > 0$. The results of unimodular Fourier multipliers on modulation spaces are also applied to the theory of nonlinear partial differential equations, for instance, nonlinear damped wave equations (see Narazaki [73]).

Since these are recent works, we expect that there are more advantages of using the modulation space. In the next section, we will state some results on modulation spaces which the author got while his doctoral course.

1.2 Results

In this subsection, we collect some results will be stated in this thesis. After we give applications of modulation spaces, we will state properties of α–modulation space, which is like a generalized modulation space.

1.2.1 Applications of modulation spaces

In this section, we consider the Cauchy problems for nonlinear dispersive equations:

$$\begin{cases} i\partial_t u + \phi(\sqrt{-\Delta})u = u^{\kappa+1} \\ u(0) = u_0 \ , \end{cases} \tag{1.9}$$

where $\phi(\sqrt{-\Delta}) = \mathcal{F}^{-1}\phi(|\xi|)\mathcal{F}$ is a Fourier multiplier. As examples of ϕ, $\phi(r) = r$, $\phi(r) = r^2$, $\phi(r) = \sqrt{1+r^2}$, and $\phi(r) = \sqrt{1+r^4}$ correspond to the wave equation, the Schrödinger equation, the Klein-Gordon equation, and the beam equation, respectively. In order to solve this Cauchy problems, time decay estimates and the Strichartz estimates of the semi–group $e^{it\phi(\sqrt{-\Delta})}$ play an important role as stated in the previous section. Let a smooth function $\phi : \mathbf{R}_+ \to \mathbf{R}$ satisfy two of the conditions (H1)–(H3):

(H1) There exists $\alpha_j > 0$ ($j \in \mathbf{Z}_+$), such that for any $m \geq 3$ and $m \in \mathbf{N}$,

$$|\phi'(r)| \sim r^{\alpha_1 - 1}, \ \ |\phi''(r)| \sim r^{\alpha_2 - 2}, \ \ |\phi^{(m)}(r)| \lesssim r^{\alpha_m - m}, \quad r \geq 1.$$

(H2) There exists $\beta_j > 0$ ($j \in \mathbf{Z}_+$), such that for any $m \geq 3$ and $m \in \mathbf{N}$,

$$|\phi'(r)| \sim r^{\beta_1 - 1}, \ \ |\phi''(r)| \sim r^{\beta_2 - 2}, \ \ |\phi^{(m)}(r)| \lesssim r^{\beta_m - m}, \quad 0 < r < 1.$$

(H3) There exists $\beta_j > 0$ ($j \in \mathbf{Z}_+$), such that for any $m \geq 2$ and $m \in \mathbf{N}$,

$$|\phi'(r)| \sim r^{\beta_1 - 1}, \ \ |\phi^{(m)}(r)| \lesssim r^{\beta_m - m}, \quad 0 < r < 1.$$

Moreover, we assume that the above indexes satisfy

$$\begin{cases} \alpha_1 \geq 1, \\ \alpha_m \leq \alpha_1 + (m-1), \\ 2n - (n-1)\alpha_1 - \alpha_2 \leq 0, \\ \beta_1 \leq \beta_m, \end{cases} \quad m \geq 2. \tag{1.10}$$

If ϕ satisfies the following conditions (A) or (B);

(A) (H1) and (H3),
(B) (H1), (H2), and $\beta_1 = \beta_2$,

then we have the time decay estimate for $s \in \mathbf{R}$

$$\|e^{it\phi(\sqrt{-\Delta})} f\|_{M^s_{p,q}} \lesssim (1+|t|)^{-\theta(1/2-1/p)} \|f\|_{M^s_{p',q}},$$

where

$$\theta \ = \ \begin{cases} \min\left(\dfrac{2n}{\beta_1}, n-1\right), & \text{if } \phi \text{ satisfies (A)}, \\ \min\left(\dfrac{2n}{\beta_1}, n\right), & \text{if } \phi \text{ satisfies (B)} \end{cases}$$

(see Proposition 3.5). We remark that if we construct the similar estimates in the frame of Besov spaces, then we need the condition satisfied that

$$\begin{cases} \alpha_m \leq \alpha_1, \\ \beta_1 \leq \beta_m, \end{cases} m \geq 2 \tag{1.11}$$

(see Remark 3.6). Comparing the conditions (assu in sec1) and (1.11), we see that the condition (1.10) for modulation spaces are relaxed more than that ones (1.11) for Besov spaces. As examples satisfying the assumption (3.4) and the conditions (A) or (B), we give $\phi(r) = r^2 + \sin r$ or $\phi(r) = r^2 + \cos r$, respectively. If we construct the time decay estimates of these two phase functions on Besov spaces by the standard argument as we will use, the regularity in the right hand side get increase (Remark 3.7). However, on modulation spaces, the regularity does not change in the both sides of time decay estimates.

Moreover, as we stated in the previous section, we also get the Strichartz estimates (1.4) on modulation spaces for all extended indexes $\gamma \geq \max(\gamma(p), 2)$. Then we have a global well-posedness in $M_{2,1}$ for the Cauchy problem (1.9):

Theorem 1.1. *Let $\kappa \in \mathbf{N}$ and $\kappa \geq \frac{4}{\theta}$, and let indexes in (H1)-(H3) satisfy (1.10). There exists $\rho > 0$ such that if $u_0 \in M_{2,1}$ satisfies $\|u_0\|_{M_{2,1}} \leq \rho$, then the Cauchy problem (1.9) has a unique global solution*

$$u \in C(\mathbf{R}, M_{2,1}) \cap \ell^1_\square(L^{2+\kappa}(\mathbf{R}, L^{2+\kappa})).$$

Here, for the frequency–uniform decomposition operator $\square_k = \mathcal{F}^{-1}\sigma_k\mathcal{F}$, the function space $\ell^1_\square(X)$ is denoted as the space of all tempered distribution $f \in \mathcal{S}'$ such that the norms

$$\|f\|_{\ell^1_\square(X)} := \sum_{k\in\mathbf{Z}^n} \|\square_k f\|_X$$

is finite. This definition can be found in Section 3.1.3.

Next, we construct a different type of a global unique solution for the Cauchy problems (1.9). In order to simplify the statement, we set $\phi(\sqrt{-\Delta}) = (-\Delta)^{m/2}$, which means the nonlinear higher order Schrödinger equations:

$$\begin{cases} i\partial_t u + (-\Delta)^{m/2} u = u^{\kappa+1}, \\ u(0) = u_0. \end{cases} \tag{1.12}$$

$m = 2$ and $m = 4$ correspond to the Schrödinger equation and the fourth order Schrödinger equation, respectively. In the previous theorem, we mainly used the Strichartz estimates to prove the global well-posedness, though we will use only time decay estimates on modulation spaces, which have the following structure: for $2 \leq p \leq \infty$ and $1 \leq q \leq \infty$

$$\left\| e^{it(-\Delta)^{m/2}} u_0 \right\|_{M_{p,q}} \lesssim (1+|t|)^{-\frac{2n}{m}\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{M_{p',q}}. \tag{1.13}$$

By using only time decay estimates (1.3), Cazenave–Weissler [12] (see also [13]) showed that there exists a unique global solution to the nonlinear Schrödinger equation such that

$$\sup_{t\in\mathbf{R}} |t|^B \big\|u(t)\big\|_{L^{2+\kappa}} < +\infty,$$

if a initial data u_0 satisfies that $\sup_{t\in\mathbf{R}} |t|^B \|e^{it\Delta}u_0\|_{L^{2+\kappa}}$ is sufficiently small. Here, $\kappa \in \mathbf{R}$, $\kappa_0 < \kappa < 4/(n-2)$ ($\kappa_0 < \kappa < \infty$ if $n = 1, 2$), $B = \frac{4-(n-2)\kappa}{2\kappa(\kappa+2)}$, and $\kappa_0 > 0$ is the positive root of $n\kappa^2 + (n-2)\kappa - 4 = 0$. Moreover, Wang–Hudzik [97] also studied the existence of a unique global solution, which satisfies

$$\sup_{t\in\mathbf{R}} \langle t\rangle^{n\kappa/(2\kappa+4)} \big\|u(t)\big\|_{M_{2+\kappa,1}} < +\infty,$$

with a sufficiently small initial data $u_0 \in M_{(2+\kappa)/(1+\kappa),1}$ for $\kappa \in \mathbf{N}$ and $\kappa > \kappa_0$. As one already knows, there is no singular point at $t = 0$ in the estimate (4.4). This property enables us to extend κ as far as infinity.

When Cazenave–Weissler [12] and Wang–Hudzik [97] showed the above existence of a global solution, they controlled the behavior of solutions by multiplying weight in time to a solution space or a initial data. On the other hands, if we control them by integrability of time decay terms $(1+|t|)^{-\theta}$, then we have the following theoreom (see also Ru–Chen [79]).

Theorem 1.2. *Let $\kappa > \kappa_0$, $p \in [2, 2+\kappa]$, and $\kappa, p \in \mathbf{N}$. There exists $\rho > 0$ such that if $u_0 \in M_{(2+\kappa)/(1+\kappa),1}$ ($\subset M_{p,1}$) satisfies $\|u_0\|_{M_{(2+\kappa)/(1+\kappa),1}} \le \rho$, then the Cauchy problem (1.12) has a unique global solution*

$$u \in C(\mathbf{R}, M_{p,1}) \cap L^{1+\kappa}(\mathbf{R}, M_{2+\kappa,1}).$$

Here, κ_0 is the positive root of $n\kappa^2 + (n-m)\kappa - 2m = 0$.

In the statement of Theorem 1.2, the persistency of solutions (that is, a solution $u \in C(\mathbf{R}, X)$ if an initial data $u_0 \in X$) does not holds strictly since an initial data $u_0 \in M_{(2+\kappa)/(1+\kappa),1}$ and a solution $u \in C(\mathbf{R}, M_{p,1})$. However, it follows from $(2+\kappa)/(1+\kappa) \le p$ that the inclusion relation of modulation spaces $M_{(2+\kappa)/(1+\kappa),1} \subset M_{p,1}$. Moreover, there is no change of regularity between the initial data class and the solution class. Thus, the initial data belong to the frame of the solution space in x–space and we can say the persistency holds in a weak sense.

In order to prove Theorem 1.2, we use the integrability of time decay terms $(1+|t|)^{-\theta}$, which is the specific characteristic of modulation spaces. Since the terms $|t|^{-\theta}$ is not integrable, we don't know whether we can show the similar argument under the other spaces (see Remark 4.5 (1)). If we measure the term $|t|^{-\theta}$ on weak L^p, of course, it takes a finite value. However, we also don't know whether iteration argument works well on $L^{p,\infty}$. On the other hand, we remark that if we assume additional condition of initial data such that the linear solution $e^{it(-\Delta)^{m/2}}u_0$ is integrable in time and space, we have the similar statement to Theorem 1.2 (see [13, Chapter 6.3, Theorem 6.3.2]).

When we prove the global well-posedness in Theorem 1.1, we assume that $\kappa \geq 2m/n$. Comparing this number and κ_0, clearly we have

$$\frac{2m}{n} > \kappa_0 := \frac{(m-n) + \sqrt{n^2 + 6mn + m^2}}{2n}.$$

Thus, Theorem 1.2 enables us to treat a wider class of nonlinearity. For example, we are able to deal with $\kappa = 3$ and $\kappa = 5, 6, 7$ for the Schrödinger and the fourth order Schrödinger equations on 1-dimension, respectively. Furthermore, the solution space in Theorem 1.2 belongs to a wider class than that in Theorem 1.1 (see Remark 4.5 (3)-(4)).

In the previous two topics, we deal with the dispersive equations with the, so–called, pure power type nonlinearity. As a next step, we study a equation with derivative nonlinear terms. In particular, we consider the Cauchy problem for the generalized Zakharov–Kuznetsov (gZK) equations:

$$\begin{cases} \partial_t u + \partial_x \Delta u = \partial_x (u^{p+1}) \\ u(0) = u_0, \end{cases}$$

where $(x, y) \in \mathbf{R} \times \mathbf{R}^{n-1}$, $t > 0$, $\Delta = \partial_x^2 + \sum_{i=1}^{n-1} \partial_{y_i}^2$, $n = 2, 3$. The cases $p = 1$ ($p = 2$) corresponds as the (modified) Zakharov–Kuznetsov. The Zakharov–Kuznetsov equation was introduced by Zakharov and Kuznetsov [101] as the generalization of the Koreteweg–de Vries (KdV) equation on one dimension into multi–dimensions. This equation describes the propagation of ion–sound waves in the magnetic field in physical meanings.

We mainly study the global well–posedness for the Cauchy problem (5.1) on two dimension in the frame of modulation spaces. In the two dimensional case, scaling critical regularity is $s_c := 1 - 2/p$ (more generally, in multi–dimension, $s_c(n, p) = n/2 - 2/p$). The following results are known:

- Linares–Pastor [62] studied the local well–posedness in the classical Sobolev spaces $H^s(\mathbf{R}^2)$ for

  $$\begin{cases} s > 3/4 & \text{if } 2 \leq p \leq 7, \\ s > 1 - 3/(2p-4) & \text{if } p \geq 8. \end{cases}$$

  Moreover, for $p \geq 3$, they also studied the global well–posedness in H^1 with small initial data.

- Farah–Linares–Pastor [21] also showed that local well–posedness in $H^s(\mathbf{R}^2)$ for $s > 1 - 2/p$ and $p > 8$.

- Ribaud–Vento [77] proved that the local well–posedness in $H^s(\mathbf{R}^2)$ for

  $$\begin{cases} s > 1/4 & \text{if } p = 2, \\ s > 5/12 & \text{if } p = 3, \\ s > 1 - 2/p & \text{if } p \geq 4. \end{cases}$$

- Grünrock [34] proved the global well–posedness in the homogeneous Sobolev spaces $\dot{H}^{s_c}(\mathbf{R}^2)$ with small initial data, where $s_c = 1 - 2/p$ and $p \geq 3$ (more precisely, in homogeneous Besov spaces).

In the paper [34], Grünrock has performed symmetrization of the gZK equation, which is innovated by Grünrock–Herr [35]. Symmetrizing the gZK equation, the equation come down to the KdV–like equation: $\partial_t v + (\partial_x^3 + \partial_y^3)v = C(\partial_x + \partial_y)v^{p+1}$. Since spacial derivatives in the linear part are separated to each spacial derivative, symmetrized gZK equation is easier to be handled. Thus, symmetrization is quite useful and powerful method for the gZK equation. However, the symmetrizing transformation defines an isomorphism on modulation spaces because of non–radial–symmetric decomposition σ_k. Thus we can not use their method in this paper, unfortunately. Symmetrization of partial differential equations can be found in several papers. For example, Linares–Ponce applied it to the Davey–Stewartson equation in [60]. This equation was studied in the frame of modulation spaces by Sugimoto–Wang–Zhang [88]. However, they also do not use the symmetrization.

In order to prove the well–posedness for the gZK equation, the Kato type smoothing, Strichartz, and maximal function estimates play an crucial role. For the Kato type smoothing and Strichartz estimates, sufficiently good estimates are already got by Ribaud–Vento [76, Proposition 3.1] and Linares–Pastor [61, Lemma 2.1 or Proposition 2.4], respectively. Therefore, we mainly improve the maximal function estimate. Linares–Pastor [61, Proposition 1.5] got the following maximal function estimate: for $s > 3/4$ and $0 \leq T \leq 1$

$$\|U(t)u_0\|_{L_x^4 L_{y,T}^\infty} \lesssim \|u_0\|_{H_{x,y}^s},$$

where the unitary group $\{U(t)\}_{t=-\infty}^{t=\infty}$ denoted by $U(t) = \mathcal{F}^{-1} e^{it(\xi^3 + \xi\eta^2)}\mathcal{F}$. On the other hand, considering the frequency spaces carefully by frequency–uniform decomposition, we have for any $k \in \mathbf{Z}^2$

$$\|\Box_k U(t)u_0\|_{L_x^4 L_{y,t}^\infty} \lesssim \langle k\rangle^{1/4} \|\Box_k u_0\|_{L_{x,y}^2}$$

(see Proposition 5.11). Comparing these two maximal function estimates, there is two advantages of regularity and time–variable. The former estimate has a larger regularity $s > 3/4$ and holds locally in time, though, the latter one has a smaller regularity $s = 1/4$ and holds globally in time. Moreover, the regularity $1/4$ is the same as that one for the KdV equation on one dimension given by Kenig–Ponce–Vega [48]. Using this maximal function estimates, we prove the global well–posedness in modulation spaces with small initial data. Let auxiliary function spaces $\mathcal{X}^\theta$ satisfy that

$$\mathcal{X}^\theta := \{u \in \mathcal{S}' : \|u\|_{X\cap Y\cap Z} \leq \rho\},$$

with

$$\|u\|_X = \sum_{|k|\gg 1} \langle k\rangle^\theta \|\Box_k u\|_{L_x^\infty L_{y,t}^2},$$

$$\|u\|_Y = \sum_{k\in\mathbf{Z}^n} \|\Box_k u\|_{L_x^p L_{y,t}^\infty},$$

$$\|u\|_Z = \sum_{k\in\mathbf{Z}^n} \langle k\rangle^\theta \|\Box_k u\|_{L_t^\infty L_{x,y}^2 \cap L_{x,y,t}^{2+p}}.$$

Then, we have the following main theorem.

Theorem 1.3. *Let integers $p \geq 4$ and $n = 2$. There exists $\rho > 0$ such that if $u_0 \in M_{2,1}^{1/p}$ satisfies that $\|u_0\|_{M_{2,1}^{1/p}} \leq \rho$, then the Cauchy problem (5.1) has a unique global solution*

$$u \in C([0,\infty), M_{2,1}^{1/p}) \cap \mathcal{X}^{1/p}.$$

From Wang–Hudzik [97] (see also Kobayashi–Sugimoto [54]), we have the sharp inclusion relations between the modulation and Sobolev or Besov spaces: for any $\varepsilon > 0$.

$$\begin{aligned} M_{2,1}^s &\not\subset H^{s+\varepsilon} \cup B_{\infty,\infty}^{s+\varepsilon}, \\ H^{s+n/2+\varepsilon} \subset B_{2,1}^{s+n/2} \subset M_{2,1}^s &\subset H^s. \end{aligned}$$

If the regularity in the right hand side are made larger, these embedding relations does not hold (see Wang [94, Appendix B]). Moreover, we also have the inclusions: $H^{s+n/2} \not\subset M_{2,1}^s$ by Wang–Zhao–Guo [99]. Thus, we see hat we deal with the global well–posedness for gZK equation in the new class of functions.

We remark that the sharpness of the regularity $1/p$ is not completely understood. For the KdV equation on one dimension, the global well–posed in $M_{2,1}$ with small initial data has been proved by Wang–Huang [96]. On the other hand, in the y–axis space, we can regard the semi–group of Zakharov–Kuznetsov equation as that of the Schrödinger equation. For the Schrödinger equation with derivative nonlinear terms: $i\partial_t u + \Delta u = \sum_{i=1}^n \partial_{x_i}(u^{p+1})$, the global well–posedness in $M_{2,1}^{1/p}$ with small initial data has been proved by Wang [94] (see also [95]). Moreover, Wang also studied the ill–posedness in $M_{2,1}^s$ for $s < 1/p$ in [94]. Thus, we can understand $1/p$ as the valid regularity.

1.2.2 Characterization of α–modulation spaces

The α–modulation space was first invented by Gröbner [30] to link Besov and modulation spaces by the parameter $0 \leq \alpha \leq 1$. Then its theory was developed by Borup–Nielsen [4, 5], Feichtinger–Gröbner [23], and Fornaisier [26]. Besov spaces $B_{p,q}^s$ and modulation spaces $M_{p,q}^s$ are constructed by using partitions of unity of frequency spaces called dyadic decomposition $\{\phi_j\}_{j\in\mathbf{N}\cup\{0\}}$ and frequency uniform decomposition $\{\sigma_k\}_{k\in\mathbf{Z}^n}$, respectively. Their partitions of unity are deduced from two well–known decomposition of $\mathbf{R}^n$, which are by Littlewood–Paley decomposition;

$$\mathbf{R}^n = \{|\xi| \leq 1\} \cup (\bigcup_{j=0}^\infty \{2^j < |\xi| \leq 2^{j+1}\})$$

and the Wiener decomposition [100];

$$\mathbf{R}^n = \bigcup_{k\in\mathbf{Z}^n} k + [-1/2, 1/2)^n.$$

In order to link these two decomposition, the so–called α–covering was introduced. Let a countable set $\mathcal{Q}$ of subsets $Q \subset \mathbf{R}^n$ be an admissible covering, that is, satisfy that

$$\mathbf{R}^n = \bigcup_{Q \in \mathcal{Q}} Q, \text{ and } \#\{Q \in \mathcal{Q} : Q \cap Q' \neq \emptyset\} \leq \exists n_0 \text{ for all } Q \in \mathcal{Q} .$$

Then, an admissible covering $\mathcal{Q}$ is called α–covering ($0 \leq \alpha \leq 1$) if

- $|Q| \sim \langle x \rangle^{\alpha n}$ for all $x \in Q$ and all $Q \in \mathcal{Q}$;
- there exists a constant $K \geq 1$ such that $R_Q / r_Q \leq K$ for all $Q \in \mathcal{Q}$,

where $r_Q = \sup\{r > 0 : B(c_r, r) \subset Q \text{ for some } c_r \in \mathbf{R}^n\}$ and $R_Q = \inf\{R > 0 : B(c_R, R) \subset Q \text{ for some } c_R \in \mathbf{R}^n\}$. As an example of α–covering ($\alpha \neq 0, 1$), we can give $\mathbf{R}^n = \{|\xi| \leq 1\} \cup (\bigcup_{j=0}^{\infty} \{j < |\xi| \leq j+1\})$, and we see that this is $(1 - 1/n)$–covering. For an α–covering $\mathcal{Q}$, we set a family of functions $\{\psi_Q\}_{Q \in \mathcal{Q}}$ satisfying

- supp $\psi_Q \subset Q$;
- $\sum_{Q \in \mathcal{Q}} \psi_Q \equiv 1$;
- $\sup_{Q \in \mathcal{Q}} |Q|^{-1+1/\min(1,p)} \left\| \mathcal{F}^{-1} \psi_Q \right\|_{L^{\min(1,p)}} < \infty$.

and denote the α–modulation spaces $M^{s,\alpha}_{p,q}$ by all tempered distribution $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the norm

$$\|f\|^{original}_{M^{s,\alpha}_{p,q}} = \left(\sum_{Q \in \mathcal{Q}} \langle \xi_Q \rangle^{sq} \left\| \mathcal{F}^{-1} \psi_Q \mathcal{F} f \right\|_p^q \right)^{1/q}$$

is finite, where a sequence $\{\xi_Q\}_{Q \in \mathcal{Q}}$ satisfies $\xi_Q \in Q$. It is known that the cases $\alpha = 1$ and $\alpha = 0$ are equivalent to Besov and modulation spaces, respectively. Moreover, Borup–Nielsen [4] proved the equivalent norm of the α–modulation space. Let a family of functions $\{\eta_k^\alpha\}_{k \in \mathbf{Z}^n}$ satisfy that

- supp $\eta_k^\alpha \subset B(\langle k \rangle^{\alpha/(1-\alpha)} k, C \langle k \rangle^{\alpha/(1-\alpha)})$;
- $\sum_{k \in \mathbf{Z}^n} \eta_k^\alpha \equiv 1$.

Then, for $0 < p, q \leq \infty$, $s \in \mathbf{R}$, and $\alpha \in [0, 1)$,

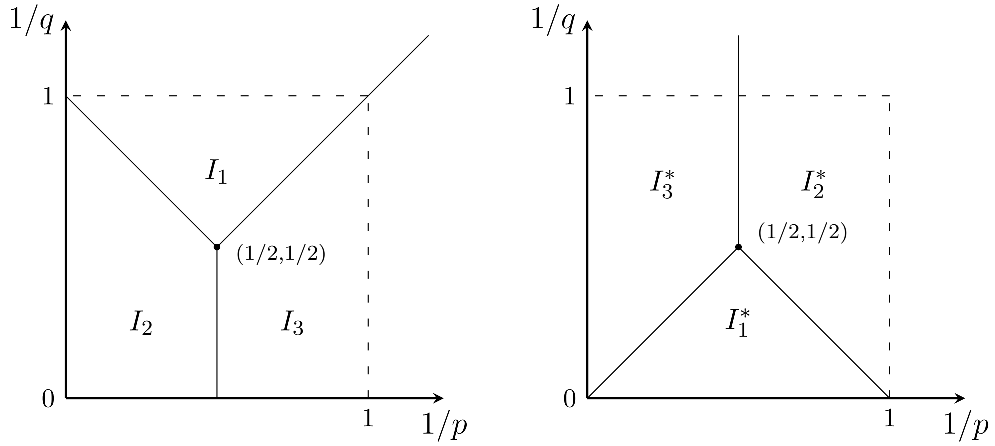
$$\|f\|^{original}_{M^{s,\alpha}_{p,q}} \sim \|f\|_{M^{s,\alpha}_{p,q}} := \left(\sum_{k \in \mathbf{Z}^n} \langle k \rangle^{sq/(1-\alpha)} \left\| \mathcal{F}^{-1} \eta_k^\alpha \mathcal{F} f \right\|_p^q \right)^{1/q} .$$

Elementary properties can be found in Section 2.2.4. We only state inclusion relations of α–modulation spaces in this section. Before that, we give notations will be used. We set for $0 < p, q \leq \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$,

$$\nu_1(p,q) = \begin{cases} 0 & \text{if } (1/p, 1/q) \in I_1^* : \min(1/p, 1/p') \geq 1/q, \\ 1/p + 1/q - 1 & \text{if } (1/p, 1/q) \in I_2^* : \min(1/q, 1/2) \geq 1/p', \\ -1/p + 1/q & \text{if } (1/p, 1/q) \in I_3^* : \min(1/q, 1/2) \geq 1/p, \end{cases}$$

$$\nu_2(p,q) = \begin{cases} 0 & \text{if } (1/p,1/q) \in I_1 : \max(1/p,1/p') \le 1/q, \\ 1/p+1/q-1 & \text{if } (1/p,1/q) \in I_2 : \max(1/q,1/2) \le 1/p', \\ -1/p+1/q & \text{if } (1/p,1/q) \in I_3 : \max(1/q,1/2) \le 1/p. \end{cases}$$

We remark that $\nu_1(p,q) = -\nu_2(p',q')$, and $\nu_1(p,q) \ge 0$, $\nu_2(p,q) \le 0$.



The index sets of ν_1 and ν_2

Then the following embedding relations between Besov and α–modulation spaces is given:

Theorem 1.4. *(See [37, Theorem 4.2.]) Let* $0 < p,q \le \infty$, $s_1, s_2 \in \mathbf{R}$, *and* $0 \le \alpha < 1$. *Then*
(1) $B^{s_1}_{p,q} \subset M^{s_2,\alpha}_{p,q}$ *holds if and only if* $s_1 \ge s_2 + n(1-\alpha)\nu_1(p,q)$ *is satisfied;*
(2) $M^{s_1,\alpha}_{p,q} \subset B^{s_2}_{p,q}$ *holds if and only if* $s_1 \ge s_2 + n(\alpha-1)\nu_2(p,q)$ *is satisfied.*

From the inclusion relation $L^p_{s+\varepsilon} \hookrightarrow B^s_{p,q} \hookrightarrow L^p_{s-\varepsilon}$ for $\varepsilon > 0$, (see [92, Remark3 in Section 2.3.2]), the similar inclusion relations between Sobolev and α–modulation spaces are deduced from Theorem 1.4. However, the critical indexes $s_1 = s_2 + n(1-\alpha)\nu_1(p,q)$ and $s_1 = s_2 + n(\alpha-1)\nu_2(p,q)$ are included in necessary conditions, though that ones are not included in sufficient conditions (see Corollary 6.2). Thus, the aim of this section is to determine whether these critical cases are needed or not. The exact answers are the following theorems:

Theorem 1.5. *Let* $1 \le p,q \le \infty$ *and* $s \in \mathbf{R}$. *Then,* $M^{0,\alpha}_{p,q}(\mathbf{R}^n) \hookrightarrow L^p_s(\mathbf{R}^n)$ *holds if and only if one of the following conditions is satisfied.*
(1) $\infty > p \ge q$ *and* $s \le n(1-\alpha)\nu_2(p,q)$;
(2) $p < q$ *and* $s < n(1-\alpha)\nu_2(p,q)$;
(3) $p = \infty$, $q = 1$, *and* $s \le n(1-\alpha)\nu_2(\infty,1)$;
(4) $p = \infty$, $q \ne 1$, *and* $s < n(1-\alpha)\nu_2(\infty,q)$.

Theorem 1.6. *Let* $1 \le p,q \le \infty$ *and* $s \in \mathbf{R}$. *Then,* $L^p_s(\mathbf{R}^n) \hookrightarrow M^{0,\alpha}_{p,q}(\mathbf{R}^n)$ *holds if and only if one of the following conditions is satisfied.*
(1) $1 < p \le q$ *and* $s \ge n(1-\alpha)\nu_1(p,q)$;
(2) $p > q$ *and* $s > n(1-\alpha)\nu_1(p,q)$;
(3) $p = 1$, $q = \infty$, *and* $s \ge n(1-\alpha)\nu_1(1,\infty)$;
(4) $p = 1$, $q \ne \infty$, *and* $s > n(1-\alpha)\nu_1(1,q)$.

Finally, we remark that for the case when $\alpha = 0$, inclusion relations between modulation spaces and Besov spaces are already studied by Gröbner [30], Toft [89], Sugimoto–Tomita [86], Wang–Huang [96]. Moreover, inclusion relations between modulation spaces and L^p–Sobolev spaces are also given by Kobayashi–Sugimoto [54].

2 Preliminaries

2.1 Basic notations

In this first subsection, we write notations will be used throughout this paper. **R**, **N**, and **Z** denote the sets of reals, positive integers, and integers, respectively. Moreover, we write $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$. We assume that the constants c and C satisfy $0 < c < 1$ and $C > 1$, which are different on each occasion. $a \lesssim b$ means that $a \le Cb$, and $a \sim b$ means that $a \lesssim b$ and $a \gtrsim b$. For $1 \le p \le \infty$, we set p' as a dual number of p, i.e. $1/p + 1/p' = 1$. For $0 < p < 1$, $p' = \infty$. We use some function spaces; Schwartz space $\mathcal{S} := \mathcal{S}(\mathbf{R}^n)$, and its dual space $\mathcal{S}' := \mathcal{S}'(\mathbf{R}^n)$. the Lebesgue spaces $L^p := L^p(\mathbf{R}^n)$ with the norm

$$\|f\|_p := \|f\|_{L^p} = \left(\int_{\mathbf{R}^n} |f(x)|^p \, dx \right)^{1/p} \quad \text{for } 0 < p < \infty.$$

If $p = \infty$, $\|f\|_\infty := \operatorname{ess.sup}_{x \in \mathbf{R}^n} |f(x)|$. L^p-Sobolev spaces $L^s_p := L^s_p(\mathbf{R}^n)$ with the norm

$$\|f\|_{L^p_s} = \|\mathcal{F}^{-1} \langle \xi \rangle^s \mathcal{F} f\|_{L^p},$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. We also set the space–time mixed norms: for $(x, y) \in \mathbf{R} \times \mathbf{R}^{n-1}$, $t \in [0, \infty)$ and $1 \le p, q < \infty$,

$$\begin{aligned} \|f\|_{L^p_t L^q_{x,y}} &:= \left\| \|f\|_{L^q_{x,y}} \right\|_{L^p_t} = \left(\int_{\mathbf{R}_t} \left(\int_{\mathbf{R}^n_{x,y}} |f(x,y,t)|^q dx dy \right)^{p/q} dt \right)^{1/p}, \\ \|f\|_{L^p_x L^q_{y,t}} &:= \left\| \|f\|_{L^q_{y,t}} \right\|_{L^p_x} = \left(\int_{\mathbf{R}_x} \left(\int_{\mathbf{R}^n_{y,t}} |f(x,y,t)|^q dt dy \right)^{p/q} dx \right)^{1/p} \end{aligned}$$

with obvious modifications if $p = \infty$ or $q = \infty$. $\mathcal{F}$ and $\widehat{\cdot}$ denote the Fourier transformation by

$$\mathcal{F} f(\xi) = \widehat{f}(\xi) := \int_{\mathbf{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

and $\mathcal{F}^{-1}$ denotes the inverse Fourier transformation by

$$\mathcal{F}^{-1} f(x) := \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i\xi \cdot x} f(\xi) d\xi.$$

Especially, to make the notations clearly, we sometimes write the Fourier transform as follows: for $f \in \mathcal{S}$ and $x, y, \xi, \eta \in \mathbf{R}$

$$\begin{aligned} \mathcal{F}_x f(\xi) &= \int_{\mathbf{R}_x} e^{-ix\xi} f(x) dx, \\ \mathcal{F}^{-1}_\xi f(x) &= \frac{1}{(2\pi)} \int_{\mathbf{R}_\xi} e^{i\xi x} f(\xi) d\xi, \end{aligned}$$

$$\begin{aligned}\mathcal{F}_{x,y}f(\xi,\eta) &= \mathcal{F}_x\mathcal{F}_y f(\xi,\eta) = \mathcal{F}_y\mathcal{F}_x f(\xi,\eta),\\ \mathcal{F}^{-1}_{\xi,\eta}f(x,y) &= \mathcal{F}^{-1}_\xi\mathcal{F}^{-1}_\eta f(x,y) = \mathcal{F}^{-1}_\eta\mathcal{F}^{-1}_\xi f(x,y).\end{aligned}$$

In the end of this subsection, we introduce the following inequality for convolution $f * g$:

Proposition 2.1. *(See [91, Section 1.5.3]) Let $0 < p \leq 1$. We set $L^p_\Omega(\mathbf{R}) = \{f \in L^p(\mathbf{R}) : supp\ \mathcal{F}f \subset \Omega\}$ and $\Omega = \{x \in \mathbf{R}^n : |x - x_0| \leq R\}$. Then there exists a constant $C = C_p > 0$ such that*

$$\|f * g\|_p \leq CR^{n(1/p-1)}\|f\|_p\|g\|_p$$

for any $f, g \in L^p_\Omega$.

2.2 Function spaces

2.2.1 Besov and Triebel–Lizorkin spaces

In this subsection, we recall Besov and Triebel–Lizorkin spaces. Let $0 < p, q \leq \infty$ and $s \in \mathbf{R}$. Set the partitions of unity $\{\phi_j\}_{j\in\mathbf{Z}_+}$ satisfying that

- supp $\phi_0 \subset B(0,2)$;
- $\phi_j = \phi(\cdot/2^j)$, where supp $\phi \subset \{\xi : 1/2 \leq |\xi| \leq 2\}$;
- $\displaystyle\sum_{j\in\mathbf{Z}_+} \phi_j \equiv 1$

We also set the the operators; for $j \in \mathbf{Z}_+$

$$\triangle_j := \mathcal{F}^{-1}\phi_j\mathcal{F}.$$

Here these operators $\triangle_j$ are called Littlewood–Paley decomposition operators (or dyadic decomposition operators). Then, we denote the Besov space $B^s_{p,q}$ and Triebel–Lizorkin $F^s_{p,q}$ as follows:

$$B^s_{p,q} := \left\{f \in \mathcal{S}'(\mathbf{R}^n) : \|f\|_{B^s_{p,q}} < +\infty\right\},$$

with the norm

$$\|f\|_{B^s_{p,q}} := \left(\sum_{j\in\mathbf{Z}_+} 2^{jsq}\,\|\triangle_j f\|^q_{L^p}\right)^{1/q},$$

and

$$F^s_{p,q} := \left\{f \in \mathcal{S}'(\mathbf{R}^n) : \|f\|_{F^s_{p,q}} < +\infty\right\},$$

with the norm

$$\|f\|_{F^s_{p,q}} := \left\|\left(\sum_{j\in\mathbf{Z}_+} 2^{jsq}|\triangle_j f|^q\right)^{1/q}\right\|_{L^p}.$$

We have the inclusion relation between these two function spaces as follows: Let $0 < p_1 < p < p_2 \le \infty$, $0 < q \le \infty$, $-\infty < s_1 < s < s_2 < \infty$, and

$$s_1 - \frac{n}{p_1} = s - \frac{n}{p} = s_2 - \frac{n}{p_2}.$$

Then,

$$B^{s_1}_{p_1,p} \hookrightarrow F^s_{p,q} \hookrightarrow B^{s_2}_{p_2,p}.$$

Moreover, the following relation is well known (see Triebel [91, Section 2.3.5]): for $1 < p < \infty$

$$\|f\|_{L^p} \sim \|f\|_{F^0_{p,2}}.$$

We also note that $B^s_{p,q}$ and $F^s_{p,q}$ are a quasi–Banach space (a Banach space if $1 \le p, q \le \infty$) and independent of the choice of window function for Littlewood–Paley decomposition operators. $\mathcal{S} \subset B^s_{p,q}, F^s_{p,q} \subset \mathcal{S}'$, and especially if $1 \le p, q < \infty$ $\mathcal{S}$ is dense in $B^s_{p,q}$ and $F^s_{p,q}$. More properties about Besov and Triebel–Lizorkin spaces can be found in Triebel [91, 92].

2.2.2 Local Hardy spaces

We state local Hardy spaces given by Goldberg [29]. Let $0 < p < \infty$. Suppose that a Schwartz function Φ satisfy that $\int_{\mathbf{R}^n} \Phi(\xi)d\xi \neq 0$. Then we denote the local Hardy space h^p as follows:

$$h^p(\mathbf{R}^n) := \left\{ f \in \mathcal{S}' : \|f\|_{h^p} = \left\| \sup_{0<t<1} \frac{1}{t^n} \left| \Phi\left(\frac{\cdot}{t}\right) * f \right| \right\|_{L^p} < \infty \right\}.$$

We note that h^p is independent of the choice of the $\Phi \in \mathcal{S}$, and satisfies that $h^1 \hookrightarrow L^1$ and $h^p = L^p$ if $1 < p < \infty$. Moreover, $h^p = F^0_{p,2}$ if $0 < p < \infty$ (see [91, Theorem 1 in Section 2.5.8]). The complex interpolation theorem is given as follows: Let $0 < \theta < 1$ and $1/p = (1-\theta)/p_1 + \theta/p_2$, then

$$(h^{p_1}, h^{p_2})_\theta = h^p$$

(see [91, Remark 1 in Section 2.4.7]). In the end of this subsection, we state the atomic decomposition of h^p (see [29, Section 4]). An h^p atom $a(x)$ is called type I if

- supp $a \subset Q$ with $|Q| < 1$;
- $\|a\|_\infty \le |Q|^{-1/p}$;
- $\displaystyle\int_{\mathbf{R}^n} x^\beta a(x)dx = 0$ for all $|\beta| \le [n(1/p-1)]$,

where Q is a cube and $|Q|$ is its Lebesgue measure, and $[\cdot] = \max\{n \in \mathbf{Z} : n \le x\}$. On the other hands, an h^p atom $a(x)$ is called type II if

- supp $a \subset Q$ with $|Q| \ge 1$;

- $\|a\|_\infty \le |Q|^{-1/p}$.

We note that all h^p atoms of type I and type II are bounded in h^p, and all $f \in h^p$ is expressed as $f = \sum_{j\in\mathbf{N}} \lambda_j a_j$, where $\{a_j\}_{j\in\mathbf{N}}$ is a family of h^p–atoms of types I and II and $\{\lambda_j\}_{j\in\mathbf{N}}$ is a complex number sequence belongs to ℓ^p. Moreover, $\|f\|_{h^p} \sim \inf \|\lambda_j\|_{\ell^p(\mathbf{N})}$, where we take the infimum on all representations $f = \sum_{j\in\mathbf{N}} \lambda_j a_j$ (see [29, Lemma 5]).

2.2.3 Modulation spaces

Next, we give the definition and some properties of the modulation spaces. Let $0 < p, q \le \infty$ and $s \in \mathbf{R}$. Let $\rho \in \mathcal{S}(\mathbf{R}^n)$ and let $\rho : \mathbf{R}^n \to [0,1]$ be a smooth and radial function supported in $B(0, \sqrt{n})$ and satisfy as follows:

$$\rho(\xi) = \begin{cases} 1\,, & |\xi| \le \sqrt{n}/2, \\ 0\,, & |\xi| \ge \sqrt{n}\,. \end{cases}$$

Since $\sum_{k\in\mathbf{Z}^n} \rho(\xi - k) \ge 1$ for all $\xi \in \mathbf{R}^n$, we set for $k \in \mathbf{Z}^n$

$$\sigma_k(\xi) = \rho(\xi - k)\Big(\sum_{m\in\mathbf{Z}^n} \rho(\xi - m)\Big)^{-1}.$$

Then we have some properties of $\{\sigma_k\}_{k\in\mathbf{Z}^n}$:

$$\begin{cases} |\sigma_k(\xi)| \ge c\,, \quad \forall \xi \in \mathcal{Q}_k\,, \\ \operatorname{supp} \sigma_k \subset \{\xi \in \mathbf{R}^n \;:\; |\xi - k| \le \sqrt{n}\}\,, \\ \sum_{k\in\mathbf{Z}^n} \sigma_k(\xi) \equiv 1, \quad \forall \xi \in \mathbf{R}^n\,, \\ |D^\alpha \sigma_k(\xi)| \le C_{|\alpha|}\,, \quad \forall \xi \in \mathbf{R}^n\,, \forall \alpha \in \mathbf{Z}^n_+ \end{cases} \tag{2.1}$$

where $\mathcal{Q}_k$ is a unit cube with the center of $k \in \mathbf{Z}^n$ and $\alpha \in \mathbf{Z}^n_+$ is a multi-index. So, the set

$$\Gamma := \{\,\{\sigma_k\}_{k\in\mathbf{Z}^n} \;:\; \{\sigma_k\}_{k\in\mathbf{Z}^n} \text{ satisfies (2.1)}\}$$

is not empty. Let $\{\sigma_k\}_{k\in\mathbf{Z}^n} \in \Gamma$. We denote operators for $k \in \mathbf{Z}^n$

$$\Box_k := \mathcal{F}^{-1}\sigma_k\mathcal{F},$$

which are called frequency–uniform decomposition operators. Then we denote

$$M^s_{p,q} := \left\{f \in \mathcal{S}'(\mathbf{R}^n) : \|f\|_{M^s_{p,q}} < \infty\right\},$$

with the norm

$$\|f\|_{M^s_{p,q}} := \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq} \|\Box_k f\|_p^q\right)^{1/q}.$$

Here $|k| = |k_1| + \cdots + |k_n|$, $\langle k\rangle = 1 + |k|$. We will simply write $M^0_{p,q} = M_{p,q}$. $M^s_{p,q}$ are called modulation spaces and these spaces are introduced by Feichtinger [22] in 1983.

We note that $M^s_{p,q}$ are a quasi–Banach space (a Banach space if $1 \le p, q \le \infty$) and independent of the choice of window function for frequency-uniform decomposition operator, namely, let $\{\sigma_k\}_{k\in\mathbf{Z}^n}$, $\{\varphi_k\}_{k\in\mathbf{Z}^n} \in \Gamma$, and let norms of $M^s_{p,q}$ with $\{\sigma_k\}_{k\in\mathbf{Z}^n}$ and $\{\varphi_k\}_{k\in\mathbf{Z}^n}$ be $\|\cdot\|^{\{\sigma_k\}}_{M^s_{p,q}}$ and $\|\cdot\|^{\{\varphi_k\}}_{M^s_{p,q}}$ respectively, then we have $\|\cdot\|^{\{\sigma_k\}}_{M^s_{p,q}} \sim \|\cdot\|^{\{\varphi_k\}}_{M^s_{p,q}}$. $\mathcal{S} \subset M^s_{p,q} \subset \mathcal{S}'$, and especially if $1 \le p, q < \infty$ $\mathcal{S}$ is dense in $M^s_{p,q}$. The dual of $M^{s,\alpha}_{p,q}$ is identified as follows:

$$\left(M^s_{p,q}\right)' = M^{-s}_{p',q'},$$

where $p' = [\max(1,p)]'$, $q' = [\max(1,q)]'$. Moreover, the complex interpolation theorem is given by Feichtinger [22, Theorem 6.1] as follows: Let $0 < \theta < 1$ and

$$s = (1-\theta)s_1 + \theta s_2, \ \ \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ \ \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2},$$

then

$$\left(M^{s_1}_{p_1,q_1}, M^{s_2}_{p_2,q_2}\right)_\theta = M^s_{p,q}.$$

The following inclusions holds true: if $s_1 \ge s_2$, $1 \le p_1 \le p_2$, and $1 \le q_1 \le q_2$, then

$$M^{s_1}_{p_1,q_1} \subset M^{s_2}_{p_2,q_2}. \tag{2.2}$$

We conclude this subsection by stating the following proposition, which will be used to estimate nonlinearity of dispersive equations:

Proposition 2.2. ([99, Lemma 4.1] on modulation spaces.) *Let $-\infty < s < \infty$, $1 \le p, q \le \infty$, and $1/p = 1/p_1 + 1/p_2$. Then, we have*

$$\|uv\|_{M^s_{p,q}} \lesssim \|u\|_{M^{\max\{s,0\}}_{p_1,1}} \|v\|_{M^{\max\{s,0\}}_{p_2,1}}.$$

Moreover, let $1/p = \sum_{j=1}^m 1/p_j$, then we have

$$\Big\|\prod_{j=1}^m u_j\Big\|_{M^s_{p,q}} \lesssim \prod_{j=1}^m \|u_j\|_{M^{\max\{s,0\}}_{p_j,1}}.$$

2.2.4 α–modulation spaces

Lastly, we introduce the definition of α–modulation space and its properties. α–modulation space is innovated by Gröbner [30] in 1992, so as to connect Besov and modulation spaces by the parameter $0 \le \alpha \le 1$. $\alpha \in [0,1]$ is the parameter to determine how to decompose the frequency space. Before starting the definition of the α–modulation spaces, let us prepare some notations. We set $B(x,r)$ as the ball with center $x \in \mathbf{R}^n$ and radius $r > 0$, and $\langle x\rangle = (1+|x|^2)^{1/2}$. A countable set $\mathcal{Q}$ of subsets $Q \subset \mathbf{R}^n$ is called an admissible covering if the both of the following statement satisfy:

- $\mathbf{R}^n = \bigcup_{Q\in\mathcal{Q}} Q$;

- there exists a constant $n_0 < \infty$ such that $\#\{Q \in \mathcal{Q} : Q \cap Q' \neq \emptyset\} \leq n_0$ for all $Q \in \mathcal{Q}$

Moreover, an admissible covering $\mathcal{Q}$ is called α–covering ($0 \leq \alpha \leq 1$) if

- $|Q| \sim \langle x \rangle^{\alpha n}$ for all $x \in Q$ and all $Q \in \mathcal{Q}$;
- there exists a constant $K \geq 1$ such that $R_Q/r_Q \leq K$ for all $Q \in \mathcal{Q}$,

where $r_Q = \sup\{r > 0 : B(c_r, r) \subset Q$ for some $c_r \in \mathbf{R}^n\}$ and $R_Q = \inf\{R > 0 : B(c_R, R) \subset Q$ for some $c_R \in \mathbf{R}^n\}$. Partitions of unity are also required to define the α–modulation spaces. Let $0 < p \leq \infty$, and let $\mathcal{Q}$ be an α–covering of $\mathbf{R}^n$. Then, we set a family of functions $\{\psi_Q\}_{Q \in \mathcal{Q}}$ satisfying

- supp $\psi_Q \subset Q$;
- $\displaystyle\sum_{Q \in \mathcal{Q}} \psi_Q \equiv 1$;
- $\displaystyle\sup_{Q \in \mathcal{Q}} |Q|^{-1+1/\min(1,p)} \left\| \mathcal{F}^{-1} \psi_Q \right\|_{L^{\min(1,p)}} < \infty$.

Now, we denote the (original) α–modulation spaces $M^{s,\alpha}_{p,q}$ by all tempered distribution $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the norm

$$\|f\|^{original}_{M^{s,\alpha}_{p,q}} = \left(\sum_{Q \in \mathcal{Q}} \langle \xi_Q \rangle^{sq} \left\| \mathcal{F}^{-1} \psi_Q \mathcal{F} f \right\|_p^q \right)^{1/q}$$

is finite, where a sequence $\{\xi_Q\}_{Q \in \mathcal{Q}}$ satisfies $\xi_Q \in Q$. It is known that the cases $\alpha = 1$ and $\alpha = 0$ are equivalent to Besov and modulation spaces, respectively. Since this definition is difficult to handle and apply to the other research fields, however, we write the equivalent definition to α–modulation spaces which is proved by Borup–Nielsen [4]. Let a Schwartz function sequence $\{\eta^\alpha_k\}_{k \in \mathbf{Z}^n}$ satisfy that

- supp $\eta^\alpha_k \subset B(\langle k \rangle^{\alpha/(1-\alpha)} k, C \langle k \rangle^{\alpha/(1-\alpha)})$;
- $\displaystyle\sum_{k \in \mathbf{Z}^n} \eta^\alpha_k \equiv 1$.

Then, we have for $0 < p, q \leq \infty$, $s \in \mathbf{R}$, and $\alpha \in [0, 1)$,

$$\|f\|^{original}_{M^{s,\alpha}_{p,q}} \sim \|f\|_{M^{s,\alpha}_{p,q}} := \left(\sum_{k \in \mathbf{Z}^n} \langle k \rangle^{sq/(1-\alpha)} \left\| \mathcal{F}^{-1} \eta^\alpha_k \mathcal{F} f \right\|_p^q \right)^{1/q},$$

which is easier to handle in the case when $0 \leq \alpha < 1$ (at least it is for this thesis). Thus, we rewrite the definition of α–modulation spaces ($0 \leq \alpha < 1$) as follows:

$$M^{s,\alpha}_{p,q} := \left\{ f \in \mathcal{S}'(\mathbf{R}^n) : \|f\|_{M^{p,q}_{s,\alpha}} < \infty \right\}.$$

We will call this equivalent definition as α–modulation spaces throughout this paper.

We note that $M^{s,\alpha}_{p,q}$ is a quasi–Banach space (a Banach space for $1 \le p, q \le \infty$) and $\mathcal{S} \subset M^{s,\alpha}_{p,q} \subset \mathcal{S}'$. Especially, for $0 < p, q < \infty$, $\mathcal{S}$ is dense in $M^{s,\alpha}_{p,q}$ (see Borup and Nielsen [5]). $M^{s,\alpha}_{p,q}$ is independent of the choice of the partitions of unity $\{\eta^\alpha_k\}_{k\in\mathbf{Z}^n}$. The dual of $M^{s,\alpha}_{p,q}$ is identified as follows (see [37, Theorem 2.1]):

$$\left(M^{s,\alpha}_{p,q}\right)' = M^{-s',\alpha}_{p',q'},$$

where $p' = [\max(1,p)]'$, $q' = [\max(1,q)]'$, and $s' = s+n\alpha(1-1/\min(1,p))$. Moreover, the complex interpolation theorem is given by Han–Wang [37, Theorem 2.2] as follows: Let $0 < \theta < 1$ and

$$s = (1-\theta)s_1 + \theta s_2, \ \ \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \ \ \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2},$$

then

$$\left(M^{s_1,\alpha}_{p_1,q_1}, M^{s_2,\alpha}_{p_2,q_2}\right)_\theta = M^{s,\alpha}_{p,q}.$$

Next, we introduce some embedding relations:

Proposition 2.3. *(See [91]) Let $\sigma \in \mathbf{R}$. Then the mapping $(I-\Delta)^{\sigma/2} : M^{s,\alpha}_{p,q} \hookrightarrow M^{s-\sigma,\alpha}_{p,q}$ is isomorphic.*

Proposition 2.4. *(See [30]) Let $0 \le \alpha < 1$ and $s \in \mathbf{R}$. Then, $M^{s,\alpha}_{2,2} = H^s$.*

Proposition 2.5. *Let $1 \le p \le \infty$, $1/p + 1/p' = 1$, and $s \in \mathbf{R}$. Then we have*

$$M^{s,\alpha}_{p,\min(p,p')} \subset L^p_s \subset M^{s,\alpha}_{p,\max(p,p')}.$$

Proof of Proposition 2.5 Obviously, we have

$$\|f\|_{L^1} \le \sum_{k\in\mathbf{Z}^n} \|\mathcal{F}^{-1}\eta^\alpha_k \mathcal{F} f\|_{L^1}.$$

Interpolating this fast and $M^{0,\alpha}_{2,2} \approx L^2$, then $M^{0,\alpha}_{p,p} \hookrightarrow L^p$ if $1 \le p \le 2$. By duality, we also have $L^p \hookrightarrow M^{0,\alpha}_{p,p}$ if $2 \le p \le \infty$. Next, we obtain from Proposition 6.7

$$\begin{aligned}
\|f\|_{M^{0,\alpha}_{1,\infty}} &\sim \sup_{k\in\mathbf{Z}^n} \|\mathcal{F}^{-1}\rho^\alpha_k \mathcal{F} f\|_{L^1} \\
&\le \sup_{k\in\mathbf{Z}^n} \|\mathcal{F}^{-1}\rho^\alpha_k\|_{L^1} \|f\|_{L^1} \\
&= \sup_{k\in\mathbf{Z}^n} \left\| \int_{\mathbf{R}^n} e^{ix\cdot\xi} \rho\left(\frac{\xi - \langle k\rangle^A k}{C\langle k\rangle^A}\right) d\xi \right\|_{L^1} \|f\|_{L^1} \\
&\sim \|f\|_{L^1}.
\end{aligned}$$

Again, interpolating this estimate and $M^{0,\alpha}_{2,2} \approx L^2$, then $L^p \hookrightarrow M^{0,\alpha}_{p,p'}$ if $1 \le p \le 2$. By duality, we also have $M^{0,\alpha}_{p,p'} \hookrightarrow L^p$ if $2 \le p \le \infty$. Therefore, by the lifting operator in Proposition 2.3 we have the desired statement. □

Proposition 2.6. *(See [37, Proposition 2.4.]) Let* $0 < p_1 \le p_2 \le \infty$, $0 < q_1, q_2 \le \infty$, $s \in \mathbf{R}$, *and* $0 \le \alpha < 1$. *Then*

$$M^{s_1,\alpha}_{p_1,q_1} \subset M^{s_2,\alpha}_{p_2,q_2}.$$

holds if either of the following conditions is satisfied:
(1) $q_1 \le q_2$ *and* $s_1 \ge s_2 + n\alpha(1/p_1 - 1/p_2)$;
(2) $q_1 > q_2$ *and* $s_1 > s_2 + n\alpha(1/p_1 - 1/p_2) + n(1-\alpha)(1/q_2 - 1/q_1)$.

Theorem 2.7. *(See [37, Theorem 4.1.]) Let* $0 < p, q \le \infty$, $s_1, s_2 \in \mathbf{R}$, *and* $0 \le \alpha_1, \alpha_2 < 1$. *Then*

$$M^{s_1,\alpha_1}_{p,q} \subset M^{s_2,\alpha_2}_{p,q}$$

holds if and only if either of the following conditions is satisfied:
(1) $\alpha_1 \ge \alpha_2$ *and* $s_1 \ge s_2 + n(\alpha_1 - \alpha_2)\nu_1(p,q)$;
(2) $\alpha_1 < \alpha_2$ *and* $s_1 \ge s_2 + n(\alpha_1 - \alpha_2)\nu_2(p,q)$.

3 The global Cauchy problems for the nonlinear dispersive equations on modulation spaces

3.1 Introduction and results

3.1.1 Introduction

In this paper, we study the Cauchy problems for nonlinear dispersive equations (NLD):

$$\begin{cases} i\partial_t u + \phi(\sqrt{-\Delta})u = f(u) \\ u(0) = u_0 \ , \end{cases} \tag{3.1}$$

where $u(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{C}$, $u_0(x) : \mathbf{R}^n \to \mathbf{C}$, $f(u) : \mathbf{C} \to \mathbf{C}$ is a nonlinear function, $i = \sqrt{-1}$, $\Delta = \partial^2/\partial^2_{x_1} + \cdots \partial^2/\partial^2_{x_n}$, $n \geq 1$, and $\phi(\sqrt{-\Delta}) = \mathcal{F}^{-1}\phi(|\xi|)\mathcal{F}$ is a Fourier multiplier. The Cauchy problem (3.1) can express many kinds of dispersive equations. For example, $\phi(r) = r$, $\phi(r) = r^2$, $\phi(r) = \sqrt{1+r^2}$, and $\phi(r) = \sqrt{1+r^4}$ correspond to the wave equation, the Schrödinger equation, the Klein-Gordon equation, and the beam equation, respectively. By Duhamel's principle, the solutions of the Cauchy problem (3.1) satisfy

$$u(t) \ = \ U(t)u_0 - i\int_0^t U(t-s)f(u)ds$$

where

$$U(t) \ = \ e^{it\phi(\sqrt{-\Delta})}.$$

So, if we get the solutions of these integral equations, then we have the solutions of the Cauchy problem (3.1).

In order to solve the Cauchy problems for nonlinear dispersive equations, decay estimates and the Strichartz estimates play an important role. As the most simple and important example of the Cauchy problem (3.1), we consider the Cauchy problem for the Schrödinger equation. Recall that the Schrödinger semi-group $e^{it\Delta}$ on Lebesgue spaces L^p has the following decay estimates:

$$\left\|e^{it\Delta}f\right\|_p \lesssim |t|^{-n(1/2-1/p)}\|f\|_{p'}, \ \ 2 \leq p \leq \infty. \tag{3.2}$$

From these estimates, we obtain the Strichartz estimates:

$$\left\|e^{it\Delta}f\right\|_{L^\gamma(\mathbf{R},L^p)} \lesssim \|f\|_2, \ \ 2 \leq p \leq \infty,$$

where $2/\gamma = n(1/2-1/p)$ and $\|f\|_{L^\gamma(\mathbf{R},L^p)} = (\int_{\mathbf{R}} \|f(t)\|_p^\gamma dt)^{1/\gamma}$. By a fixed point theorem and these two estimates, we can show the well-posedness of nonlinear Schrödinger equations (cf. [10, 11, 13, 41]). There are two kinds of the well-posedness : the one obtained from the decay estimates and the one from the Strichartz estimates. For example, Cazenave and Weissler [12] showed by the decay estimates that there exists a solution u of the nonlinear Schrödinger equation with $f(u) = |u|^\kappa u$ and sufficiently small u_0 ; $\sup_{t\in\mathbf{R}} |t|^B \|e^{it\Delta}u_0\|_{L^{2+\kappa}} < +\infty$ such that

$$\|u\|_X = \sup_{t\in\mathbf{R}} |t|^B \big\|u(t)\big\|_{L^{2+\kappa}} < +\infty,$$

where $\kappa \in \mathbf{R}$, $\kappa_0 < \kappa < 4/(n-2)$ ($\kappa_0 < \kappa < \infty$ if $n = 1$), $B = \frac{4-(n-2)\kappa}{2\kappa(\kappa+2)}$, $0 < B < n\kappa/2(2+\kappa)$, and $B(1+\kappa) < 1$. κ_0 is the positive root of the equation: $n\kappa_0^2 + (n-2)\kappa_0 - 4 = 0$. On the other hand, using the Strichartz estimates, Cazenave and Weissler [10] studied the local L^2 critical case, i.e. $\kappa = 4/n$, and the local H_2^1 critical case, i.e. $\kappa = 4/(n-2)$, for the Schrödinger equation. Moreover, in [11], they discussed the local H_2^s critical case, i.e. $\kappa = 4/(n-2s)$, and obtained the global solution for sufficiently small initial data in $\dot{H}_2^s$.

Wang and Hudzik [97] obtained the global well-posedness of the nonlinear Schrödinger equation on modulation spaces $M_{p,q}^s$. On modulation spaces, decay estimates of Schrödinger semi-group have the following forms; for $2 \le p \le \infty$, $1 \le q \le \infty$, and $s \in \mathbf{R}$

$$\|e^{it\Delta}f\|_{M_{p,q}^s} \lesssim (1+|t|)^{-n(1/2-1/p)}\|f\|_{M_{p',q}^s}. \tag{3.3}$$

In comparison to the decay estimates (3.2), estimates (3.3) do not have a singular point at $t = 0$. Moreover, the decay rate of the estimates (3.3) at $t = \infty$ is the same as that of the estimates (3.2). By virtue of this property, if we consider the global well-posedness on modulation spaces when $f(u)$ is a $(\kappa+1)$-time product of u and $\overline{u}$, we can show that there exists a solution u such that

$$\|u\|_X = \sup_{t\in\mathbf{R}} \langle t\rangle^{2/\gamma} \big\|u(t)\big\|_{M_{2+\kappa,1}} < +\infty$$

where $\kappa \in \mathbf{N}$, $\kappa_0 < \kappa$ and $2/\gamma = n\kappa/2(2+\kappa)$. That is, the use of modulation spaces enables us to deal with the case $\kappa \ge 4/(n-2)$ (cf. [97]). In recent works, inclusion relations between modulation and the other spaces have been studied. For example, we have easily $M_{p,1} \subset L^p \subset M_{p,\infty}$, for any $1 \le p \le \infty$. In addition, embedding between modulation and Besov spaces is studied by Gröbner [30], Toft [89], Sugimoto and Tomita [86]. Moreover, Sugimoto and Kobayashi [54] studied embedding between modulation and L^p-Sobolev spaces. Some other properties of Besov spaces can be found in [91].

3.1.2 Main theorems for NLD

In this paper, we generalize the Cauchy problem for the Schrödinger equation to (3.1) and discuss the well-posedness on modulation spaces, where a smooth function $\phi : \mathbf{R}_+ \to \mathbf{R}$ satisfies (H1)-(H3):

(H1) There exists $\alpha_j > 0$ ($j \in \mathbf{Z}_+$), such that for any $m \ge 3$ and $m \in \mathbf{N}$,

$$|\phi'(r)| \sim r^{\alpha_1 - 1}, \;\; |\phi''(r)| \sim r^{\alpha_2 - 2}, \;\; |\phi^{(m)}(r)| \lesssim r^{\alpha_m - m}, \quad r \ge 1.$$

(H2) There exists $\beta_j > 0$ ($j \in \mathbf{Z}_+$), such that for any $m \ge 3$ and $m \in \mathbf{N}$,

$$|\phi'(r)| \sim r^{\beta_1 - 1}, \;\; |\phi''(r)| \sim r^{\beta_2 - 2}, \;\; |\phi^{(m)}(r)| \lesssim r^{\beta_m - m}, \quad 0 < r < 1.$$

(H3) There exists $\beta_j > 0$ ($j \in \mathbf{Z}_+$), such that for any $m \geq 2$ and $m \in \mathbf{N}$,

$$|\phi'(r)| \sim r^{\beta_1 - 1}, \ \ |\phi^{(m)}(r)| \lesssim r^{\beta_m - m}, \quad 0 < r < 1.$$

Since ϕ is not necessarily a homogeneous function, i.e. $\phi(\lambda^m r) \neq \lambda^m \phi(r)$ for any $m \in \mathbf{Z}_+$ and $\lambda > 0$, the behavior of $\phi(r)$ for large r is not the same as that for small r. Decay estimates of these extended dispersive equations which satisfy $\alpha_1 = \alpha_m$ and $\beta_1 = \beta_m$ ($m \geq 2$) on Besov spaces have been already studied by Guo, Peng, Wang [96]. The main contribution of this paper is to give similar estimates on modulation spaces under weaker conditions:

$$\begin{cases} \alpha_1 \geq 1, \\ \alpha_m \leq \alpha_1 + (j-1), \\ 2n - (n-1)\alpha_1 - \alpha_2 \leq 0, \\ \beta_1 \leq \beta_m, \end{cases} \quad m \geq 2. \tag{3.4}$$

In fact, if ϕ satisfies the following conditions (A) or (B);

(A) (H1) and (H3),

(B) (H1), (H2), and $\beta_1 = \beta_2$,

we have the following estimate for $s \in \mathbf{R}$, $0 \leq \varepsilon \leq 1$,

$$\|e^{it\phi(\sqrt{-\Delta})} f\|_{M^s_{p,q}} \lesssim (1+|t|)^{-\theta\varepsilon(1/2-1/p)} \|f\|_{M^s_{p',q}} \tag{3.5}$$

where

$$\theta = \begin{cases} \min\left(\dfrac{2n}{\beta_1}, n-1\right), & \text{if } \phi \text{ satisfies (A)}, \\ \min\left(\dfrac{2n}{\beta_1}, n\right), & \text{if } \phi \text{ satisfies (B)} \end{cases}$$

(see Proposition 3.5). As examples satisfying the assumption (3.4) and the conditions (A) or (B), we give $\phi(r) = r^2 + \sin r$ or $\phi(r) = r^2 + \cos r$, respectively. If we construct the time decay estimates of these two phase functions on Besov spaces by the standard argument as we will use, the regularity in the right hand side get increase (Remark 3.7). However, on modulation spaces, the regularity does not change in the both sides of time decay estimates.

Since the estimate (3.5) does not have the singularity at $t = 0$, we can treat the nonlinear terms $f(u)$ with $\kappa \geq 4/(n-2)$ in the Cauchy problems (3.1) by the same argument for the Schrödinger equation. Now, we state our main theorems. Let $\pi(u^{\kappa+1})$ be any $(\kappa+1)$-time product of u and $\overline{u}$, and let $\kappa > \kappa_0$ where κ_0 is the positive root of the equation $\theta\varepsilon\kappa_0^2 + (\theta\varepsilon - 2)\kappa_0 - 4 = 0$. Then we have the following main theorem:

Theorem 3.1. *Let $f(u) = \pi(u^{\kappa+1})$, $\kappa \in \mathbf{N}$, $\kappa > \kappa_0$, $\delta(p) = 1/2 - 1/p$, $0 < \varepsilon \leq 1$, and let indexes in (H1)-(H3) satisfy (3.4). There exists a sufficiently small $\rho > 0$ and if an initial data $u_0 \in M_{2+\kappa/1+\kappa,1}$ satisfies that $\|u_0\|_{M_{2+\kappa/1+\kappa,1}} \leq \rho$, then the Cauchy problem (3.1) has a unique global solution*

$$u \in X := \left\{ u \ ; \ \sup_{t \in \mathbf{R}} (1+|t|)^{\theta\varepsilon\delta(2+\kappa)} \|u(t)\|_{M_{2+\kappa,1}} < \infty \right\}.$$

We can consider more generalized nonlinear terms of exponential functions, namely, $\lambda(e^{\varrho|u|^2} - \sum_{k<k_0} \varrho^k|u|^{2k}/k!)u$ where $k_0 \in \mathbf{N}$ is the minimum positive integer which satisfies $k_0 > \kappa_0/2$ (see Remark 3.17). Since the assumption $\kappa > \kappa_0$ is essential in Theorem 5.1, we need to remove lower terms $\lambda \sum_{k\leq k_0} |u|^{2k}u/k!$ from the exponential function.

Theorem 3.2. *Let $f(u) = \lambda(e^{\varrho|u|^2} - \sum_{k<k_0} \varrho^k|u|^{2k}/k!)u$ $(\lambda \in \mathbf{C})$, $\delta(p) = 1/2 - 1/p$, $p \in \mathbf{N}$, $p \in [3, 2k_0 + 2]$, $0 < \varepsilon \leq 1$, and let indexes in (H1)-(H3) satisfy (3.4). There exists a sufficiently small $\rho > 0$ and if an initial data $u_0 \in M_{p',1}$ satisfies that $\|u_0\|_{M_{p',1}} \leq \rho$, then the Cauchy problem (3.1) has a unique global solution*

$$u \in Y := \Big\{u \ ; \ \sup_{t\in\mathbf{R}}(1 + |t|)^{\theta\varepsilon\delta(p)}\|u(t)\|_{M_{p,1}} < \infty\Big\}.$$

We note that Theorems 3.1 and 3.2 are deduced from only decay estimates. The results by the Strichartz estimates for (3.1) are stated in next section.

3.1.3 Additional theorems for NLD

We will present here some additional theorems for the Cauchy problem (3.1). As is explained in the last section, we use decay estimates to deduce Theorems 3.1 and 3.2. If we use Strichartz estimates, we have the following theorems, which treat the well-posedness of the Cauchy problem (3.1) with small initial data on modulation spaces. Before we state main theorems, we introduce function spaces:

$$\ell^{s,q}_{\square}(X) := \left\{f \in \mathcal{S}'(\mathbf{R}^{n+1}) : \|f\|_{\ell^{s,q}_{\square}(X)} < \infty\right\},$$

where

$$\|f\|_{\ell^{s,q}_{\square}(X)} := \Big(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq}\|\square_k f\|^q_X\Big)^{1/q}$$

and

$$C^s_{\square}(X) := \left\{f \in \mathcal{S}'(\mathbf{R}^{n+1}) : \|f\|_{C^s_{\square}(X)} < \infty, \lim_{|k|\to\infty} \langle k\rangle^s\|\square_k f\|_X = 0\right\},$$

where

$$\|f\|_{C^s_{\square}(X)} := \sup_{k\in\mathbf{Z}^n} \langle k\rangle^s\|\square_k f\|_X.$$

We will simply write $\ell^{0,q}_{\square}(X) = \ell^q_{\square}(X)$. Let $X = L^\gamma(\mathbf{R}, L^p)$. It was shown that for any $1 \leq p, q, \gamma < \infty$, $C^\infty_0(\mathbf{R}, \mathcal{S}(\mathbf{R}^n))$ is dense in $\ell^{s,q}_{\square}(L^\gamma(\mathbf{R}, L^p))$ and $C^s_{\square}(L^\gamma(\mathbf{R}, L^p))$. Moreover, we have

$$\big(\ell^{s,q}_{\square}(L^\gamma(\mathbf{R}, L^p))\big)^* = \ell^{-s,q'}_{\square}(L^{\gamma'}(\mathbf{R}, L^{p'}))$$

and

$$\big(C^s_{\square}(L^\gamma(\mathbf{R}, L^p))\big)^* = \ell^{-s,1}_{\square}(L^{\gamma'}(\mathbf{R}, L^{p'})).$$

All these properties can be found in [97, 98]. Using these spaces, we state theorems:

Theorem 3.3. *Let $f(u) = \pi(u^{\kappa+1})$, $\kappa \in \mathbf{N}$, $\kappa \geq \frac{4}{\theta\varepsilon}$, $\delta(p) = 1/2 - 1/p$, $0 < \varepsilon \leq 1$, and let indices in (H1)-(H3) satisfy (3.4). There exists a sufficiently small $\rho > 0$ and if an initial data $u_0 \in M_{2,1}$ satisfies that $\|u_0\|_{M_{2,1}} \leq \rho$, then the Cauchy problem (3.1) has a unique global solution*

$$u \in C(\mathbf{R}, M_{2,1}) \cap \ell^1_\square(L^{2+\kappa}(\mathbf{R}, L^{2+\kappa})).$$

By the same argument of Theorem 3.2 in Section 3.1.2, we remove lower terms $\sum_{k<k_0} \varrho^k |u|^{2k} u/k!$ where $k_0 \in \mathbf{N}$ depends on $4/\theta\varepsilon$ in Theorem 3.3 (see Remark 3.20). Then, we have the following theorem:

Theorem 3.4. *Let $f(u) = \lambda(e^{\varrho|u|^2} - \sum_{k<k_0} \varrho^k |u|^{2k}/k!)u$ $(\lambda \in \mathbf{C})$, $\delta(p) = 1/2 - 1/p$, $p \in \mathbf{N}$, $p = [3, 2k_0 + 2]$, $0 < \varepsilon \leq 1$, and let indices in (H1)-(H3) satisfy (3.4). There exists a sufficiently small $\rho > 0$ and if $u_0 \in M_{p',1}$ satisfies that $\|u_0\|_{M_{p',1}} \leq \rho$, then the Cauchy problem (3.1) has a unique global solution*

$$u \in C\big(\mathbf{R}, M_{2,1}\big) \cap \ell^1_\square\big(L^p(\mathbf{R}, L^p)\big).$$

3.2 Some estimates on modulation spaces

We begin with decay estimates for the semi-group $U(t) = e^{it\phi(\sqrt{-\Delta})}$ where $\phi(r)$ satisfies (H1)-(H3) in Section 3.1.2. In [97], decay estimates for the Schrödinger equation and the Klein-Gordon equation on modulation spaces were studied. We extend them and have the following estimates:

Proposition 3.5. *Let indices in (H1)-(H3) satisfy (3.4), and let ϕ satisfy the following conditions (A) or (B);*

(A) (H1) and (H3),

(B) (H1), (H2), and $\beta_1 = \beta_2$.

Let $s \in \mathbf{R}$, $1 \leq p, q \leq \infty$, and $0 \leq \varepsilon \leq 1$. Then, we have

$$\|U(t)f\|_{M^s_{p,q}} \lesssim (1 + |t|)^{-\theta\varepsilon\delta} \|f\|_{M^s_{p',q}} \tag{3.6}$$

where

$$\begin{aligned} \theta &= \begin{cases} \min\left(\dfrac{2n}{\beta_1}, n-1\right), & \textit{if } \phi \textit{ satisfies (A)}, \\ \min\left(\dfrac{2n}{\beta_1}, n\right), & \textit{if } \phi \textit{ satisfies (B)}, \end{cases} \\ \delta &= \delta(p) = \frac{1}{2} - \frac{1}{p}. \end{aligned}$$

Proof. Let $\{\sigma_k\}_{k\in\mathbf{Z}^n} \in \Gamma$. It follows from Young's inequality that

$$\|U(t)\square_k f\|_\infty \lesssim \sum_{\ell\in\Lambda} \|\mathcal{F}^{-1} e^{it\phi(|\cdot|)} \sigma_k(\cdot)\|_\infty \|\square_{k+\ell} f\|_1$$

$$= \sum_{\ell\in\Lambda} \Big\| \int_{\mathbf{R}^n} e^{it(x\cdot\xi+\phi(|\xi+k|))}\sigma_0(\xi)d\xi \Big\|_\infty \|\Box_{k+\ell} f\|_1$$

where $\Lambda = \{\ell \in \mathbf{Z}^n : B(\ell, \sqrt{n}) \cap B(0, \sqrt{n}) \text{ is not empty}\}$. We immediately have

$$\|U(t)\Box_k f\|_\infty \lesssim \|\Box_k f\|_1, \;\; k \in \mathbf{Z}^n. \tag{3.7}$$

In particular, for the case $k \in \mathbf{Z}^n$ such that the support of σ_k includes the origin, we have the estimate:

$$\|U(t)\Box_k f\|_\infty \lesssim (1+|t|)^{-\theta/2}\|\Box_k f\|_1 \tag{3.8}$$

by Guo [96, Theorem 1 (c)]. We consider the case $k \in \mathbf{Z}^n$ such that the support of σ_k does not include the origin. We set $\phi_k(x,\xi) = x\cdot\xi + \phi(|\xi+k|)$. First, in the case $|x| \le \frac{1}{2}\inf_{\xi\in[-1,1]^n}|\text{grad }\phi\big(|\xi+k|\big)|$ or $|x| \ge 2\sup_{\xi\in[-1,1]^n}|\text{grad }\phi\big(|\xi+k|\big)|$, we have

$$\begin{aligned} |\text{grad }\phi_k| &= |x + \text{grad }\phi(|\xi+k|)| \\ &\gtrsim |\text{grad }\phi(|\xi+k|)| \\ &\sim |\xi+k|^{\alpha_1-1} \sim \langle k\rangle^{\alpha_1-1} > 0 \end{aligned}$$

on the support of σ_0. So the operator:

$$\mathbf{L} = \frac{1}{|\text{grad }\phi_k|^2}\sum_{j=1}^n \frac{\partial\phi_k}{\partial\xi_j}\frac{\partial}{\partial\xi_j}$$

makes sense, and then we have

$$\mathbf{L}e^{it\phi_k} = ite^{it\phi_k}.$$

Using integration by parts, we have for any $N \in \mathbf{Z}_+$,

$$\begin{aligned} \Big|\int_{\mathbf{R}^n} e^{it(x\cdot\xi+\phi(|\xi+k|))}\sigma_0(\xi)d\xi\Big| &= \frac{1}{|t|^N}\Big|\int_{\mathbf{R}^n}\Big(\mathbf{L}^N e^{it\phi_k(x,\xi)}\Big)\sigma_0(\xi)d\xi\Big| \\ &= \frac{1}{|t|^N}\Big|\int_{\mathbf{R}^n} e^{it\phi_k(x,\xi)}\Big((\mathbf{L}^*)^N\sigma_0(\xi)\Big)d\xi\Big| \end{aligned}$$

where

$$\mathbf{L}^* = \frac{1}{|\text{grad }\phi_k|^2}\Big(-\sum_{j=1}^n \frac{\partial\phi_k}{\partial\xi_j}\frac{\partial}{\partial\xi_j}\Big) + \text{div}\Big(\frac{\text{grad }\phi_k}{|\text{grad }\phi_k|^2}\Big)$$

We set $g = (g_1, \cdots, g_n) = \text{grad }\phi_k$, then we have

$$\begin{aligned} (\mathbf{L}^*)^N\sigma_0 &= \Big\{-\frac{g}{|g|^2}\cdot\nabla + \text{div}\Big(\frac{g}{|g|^2}\Big)\Big\}^N\sigma_0 \\ &= (\mathbf{L}^*)^{N-1}\Big\{-\frac{g}{|g|^2}\cdot\nabla\sigma_0 + \text{div}\Big(\frac{g}{|g|^2}\Big)\sigma_0\Big\} \end{aligned}$$

$$= \quad (\mathbf{L}^*)^{N-2}\Big\{\big(\frac{g}{|g|^2}\cdot\nabla\big)\big(\frac{g}{|g|^2}\cdot\nabla\big)\sigma_0 - \big[\mathrm{div}\big(\frac{g}{|g|^2}\big)\big]\big(\frac{g}{|g|^2}\cdot\nabla\big)\sigma_0$$
$$-\big(\frac{g}{|g|^2}\cdot\nabla\big)\big[\mathrm{div}\big(\frac{g}{|g|^2}\big)\sigma_0\big] + \big[\mathrm{div}\big(\frac{g}{|g|^2}\big)\big]^2\sigma_0\Big\}$$
$$\vdots$$

Moreover, for any $j = 1, \cdots, n$ and $\alpha \in \mathbf{Z}^n_+$, we have

$$\begin{aligned}\big|\partial^\alpha \frac{g_j}{|g|^2}\big| &= \Big|\partial^\alpha \frac{(\mathrm{grad}\ \phi_k)_j}{|\mathrm{grad}\ \phi_k|^2}\Big| \\ &\lesssim |\xi + k|^{-(\alpha_1-1)} \\ &\sim \langle k\rangle^{-(\alpha_1-1)}\end{aligned}$$

under the assumption $\alpha_j \le \alpha_1 + (j-1)$. So, we have

$$\big|(\mathbf{L}^*)^N\sigma_0(\xi)\big| \lesssim \langle k\rangle^{-N(\alpha_1-1)},\ \ N \in \mathbf{Z}_+,$$

and we have

$$\Big|\int_{\mathbf{R}^n} e^{it(x\cdot\xi+\phi(|\xi+k|))}\sigma_0(\xi)d\xi\Big| \lesssim |t|^{-N}\langle k\rangle^{-N(\alpha_1-1)},\ \ N \in \mathbf{Z}_+. \tag{3.9}$$

Interpolating (3.7) with (3.9), we obtain for any $M \ge 0$,

$$\Big|\int_{\mathbf{R}^n} e^{it(x\cdot\xi+\phi(|\xi+k|))}\sigma_0(\xi)d\xi\Big| \lesssim |t|^{-M}\langle k\rangle^{-M(\alpha_1-1)} \lesssim |t|^{-M}. \tag{3.10}$$

Next, we consider the case $\frac{1}{2}\mathrm{inf}_{\xi\in[-1,1]^n}|\mathrm{grad}\ \phi\big(|\xi+k|\big)| \le |x| \le 2\mathrm{sup}_{\xi\in[-1,1]^n}|\mathrm{grad}\ \phi\big(|\xi+k|\big)|$, which is bounded subsets of $\mathbf{R}^n_x$. By elementary matrix operations, we have the Hessian of ϕ_k;

$$\nabla^2\phi_k = \begin{pmatrix} \frac{\phi'}{|\xi_k|} + \frac{\xi_{k1}^2}{|\xi_k|^2}f & \frac{\xi_{k1}\xi_{k2}}{|\xi_k|^2}f & \cdots & \frac{\xi_{k1}\xi_{kn}}{|\xi_k|^2}f \\ \frac{\xi_{k2}\xi_{k1}}{|\xi_k|^2}f & \frac{\phi'}{|\xi_k|} + \frac{\xi_{k2}^2}{|\xi_k|^2}f & \cdots & \frac{\xi_{k2}\xi_{kn}}{|\xi_k|^2}f \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\xi_{kn}\xi_{k1}}{|\xi_k|^2}f & \frac{\xi_{kn}\xi_{k2}}{|\xi_k|^2}f & \cdots & \frac{\phi'}{|\xi_k|} + \frac{\xi_{kn}^2}{|\xi_k|^2}f \end{pmatrix}$$
$$\to \begin{pmatrix} \frac{\phi'}{|\xi_k|} + \frac{\xi_{k1}^2}{|\xi_k|^2}f & \frac{\xi_{k1}\xi_{k2}}{|\xi_k|^2}f & \frac{\xi_{k1}\xi_{k3}}{|\xi_k|^2}f & \cdots & \frac{\xi_{k1}\xi_{kn}}{|\xi_k|^2}f \\ -\frac{\xi_{k2}}{\xi_{k1}}\frac{\phi'}{|\xi_k|} & \frac{\phi'}{|\xi_k|} & 0 & \cdots & 0 \\ -\frac{\xi_{k3}}{\xi_{k1}}\frac{\phi'}{|\xi_k|} & 0 & \frac{\phi'}{|\xi_k|} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\xi_{kn}}{\xi_{k1}}\frac{\phi'}{|\xi_k|} & 0 & 0 & \cdots & \frac{\phi'}{|\xi_k|} \end{pmatrix}$$

$$
\to \begin{pmatrix} \frac{\phi'}{|\xi_k|}+f & \frac{\xi_{k1}\xi_{k2}}{|\xi_k|^2}f & \frac{\xi_{k1}\xi_{k3}}{|\xi_k|^2}f & \cdots & \frac{\xi_{k1}\xi_{kn}}{|\xi_k|^2}f \\ 0 & \frac{\phi'}{|\xi_k|} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\phi'}{|\xi_k|} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\phi'}{|\xi_k|} \end{pmatrix}
$$

$$
= \begin{pmatrix} \phi'' & \frac{\xi_{k1}\xi_{k2}}{|\xi_k|^2}f & \frac{\xi_{k1}\xi_{k3}}{|\xi_k|^2}f & \cdots & \frac{\xi_{k1}\xi_{kn}}{|\xi_k|^2}f \\ 0 & \frac{\phi'}{|\xi_k|} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\phi'}{|\xi_k|} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\phi'}{|\xi_k|} \end{pmatrix},
$$

where $\xi_k = \xi + k \in \mathbf{R}^n$, $\xi_{kj} = \xi_j + k_j \in \mathbf{R}$ ($j \in \mathbf{N}$), and $f = -\frac{\phi'}{|\xi_k|} + \phi''$. In the second arrow, we add $(-\xi_{kj}/\xi_{k1})$ times column 1 to column j. In the third arrow, we add (ξ_{kj}/ξ_{k1}) times row j to row 1. Then, by cofactor expansions, we have $|\nabla^2\phi_k| = |\xi+k|^{-(n-1)}|(\phi')^{n-1}\phi''| \sim \langle k\rangle^{-2n+(n-1)\alpha_1+\alpha_2} \geq 1$ on the support of σ_0. Thus, by stationary phase methods (see [27, 39, 64, 78]), there exists the critical point $\xi_0 \in \mathbf{R}^n$ of ϕ_k in support of σ_0, i.e. grad $\phi_k(\xi_0) = 0$:

$$
\begin{aligned}
\Big|\int_{\mathbf{R}^n} e^{it(x\cdot\xi+\phi(|\xi+k|))}\sigma_0(\xi)d\xi\Big| &\lesssim |t|^{-\frac{n}{2}}|\nabla^2\phi_k(\xi_0)|^{-\frac12} \\
&\lesssim |t|^{-\frac{n}{2}}\langle k\rangle^{\frac12(2n-(n-1)\alpha_1-\alpha_2)} \\
&\lesssim |t|^{-\frac{n}{2}}. \qquad (3.11)
\end{aligned}
$$

Comparing (3.10) with (3.11), we have

$$
\Big|\int_{\mathbf{R}^n} e^{it(x\cdot\xi+\phi(|\xi+k|))}\sigma_0(\xi)d\xi\Big| \lesssim |t|^{-\frac{n}{2}}, \quad x \in \mathbf{R}^n. \qquad (3.12)
$$

Collecting estimates (3.7), (3.8) and (3.12), we have

$$
\|U(t)\Box_k f\|_\infty \lesssim (1+|t|)^{-\theta/2}\|\Box_k f\|_1, \;\; k \in \mathbf{Z}^n. \qquad (3.13)
$$

On the other hand, by Plancherel's theorem, we have

$$
\|U(t)\Box_k f\|_2 \sim \|\Box_k f\|_2. \qquad (3.14)
$$

Interpolating estimates (3.13) and (3.14), we have

$$\|U(t)\Box_k f\|_p \sim (1+|t|)^{-\theta\delta}\|\Box_k f\|_{p'},\ \ k\in \mathbf{Z}^n,\ \ 2\le p\le\infty. \tag{3.15}$$

where $\delta = \delta(p) = 1/2 - 1/p$. Furthermore, by Hausdorff-Young's and Hölder's inequalities, we have

$$\begin{aligned}\|U(t)\Box_k f\|_p &\lesssim \sum_{\ell\in\Lambda}\|\sigma_{k+\ell}e^{it\phi(|\xi|)}\mathcal{F}\Box_k f\|_{p'}\\ &\lesssim \|\mathcal{F}\Box_k f\|_p\\ &\lesssim \|\Box_k f\|_{p'}.\end{aligned} \tag{3.16}$$

Again, from interpolation between estimates (3.15) and (3.16), for any $0\le\varepsilon\le 1$, we have

$$\|U(t)\Box_k f\|_p \sim (1+|t|)^{-\theta\varepsilon\delta}\|\Box_k f\|_{p'}. \tag{3.17}$$

Multiplying $\langle k\rangle^s$ to both sides of (3.17) and taking the ℓ^q-norm, we have the desired result. □

Remark 3.6. We remark that Guo–Peng–Wang [96, Theorem 1] gives similar estimates on Besov spaces under some assumptions on ϕ, which are stronger than (H1)-(H3). For example, for large $j\in\mathbf{Z}_+$, we have for $0\le\theta\le(n-1)/2$

$$\|e^{it\phi(\sqrt{-\Delta})}\triangle_j u_0\|_\infty \lesssim |t|^{-\theta}2^{j(n-\alpha_1\theta)}\|\triangle_j u_0\|_1, \tag{3.18}$$

where $\triangle_j$ are the Littlewood-Paley projectors, if ϕ satisfies (H1)$'$:

(H1)$'$ There exists $\alpha_1>0$, such that for any $m\ge 2$ and $m\in\mathbf{N}$,

$$|\phi'(r)|\sim r^{\alpha_1-1}\ ,\ \ |\phi^{(m)}(r)|\lesssim r^{\alpha_1-m}\ ,\ \ r\ge 1,$$

and have for $0\le\theta\le 1$

$$\|e^{it\phi(\sqrt{-\Delta})}\triangle_j u_0\|_\infty \lesssim |t|^{-\frac{n-1+\theta}{2}}2^{j(n-\frac{\alpha_1(n-1+\theta)}{2}-\frac{\theta(\alpha_2'-\alpha_1)}{2})}\|\triangle_k u_0\|_1, \tag{3.19}$$

if in addition ϕ satisfies $\phi'(r)\sim r^{\alpha_2'-2}$ for $r\ge 1$. If we follow the proof carefully, assumptions in [96, Theorem 1] can be relaxed to (H1)-(H3) with

$$\begin{cases}\alpha_m\le\alpha_1,\\ \beta_1\le\beta_m,\end{cases}\ m\ge 2, \tag{3.20}$$

which are still stronger than the assumptions (3.4). We remark that estimates on modulation spaces can be also deduced from those on Besov spaces. For example, for large $|k|$, we have for $0\le\theta\le(n-1)/2$

$$\|e^{it\phi(\sqrt{-\Delta})}\Box_k u_0\|_\infty \lesssim |t|^{-\theta}\langle k\rangle^{n-\alpha_1\theta}\|\Box_k u_0\|_1, \tag{3.21}$$

if ϕ satisfies (H1)$'$. In addition, if ϕ satisfies $\phi'(r) \sim r^{\alpha_2'-2}$ for $r \geq 1$, then we have for $0 \leq \theta \leq 1$

$$\|e^{it\phi(\sqrt{-\Delta})}\Box_k u_0\|_\infty \lesssim |t|^{-\frac{n-1+\theta}{2}} \langle k\rangle^{(n-\frac{\alpha_1(n-1+\theta)}{2}-\frac{\theta(\alpha_2'-\alpha_1)}{2})} \|\Box_k u_0\|_1. \tag{3.22}$$

In fact, let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ be a radial function and satisfy

$$\varphi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2. \end{cases}$$

We write $\psi(\xi) = \varphi(\xi) - \varphi(2\xi)$ and set $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \in \mathbf{Z}$. First, we consider the case $|k| \geq k_0 \gg 1$. If we choose a constant $N \in \mathbf{N}$ which satisfies $\sum_{\ell=-N}^{N} \psi_{j+\ell}(\xi) = 1$ on the support of σ_k for $|k| \in [2^{j-1}, 2^j]$, then we have

$$\begin{aligned} \|e^{it\phi(\sqrt{-\Delta})}\Box_k u_0\|_\infty &= \|\mathcal{F}^{-1} e^{it\phi(|\xi|)}\sigma_k \sum_{\ell=-N}^{N} \psi_{j+\ell}(\xi)\mathcal{F}u_0\|_\infty \\ &\lesssim \sum_{\ell=-N}^{N} \|\mathcal{F}^{-1} e^{it\phi(|\xi|)}\psi_{j+\ell}(\xi)\|_\infty \|\Box_k u_0\|_1. \end{aligned}$$

Since $\langle k\rangle \sim 2^j$ for large $|k|$, we have the estimates (3.21) and (3.22) by (3.18) and (3.19).

Remark 3.7. As examples satisfying the assumptions (3.4), we give $\phi(r) = r^2 + \cos r$ and $\phi(r) = r^2 + \sin r$, where $r = |\xi| > 0$. These are regarded as the Schrödinger operators with the perturbation, which oscillates in frequency spaces. If we consider $\phi(r) = r^2 + \cos r$, then we have for $r > 0$ and $j \geq 3$

$$|\phi'| \sim r, \quad |\phi''| \sim 1, \quad |\phi^{(j)}| \lesssim 1.$$

In fact, we get these conclusions from the following calculations; for $r \geq 1$,

$$\begin{aligned} |\phi'| &= |2r - \sin r| \leq 2r + 1 \leq 3r, \\ |\phi'| &= |2r - \sin r| \geq |2r - 1| \geq r. \end{aligned}$$

For $0 < r < 1$,

$$\begin{aligned} |\phi'| &= |2r - \sin r| \leq 2r + r \leq 3r, \\ |\phi'| &= |2r - \sin r| \geq 2r - r = r. \end{aligned}$$

For $r > 0$ and $j \geq 3$,

$$\begin{aligned} |\phi''| &= |2 - \cos r| \sim 1, \\ |\phi^{(j)}| &= |\cos r| \text{ or } |\sin r| \lesssim 1. \end{aligned}$$

Thus, since we have for $j \geq 2$

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 2,$$

$$
\begin{aligned}
&2n-(n-1)\alpha_1-\alpha_2=0,\\
&j=\alpha_j\le\alpha_1+(j-1)=j+1,\\
&2=\beta_1\le\beta_j=j,
\end{aligned}
$$

we see that $\phi(r)=r^2+\cos r$ satisfies the the conditions (3.4) and (B) in Proposition 3.5, and then we have

$$\|e^{it(-\Delta+\cos(\sqrt{-\Delta}))}f\|_{M^s_{p,q}}\lesssim(1+|t|)^{-n(1/2-1/p)}\|f\|_{M^s_{p',q}}.$$

On the other hands, considering $\phi(r)=r^2+\sin r$, we have the following expressions; for $r\ge1$

$$|\phi'|\sim r,\quad |\phi''|\sim1,\quad |\phi^{(j)}|\lesssim1,$$

and for $0<r<1$

$$|\phi'|\sim 1,\quad |\phi''|\sim1,\quad |\phi^{(j)}|\lesssim1.$$

Thus,

$$
\begin{aligned}
&\alpha_1=\alpha_2=\beta_2=2,\\
&\beta_1=1\\
&2n-(n-1)\alpha_1-\alpha_2=0,\\
&j=\alpha_j\le\alpha_1+(j-1)=j+1,\\
&1=\beta_1\le\beta_j=j.
\end{aligned}
$$

Since $\beta_1\neq\beta_2$ and $\phi(r)=r^2+\sin r$ does not satisfy the condition (B), the time decay term is $(1+|t|)^{-(n-1)\delta}$ and we have

$$\|e^{it(-\Delta+\sin(\sqrt{-\Delta}))}f\|_{M^s_{p,q}}\lesssim(1+|t|)^{-(n-1)(1/2-1/p)}\|f\|_{M^s_{p',q}}.$$

Moreover, if we construct the time decay estimates of the phase function $\phi(r)=r^2+\cos r$ on Besov spaces by the standard argument as we did, we have

$$\|e^{it(-\Delta+\cos(\sqrt{-\Delta}))}u_0\|_{B^s_{p,q}}\lesssim|t|^{-n(1/2-1/p)}\|u_0\|_{B^{s+(n-1)(1/2-1/p)}_{p',q}}.$$

This estimate has a big problem, which the regularity in the right hand side increases in higher dimensions $n\ge2$. From this fact, we don't know whether we can show the existence of a solution by the fixed point argument. This problem occurs from the fact that Besov spaces cut frequency spaces by dyadic decomposition, whose supports get wider as the decomposition gets far away from the origin.

Remark 3.8. Time decay estimates for many partial differential equations have been studied by many researchers. Those for the wave equations ($\phi(r)=r$) can be found in Strichartz [82], and those for the wave type equations can be found in Brenner [7] and Pecher [75]. Time decay estimates for the Klein–Gordon equations was proved by Marshall et. al. [65] (the paper by Brenner [8] may be also helpful – at least it was for the author). One can see the estimates for the beam equations ($\phi(r)=\sqrt{1+r^4}$) in Levandosky [59]. For the fourth order Schrödinger equations ($\phi(r)=r^4\pm\varepsilon r^2$ where $\varepsilon=\{-1,0,1\}$), Ben–Artzi et. al. showed in [1].

Remark 3.9. We also remark that the sharpness of the results in Proposition 3.5 is not completely understood. However, in the case when ϕ are homogeneous functions, these results are also sharp (see [97] for the Schrödinger equation, for example). Furthermore, for the beam equation, the decay rate in Proposition 3.5 is known to be sharp by [59].

Using these decay estimates in Proposition 3.5, we have the following Strichartz estimates of semi-group $U(t) = e^{it\phi(\sqrt{-\Delta})}$ on modulation spaces (see also [102]) and their applications:

Proposition 3.10. *Let indices in (H1)-(H3) satisfy (3.4), and let* $n \geq 1$*,* $2 \leq p < \infty$*,* $1 \leq q < \infty$*,* $s \in \mathbf{R}$*, and* $0 < \varepsilon \leq 1$*. Assume that*

$$\begin{cases} \dfrac{2}{\gamma} \leq \theta\varepsilon\delta & \textit{if } \ 0 < \theta\varepsilon\delta \leq 1, \\ \\ \dfrac{2}{\gamma} < \theta\varepsilon\delta & \textit{if } \ \theta\varepsilon\delta > 1. \end{cases}$$

Then, we have

$$\|U(t)f\|_{\ell_\square^{s,q}(L^\gamma(\mathbf{R},L^p))} \lesssim \|f\|_{M^s_{2,q}} \tag{3.23}$$

where θ *and* δ *are the same parameters in Proposition 3.5.*

Proof. We use the dual estimate method (cf. [11, 13]). Let $f \in \mathcal{S}(\mathbf{R}^n)$, $g \in C_0^\infty(\mathbf{R}, \mathcal{S}(\mathbf{R}^n))$. First, we state the case $1 < q < \infty$ and $2 < p < \infty$. By Fubini's theorem, Lebesgue's dominated convergence theorem, and Hölder's inequality, we have

$$\begin{aligned} & \left| \int_{\mathbf{R}} \big(U(t)f, g(t)\big) dt \right| \\ \lesssim \ & \sum_{\substack{k\in\mathbf{Z}^n \\ \ell\in\Lambda}} \left| \left(\square_k f, \square_{k+\ell} \int_{\mathbf{R}} U(-t)g(t)dt\right) \right| \langle k\rangle^s \langle k\rangle^{-s} \\ \leq \ & \|f\|_{M^s_{2,q}} \left\| \int_{\mathbf{R}} U(-t)g(t)dt \right\|_{M^{-s}_{2,q'}} \end{aligned} \tag{3.24}$$

where $(a, b) = \int_{\mathbf{R}^n} a(x)\overline{b(x)}dx$, and $1/q + 1/q' = 1$. For any $k \in \mathbf{Z}^n$, we have

$$\begin{aligned} & \left\| \square_k \int_{\mathbf{R}} U(-t)g(t)dt \right\|_2^2 \\ = \ & \int_{\mathbf{R}} \left(\square_k g(t), \square_k \int_{\mathbf{R}} U(t-s)g(s)ds\right) dt \\ \leq \ & \|\square_k g\|_{L^{\gamma'}(\mathbf{R},L^{p'})} \left\| \square_k \int_{\mathbf{R}} U(t-s)g(s)ds \right\|_{L^\gamma(\mathbf{R},L^p)} \end{aligned} \tag{3.25}$$

where $1/p + 1/p' = 1$ and $1/\gamma + 1/\gamma' = 1$.

Step 1. $0 < \theta\varepsilon\delta < 1$ and $2/\gamma = \theta\epsilon\delta$. By the decay estimates (3.6) and Hardy-Littlewood-Sobolev's inequality, we have

$$\begin{aligned} & \left\| \Box_k \int_{\mathbf{R}} U(t-s)g(s)ds \right\|_{L^\gamma(\mathbf{R},L^p)} \\ \lesssim & \left\| \int_{\mathbf{R}} |t-s|^{-\theta\epsilon\delta} \|\Box_k g(s)\|_{L^{p'}} ds \right\|_{L^\gamma(\mathbf{R})} \\ \lesssim & \|\Box_k g\|_{L^{\gamma'}(\mathbf{R},L^{p'})}. \end{aligned} \tag{3.26}$$

Step 2. $\theta\varepsilon\delta = 1$ and $\gamma > 2$. By the decay estimates (3.6) and Young's inequality, we have

$$\begin{aligned} & \left\| \Box_k \int_{\mathbf{R}} U(t-s)g(s)ds \right\|_{L^\gamma(\mathbf{R},L^p)} \\ \lesssim & \left\| \langle\cdot\rangle^{-1} * \|\Box_k g(\cdot)\|_{L^{p'}} \right\|_{L^\gamma(\mathbf{R})} \\ \lesssim & \|\langle\cdot\rangle^{-1}\|_{L^{\gamma/2}(\mathbf{R})} \|\Box_k g\|_{L^{\gamma'}(\mathbf{R},L^{p'})} \\ \lesssim & \|\Box_k g\|_{L^{\gamma'}(\mathbf{R},L^{p'})} \end{aligned} \tag{3.27}$$

where $\dfrac{1}{\gamma} = \dfrac{2}{\gamma} + \dfrac{1}{\gamma'} - 1$.

Step 3. $\theta\varepsilon\delta = 1$ and $2/\gamma = 1$. In this case, we cannot use the two inequalities as shown in the two steps above. So, by Keel-Tao's endpoint Strichartz estimates [46], we have

$$\left\| \Box_k \int_{\mathbf{R}} U(t-s)g(s)ds \right\|_{L^\gamma(\mathbf{R},L^p)} \lesssim \|\Box_k g\|_{L^{\gamma'}(\mathbf{R},L^{p'})}. \tag{3.28}$$

Step 4. $\theta\varepsilon\delta > 1$ and $2/\gamma < \theta\epsilon\delta$. By the decay estimates (3.6) and Young's inequality, we have

$$\begin{aligned} & \left\| \Box_k \int_{\mathbf{R}} U(t-s)g(s)ds \right\|_{L^\gamma(\mathbf{R},L^p)} \\ \lesssim & \left\| \langle\cdot\rangle^{-\theta\varepsilon\delta} * \|\Box_k g(\cdot)\|_{L^{p'}} \right\|_{L^\gamma(\mathbf{R})} \\ \lesssim & \|\Box_k g\|_{L^{\gamma'}(\mathbf{R},L^{p'})} \end{aligned} \tag{3.29}$$

where $\dfrac{1}{\gamma} = \dfrac{2}{\gamma} + \dfrac{1}{\gamma'} - 1$.

Step 5. From the interpolation between Step 1 and Step 2, the inequality is true when $0 < \theta\varepsilon\delta < 1$ and $2/\gamma \le \theta\varepsilon\delta$. Multiplying $\langle k\rangle^{-s}$ and taking $\ell^{q'}$-norm to (3.24)-(3.29) , we have

$$\left| \int_{\mathbf{R}} \big(U(t)f, g(t)\big) dt \right| \lesssim \|f\|_{M^s_{2,q}} \|g\|_{\ell_\Box^{-s,q}(L^{\gamma'}(\mathbf{R},L^{p'}))}. \tag{3.30}$$

Then, by density, we have the desired result.

Step 6. In particular, if $p = 2$, then $p' = 2$, $\gamma = \infty$, and $\gamma' = 1$. Since (3.26) is clear in this case, we have (3.30) immediately. In the case $q = 1$, using the properties that $\left(C^s_\square(L^\gamma(\mathbf{R}, L^p))\right)^* = \ell^{-s,1}_\square(L^{\gamma'}(\mathbf{R}, L^{p'}))$, and $C^\infty_0(\mathbf{R}, \mathcal{S}(\mathbf{R}^n))$ is dense in $C^s_\square(L^\gamma(\mathbf{R}, L^p))$, it is almost same with the argument which we used in this proof. □

Remark 3.11. We can always use Hardy-Littlewood-Sobolev's inequality because we can choose $0 < \varepsilon \le 1$ satisfying $0 < \theta\epsilon\delta < 1$. If $\varepsilon = 0$, then $\theta\varepsilon\delta = 0$, so the decay estimates (3.6) do not make sense, and we need an assumption $0 < \varepsilon \le 1$ instead. We can set the assumption $\gamma \ge \max\{2, 2/(\theta\varepsilon\delta)\}$ instead.

Proposition 3.12. *If $\gamma \ge q$, then we have*

$$\|U(t)f\|_{L^\gamma(\mathbf{R}, M^s_{p,q})} \lesssim \|f\|_{M^s_{2,q}}.$$

Proof. Applying Minkowski's inequality to left side of (3.23), we have the desired result. □

We set

$$(\mathcal{U}f)(t) = \int_0^t U(t-s)f(s)ds. \tag{3.31}$$

Using the Strichartz estimates, we introduce some properties of this operator:

Proposition 3.13. *Under the same assumptions in Proposition 3.10, we have*

$$\|\mathcal{U}f\|_{\ell^{s,q}_\square(L^\infty(\mathbf{R}, L^2))} \lesssim \|f\|_{\ell^{s,q}_\square(L^{\gamma'}(\mathbf{R}, L^{p'}))}. \tag{3.32}$$

Moreover, if $\gamma' \le q$, then we have

$$\|\mathcal{U}f\|_{L^\infty(\mathbf{R}, M^s_{2,q})} \lesssim \|f\|_{L^{\gamma'}(\mathbf{R}, M^s_{p',q})}. \tag{3.33}$$

Proof. It is clear from (3.25) and (3.26) in the former. Applying Minkowski's inequality to the right side of (3.32), we have the desired result. □

Proposition 3.14. *Under the same assumptions in Proposition 3.10, we have*

$$\|\mathcal{U}f\|_{\ell^{s,q}_\square(L^\gamma(\mathbf{R}, L^p))} \lesssim \|f\|_{\ell^{s,q}_\square(L^{\gamma'}(\mathbf{R}, L^{p'}))}. \tag{3.34}$$

Moreover, if $\gamma' \le q$, then we have

$$\|\mathcal{U}f\|_{L^\gamma(\mathbf{R}, M^s_{p,q})} \lesssim \|f\|_{L^{\gamma'}(\mathbf{R}, M^s_{p',q})}. \tag{3.35}$$

Proof. In the former, it follows immediately from (3.26). In the later, from the decay estimate(3.6) and Hardy-Littlewood-Sobolev's inequality, we obtain the desired result. □

Proposition 3.15. *Under the same assumptions in Proposition 3.10, we have*

$$\|\mathcal{U}f\|_{\ell_\square^{s,q}(L^\gamma(\mathbf{R},L^p))} \lesssim \|f\|_{\ell_\square^{s,q}(L^1(\mathbf{R},L^2))}. \tag{3.36}$$

Moreover, if $\gamma \geq q$, *then we have*

$$\|\mathcal{U}f\|_{L^\gamma(\mathbf{R},M^s_{p,q})} \lesssim \|f\|_{L^1(\mathbf{R},M^s_{2,q})}. \tag{3.37}$$

Proof. We use the dual estimate method. Let $f, g \in C_0^\infty(\mathbf{R}, \mathcal{S}(\mathbf{R}^n))$, $p \neq 2$, and $q \neq 1$. By transformed variable and Hölder's inequality,

$$\begin{aligned}
&\left|\int_{\mathbf{R}}\left(\int_0^t U(t-s)f(s)ds, g(s)\right)dt\right| \\
\leq\ & \sum_{\substack{k\in\mathbf{Z}^n\\ \ell\in\Lambda}}\left|\left(\int_0^\infty + \int_{-\infty}^0\right)\int_0^t \Big(\square_{k+\ell}f(s), \square_k U(s-t)g(t)\Big)dsdt\right| \\
=\ & \sum_{\substack{k\in\mathbf{Z}^n\\ \ell\in\Lambda}}\left\{\left|\int_0^\infty \Big(\square_{k+\ell}f(s), \square_k\int_s^\infty U(s-t)g(t)dt\Big)ds\right|\right. \\
&\qquad\left.+\left|\int_{-\infty}^0 \Big(\square_{k+\ell}f(s), \square_k\int_{-\infty}^s U(s-t)g(t)dt\Big)ds\right|\right\} \\
\lesssim\ & \|f\|_{\ell_\square^{s,q}(L^1(\mathbf{R},L^2))}\|g\|_{\ell_\square^{-s,q'}(L^{\gamma'}(\mathbf{R},L^{p'}))}.
\end{aligned}$$

If $p=2$ and $q=1$, then the argument is the same as used in the proof of Proposition 3.10. Thus, we have (3.36) by density. Moreover, we have (3.37) by Minkowski's inequality. □

3.3 Proofs of main theorems

3.3.1 Proof of Theorems 3.1 and 3.2

Since we use a fixed-point theorem, we need to verify that spaces we work on are Banach spaces. We only explain completeness of spaces X in Theorem 3.1. Indeed, spaces X are Banach spaces equipped with the norms

$$\|u\|_X = \sup_{t\in\mathbf{R}}\big(1+|t|\big)^{\theta\varepsilon\delta(2+\kappa)}\|u(t)\|_{M_{2+\kappa,1}}.$$

Suppose that $\{v_j\}_{j\in\mathbf{Z}_+}$ is a Cauchy sequence. It follows that $\{w_j\}_{j\in\mathbf{Z}_+}$ defined by $w_j = \big(1+|t|\big)^\beta v_j$ is a Cauchy sequence in $L^\infty(\mathbf{R}, M_{2+\kappa,1})$. Then there exists a w such that $\|w-w_j\|_{L^\infty(\mathbf{R},M_{2+\kappa,1})} \to 0$ $(j\to\infty)$. Let $v = \big(1+|t|\big)^{-\beta}w$, then we get $v\in X$ and $\|v-v_j\|_X = \|w-w_j\|_{L^\infty(\mathbf{R},M_{2+\kappa,1})} \to 0$ $(j\to\infty)$. Using the same argument, we have completeness of spaces Y in Theorem 3.2. We explain a brief proof of Theorem 3.1:

Proof of Theorem 3.1. This theorem follows from a fixed-point theorem. We set

$$X = \Big\{ u \ ; \ \|u\|_X = \sup_{t\in\mathbf{R}} (1+|t|)^{\theta\varepsilon\delta(2+\kappa)} \|u(t)\|_{M_{2+\kappa,1}} \le \rho \Big\}.$$

Given $u_0 \in M_{2+\kappa/1+\kappa,1}$, we set

$$\mathcal{N}[u](t) = U(t)u_0 - i\int_0^t U(t-s) f(u)(s)ds\ , \quad u \in X \text{ and } t \in \mathbf{R}.$$

From decay estimate (3.6), we obtain for the first terms,

$$\begin{aligned} \big\|U(t)u_0\big\|_X &\lesssim \sup_{t\in\mathbf{R}} \langle t\rangle^{\theta\varepsilon\delta-\theta\varepsilon\delta} \|u_0\|_{M_{2+\kappa/1+\kappa,1}} \\ &\lesssim \|u_0\|_{M_{2+\kappa/1+\kappa,1}}. \end{aligned}$$

For the second terms, using Proposition 2.2, then we have

$$\begin{aligned} &\Big\| i\int_0^t U(t-s) f(u)(s)ds \Big\|_X \\ \lesssim\ & \sup_{t\in\mathbf{R}} \langle t\rangle^{\theta\varepsilon\delta} \int_0^t \langle t-s\rangle^{-\theta\varepsilon\delta} \|f(u)(s)\|_{M_{2+\kappa/1+\kappa,1}} ds \\ \lesssim\ & \|u\|_X^{1+\kappa} \sup_{t\in\mathbf{R}} \langle t\rangle^{\theta\varepsilon\delta} \int_0^t \langle t-s\rangle^{-\theta\varepsilon\delta} \langle s\rangle^{-(1+\kappa)\theta\varepsilon\delta} ds, \end{aligned} \tag{3.38}$$

where $\langle t\rangle = 1+|t|$ and $\delta = \delta(2+\kappa)$. Since $\kappa > \kappa_0$, we have $(1+\kappa)\theta\varepsilon\delta > 1$. So, it follows that

$$\begin{aligned} &\int_0^{t/2} \langle t-s\rangle^{-\theta\varepsilon\delta} \langle s\rangle^{-(1+\kappa)\theta\varepsilon\delta} ds \\ \lesssim\ & \langle t/2\rangle^{-\theta\varepsilon\delta} \frac{1}{1-(1+\kappa)\theta\varepsilon\delta} \Big(\langle t/2\rangle^{1-(1+\kappa)\theta\varepsilon\delta} - 1 \Big) \\ \lesssim\ & \langle t\rangle^{-\theta\varepsilon\delta} \end{aligned}$$

and

$$\int_{t/2}^t \langle t-s\rangle^{-\theta\varepsilon\delta} \langle s\rangle^{-(1+\kappa)\theta\varepsilon\delta} ds \quad \lesssim \quad \langle t/2\rangle^{-\theta\varepsilon\delta} \int_0^{t/2} \langle s\rangle^{-\theta\varepsilon\delta} ds.$$

Dividing into three cases that $\theta\varepsilon\delta < 1$, $\theta\varepsilon\delta > 1$, and $\theta\varepsilon\delta = 1$, then we have

$$\int_{t/2}^t \langle t-s\rangle^{-\theta\varepsilon\delta} \langle s\rangle^{-(1+\kappa)\theta\varepsilon\delta} ds \lesssim \langle t/2\rangle^{-\theta\varepsilon\delta} \lesssim \langle t\rangle^{-\theta\varepsilon\delta}.$$

From these argument, we obtain

$$\|\mathcal{N}[u]\|_X \lesssim \|u_0\|_{M_{2+\kappa/1+\kappa,1}} + \|u\|_X^{1+\kappa}.$$

On the other hand, for any $u, v \in X$, we have

$$\|\mathcal{N}[u-v]\|_X \lesssim \left(\|u\|_X^\kappa + \|v\|_X^\kappa\right)\|u-v\|_X.$$

If we assume that $\|u_0\|_{M_{2+\kappa/1+\kappa,1}} \lesssim \rho/2$ and $\rho > 0$ is small enough, then $\mathcal{N} : X \to X$ is a strict contraction. Thus $\mathcal{N}$ has a unique fixed-point, and we have the desired result. □

Proof of Theorem 3.2. The proof is similar to that of Theorem 3.1. Put

$$Y = \left\{u \ ; \ \|u\|_Y = \sup_{t\in\mathbf{R}}(1+|t|)^{\theta\varepsilon\delta(p)}\|u(t)\|_{M_{p,1}} \le \rho\right\}$$

where $p \in \mathbf{N}$ and $p \in [3, 2k_0+2]$. On the other hand, using Taylor's expansion of $f(u)$, we have

$$f(u) = \sum_{k\ge k_0}\frac{\varrho^k}{k!}|u|^{2k}u.$$

From the decay estimates (3.6) and Proposition 2.2, we obtain

$$\begin{aligned}\big\|\mathcal{N}[u]\big\|_Y &\lesssim \|u_0\|_{M_{p',1}} + \sum_{k\ge k_0}\frac{\varrho^k}{k!}\sup_{t\in\mathbf{R}}\langle t\rangle^{\theta\varepsilon\delta}\int_0^t\langle t-s\rangle^{-\theta\varepsilon\delta}\big\||u|^{2k}u\big\|_{M_{p',1}}ds \\ &\lesssim \|u_0\|_{M_{p',1}} + \sum_{k\ge k_0}\frac{\varrho^k}{k!}\|u\|_Y^{2k+1}\sup_{t\in\mathbf{R}}\langle t\rangle^{\theta\varepsilon\delta}\int_0^t\langle t-s\rangle^{-\theta\varepsilon\delta}\langle s\rangle^{-(2k+1)\theta\varepsilon\delta}ds\end{aligned}$$

where $\delta = \delta(p)$. In the second inequality, we used

$$\big\||u|^{2k}u\big\|_{M_{p',1}} \lesssim \|u\|_{M_{p,1}}^{p-1}\|u\|_{M_{\infty,1}}^{2k+2-p} \lesssim \|u\|_{M_{p,1}}^{2k+1}$$

where

$$\frac{1}{p'} = \frac{p-1}{p} + \frac{2k+2-p}{\infty},$$

which is given by the embedding $M_{p_1,q} \subset M_{p_2,q}$ $(p_1 \le p_2)$ and Proposition 2.2. Thus, using the same argument used in the proof of Theorem 3.1, we have

$$\big\|\mathcal{N}[u]\big\|_Y \lesssim \|u_0\|_{M_{p',1}} + \sum_{k\ge k_0}\frac{\varrho^k}{k!}\|u\|_Y^{2k+1}.$$

If we assume that $\|u_0\|_{M_{p',1}} \lesssim \rho/2$ and $\rho > 0$ is small, then $\mathcal{N} : Y \to Y$ is a strict contraction by the same argument we used in the proof of Theorem 3.1. Thus $\mathcal{N}$ has a unique fixed-point, and we have the desired result. □

Remark 3.16. In Theorem 3.2, we need to assume that $p \in \mathbf{N}$ satisfies $p \in [3, 2k_0+2]$. From $1/p' = (p-1)/p + (2k+2-p)/\infty$, we obtain $p \in [2, 2k_0+2]$. But if $p = 2$, then $\delta(p) = 0$ and

$$\int_0^t\langle t-s\rangle^{-\theta\varepsilon\delta}\langle s\rangle^{-(2k+1)\theta\varepsilon\delta}ds = t.$$

Hence, this case $p = 2$ must be excluded.

Remark 3.17. Since we have $\kappa > \kappa_0$ for nonlinear terms $f(u) = |u|^\kappa u$ in Theorem 3.1, we need to remove the lower terms, namely, $\sum_{k<k_0} \varrho^k |u|^{2k} u/k!$ from $f(u)$. Then, in Theorem 3.2, we have

$$f(u) = \lambda \sum_{k \ge k_0} \frac{\varrho^k}{k!} |u|^{2k} u$$

where $k_0 \in \mathbf{N}$ depends on κ_0. Hence, we need to set $k \in \mathbf{N}$ and $k > \kappa_0/2$ i.e. k_0 is the minimum positive integer which satisfies $k_0 > \kappa_0/2$. We assume that $\varepsilon = 1$ to simplify arguments. κ_0 is monotonic decreasing in $\theta\varepsilon$, hence it suffices to consider the case $n = 1$. If $\beta_1 \le 2$, then we have $\theta\varepsilon = n$ and $\kappa_0 = (1 + \sqrt{17})/2$ for $n = 1$. From these arguments, we obtain

$$f(u) = \lambda(e^{\varrho|u|^2} - 1 - \varrho|u|^2)u = \lambda \sum_{k \ge 2} \frac{\varrho^k}{k!} |u|^{2k} u, \;\; n \ge 1.$$

On the other hand, if $\beta_1 \ge 2$, then $\theta\varepsilon = 2n/\beta_1$. Using the same arguments which we have used here, we can see that k_0 is the minimum positive integer which satisfies

$$k_0 > \frac{1}{2} \frac{\beta_1 - 1 + \sqrt{\beta_1^2 + 6\beta_1 + 1}}{2}.$$

3.3.2 Proofs of Theorems 3.3 and 3.4

First, we state completeness of function spaces that we use in the proofs of Theorems 3.3 and 3.4 before we state the proofs. Let $X_1 = \ell^1_\square\big(L^\infty(\mathbf{R}, L^2)\big)$ and $X_2 = \ell^1_\square\big(L^{2+\kappa}(\mathbf{R}, L^{2+\kappa})\big)$, then $X_1 \cap X_2$ is a Banach space equipped with the norm

$$\|u\|_{X_1 \cap X_2} = \|u\|_{X_1} + \|u\|_{X_2}.$$

We only state that this space is complete. Indeed, suppose that $\{v_j\}_{\mathbf{Z}_+}$ is a Cauchy sequence in $X_1 \cap X_2$. Then $\{v_j\}_{\mathbf{Z}_+}$ is also a Cauchy sequence in X_1 and X_2. Since X_1 and X_2 are complete, there exist $v \in X_1$, $w \in X_2$ such that $\lim_{j\to\infty} v_j = v$ in X_1, $\lim_{j\to\infty} v_j = w$ in X_2. Let a subsequence $\{v_{j(m)}\}_{m\in\mathbf{Z}_+}$ of $\{v_j\}_{j\in\mathbf{Z}_+}$ satisfy $|v_{j(1)} - v| \le |v_{j(2)} - v| \le \cdots$. Then, by the monotone convergence theorem, we have $\lim_{k\to\infty} v_{j(k)} = v$(a.e.). Similarly, letting a subsequence $\{v_{j(\ell)}\}_{\ell\in\mathbf{Z}_+}$ of $\{v_{j(k)}\}_{k\in\mathbf{Z}_+}$ satisfy the same property as above, we get $\lim_{\ell\to\infty} v_{j(\ell)} = w$(a.e.). From consistency of the limit, we obtain $v = w$ in $X_1 \cap X_2$(a.e.). Therefore, we get $\lim_{j\to\infty} v_j = v$ in $X_1 \cap X_2$. From the same argument, we can show completeness of $Y_1 \cap Y_2$ in the proof of Theorem 3.4. Under this completeness, we use a fixed-point theorem and show Theorems 3.3 and 3.4:

Lemma 3.18. (See [97, Lemma 8.2]) *Let* $1 \le p, p_j, \gamma, \gamma_j \le \infty$ *(*$j = 1, \cdots, m$*) satisfy*

$$\frac{1}{p'} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \;\; \textit{and} \;\; \frac{1}{\gamma'} = \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_m}.$$

Let $\alpha = 0$ and $q = 1$, or $\alpha > 0$ and $q'\alpha > nm$. Then

$$\Big\| \prod_{j=1}^{m} u_j \Big\|_{\ell_{\square}^{-\alpha,q}(L^{\gamma'}(\mathbf{R},L^{p'}))} \lesssim \prod_{j=1}^{m} \|u_j\|_{\ell_{\square}^{q}(L^{\gamma_j}(\mathbf{R},L^{p_j}))}.$$

The function spaces $\ell_{\square}^{-\alpha,q}(L^{\gamma}(\mathbf{R}, L^{p}))$ can be found in Section 3.1.3. Now, we begin with the proofs of Theorems 3.3 and 3.4.

Proof of Theorem 3.3. We set

$$\mathcal{X} = \{u \ ; \ \|u\|_{X_1 \cap X_2} \leq \rho\}.$$

From estimates (3.23) and (3.32), we obtain

$$\begin{aligned} \big\|\mathcal{N}[u]\big\|_{X_1} &\lesssim \big\|U(\cdot)u_0\big\|_{X_1} + \big\|\mathcal{U}f(u)\big\|_{X_1} \\ &\lesssim \|u_0\|_{M_{2,1}} + \big\|f(u)\big\|_{\ell_{\square}^{1}\left(L^{\frac{1+\kappa}{2+\kappa}}(\mathbf{R},L^{\frac{1+\kappa}{2+\kappa}})\right)} \end{aligned}$$

Using Lemma 3.18 as $q' = \infty$, we have

$$\big\|f(u)\big\|_{\ell_{\square}^{1}\left(L^{\frac{1+\kappa}{2+\kappa}}(\mathbf{R},L^{\frac{1+\kappa}{2+\kappa}})\right)} \lesssim \|u\|_{X_2}^{1+\kappa}$$

where

$$\frac{1+\kappa}{2+\kappa} = \underbrace{\frac{1}{2+\kappa} + \cdots + \frac{1}{2+\kappa}}_{1+\kappa}.$$

Similarly, we have

$$\begin{aligned} \big\|\mathcal{N}[u]\big\|_{X_2} &\lesssim \|u_0\|_{M_{2,1}} + \big\|f(u)\big\|_{\ell_{\square}^{1}\left(L^{\frac{1+\kappa}{2+\kappa}}(\mathbf{R},L^{\frac{1+\kappa}{2+\kappa}})\right)} \\ &\lesssim \|u_0\|_{M_{2,1}} + \|u\|_{X_2}^{1+\kappa}. \end{aligned}$$

It follows from two estimates as above that

$$\big\|\mathcal{N}[u]\big\|_{X_1 \cap X_2} \lesssim \|u_0\|_{M_{2,1}} + \|u\|_{X_1 \cap X_2}^{1+\kappa},$$

If we assume that $\|u_0\|_{M_{2+\kappa/1+\kappa,1}} \lesssim \rho/2$ and $\rho > 0$ is sufficiently small, then $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ is a strict contraction by the same argument we used in the above proofs. Thus, since $\mathcal{N}$ has a unique fixed-point, we have the desired result. $\square$

Remark 3.19. In Theorem 3.3, since we use the Strichartz estimate, we need to assume that $\kappa \in \mathbf{N}$ satisfies $\kappa \geq 4/\theta\varepsilon$. Indeed, from the conditions for the Strichartz estimates, we need to set $\kappa \in \mathbf{N}$ to satisfy

$$\frac{2}{\kappa+2} \leq \theta\varepsilon(\frac{1}{2} - \frac{1}{\kappa+2}).$$

Proof of Theorem 3.4. Let $Y_1 = \ell^1_\Box\big(L^\infty(\mathbf{R}, L^2)\big)$ and $Y_2 = \ell^1_\Box\big(L^p(\mathbf{R}, L^p)\big)$. Since the proofs that Y_1 and Y_2 are Banach spaces are the same with the argument which we used in the proofs of Theorem 3.3, we omit them. We set

$$\mathcal{Y} = \{u \ ; \ \|u\|_{Y_1\cap Y_2} \le \rho\}.$$

From estimates (3.23) and (3.32), we obtain

$$\big\|\mathcal{N}[u]\big\|_{Y_1} \lesssim \|u_0\|_{M_{2,1}} + \big\|f(u)\big\|_{\ell^1_\Box\left(L^{p'}(\mathbf{R},L^{p'})\right)}$$

Using Lemma 3.18 as $q' = \infty$, we have

$$\begin{aligned}
&\big\|f(u)\big\|_{\ell^1_\Box\left(L^{p'}(\mathbf{R},L^{p'})\right)} \\
\lesssim\ & \lambda \sum_{k\ge k_0} \frac{\varrho^k}{k!} \|u\|^{p-1}_{\ell^1_\Box\left(L^p(\mathbf{R},L^p)\right)} \|u\|^{2k+2-p}_{\ell^1_\Box\left(L^\infty(\mathbf{R},L^\infty)\right)} \\
\lesssim\ & \lambda \sum_{k\ge k_0} \frac{\varrho^k}{k!} \|u\|^{p-1}_{\ell^1_\Box\left(L^p(\mathbf{R},L^p)\right)} \|u\|^{2k+2-p}_{\ell^1_\Box\left(L^\infty(\mathbf{R},L^2)\right)} \\
\lesssim\ & \lambda \sum_{k\ge k_0} \frac{\varrho^k}{k!} \|u\|^{2k+1}_{Y_1\cap Y_2}
\end{aligned}$$

where

$$\frac{1}{p'} = \frac{p-1}{p} + \frac{2k+2-p}{\infty}$$

and

$$\|\Box_k u\|_\infty \lesssim \|\Box_k u\|_2.$$

Similarly, from inclusion relation $M_{p_1,q} \subset M_{p_2,q}$ $(p_1 \le p_2)$, we have

$$\begin{aligned}
\big\|\mathcal{N}[u]\big\|_{Y_2} &\lesssim \|u_0\|_{M_{2,1}} + \big\|f(u)\big\|_{\ell^1_\Box\left(L^{p'}(\mathbf{R},L^{p'})\right)} \\
&\lesssim \|u_0\|_{M_{2,1}} + \lambda \sum_{k\ge k_0} \frac{\varrho^k}{k!} \|u\|^{2k+1}_{Y_1\cap Y_2}.
\end{aligned}$$

It follows from two estimates as above that

$$\big\|\mathcal{N}[u]\big\|_{Y_1\cap Y_2} \lesssim \|u_0\|_{M_{2,1}} + \sum_{k\ge k_0} \frac{\varrho^k}{k!} \|u\|^{2k+1}_{Y_1\cap Y_2},$$

From this argument, we can easily show that Theorem 3.4 holds. □

Remark 3.20. From the argument we have used in Theorem 3.2, we have

$$f(u) = \lambda \sum_{k\ge k_0} \frac{\varrho^k}{k!} |u|^{2k} u.$$

Hence, we need to set $k \in \mathbf{N}$ and $k \geq 4/(2\theta\varepsilon)$ i.e. k_0 is the minimum positive integer which satisfies $k_0 \geq 4/(2\theta\varepsilon)$. We also assume that $\varepsilon = 1$ to simplify arguments. From the argument we have used in Theorem 3.2, we need to set $k_0 = 2$ if $\beta_1 \leq 2$, and set k_0 to be the minimum positive integer which satisfies

$$k_0 \geq \left.\frac{4}{2\theta\varepsilon}\right|_{n=1,\varepsilon=1} = \beta_1, \text{ if } \beta_1 \geq 2.$$

Remark 3.21. From the setting $1/p' = (p-1)/p + (2k+2-p)/\infty$ in this proof and the assumption for the Strichartz estimates, we have $p \in \mathbf{N}$ satisfies $p \in [2 + 4/(\theta\varepsilon), 2 + 2k_0]$ where $k_0 \geq 4/(2\theta\varepsilon)$. If we set $\varepsilon = 1$ and $\beta_1 \leq 2$, then we have only $p = 4$.

4 Solutions to nonlinear higher order Schrödinger equations with small initial data on modulation spaces

4.1 Introduction

4.1.1 Higher order Schrödinger equations

In this paper, we consider the following Cauchy problems for nonlinear higher order Schrödinger equations (NLHS):

$$\begin{cases} i\partial_t u + (-\Delta)^{m/2} u = f(u), & (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0) = u_0, & x \in \mathbf{R}^n. \end{cases} \tag{4.1}$$

Here differential operator $(-\Delta)^{m/2} = \mathcal{F}^{-1}|\xi|^m \mathcal{F}$ is Fourier multiplier with $m \geq 2$. For example, $m = 2$ and $m = 4$ correspond to the Schrödinger equation and the fourth order Schrödinger equation, respectively. By Duhamel's principle, we express the solution to the Cauchy problem (4.1) as the following equivalent integral equation

$$u(t) = U(t)u_0 - i\int_0^t U(t-s)f(u)ds, \tag{4.2}$$

where

$$U(t) = e^{it(-\Delta)^{m/2}},$$

and solve this integral equation by a fixed point argument. In order to solve this problem, time decay estimates (dispersion estimates):

$$\|U(t)u_0\|_{L^p} \lesssim |t|^{-\frac{2n}{m}\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{L^{p'}} \tag{4.3}$$

and the Strichartz estimates:

$$\|U(t)u_0\|_{L^\gamma_t L^\rho_x} \lesssim \|u_0\|_{L^2}$$

are useful. Here, $1/p + 1/p' = 1$, $2/\gamma = 2n/m(1/2 - 1/\rho)$, $2 \leq p, \rho \leq \infty$. Invoking these two estimates, we can show well-posedness for the problem (4.1). In this paper, we mainly compose a unique global solution to the equation in the frame of modulation spaces $M^s_{p,q}$. One can find the definition and some properties of modulation spaces in Section 2.2.3. Especially, the inclusion relation $M_{p,1} \subset L^p \subset M_{p,\infty}$ holds for $1 \leq p \leq \infty$, where we write $M_{p,q} = M^0_{p,q}$ for simplicity.

In the following, as a most important example of (4.1), we state the Cauchy problem for the nonlinear Schrödinger equation, that is, $m = 2$. Cazenave–Weissler [11] studied the global well-posedness in H^s $(0 \leq s < n/2)$ if initial data $u_0 \in \dot{H}^s$ is sufficient small, when the nonlinearity $f(u) = |u|^\kappa u$ $(\kappa = 4/(n-2s) \Rightarrow \kappa \geq 4/n)$. When $f(u) = u^{\kappa+1}$ $(\kappa \in \mathbf{N}$ and $\kappa \geq 4/n)$, Wang–Hudzik [97] showed the global well-posedness in $M_{2,1}$ with a small initial data. These two results are got by using

Strichartz type estimates. On the other hand, by using only time decay estimates, Cazenave–Weissler [12] showed that there exists a unique global solution to the nonlinear Schrödinger equation with $f(u) = |u|^\kappa u$ such that

$$\sup_{t\in\mathbf{R}} |t|^B \big\|u(t)\big\|_{L^{2+\kappa}} < +\infty,$$

if a initial data u_0 satisfying $\sup_{t\in\mathbf{R}} |t|^B \|e^{it\Delta}u_0\|_{L^{2+\kappa}} < +\infty$ is sufficiently small. Here, $\kappa \in \mathbf{R}$, $\kappa_0 < \kappa < 4/(n-2)$ ($\kappa_0 < \kappa < \infty$ if $n = 1, 2$), $B = \frac{4-(n-2)\kappa}{2\kappa(\kappa+2)}$, and $\kappa_0 > 0$ is the positive root of $n\kappa^2 + (n-2)\kappa - 4 = 0$. Moreover, Wang–Hudzik [97] also studied the existence of a unique global solution, which satisfies

$$\sup_{t\in\mathbf{R}} \langle t\rangle^{n\kappa/(2\kappa+4)} \big\|u(t)\big\|_{M_{2+\kappa,1}} < +\infty,$$

with a sufficiently small initial data $u_0 \in M_{(2+\kappa)/(1+\kappa),1}$ when the nonlinearity $f(u) = u^{\kappa+1}$ for $\kappa \in \mathbf{N}$ and $\kappa > \kappa_0$. In [97], they have used time decay estimates on modulation spaces:

$$\|U(t)u_0\|_{M_{p,q}} \lesssim (1+|t|)^{-\frac{2n}{m}\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{M_{p',q}}, \tag{4.4}$$

instead of (4.3). As one might see from comparison between time decay estimates (4.3) in L^p spaces and (4.4) in modulation spaces, there is no singular point at $t = 0$ in the estimate (4.4). This property enables us to extend κ as far as infinity.

When Cazenave–Weissler [12] and Wang–Hudzik [97] have shown the existence of a global solution by using only time decay estimates, they control the behavior of solutions by multiplying weight in time to a solution space or a initial data. On the other hands, if we control them by integrability of time decay terms $(1+|t|)^{-\theta}$, then we have the following theorem (see also Ru–Chen [79]).

Theorem 4.1. *Let $f(u) = \pi(u^{\kappa+1})$ be any $(\kappa+1)$-time product of u and $\overline{u}$ ($\kappa \in \mathbf{N}$). Let $\kappa > \kappa_0$, $p \in \mathbf{N}$, and $p \in [2, 2+\kappa]$. There exists $\rho > 0$ such that if $u_0 \in M_{(2+\kappa)/(1+\kappa),1}$ ($\subset M_{p,1}$) satisfies $\|u_0\|_{M_{(2+\kappa)/(1+\kappa),1}} \le \rho$, then the Cauchy problem (4.1) has a unique global solution*

$$u \in C(\mathbf{R}, M_{p,1}) \cap L^{1+\kappa}(\mathbf{R}, M_{2+\kappa,1}).$$

Here, κ_0 is the positive root of $n\kappa^2 + (n-m)\kappa - 2m = 0$.

In the statement of Theorem 4.1, the persistency of solutions (that is, a solution $u \in C(\mathbf{R}, X)$ if an initial data $u_0 \in X$) does not holds strictly since an initial data $u_0 \in M_{(2+\kappa)/(1+\kappa),1}$ and a solution $u \in C(\mathbf{R}, M_{p,1})$. However, it follows from $(2+\kappa)/(1+\kappa) \le p$ that the inclusion relation of modulation spaces $M_{(2+\kappa)/(1+\kappa),1} \subset M_{p,1}$. Moreover, there is no change of regularity between the initial data class and the solution class. So, the initial data belong to the frame of the solution space in $\mathbf{R}^n$–space and we can say the persistency holds in this sense.

In order to prove Theorem 4.1, we use the integrability of time decay terms $(1+|t|)^{-\theta}$, which is the specific characteristic of modulation spaces. Since $|t|^{-\theta}$ is not integrable for any $t \neq 0$, we don't know whether we can show the similar argument under L^p, Besov, or Sobolev spaces (see Remark 4.5 (1)). However, we remark that if we assume additional condition of initial data such that the linear solution $U(t)u_0$ is integrable in time and space, we have the similar statement to Theorem 4.1 (see [13, Chapter 6.3, Theorem 6.3.2]).

When we prove the global well-posedness by the Strichartz estimates, we assume that $\kappa \geq 4/n$ as mentioned above. For general, $\kappa \geq 2m/n$ (see Kato [41]). Comparing this number and κ_0, clearly we have

$$\frac{2m}{n} > \kappa_0 := \frac{(m-n)+\sqrt{n^2+6mn+m^2}}{2n}.$$

Thus, Theorem 5.1 enables us to treat a wider class of power type nonlinearity $\pi(u^{\kappa+1})$. In particular, for the Schrödinger equation on 1-dimension, we are able to deal with $\kappa = 3$. Furthermore, the solution space in Theorem 4.1 belongs to a wider class than that in Wang-Hudzik [97] (see Remark 4.5 (3)-(4)).

Moreover, we also show the existence of a global solution for an exponential growth nonlinearity as Nakamura–Ozawa [72] studied.

Theorem 4.2. *Let $f(u) = (e^{\lambda|u|^2} - \sum_{k<k_0} \lambda^k |u|^{2k}/k!)u$ ($\lambda > 0$), $p \in \mathbf{N}$, $p \in [2, 2+2k_0]$. There exists $\rho > 0$ such that if $u_0 \in M_{(1+2k_0)/(2+2k_0),1}$ ($\subset M_{p,1}$) satisfies $\|u_0\|_{M_{(1+2k_0)/(2+2k_0),1}} \leq \rho$, then the Cauchy problem (4.1) has a unique global solution*

$$u \in C(\mathbf{R}, M_{p,1}) \cap L^{1+2k_0}(\mathbf{R}, M_{2+2k_0,1}).$$

Here, $k_0 \in \mathbf{N}$ is the smallest integer such that $k_0 > \kappa_0/2$ and $\kappa_0 > 0$ is the positive number in Theorem 4.1.

In Theorem 4.2 we remove the lower terms $\sum_{k<k_0} \lambda^k |u|^{2k}/k!$ from an exponential growth nonlinearity, since we assume that $\kappa > \kappa_0$ in Theorem 4.1 to get a global solution. For instance, we have for the Schrödinger equation

$$\begin{aligned} f(u) &= (e^{\lambda|u|^2} - 1 - \lambda|u|^2)u, \text{ if } n = 1, \\ f(u) &= (e^{\lambda|u|^2} - 1)u, \text{ if } n \geq 2, \end{aligned}$$

since $\kappa_0 = \frac{(2-n)+\sqrt{n^2+12n+4}}{2n}$.

4.1.2 Inhomogeneou type of NLHS

Next, we consider the following Cauchy problems for nonlinear dispersive equations:

$$\begin{cases} i\partial_t u + \phi(\sqrt{-\Delta})u = f(u), & (t,x) \in \mathbf{R} \times \mathbf{R}^n, \\ u(0) = u_0, & x \in \mathbf{R}^n. \end{cases} \tag{4.5}$$

Here differential operators $\phi(\sqrt{-\Delta}) = \mathcal{F}^{-1}\phi(|\xi|)\mathcal{F}$ is the Fourier multiplier with ϕ, and ϕ is a polynomial. For example, $\phi(r) = r^2$ and $\phi(r) = r^4 + r^2$ corresponds to the Schrödinger equation and the fourth order Schrödinger equation, respectively. For this problem, we solve the equivalent integral form (4.2), where $U(t) = e^{it\phi(\sqrt{-\Delta})}$. As mentioned at the above subsection, a time decay estimate plays an important role to constitute a solution. In this subsection, we firstly establish a time decay estimate, and then show the existence of a global solution on modulation spaces as well as we studied in the previous subsection. By the Fourier transform, the linear solution $e^{it\phi(\sqrt{-\Delta})}u_0$ is expressed as

$$e^{it\phi(\sqrt{-\Delta})}u_0 = \mathcal{F}^{-1}e^{it\phi(|\xi|)} * u_0,$$

where

$$\mathcal{F}^{-1}e^{it\phi(|\xi|)} = \int_{\mathbf{R}^n} e^{ix\cdot\xi+it\phi(|\xi|)}d\xi.$$

Thus it suffices to show that this oscillatory integral $\mathcal{F}^{-1}e^{it\phi(|\xi|)}$ has time decay. Analysis of oscillatory integral is studied by many researchers. See [1, 2, 15, 19, 20, 32, 42, 49, 68, 69, 83, 84, 85, 103]. For example of these researches, Guo-Peng-Wang [32] and Chen-Miao-Yao [15] constructed decay estimates if the symbols ϕ are algebraic functions, for instance, $\phi(|\xi|) = |\xi|^2 + |\xi|$. Kato [42] established them if ϕ has perturbations of elementary functions, for instance, $\phi(|\xi|) = |\xi|^2 + \sin(|\xi|)$.

In this subsection, we assume that ϕ is a polynomial. Thus, we can express ϕ as the following form: for $M \geq 2$ and $0 \leq m < M$

$$\phi(|\xi|) = \sum_{j=m}^{M} \alpha_j|\xi|^j \text{ for } |\xi| > 0,$$

where $\alpha_j \in \mathbf{R}$. Since ϕ is a radial function, it suffices to consider the 1-dimension case. Then, we can classify $\phi(r)$ as four cases (A)–(D): For $r > 0$,

(A) ϕ has NO critical points and NO inflection points.

(B) ϕ has NO critical points, but have inflection points.

(C) ϕ has a critical point, but have NO inflection points.

(D) ϕ has critical points and inflection points.

In particular, for the case (c), ϕ obviously have only one critical point. Here, a critical point is a point such that $\phi'(r) = 0$ and a inflection point is a point such that $\phi''(r) = 0$. In all of the previous results, decay rates of a time decay estimate depend on the largest order M and smallest order m of ϕ. We will present here time decay estimates with better decay rates than the previous results by considering multiplicities of critical and inflection points of ϕ carefully (see Remark 4.9 (3)). In fact, we have the following time decay estimates:

Proposition 4.3. *Let $n \geq 1$, $2 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have*

$$\left\| e^{it\phi(\sqrt{-\Delta})}u_0 \right\|_{M_{p,q}} \lesssim P(t) \left\| u_0 \right\|_{M_{p',q}}.$$

Here, $P(t)$ are as follows:

(a) *For the case when ϕ satisfies (A),*

$$P(t) = \begin{cases} (1+|t|)^{-(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)}, & \text{for } m = 1, \\ (1+|t|)^{-\frac{2n}{m}\left(\frac{1}{2}-\frac{1}{p}\right)}, & \text{for } m \geq 2. \end{cases}$$

(b) *For the case when ϕ satisfies (B),*

$$P(t) = \begin{cases} (1+|t|)^{-(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)}, & \text{for } m = 1, \\ (1+|t|)^{-\min\left\{\frac{2n}{m}, 2\Theta_2\right\}\left(\frac{1}{2}-\frac{1}{p}\right)}, & \text{for } m \geq 2, \end{cases}$$

where $\Theta_2 = \min_k \left\{\frac{n-1}{2} + \frac{1}{\ell_{2,k}+2}\right\}$ *and* $\{\ell_{2,k}\}$ *are multiplicities of inflection points of* ϕ.

(c) *For the case when ϕ satisfies (C),*

$$P(t) = \begin{cases} (1+|t|)^{-\min\{n-1,1\}\left(\frac{1}{2}-\frac{1}{p}\right)}, & \text{for } m = 1, \\ (1+|t|)^{-\min\left\{\frac{2n}{m}, 1\right\}\left(\frac{1}{2}-\frac{1}{p}\right)}, & \text{for } m \geq 2, \end{cases}$$

(d) *For the case when ϕ satisfies (D),*

$$P(t) = \begin{cases} (1+|t|)^{-\min\{n-1,2\Theta_1\}\left(\frac{1}{2}-\frac{1}{p}\right)}, & \text{for } m = 1, \\ (1+|t|)^{-\min\left\{\frac{2n}{m}, 2\Theta_1, 2\Theta_2\right\}\left(\frac{1}{2}-\frac{1}{p}\right)}, & \text{for } m \geq 2, \end{cases}$$

where $\Theta_1 = \min_k \left\{\frac{1}{\ell_{1,k}+1}\right\}$ *and* $\{\ell_{1,k}\}$ *are multiplicities of critical points of* ϕ.

Furthermore, for the case $m = 0$, the decay rates depend on the second smallest order of ϕ.

In this proposition, the largest order $M \in \mathbf{N}$ of ϕ disappears from every time decay factors $P(t)$. Actually, the largest order M and the smallest order m affect time decay factors for small time $|t| \leq 1$ and for large time $|t| \geq 1$, respectively (see Proposition 4.10). However, since time decay factors $|t|^{-\theta}$ can be changed to $(1+|t|)^{-\theta}$ on modulation spaces, it suffices to consider only large time. Thus, only the smallest order m appears in all time decay rates in Proposition 4.3.

In Proposition 4.3, we see that $e^{i\phi(\sqrt{-\Delta})} : M_{p',q} \to M_{p,q}$ with $M \geq 2$ and $t = 1$. On the other hands, uniform boundedness of $e^{i(-\Delta)^{M/2}}$ in $M^s_{p,q}$ with $0 \leq M \leq 2$, i.e.

$$e^{i(-\Delta)^{M/2}} : M_{p,q} \hookrightarrow M_{p,q} \text{ for } 1 \leq p, q \leq \infty,$$

was given by Bényi, et al. [3]. Moreover, for $M > 2$, Miyachi, et al. [70] showed that $e^{i(-\Delta)^{M/2}}$ is bounded from $M^s_{p,q}$ to $M_{p,q}$ if and only if $s \geq (M-2)n|1/p-1/2|$. Since we know that $e^{i(-\Delta)^{M/2}}$ is bounded from L^p_s to L^p if and only if $s \geq Mn|1/p-1/2|$ (see Miyachi [67]), we see that a loss of regularity on modulation spaces is smaller than that on L^p spaces.

From these estimates in Proposition 4.3, we obtain the following theorems:

Theorem 4.4. *Let $f(u) = \pi(u^{\kappa+1})$ be any $(\kappa+1)$-time product of u and $\overline{u}$ ($\kappa \in \mathbf{N}$). Let $\kappa > \kappa_0$ and $p \in [2, 2+\kappa] \cap \mathbf{N}$. There exists $\rho > 0$ such that if $u_0 \in M_{(2+\kappa)/(1+\kappa),1}$ satisfies $\|u_0\|_{M_{(2+\kappa)/(1+\kappa),1}} \le \rho$, then the Cauchy problem (4.5) has a unique global solution*

$$u \in C(\mathbf{R}, M_{p,1}) \cap L^{1+\kappa}(\mathbf{R}, M_{2+\kappa,1}).$$

Here, κ_0 are the following constants:

(a-i) *For the case when ϕ satisfies (A), $m = 1$, and $n \ge 2$,*

$$\kappa_0 \text{ is the positive root of } (n-1)\kappa^2 + (n-3)\kappa - 4 = 0.$$

(a-ii) *For the case when ϕ satisfies (A), $m \ge 2$, and $n \ge 1$,*

$$\kappa_0 \text{ is the positive root of } n\kappa^2 + (n-m)\kappa - 2m = 0.$$

(b-i) *For the case when ϕ satisfies (B), $m = 1$, and $n \ge 2$,*

$$\kappa_0 \text{ is the positive root of } (n-1)\kappa^2 + (n-3)\kappa - 4 = 0,$$

(b-ii) *For the case when ϕ satisfies (B), $m \ge 2$, and $n \ge 1$,*

$$\kappa_0 \text{ is the positive root of } \theta\kappa^2 + (\theta-2)\kappa - 4 = 0,$$

where $\theta = \min\left\{\frac{2n}{m}, 2\Theta_2\right\}$ and $\Theta_2 = \min_k \left\{\frac{n-1}{2} + \frac{1}{\ell_{2,k}+2}\right\}$, and $\{\ell_{2,k}\}$ are multiplicities of inflection points of ϕ.

(c-i) *For the case when ϕ satisfies (C), $m = 1$, and $n \ge 2$,*

$$\kappa_0 \text{ is the positive root of } \kappa^2 - \kappa - 4 = 0,$$

(c-ii) *For the case when ϕ satisfies (C), $m \ge 2$, and $n \ge 1$,*

$$\kappa_0 \text{ is the positive root of } \theta\kappa^2 + (\theta-2)\kappa - 4 = 0,$$

where $\theta = \min\left\{\frac{2n}{m}, 1\right\}$.

(d-i) *For the case when ϕ satisfies (D), $m = 1$, and $n \ge 2$,*

$$\kappa_0 \text{ is the positive root of } \Theta_1\kappa^2 + (\Theta_1 - 2)\kappa - 4 = 0,$$

where $\Theta_1 = \min_k \left\{\frac{1}{\ell_{1,k}+1}\right\}$ and $\{\ell_{1,k}\}$ are multiplicities of critical points of ϕ.

(d-ii) *For the case when ϕ satisfies (D), $m \ge 2$, and $n \ge 1$,*

$$\kappa_0 \text{ is the positive root of } \theta\kappa^2 + (\theta-2)\kappa - 4 = 0,$$

where $\theta = \min\left\{\frac{2n}{m}, 2\Theta_1, 2\Theta_2\right\}$.

For the case $m = 0$, $\kappa_0 > 0$ is determined by the second smallest order of ϕ. For example, if we consider $\phi(r) = r^4 - r^2 + 1$, $\kappa_0 > 0$ is the positive root of (d-ii) in Theorem 4.4.

4.2 Proofs for the existence of a solution to NLHS

In this section, we prove the main Theorems 4.1 and 4.2 in Subsection 4.1.1. In order to show these theorems, in addition to estimates (4.4), we use

$$\left\| e^{it(-\Delta)^{m/2}} u_0 \right\|_{M_{p,q}} \lesssim \|u_0\|_{M_{p',q}}, \tag{4.6}$$

where $2 \le p \le \infty$, $1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. This estimates are given by the Hausdorff–Young inequality and the Hölder inequality, immediately. Now, we begin with the proof.

Proof of Theorem 4.1. We set

$$\mathcal{X} = \{u \ ; \ \|u\|_{X_1 \cap X_2} \le \rho\},$$

where $p \in [2, 2+\kappa]$,

$$X_1 := L^{1+\kappa}(\mathbf{R}, M_{2+\kappa,1}), \text{ and } X_2 := L^\infty(\mathbf{R}, M_{p,1}).$$

We set

$$\mathcal{N}[u](t) = U(t)u_0 - i\int_0^t U(t-s)f(u)(s)ds\ , \quad u \in X \text{ and } t \in \mathbf{R}$$

where $U(t) := e^{it(-\Delta)^{m/2}}$. From estimates (4.4), we obtain

$$\|U(t)u_0\|_{X_1} \lesssim \left\| (1+|t|)^{-\frac{2n}{m}\frac{\kappa}{2(2+\kappa)}} \|u_0\|_{M_{(2+\kappa)/(1+\kappa),1}} \right\|_{L^{1+\kappa}(\mathbf{R})}.$$

Since $\kappa > \kappa_0$, we have

$$\frac{2n}{m}\frac{\kappa}{2(2+\kappa)}(1+\kappa) > 1.$$

Thus we have

$$(1+|t|)^{-\frac{2n}{m}\frac{\kappa}{2(2+\kappa)}} \in L^{1+\kappa}(\mathbf{R}), \tag{4.7}$$

and

$$\|U(t)u_0\|_{X_1} \lesssim \|u_0\|_{M_{(2+\kappa)/(1+\kappa),1}}.$$

Next, we consider the Duhamel terms. By Proposition 2.2, the Young inequality and the Hölder inequality, we have

$$\begin{aligned} & \left\| \int_0^t U(t-s)f(u)(s)ds \right\|_{X_1} \\ \le\ & \left\| \int_0^t (1+|t-s|)^{-\frac{2n}{m}\frac{\kappa}{2(2+\kappa)}} \left\|\pi(u^{1+\kappa})\right\|_{M_{(2+\kappa)/(1+\kappa),1}} ds \right\|_{L^{1+\kappa}(\mathbf{R})} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\| \int_{\mathbf{R}} (1+|t-s|)^{-\frac{2n}{m}\frac{\kappa}{2(2+\kappa)}} \|u\|_{M_{2+\kappa,1}}^{1+\kappa} ds \right\|_{L^{1+\kappa}(\mathbf{R})} \\
&\lesssim \left\| (1+|t|)^{-\frac{2n}{m}\frac{\kappa}{2(2+\kappa)}} \right\|_{L^{1+\kappa}(\mathbf{R})} \left\| \|u\|_{M_{2+\kappa,1}}^{1+\kappa} \right\|_{L^1(\mathbf{R})} \\
&\lesssim \left\| \|u\|_{M_{2+\kappa,1}} \right\|_{L^{1+\kappa}(\mathbf{R})}^{1+\kappa} \\
&= \|u\|_{L^{1+\kappa}(\mathbf{R},M_{2+\kappa,1})}^{1+\kappa}.
\end{aligned}
$$

In the second inequality, we used Proposition 2.2 as

$$
\frac{1+\kappa}{2+\kappa} = \underbrace{\frac{1}{2+\kappa} + \cdots + \frac{1}{2+\kappa}}_{1+\kappa}.
$$

From the estimates (4.6), we obtain

$$
\|U(t)u_0\|_{X_2} \lesssim \left\| \|u_0\|_{M_{(2+\kappa)/(1+\kappa),1}} \right\|_{L^\infty(\mathbf{R})} \lesssim \|u_0\|_{M_{(2+\kappa)/(1+\kappa),1}}.
$$

Finally, We consider the Duhamel terms under X_2. By the estimates (4.6) and the inclusion relation on modulation spaces (2.2), we have

$$
\begin{aligned}
&\left\| \int_0^t U(t-s) f(u)(s) ds \right\|_{L^\infty(\mathbf{R},M_{p,1})} \\
\lesssim\ & \left\| \int_0^t \left\| \pi(u^{1+\kappa})(s) \right\|_{M_{p',1}} ds \right\|_{L^\infty(\mathbf{R})} \\
\lesssim\ & \left\| \|u\|_{M_{p'(1+\kappa),1}}^{1+\kappa} \right\|_{L^1(\mathbf{R})} \\
\lesssim\ & \left\| \|u\|_{M_{2+\kappa,1}}^{1+\kappa} \right\|_{L^1(\mathbf{R})} \\
\lesssim\ & \left\| \|u\|_{M_{2+\kappa,1}} \right\|_{L^{1+\kappa}(\mathbf{R})}^{1+\kappa} \\
=\ & \|u\|_{L^{1+\kappa}(\mathbf{R},M_{2+\kappa,1})}^{1+\kappa}.
\end{aligned}
$$

In the third inequality, we used the inclusion relation (2.2) for $2+\kappa \le p'(1+\kappa)$ and $p \in [2, 2+\kappa]$.

Thus, we have

$$
\begin{aligned}
\|\mathcal{N}[u]\|_{\mathcal{X}} &\lesssim \|u_0\|_{M_{(2+\kappa)/(1+\kappa),1}} + \|u\|_{L^{1+\kappa}(\mathbf{R},M_{2+\kappa,1})}^{1+\kappa} \\
&\lesssim \|u_0\|_{M_{(2+\kappa)/(1+\kappa),1}} + \|u\|_{\mathcal{X}}^{1+\kappa}.
\end{aligned}
$$

On the other hand, for any $u, v \in \mathcal{X}$, we have

$$
\|\mathcal{N}[u-v]\|_{\mathcal{X}} \lesssim \left(\|u\|_{\mathcal{X}}^{\kappa} + \|v\|_{\mathcal{X}}^{\kappa} \right) \|u-v\|_{\mathcal{X}}.
$$

If we assume that $\rho > 0$ is sufficiently small and $\|u_0\|_{M_{2+\kappa/1+\kappa,1}} \lesssim \rho/2$, then $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ is a strict contraction. Therefore, $\mathcal{N}$ has a unique fixed-point and we have the desired result. □

Remark 4.5. We have four remarks for Theorem 4.1.
(1) Since

$$|t|^{-\frac{2n}{m}\frac{\kappa}{2(2+\kappa)}} \notin L^{1+\kappa}(\mathbf{R}),$$

we can not apply the same argument on L^p spaces.
(2) If we use the Strichartz estimates to show the well-posedness, we need the assumption that $\kappa \geq 2m/n$ (See [41]). For the Schrödinger equations, we need $\kappa \geq 4/n$ (See [97]). Comparing between $2m/n$ and κ_0, we have

$$2m/n > \kappa_0 = \frac{(m-n)+\sqrt{n^2+6mn+m^2}}{2n}.$$

In fact, we have

$$\begin{aligned}\frac{2m}{n}-\kappa_0 &= \frac{2m}{n}-\frac{(m-n)+\sqrt{n^2+6mn+m^2}}{2n} \\ &= \frac{(3m+n)-\sqrt{(n+3m)^2-8m^2}}{2n} > 0.\end{aligned}$$

Thus, we can deal with a wider class of power type nonlinearity $\pi(u^{\kappa+1})$.
(3) In [97, Theorem 1.3], Wang-Hudzik constructed the solution spaces in $C(\mathbf{R}, M_{2,1})$ by using the Strichartz estimates. From the inclusion relation: $M_{p_1,1} \subset M_{p_2,1}$ $(1 \leq p_1 \leq p_2 \leq \infty)$, we see that $C(\mathbf{R}, M_{2,1}) \subset C(\mathbf{R}, M_{p,1})$ for $p \in [2, 2+\kappa]$.
(4) In [97, Theorem 1.1], Wang-Hudzik also showed that there exists a global unique solution for the Schrödinger equation, which belongs to

$$X := \Big\{ u : \sup_{t\in\mathbf{R}} (1+|t|)^{n\kappa/(4+2\kappa)} \|u(t)\|_{M_{2+\kappa,1}} < \infty \Big\}.$$

The solution space $L^{1+\kappa}(\mathbf{R}, M_{2+\kappa,1})$ in Theorem 4.1 is a wider class than the space X. In fact, let $u \in X$. Then we have

$$\begin{aligned}&\|u\|^{1+\kappa}_{L^{1+\kappa}(\mathbf{R},M_{2+\kappa,1})} \\ =\ & \int_{\mathbf{R}^n} \|u(t,\cdot)\|^{1+\kappa}_{M_{2+\kappa,1}}\, dt \\ =\ & \int_{\mathbf{R}^n} (1+|t|)^{-n\kappa(1+\kappa)/(4+2\kappa)} \Big((1+|t|)^{n\kappa/(4+2\kappa)} \|u(t,\cdot)\|_{M_{2+\kappa,1}} \Big)^{1+\kappa} dt \\ \lesssim\ & \|u\|_X \int_{\mathbf{R}^n} (1+|t|)^{-n\kappa(1+\kappa)/(4+2\kappa)} dt < \infty.\end{aligned}$$

Here, we used the assumption $\kappa > \kappa_0$ in the fourth inequality.

Next, we prove Theorem 4.2.

Proof of Theorem 4.2. In this theorem, we have by the Taylor expansion

$$f(u) = \sum_{k\geq k_0} \frac{\lambda^k}{k!} |u|^{2k} u.$$

We set

$$\mathcal{Y} = \{u \ ; \ \|u\|_{Y_1 \cap Y_2} \le \rho\},$$

where $p \in [2, 2+2k_0]$,

$$Y_1 := L^{1+2k_0}(\mathbf{R}, M_{2+2k_0,1}), \text{ and } Y_2 := L^\infty(\mathbf{R}, M_{p,1}).$$

We set

$$\mathcal{N}[u](t) = U(t)u_0 - i\int_0^t U(t-s)f(u)(s)ds\ , \quad u \in X \text{ and } t \in \mathbf{R}$$

where $U(t) := e^{it(-\Delta)^{m/2}}$. From estimates (4.4), we obtain

$$\|U(t)u_0\|_{Y_1} \lesssim \left\| (1+|t|)^{-\frac{n}{m}\frac{k_0}{1+k_0}} \|u_0\|_{M_{(2+2k_0)/(1+2k_0),1}} \right\|_{L^{1+2k_0}(\mathbf{R})}.$$

Since $k_0 > \kappa_0/2$, we have

$$\frac{n}{m}\frac{k_0}{1+k_0}(1+2k_0) > 1 \text{ and } (1+|t|)^{-\frac{n}{m}\frac{k_0}{1+k_0}} \in L^{1+2k_0}(\mathbf{R}).$$

Thus, we have

$$\|U(t)u_0\|_{Y_1} \lesssim \|u_0\|_{M_{(2+2k_0)/(1+k_0),1}}.$$

Next, we consider the Duhamel terms. By Proposition 2.2, the Young inequality and the Hölder inequality, we have

$$\begin{aligned}
&\left\| \int_0^t U(t-s)f(u)(s)ds \right\|_{Y_1} \\
\le\ & \sum_{k \ge k_0} \frac{\lambda^k}{k!} \left\| \int_0^t (1+|t-s|)^{-\frac{n}{m}\frac{k_0}{1+k_0}} \left\| |u|^{2k}u \right\|_{M_{(2+2k_0)/(1+2k_0),1}} ds \right\|_{L^{1+2k_0}(\mathbf{R})} \\
\lesssim\ & \left\| (1+|t|)^{-\frac{n}{m}\frac{k_0}{1+k_0}} \right\|_{L^{1+2k_0}(\mathbf{R})} \sum_{k \ge k_0} \frac{\lambda^k}{k!} \left\| \left\| |u|^{2k}u \right\|_{M_{(2+2k_0)/(1+2k_0),1}} \right\|_{L^1(\mathbf{R})} \\
\lesssim\ & \sum_{k \ge k_0} \frac{\lambda^k}{k!} \|u\|_{Y_1 \cap Y_2}^{1+2k}.
\end{aligned}$$

In the third inequality, we used the following argument:

$$\begin{aligned}
&\left\| \left\| |u|^{2k}u \right\|_{M_{(2+2k_0)/(1+2k_0),1}} \right\|_{L^1(\mathbf{R})} \\
\lesssim\ & \left\| \|u\|_{M_{2+2k_0,1}}^{1+2k_0} \|u\|_{M_{\infty,1}}^{2k-2k_0} \right\|_{L^1(\mathbf{R})} \\
\le\ & \left\| \|u\|_{M_{2+2k_0,1}}^{1+2k_0} \right\|_{L^1(\mathbf{R})} \left\| \|u\|_{M_{p,1}}^{2k-2k_0} \right\|_{L^\infty(\mathbf{R})} \\
\le\ & \|u\|_{L^{1+2k_0}(\mathbf{R}, M_{2+2k_0,1})}^{1+2k_0} \|u\|_{L^\infty(\mathbf{R}, M_{p,1})}^{2k-2k_0}
\end{aligned}$$

$$\leq \|u\|_{Y_1\cap Y_2}^{1+2k}$$

by the inclusion relation (2.2) and Proposition 2.2 with

$$\frac{1+2k_0}{2+2k_0} = \frac{(2+2k_0)-1}{2+2k_0} + \frac{2k-2k_0}{\infty}.$$

On the other hands, from the estimates (4.6), we obtain

$$\begin{aligned}\|U(t)u_0\|_{Y_2} &\lesssim \left\| \|u_0\|_{M_{(2+2k_0)/(1+2k_0),1}} \right\|_{L^\infty(\mathbf{R})} \\ &\lesssim \|u_0\|_{M_{(2+2k_0)/(1+2k_0),1}}.\end{aligned}$$

Finally, We consider the Duhamel terms under Y_2. Using the estimates (4.6) and Proposition 2.2 as

$$\frac{1}{p'} = \underbrace{\frac{1}{p'(1+2k)} + \cdots + \frac{1}{p'(1+2k)}}_{1+2k}.$$

we have

$$\begin{aligned}&\left\| \int_0^t U(t-s)f(u)(s)ds \right\|_{L^\infty(\mathbf{R},M_{p,1})} \\ \lesssim& \sum_{k\geq k_0} \frac{\lambda^k}{k!} \left\| \int_0^t \left\| |u|^{2k}u \right\|_{M_{p',1}} ds \right\|_{L^\infty(\mathbf{R})} \\ \leq& \sum_{k\geq k_0} \frac{\lambda^k}{k!} \left\| \|u\|_{M_{p'(1+2k),1}}^{1+2k} \right\|_{L^1(\mathbf{R})}.\end{aligned}$$

Since $p \leq 2+2k_0 \leq p'(1+2k)$ from $p \in [2, 2+2k_0] \cap \mathbf{N}$, the inclusion relation $M_{p,1} \subset M_{2+2k_0,1} \subset M_{p'(1+2k),1}$ holds. So,

$$\begin{aligned}&\left\| \|u\|_{M_{p'(1+2k),1}}^{1+2k} \right\|_{L^1(\mathbf{R})} \\ \lesssim& \left\| \|u\|_{M_{p,1}}^{2k-2k_0} \|u\|_{M_{2+2k_0,1}}^{1+2k_0} \right\|_{L^1(\mathbf{R})} \\ \lesssim& \|u\|_{L^\infty(\mathbf{R},M_{p,1})}^{2k-2k_0} \|u\|_{L^{1+2k_0}(\mathbf{R},M_{2+2k_0,1})}^{1+2k_0} \\ \leq& \|u\|_{Y_1\cap Y_2}^{1+2k}.\end{aligned}$$

Thus, we have

$$\|\mathcal{N}[u]\|_{\mathcal{Y}} \lesssim \|u_0\|_{M_{(2+2k_0)/(1+2k_0),1}} + \sum_{k\geq k_0} \frac{\lambda^k}{k!} \|u\|_{\mathcal{Y}}^{1+2k}.$$

On the other hand, for any $u, v \in \mathcal{Y}$, we have

$$\|\mathcal{N}[u-v]\|_{\mathcal{Y}} \lesssim \left(\|u\|_{\mathcal{Y}}^{2k_0} + \|v\|_{\mathcal{Y}}^{2k_0} \right) \|u-v\|_{\mathcal{Y}}.$$

If we assume that $\rho > 0$ is sufficiently small and $\|u_0\|_{M_{(2+2k_0)/(1+2k_0),1}} \lesssim \rho/2$, then $\mathcal{N} : \mathcal{Y} \to \mathcal{Y}$ is a strict contraction. Therefore, $\mathcal{N}$ has a unique fixed-point and we have the desired result. □

4.3 Analysis of oscillatory integrals

From this subsection, we consider the Cauchy problem (4.5). In this section, we prepare to establish time decay estimates in Proposition 4.3. We construct the estimates on Besov spaces $B^s_{p,q}$, and then change them to those on modulation spaces $M^s_{p,q}$. As mentioned in Subsection 4.1.2, it follows that

$$e^{it\phi(\sqrt{-\Delta})}u_0 = \mathcal{F}^{-1}e^{it\phi(|\xi|)} * u_0,$$

where

$$\mathcal{F}^{-1}e^{it\phi(|\xi|)} = \int_{\mathbf{R}^n} e^{ix\cdot\xi+it\phi(|\xi|)}d\xi.$$

In order to show this oscillatory integral $\mathcal{F}^{-1}e^{it\phi(|\xi|)}$ has time decay, we precisely analyze the oscillatory factor $e^{ix\cdot\xi+it\phi(|\xi|)}$ by Littlewood–Paley decomposition ψ_j (see Subsection 2.2.1). Thus, we consider

$$\begin{aligned} I_j(x) &:= \mathcal{F}^{-1}\left[e^{it\phi(|\cdot|)}\psi_j(\cdot)\right](x) \\ &= \int_{\mathbf{R}^n} e^{ix\cdot\xi+it\phi(|\xi|)}\psi(2^{-j}\xi)d\xi. \end{aligned}$$

Furthermore, we estimate this integral under L^∞-norm, it is equivalent to estimate the following oscillatory integral by change of variable:

$$I_j(2^{-j}x) = 2^{jn}\int_{\mathbf{R}^n} e^{ix\cdot\xi}e^{it\phi(2^j|\xi|)}\psi(\xi)d\xi.$$

We reset $I_j(2^{-j}x)$ as

$$I_j(x) = 2^{jn}\int_{\mathbf{R}^n} e^{ix\cdot\xi}e^{it\phi(2^j|\xi|)}\psi(\xi)d\xi. \tag{4.8}$$

Now, since $\phi : \mathbf{R}^n \to \mathbf{R}$ is a radial polynomial with the largest order $2 \le M$ and the smallest order $0 \le m < M$, we can write ϕ as

$$\phi(r) = \sum_{j=m}^{M}\alpha_j r^j \text{ for } r = |\xi| > 0. \tag{4.9}$$

Moreover, there exists $P > 0$ such that

$$|\phi'(r)| \sim r^{M-1} \text{ and } |\phi''(r)| \sim r^{M-2} \text{ for } r \ge P > 0,$$

and there also exists $p > 0$ such that

$$|\phi'(r)| \sim r^{m_1-1} \text{ and } |\phi''(r)| \sim r^{m_2-2} \text{ for } 0 < r \le p.$$

Here, $m_1, m_2 \in \mathbf{N}$ satisfy that $m_1 = 1$ and $m_2 \ge 2$ if $m = 1$ and $m_1 = m_2 = m$ if $m \ge 2$.

We consider the oscillatory integral $I_j(x)$ by relating $P, p > 0$ and the supports of ψ_j. More precisely, setting $j_0 \in \mathbf{Z}$ is the largest integer such that for all $j \le j_0$ supp $\psi_j \subset (0, p]$ and $J_0 \in \mathbf{Z}$ is the smallest integer such that for all $j \ge J_0$ supp $\psi_j \subset [P, \infty]$, we consider $I_j(x)$ for three cases: $j \le j_0$, $j_0 < j < J_0$, and $J_0 \le j$. Then, we have the following statement.

Proposition 4.6. *Let ϕ be a polynomial (4.9) and $j_0, J_0 \in \mathbf{Z}$ be the integers as is set above. Then we have the following estimates for oscillatory integrals (4.8):*
(a) *For the case when ϕ satisfies (A), we have*

$$|I_j(x)| \lesssim |t|^{-\theta} 2^{j(n-M\theta)}, \text{ for } j \geq 0 \text{ and } 0 \leq \theta \leq n/2,$$

$$\sum_{j<0} |I_j(x)| \lesssim \begin{cases} (1+|t|)^{-\frac{n-1}{2}}, & \text{if } m = 1, \\ (1+|t|)^{-\frac{n}{m}}, & \text{if } m \geq 2. \end{cases}$$

(b) *For the case when ϕ satisfies (B), we have*

$$|I_j(x)| \lesssim |t|^{-\theta} 2^{j(n-M\theta)}, \text{ for } j \geq J_0 \text{ and } 0 \leq \theta \leq n/2,$$

$$|I_j(x)| \lesssim |t|^{-\theta}, \text{ for } j_0 < j < J_0 \text{ and } 0 \leq \theta \leq \Theta_2,$$

$$\sum_{j \leq j_0} |I_j(x)| \lesssim \begin{cases} (1+|t|)^{-\frac{n-1}{2}}, & \text{if } m = 1, \\ (1+|t|)^{-\frac{n}{m}}, & \text{if } m \geq 2. \end{cases}$$

Here, $\Theta_2 = \min_k \left\{ \frac{n-1}{2} + \frac{1}{\ell_{2,k}+2} \right\}$ and $\{\ell_{2,k}\}$ are multiplicities of inflection points of ϕ.

(c) *For the case when ϕ satisfies (C), we have*

$$|I_j(x)| \lesssim |t|^{-\theta} 2^{j(n-M\theta)}, \text{ for } j \geq J_0 \text{ and } 0 \leq \theta \leq n/2,$$

$$|I_j(x)| \lesssim |t|^{-\theta}, \text{ for } j_0 < j < J_0 \text{ and } 0 \leq \theta \leq 1/2,$$

$$\sum_{j \leq j_0} |I_j(x)| \lesssim \begin{cases} (1+|t|)^{-\frac{n-1}{2}}, & \text{if } m = 1, \\ (1+|t|)^{-\frac{n}{m}}, & \text{if } m \geq 2. \end{cases}$$

(d) *For the case when ϕ satisfies (D), we have*

$$|I_j(x)| \lesssim |t|^{-\theta} 2^{j(n-M\theta)}, \text{ for } j \geq J_0 \text{ and } 0 \leq \theta \leq n/2,$$

$$|I_j(x)| \lesssim |t|^{-\theta}, \text{ for } j_0 < j < J_0 \text{ and } 0 \leq \theta \leq \min\{\Theta_1, \Theta_2\},$$

$$\sum_{j \leq j_0} |I_j(x)| \lesssim \begin{cases} (1+|t|)^{-\frac{n-1}{2}}, & \text{if } m = 1, \\ (1+|t|)^{-\frac{n}{m}}, & \text{if } m \geq 2. \end{cases}$$

Here, $\Theta_1 = \min_k \left\{ \frac{1}{\ell_{1,k}+1} \right\}$ and $\{\ell_{1,k}\}$ are multiplicities of critical points of ϕ.

Furthermore, for the case $m = 0$, the decay rates depend on the second smallest order of ϕ.

Before we prove Proposition 4.6, we prepare two lemmas.

Lemma 4.7. *Let $F(r)$ be a polynomial of degree $M \in \mathbf{N}$. Let a point $a \in \mathbf{R}$ satisfy $F(a) = 0$ and be a root of multiplicity $m \in \mathbf{N}$. Then we have $F^{(m)}(a) \neq 0$.*

Proof. Since F is a polynomial, we set $F(r) = (r-a)^m f(r)$ where $f(r) = \mathcal{O}(r^{M-m})$ and $f(a) \neq 0$ without loss of generality. By Lipnitz rule, for all $m \in \{1, \cdots, M\}$, we have

$$\begin{aligned} F^{(m)}(r) &= \sum_{j=0}^{m} \frac{m!}{(m-j)!}(r-a)^{m-j} f^{(m-j)}(r) \\ &= m! f(r) + \sum_{j=0}^{m-1} \frac{m!}{(m-j)!}(r-a)^{m-j} f^{(m-j)}(r) \end{aligned}$$

Thus, we have $F^{(m)}(a) \neq 0$. □

Lemma 4.8. *(Stein [80, 81]) Let ϕ be a real valued and smooth in (a,b). If $\left|\phi^{(m)}(r)\right| \gtrsim 1$ for some $m \geq 2$ and $m \in \mathbf{N}$, then we have*

$$\int_a^b e^{i\lambda\phi(r)} f(r) dr \lesssim \lambda^{-1/m} \left\{ \|f\|_\infty + \|f'\|_1 \right\}.$$

Now we begin with the proof of Proposition 4.6.

Proof of (a) in Proposition 4.6. Since ϕ has no critical points and inflection points, we set $P = p = 1$, $J_0 = 0$, and $j_0 = -1$. The detail proof can be found in [32].

Proof of (b) in Proposition 4.6. We divide the proof into the case $n = 1$ and $n \geq 2$.
STEP 1. First, we state the case $n = 1$. For the cases $j \geq J_0$, by Lemma 4.8, we have

$$\begin{aligned} |I_j(x)| &\lesssim 2^j \left(|t| 2^{jM}\right)^{-1/2} \left\{ \|\psi\|_\infty + \|\psi'\|_1 \right\} \\ &\sim |t|^{-1/2} 2^{j(1-M/2)}. \end{aligned}$$

For the cases $j \leq j_0$, from Theorem 1 in Guo-Peng-Wang [32], we obtain

$$\sum_{j \leq j_0} |I_j(x)| \lesssim \begin{cases} 1, & \text{if } m_1 \neq m_2, \\ (1+|t|)^{-\min(\frac{1}{2}, \frac{1}{m_1})}, & \text{if } m_1 = m_2, \end{cases}$$

so we have

$$\sum_{j \leq j_0} |I_j(x)| \lesssim \begin{cases} 1, & \text{if } m = 1, \\ (1+|t|)^{-\frac{1}{m}}, & \text{if } m \geq 2. \end{cases}$$

Next, we consider the cases $j_0 < j < J_0$. In these cases, ϕ has inflection points. Let $R_{2,k} > 0$ ($k \leq M-2$) be positive inflection points of ϕ which satisfy $0 < R_{2,1} <$

$R_{2,2} < \cdots < R_{2,\beta}$ ($\beta \le M-2$) with repetitions and let $\ell_{2,k} \in \mathbf{N}$ be multiplicities of positive inflection points $R_{2,k}$ for all $k \in \{1, 2, \cdots, \beta\}$. We decompose ψ into

$$\psi(r) = \psi_0(r) + \psi_1(r) + \cdots + \psi_\beta(r). \tag{4.10}$$

Here, $\psi_0, \psi_k \in C_0^\infty(\mathbf{R}_+)$, $2^{-j}R_{2,k} \notin \mathrm{supp}\, \psi_0$, and $\mathrm{supp}\, \psi_k = [2^{-j}R_{2,k} - \varepsilon, 2^{-j}R_{2,k} + \varepsilon]$ for the suitable $\varepsilon > 0$ and $1 \le k \le \beta$. Then we have

$$\begin{aligned} & I_j(x) \\ = \;& 2^j \int_{\mathbf{R}} e^{ix\xi} e^{it\phi(2^j|\xi|)} \psi(\xi) d\xi \\ = \;& 2^j \int_{\mathbf{R}_+} e^{ix\xi + it\phi(2^j|\xi|)} \psi(\xi) d\xi + 2^j \int_{\mathbf{R}_+} e^{-ix\xi + it\phi(2^j|\xi|)} \psi(\xi) d\xi \\ = \;& 2^j \sum_{k=0}^{\beta} \left\{ \int_{\mathbf{R}_+} e^{ix\xi + it\phi(2^j|\xi|)} \psi_k(\xi) d\xi + \int_{\mathbf{R}_+} e^{-ix\xi + it\phi(2^j|\xi|)} \psi_k(\xi) d\xi \right\} \\ =: \;& \sum_{k=0}^{\beta} I_{j,k}^+(x) + I_{j,k}^-(x). \end{aligned}$$

Since $\phi^{(\ell_{2,k}+2)}(R_{2,k}) \neq 0$ for all $1 \le k \le \beta$ from Lemma 4.7, it follows by Lemma 4.8 that

$$\left| I_{j,k}^{\pm}(x) \right| \lesssim |t|^{-1/(\ell_{2,k}+2)}.$$

On the other hands, since $\phi', \phi'' \neq 0$ on $\mathrm{supp}\, \psi_0$, we have by Lemma 4.8 as $m = 2$

$$\left| I_{j,0}^{\pm}(x) \right| \lesssim |t|^{-1/2}.$$

STEP 2. Next, we consider the case $n \ge 2$. In these cases, by changing to polar coordinates, we have

$$\begin{aligned} I_j(x) \;:=\;& I_j(s) && (4.11) \\ =\;& 2^{jn} \left\{ \int_0^\infty e^{irs + it\phi(2^j r)} \psi(r) r^{n-1} h_+(rs) dr \right. \\ & \left. + \int_0^\infty e^{-irs + it\phi(2^j r)} \psi(r) r^{n-1} h_-(rs) dr \right\} && (4.12) \\ =:\;& I_j^+(s) + I_j^-(s). \end{aligned}$$

Here, $s = |x|$, $h_-(r)$ is the complex conjugate of $h_+(r)$, where for the Bessel function $J_{(n-2)/2}(r)$

$$\mathcal{R}(e^{ir} h_+(r)) = C_n r^{-(n-2)/2} J_{(n-2)/2}(r), \tag{4.13}$$

and $h_\pm(r)$ satisfy that

$$\left| \partial_r^k h_\pm(r) \right| \lesssim (1+r)^{-(n-1)/2-k} \tag{4.14}$$

(see [40, Chapter 1, Equation (1.5)]). Some precise properties can be found in [1, 32]. We begin with $j \geq J_0$ and $j \leq j_0$. It follows that $|\phi'(r)| \neq 0$ and $|\phi''(r)| \neq 0$ on $\operatorname{supp} \psi_j$. By Theorem 1 in [32], we have for $j \geq J_0$

$$|I_j(s)| \lesssim |t|^{-\theta} 2^{j(n-M\theta)}, \text{ for } 0 \leq \theta \leq \frac{n}{2}.$$

For $j \leq j_0$, we have

$$\sum_{j \leq j_0} |I_j(s)| \lesssim \begin{cases} (1+|t|)^{-\min(\frac{n-1}{2}, \frac{n}{m_1})}, & \text{if } m_1 \neq m_2, \\ (1+|t|)^{-\min(\frac{n}{2}, \frac{n}{m_1})}, & \text{if } m_1 = m_2. \end{cases}$$

Thus we have

$$\sum_{j \leq j_0} |I_j(s)| \lesssim \begin{cases} (1+|t|)^{-\frac{n-1}{2}}, & \text{if } m = 1, \\ (1+|t|)^{-\frac{n}{m}}, & \text{if } m \geq 2. \end{cases}$$

Next, we consider $j_0 < j < J_0$. Since $\phi'(r) \neq 0$ for any $r > 0$, we have

$$\frac{1}{it2^j \phi'(2^j r)} \frac{d}{dr} e^{it\phi(2^j r)} = e^{it\phi(2^j r)},$$

and set a differential operator

$$\tilde{D} = \frac{1}{it2^j \phi'(2^j r)} \frac{d}{dr},$$

and its transpose

$$\tilde{D}^* f = -\frac{1}{it2^j} \frac{d}{dr} \left(\frac{f}{\phi'(2^j r)} \right).$$

For $s \leq 1$, using integration by parts, we have for any $N \in \mathbf{N}$

$$\begin{aligned} \left| I_j^{\pm}(s) \right| &= 2^{jn} \left| \int_0^\infty e^{\pm irs + it\phi(2^j r)} \psi(r) r^{n-1} h_{\pm}(rs) dr \right| \\ &= 2^{jn} \left| \int_0^\infty e^{it\phi(2^j r)} (\tilde{D}^*)^N \left(e^{\pm irs} \psi(r) r^{n-1} h_{\pm}(rs) \right) dr \right|. \end{aligned}$$

By the Leipnitz rule, we have

$$(\tilde{D}^*)^N (f) = \sum_{|\gamma| = N} C_\gamma g^{(\gamma_1)} \cdots g^{(\gamma_j)} f^{(\gamma_{j+1})},$$

where $f = e^{\pm irs} \psi(r) r^{n-1} h_{\pm}(rs)$, $g = 1/(it2^j \phi'(2^j r))$, and $\gamma = (\gamma_1, \cdots, \gamma_{j+1}) \in \mathbf{Z}_+^{j+1}$, and $0 \leq \gamma_1 \leq \cdots \leq \gamma_j$. For any $k \in \mathbf{Z}_+$, we have

$$\left| \frac{d^k}{dr^k} \left(\frac{1}{\phi'(2^j r)} \right) \right| \lesssim 2^{-j(M-1)} \lesssim 1$$

and

$$\left|\frac{d^k}{dr^k}f\right| \lesssim 1,$$

on supp ψ for $j_0 < j < J_0$. Thus we have

$$\left|(\tilde{D}^*)^N(f)\right| \lesssim 1,$$

and

$$\left|I_j^{\pm}(s)\right| \lesssim |t|^{-N}. \tag{4.15}$$

Furthermore, since we immediately have

$$\left|I_j^{\pm}(s)\right| \lesssim 1, \tag{4.16}$$

interpolating (4.15) with (4.16), we have for any $\theta \geq 0$

$$\left|I_j^{\pm}(s)\right| \lesssim |t|^{-\theta}.$$

Next, we consider the case $s \geq 1$. If we assume that $\phi' > 0$ and $t > 0$, we don't lose generality. So, it follows that $|s + t2^j\phi'(2^jr)| \geq |t2^j\phi'(2^jr)| \sim |t|$. Using integration by parts as well as we did for $s \leq 2$, we have for any $\theta \geq 0$

$$\left|I_j^{+}(s)\right| \lesssim |t|^{-\theta}.$$

Next, we consider $I_j^-(s)$. If $s \not\sim t2^j\phi'(2^jr)$, we have $|-s+t2^j\phi'(2^jr)| \sim |t2^j\phi'(2^jr)| \sim |t|$. Then, integrating by parts, we have for any $\theta \geq 0$

$$\left|I_j^{-}(s)\right| \lesssim |t|^{-\theta}.$$

For the case $s \sim t2^j\phi'(2^jr)$, we will use Lemma 4.7 and Lemma 4.8. As is done in the case $n = 1$, we decompose the support of ψ into (4.10). We set

$$\begin{aligned} I_j^-(s) &= 2^{jn}\int_0^\infty e^{-irs+it\phi(2^jr)}\psi(r)r^{n-1}h_-(rs)dr \\ &= 2^{jn}\sum_{k=0}^{\beta}\int_0^\infty e^{-irs+it\phi(2^jr)}\psi_k(r)r^{n-1}h_-(rs)d\xi \\ &=: \sum_{k=0}^{\beta} I_{j,k}^-(s). \end{aligned}$$

Since $\phi' \neq 0$, $\phi'' \neq 0$, and $|\phi''(2^jr)| \sim 1$ on supp ψ_0, we have by Lemma 4.8 as $m = 2$

$$\left|I_{j,0}^-(s)\right| \lesssim |t|^{-1/2}\left\{\left\|\psi_0(r)r^{n-1}h_-(rs)\right\|_{L^\infty} + \left\|\left(\psi_0(r)r^{n-1}h_-(rs)\right)'\right\|_{L^1}\right\}.$$

Since $s \sim t2^j\phi'(2^jr) \sim t$ and $r \sim 1$ on supp ψ_0, we obtain from (4.14)

$$\left\|\psi_0(r)r^{n-1}h_-(rs)\right\|_{L^\infty} \lesssim \frac{1}{(1+rs)^{\frac{n-1}{2}}} \lesssim |t|^{-(n-1)/2}$$

and

$$
\begin{aligned}
& \left\|\left(\psi_0(r)r^{n-1}h_-(rs)\right)'\right\|_{L^1} \\
= & \int_0^\infty \left|\frac{d}{dr}\left(\psi_0(r)r^{n-1}\right)h_-(rs)+\psi_0(r)r^{n-1}\frac{d}{dr}\left(h_-(rs)\right)\right| dr \\
\lesssim & \frac{1}{(1+s)^{\frac{n-1}{2}}}+\frac{s}{(1+s)^{\frac{n-1}{2}+1}} \\
\lesssim & |t|^{-(n-1)/2}.
\end{aligned}
$$

Therefore, we have

$$
\left|I_{j,0}^-(s)\right| \lesssim |t|^{-n/2}. \tag{4.17}
$$

By interpolation (4.16) with (4.17), we have for $0 \le \theta \le n/2$

$$
\left|I_{j,0}^-(s)\right| \lesssim |t|^{-\theta}.
$$

On the other hands, since we have $\phi^{(\ell_{2,k}+2)}(R_{2,k}) \neq 0$ for $1 \le k \le \beta$ by Lemma 4.7, it follows that $\phi^{(\ell_{2,k}+2)}(r) \sim 1$ on the support of ψ_k. Using Lemma 4.8, we have

$$
\begin{aligned}
& \left|I_{j,k}^-(s)\right| \\
\lesssim & |t|^{-\frac{1}{\ell_{2,k}+2}}\left\{\left\|\psi_k(r)r^{n-1}h_-(rs)\right\|_{L^\infty}+\left\|\left(\psi_k(r)r^{n-1}h_-(rs)\right)'\right\|_{L^1}\right\} \\
\lesssim & |t|^{-\frac{1}{\ell_{2,k}+2}-\frac{n-1}{2}}.
\end{aligned}
$$

Since $\frac{1}{\ell_{2,k}+2}+\frac{n-1}{2} \le n/2$, so we have for $j_0 < j < J_0$ and $0 \le \theta \le \Theta_2 := \min_k\left\{\frac{n-1}{2}+\frac{1}{\ell_{2,k}+2}\right\}$,

$$
|I_j(s)| \le |t|^{-\theta}.
$$

Therefore we have the desired results. $\square$

Proof of (c) in Proposition 4.6. In this case, since there exists a critical point of ϕ, we can not use integration by parts. We divide two cases.
STEP 1. First, we consider the case $n=1$. Using the same argument as is used in the proof of Proposition 4.6 (b), we have for $j \ge J_0$ and $j \le j_0$

$$
|I_j(x)| \lesssim |t|^{-1/2}2^{j(1-M/2)}, \text{ for } 0 \le \theta \le n/2,
$$

and

$$
\sum_{j \le j_0}|I_j(x)| \lesssim \begin{cases} 1, & \text{if } m=1, \\ (1+|t|)^{-\frac{1}{m}}, & \text{if } m \ge 2, \end{cases}
$$

For $j_0 < j < J_0$, we only use Lemma 4.8 as $m=2$ because there exists no inflection points. So, we have

$$
|I_j(x)| \lesssim |t|^{-1/2}.
$$

STEP 2. Next, we consider $n \geq 2$. Let $I_j(s)$ and $I_j^{\pm}(s)$ be the oscillatory integrals in (4.12). For $j \geq J_0$ and $j \leq j_0$, we have

$$|I_j(s)| \lesssim |t|^{-\theta} 2^{j(n-M\theta)} \text{ for } 0 \leq \theta \leq \frac{n}{2}.$$

and

$$\sum_{j \leq j_0} |I_j(s)| \lesssim \begin{cases} (1+|t|)^{-\frac{n-1}{2}}, & \text{if } m=1, \\ (1+|t|)^{-\frac{n}{m}}, & \text{if } m \geq 2. \end{cases}$$

For $j_0 < j < J_0$, by Lemma 4.8, we have

$$\begin{aligned} |I_j(s)| &\lesssim |t|^{-1/2} \left\{ \left\| \psi(r) r^{n-1} h_{\pm}(rs) \right\|_{\infty} + \left\| \frac{d}{dr} \left(\psi(r) r^{n-1} h_{\pm}(rs) \right) \right\|_{1} \right\} \\ &\lesssim |t|^{-1/2}. \end{aligned}$$

Here, we used $|\partial_r h_{\pm}(rs)| \lesssim 1$ on compact supports by (4.14). □

Proof of (d) in Proposition 4.6. Since there exists both of critical and inflection points of ϕ in this case, we can not use integration by parts near critical points. First, we consider the case $n=1$. We have the same results for $j \leq j_0$ and $j \geq J_0$. So we omit the detail.

For $j_0 < j < J_0$, we only use Lemma 4.7 and 4.8. Let $R_{1,k} > 0$ ($k \leq M-1$) be positive critical points of ϕ which satisfy $0 < R_{1,1} < R_{1,2} < \cdots < R_{1,\alpha}$ ($\alpha \leq M-1$) with repetitions. Let $\ell_{1,k} \in \mathbf{N}$ be multiplicities of positive critical points $R_{1,k} > 0$ for $1 \leq k \leq \alpha$. Moreover, we decompose ψ into

$$\psi(r) = \psi_0 + \psi_1(r) + \cdots + \psi_\alpha(r).$$

Here, $\psi_0, \psi_k \in C_0^\infty(\mathbf{R}_+)$, $2^{-j} R_{1,k} \notin \operatorname{supp} \psi_0$, and $\operatorname{supp} \psi_k = [2^{-j} R_{1,k} - \varepsilon, 2^{-j} R_{1,k} + \varepsilon]$ for the suitable $\varepsilon > 0$ and $1 \leq k \leq \alpha$. We set

$$\begin{aligned} I_j(x) &= 2^j \int_{\mathbf{R}_+} e^{ix\xi + it\phi(2^j|\xi|)} \psi(\xi) d\xi + 2^j \int_{\mathbf{R}_+} e^{-ix\xi + it\phi(2^j|\xi|)} \psi(\xi) d\xi \\ &= 2^j \sum_{k=0}^{\alpha} \left\{ \int_{\mathbf{R}_+} e^{ix\xi + it\phi(2^j|\xi|)} \psi_k(\xi) d\xi + \int_{\mathbf{R}_+} e^{-ix\xi + it\phi(2^j|\xi|)} \psi_k(\xi) d\xi \right\} \\ &=: \sum_{k=0}^{\alpha} I_{j,k}^{+}(x) + I_{j,k}^{-}(x). \end{aligned}$$

Using Lemma 4.7 and 4.8, we have for $1 \leq k \leq \alpha$

$$\left| I_{j,k}^{\pm}(x) \right| \lesssim |t|^{-1/(\ell_{1,k}+1)}.$$

Next we consider $I_{j,0}^{\pm}(x)$. Let $R_{2,k} > 0$ ($k \leq M-2$) be positive inflection points of ϕ which satisfy $0 < R_{2,1} < R_{2,2} < \cdots < R_{2,\beta}$ ($\beta \leq M-2$) with repetitions and let

$\ell_{2,k} \in \mathbf{N}$ be multiplicities of positive inflection points $R_{2,k}$ for all $k \in \{1, 2, \cdots, \beta\}$. We decompose ψ into

$$\psi_0(r) = \tilde{\psi_0}(r) + \tilde{\psi_1}(r) + \cdots + \tilde{\psi_\beta}(r).$$

Here, $\tilde{\psi_0}, \tilde{\psi_k} \in C_0^\infty(\mathbf{R}_+)$, $2^{-j}R_{2,k} \notin \operatorname{supp} \tilde{\psi_0}$, and $\operatorname{supp} \tilde{\psi_k} = [2^{-j}R_{2,k} - \varepsilon, 2^{-j}R_{2,k} + \varepsilon]$ for the suitable $\varepsilon > 0$ and $1 \le k \le \beta$. We decompose $I^{\pm}_{j,k_0}(x)$ into

$$I^{\pm}_{j,0}(x) = \sum_{k=0}^{\beta} II^{\pm}_{j,k}(x),$$

where

$$II^{\pm}_{j,k}(x) = 2^j \int_{\mathbf{R}_+} e^{\pm ix\xi + it\phi(2^j|\xi|)} \tilde{\psi_k}(\xi) d\xi.$$

As we showed in (b) of Proposition 4.6, using integration by parts and Lemma 4.7 and 4.8, we have for $1 \le k \le \beta$

$$\left| II^{\pm}_{j,k}(x) \right| \lesssim |t|^{-\frac{1}{\ell_{2,k}+2}},$$

and

$$\left| II^{\pm}_{j,0}(x) \right| \lesssim |t|^{-1/2},$$

For the case $n \ge 2$, the proof is almost the same, so we omit it. □

Remark 4.9. We have the five remarks about Proposition 4.6.
(1) If the maximum degree of ϕ is equal to 1, then dispersive equations (4.5) are equivalent to wave equations. So we assume that $M \ge 2$.

(2) We introduce examples for each case in Proposition 4.6. Those of (a), (b), (c), and (d) are $\phi(r) = r^4 + r^2$, $r^4 - 5r^3/3 + r^2$, $r^3 - 3r^2 + 9r$, and $r^4 - r^2$, respectively. In particular, $\phi(r) = r^4 \pm r^2$ are called the fourth order Schrödinger equation [1].

(3) In particular, we consider $\phi(r) = r^4 - 5r^3/3 + r^2$ in the just above remark. We have $\phi'(r) = 4r^3 - 5r^2 + 2r$, $\phi''(r) = 12r^2 - 10r + 2$, $\phi^{(3)}(r) = 24r - 10$, and $\phi^{(4)}(r) = 24$. This function has no critical point, but has an inflection point $r = 1/2, 1/3$ for $r > 0$. So, by Guo-Peng-Wang [32], the time decay order of $I_j(x)$ for $j \ge 0$ is $-(n-1)/2$. Furthermore, by Chen-Miao-Yao [15], we have the time decay order is $-(n-1)/2 - 1/4$. However, since $\phi''(r) = 12(r-1/2)(r-1/3)$, the order is $-(n-1)/2 - 1/3$ by Proposition 4.6. This order is better than the other orders for large t.

(4) We actually calculate the time decay factor for the fourth order Schrödinger equation with $\phi(r) = r^4 - r^2$ in [1]. Since $\phi'(r) = 4r^3 - 2r$ and $\phi''(r) = 12r^2 - 2$, so the critical and inflection points are $r = 0$, $\pm 1/\sqrt{2}$ and $r = \pm 1/\sqrt{6}$, respectively. Since the support of ψ_j contains these points when $j = -2, -1, 0$, so it suffices to only consider the case which $j=-2,-1,0$. Moreover, it clearly follows that $|\phi'(r)|\sim r^3$ and $|\phi''(r)|\sim r^2$ on supp ψ_j for all $j\geq 1$, and $|\phi'(r)|\sim r$ and $|\phi''(r)|\sim 1$ on supp ψ_j for all $j\leq -3$. Therefore, by Guo-Peng-Wang [32], we have,

$$|I_j(x)|\lesssim |t|^{-\theta}2^{j(n-4\theta)},\ \text{ for } 0\leq\theta\leq n/2 \text{ and } j\geq 1,$$

and

$$\sum_{j\leq -3}|I_j(x)|\lesssim |t|^{-n/2}.$$

Now we consider the case $j=-2,-1,0$. For $j=-2$, $\phi'(r)\neq 0$ and $\phi^{(3)}(r)\neq 0$, but there exists a point such that $\phi''(r)=0$ on the support of ψ_j. Using integration by parts and Lemma 4.8, we have

$$|I_{-2}(x)|\lesssim |t|^{-(n-1)/2-1/3}=|t|^{-n/2+1/6}.$$

Next case $j=0$. Since $\phi''(r)\neq 0$ on the support of ψ_0 but there exists a point such that $\phi'(r)=0$ on the support of ψ_0, we can not use integration by parts. By Lemma 4.8, we have

$$|I_0(x)|\lesssim |t|^{-1/2}.$$

Finally, we state the case $j=-1$. Since $1/4\leq |1/\sqrt{6}|,\ |1/\sqrt{2}|\leq 1$, we decompose $\psi_{-1}=\psi_{-1,1}+\psi_{-1,2}$ where $\psi_{-1,1}$, $\psi_{-1,2}\in C_0^\infty(\mathbf{R})$ and supp $\psi_{-1,1}\in[1/4,1/2+\varepsilon]$, supp $\psi_{-1,2}\in[1/2-\varepsilon,1]$. It is clear that $1/\sqrt{6}\in[1/4,1/2+\varepsilon]$ and $1/\sqrt{2}\in[1/2-\varepsilon,1]$. So, from the same argument, we have the time decay order $|t|^{-n/2+1/6}$ and $|t|^{-1/2}$ on the support of $\psi_{-1,1}$ and $\psi_{-1,2}$, respectively. This is the same time decay factors as those in Proposition 4.6 and in Ben-Artzi-Koch-Saut [1].

(5) Although we assume that the symbols ϕ have rotational symmetry in this paper, we can generalize it more. Actually, if we change $\phi(|\xi|)$ to $\phi(F(\xi))$ and we change Besov spaces to homogeneous Besov spaces, then we can get the similar results to Proposition 4.6. Here, functions $F:\mathbf{R}^n\to\mathbf{R}$ are homogeneous functions of degree 1 and satisfy a geometry condition called finite type. As the examples of $F(\xi)$, we have $F(\xi)=(\xi_1^m+\cdots+\xi_n^m)^{1/m}$ (m are even and greater than 4) and $(\xi_1^4+6\xi_1^2\xi_2^2+\xi_2^4)^{1/4}$. More precise definition and properties about finite type are written in Bruna-Nagel-Wainger [9], Chen-Miao-Yao [15], Zheng-Yao-Fan [103].

4.4 Time decay estimates

In this section, by using Proposition 4.6, we establish time decay estimates on Besov spaces and modulation spaces. First, we construct them on Besov spaces.

Proposition 4.10. *Let $2\leq p\leq\infty$, $1\leq q\leq\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, and $\delta=\delta(p)=\frac{1}{2}-\frac{1}{p}$. Then we have*

$$\left\|e^{it\phi(\sqrt{-\Delta})}u_0\right\|_{B^s_{p,q}}\lesssim P(t)\left\|u_0\right\|_{B^{s'}_{p',q}},$$

where $P(t)$ and $s, s' \in \mathbf{R}$ are the followings.

(a) *For the case when ϕ satisfies (A) and $s' - s \geq (2-M)n\delta$,*

$$P(t) = \begin{cases} |t|^{\frac{1}{M}\min(s'-s-2n\delta,0)}, & |t| \leq 1, \\ |t|^{-(n-1)\delta}, & |t| \geq 1, \end{cases} \quad \text{for } m = 1,$$

$$P(t) = \begin{cases} |t|^{\frac{1}{M}\min(s'-s-2n\delta,0)}, & |t| \leq 1, \\ |t|^{-\frac{2n}{m}\delta}, & |t| \geq 1, \end{cases} \quad \text{for } m \geq 2.$$

(b) *For the case when ϕ satisfies (B) and $s' - s \geq (2-M)n\delta$,*

$$P(t) = \begin{cases} |t|^{\frac{1}{M}\min(s'-s-2n\delta,0)}, & |t| \leq 1, \\ |t|^{-(n-1)\delta}, & |t| \geq 1, \end{cases} \quad \text{for } m = 1,$$

$$P(t) = \begin{cases} |t|^{\frac{1}{M}\min(s'-s-2n\delta,0)}, & |t| \leq 1, \\ |t|^{-\min\left(\frac{2n}{m},2\Theta_2\right)\delta}, & |t| \geq 1, \end{cases} \quad \text{for } m \geq 2,$$

where Θ_2 is the same as in (b) of Proposition 4.6.

(c) *For the case when ϕ satisfies (C) and $s' - s \geq (2-M)n\delta$,*

$$P(t) = \begin{cases} |t|^{\frac{1}{M}\min(s'-s-2n\delta,0)}, & |t| \leq 1, \\ |t|^{-\min(n-1,1)\delta}, & |t| \geq 1, \end{cases} \quad \text{for } m = 1,$$

$$P(t) = \begin{cases} |t|^{\frac{1}{M}\min(s'-s-2n\delta,0)}, & |t| \leq 1, \\ |t|^{-\min\left(\frac{2n}{m},1\right)\delta}, & |t| \geq 1, \end{cases} \quad \text{for } m \geq 2.$$

(d) *For the case when ϕ satisfies (D) and $s' - s \geq (2-M)n\delta$,*

$$P(t) = \begin{cases} |t|^{\frac{1}{M}\min(s'-s-2n\delta,0)}, & |t| \leq 1, \\ |t|^{-\min(n-1,2\Theta_1)\delta}, & |t| \geq 1, \end{cases} \quad \text{for } m = 1,$$

$$P(t) = \begin{cases} |t|^{\frac{1}{M}\min(s'-s-2n\delta,0)}, & |t| \leq 1, \\ |t|^{-\min\left(\frac{2n}{m},2\Theta_1,2\Theta_2\right)\delta}, & |t| \geq 1. \end{cases} \quad \text{for } m \geq 2,$$

where Θ_1 and Θ_2 are the same as in (d) of Proposition 4.6.

Proof of (a) in Proposition 4.10. First, we state the proof for $|t| \leq 1$. By (a) in Proposition 4.6 and interpolation theorem, we immediately have

$$\left\| e^{it\phi(\sqrt{-\Delta})} P_{\leq 0} u_0 \right\|_p \lesssim \| P_{\leq 0} u_0 \|_{p'}$$

and

$$\left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p \lesssim |t|^{-2\theta\delta} 2^{2j(n-M\theta)\delta} \| \triangle_j u_0 \|_{p'}, \text{ for } j \geq 0.$$

Then we have for $j \geq 0$

$$2^{js} \left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p \lesssim |t|^{-2\theta\delta} 2^{-j(s'-s-2(n-M\theta)\delta)} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'}, \text{ for } j \geq 0.$$

If $(2-M)n\delta \leq s'-s \leq 2n\delta$, then there exists $0 \leq \theta \leq n/2$ such that $s'-s-2(n-M\theta)\delta = 0$. Since $s'-s-2(n-M\theta)\delta \geq 0$ if $2n\delta \leq s'-s$, so we set $\theta = 0$. Then we have

$$2^{js} \left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p \lesssim |t|^{\frac{1}{M}\min(s'-s-2n\delta,0)} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'}.$$

Next, for the case $|t| \geq 1$, we have

$$\left\| e^{it\phi(\sqrt{-\Delta})} P_{\leq 0} u_0 \right\|_p \lesssim \begin{cases} |t|^{-(n-1)\delta} \left\| P_{\leq 0} u_0 \right\|_{p'}, & m = 1, \\ |t|^{-\frac{2n}{m}\delta} \left\| P_{\leq 0} u_0 \right\|_{p'}, & m \geq 2, \end{cases}$$

and for $j \geq 0$

$$\left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p \lesssim |t|^{-n\delta} 2^{j(2n-Mn)\delta} \left\| \triangle_j u_0 \right\|_{p'}.$$

Then it follows that from $s'-s-(2-M)n\delta \geq 0$

$$\begin{aligned} 2^{js} \left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p &\lesssim |t|^{-n\delta} 2^{-j(s'-s-(2-M)n\delta)} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'} \\ &\lesssim |t|^{-n\delta} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'}. \end{aligned}$$

Thus we have the desired results. □

Proof of (b) in Proposition 4.10. We consider the case $|t| \leq 1$. We have for $j_0 < j < J_0$

$$\left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p \lesssim \left\| \triangle_j u_0 \right\|_{p'},$$

and

$$\left\| e^{it\phi(\sqrt{-\Delta})} P_{\leq 0} u_0 \right\|_p \lesssim \left\| P_{\leq 0} u_0 \right\|_{p'},$$

if we set $\theta = 0$. The proof for the case $j \geq J_0$ is the same as the above.

Next, for the case $|t| \geq 1$, we have

$$\left\| e^{it\phi(\sqrt{-\Delta})} P_{\leq 0} u_0 \right\|_p \lesssim \begin{cases} |t|^{-(n-1)\delta} \left\| P_{\leq 0} u_0 \right\|_{p'}, & m = 1, \\ |t|^{-\frac{2n}{m}\delta} \left\| P_{\leq 0} u_0 \right\|_{p'}, & m \geq 2, \end{cases}$$

and for $j_0 < j < J_0$,

$$2^{js} \left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p \lesssim |t|^{-2\Theta_2\delta} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'},$$

and for $j \geq J_0$

$$\begin{aligned} 2^{js} \left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p &\lesssim |t|^{-n\delta} 2^{-j(s'-s-(2-M)n\delta)} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'} \\ &\lesssim |t|^{-n\delta} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'} . \end{aligned}$$

Thus we have the desired results. □

Proofs of (c) in Proposition 4.10. If $|t| \leq 1$, we use the same argument as above. For $|t| \geq 1$, we have

$$\left\| e^{it\phi(\sqrt{-\Delta})} P_{\leq 0} u_0 \right\|_p \lesssim \begin{cases} |t|^{-(n-1)\delta} \left\| P_{\leq 0} u_0 \right\|_{p'} , & m = 1, \\ |t|^{-\frac{2n}{m}\delta} \left\| P_{\leq 0} u_0 \right\|_{p'} , & m \geq 2, \end{cases}$$

and for $j_0 < j < J_0$,

$$2^{js} \left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p \lesssim |t|^{-\delta} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'} ,$$

and for $j \geq J_0$

$$\begin{aligned} 2^{js} \left\| e^{it\phi(\sqrt{-\Delta})} \triangle_j u_0 \right\|_p &\lesssim |t|^{-n\delta} 2^{-j(s'-s-(2-M)n\delta)} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'} \\ &\lesssim |t|^{-n\delta} 2^{js'} \left\| \triangle_j u_0 \right\|_{p'} , \end{aligned}$$

Thus we have the desired results. □

Proofs of (d) in Proposition 4.10. Almost the proof is the same as the other ones. For $m = 1$, we have

$$\min_k \left(\frac{n-1}{2}, \frac{1}{\ell_{1,k}+1}, \frac{n-1}{2} + \frac{1}{\ell_{2,k}+2} \right) = \min_k \left(\frac{n-1}{2}, \frac{1}{\ell_{1,k}+1} \right).$$

Thus we have the desired results. □

Finally, we show Proposition 4.3 in the Subsection 4.1.2.

Proof of Proposition 4.3. From the definitions of Besov spaces and modulation spaces, we obtain for large $|k|$

$$\begin{aligned} \| e^{it\phi(\sqrt{-\Delta})} \Box_k u_0 \|_\infty &= \| \mathcal{F}^{-1} e^{it\phi(|\xi|)} \sigma_k \sum_{\ell=-N}^{N} \psi_{j+\ell}(\xi) \mathcal{F} u_0 \|_\infty \\ &\lesssim \sum_{\ell=-N}^{N} \| \mathcal{F}^{-1} e^{it\phi(|\xi|)} \psi_{j+\ell}(\xi) \|_\infty \| \Box_k u_0 \|_1, \end{aligned}$$

where $\sum_{\ell=-N}^{N} \psi_{j+\ell}(\xi) = 1$ on the support of σ_k for $|k| \in [2^{j-1}, 2^j]$ and $N \in \mathbf{N}$ is a universal constant (See the proof of Lemma 2.13 in [97]). Then, by the same argument as in Proposition 4.10, we have the following dispersive estimates for $s' - s \geq (2-M)n\delta$

$$\left\| e^{it\phi(\sqrt{-\Delta})} u_0 \right\|_{M^s_{p,q}} \lesssim P(t) \left\| u_0 \right\|_{M^{s'}_{p',q}},$$

where $P(t)$ are exactly the same as in Proposition 4.10 for every cases. Thus, setting as $s = s' = 0$, we obtain Proposition 4.3 (a)–(d) from the estimate (4.6) and $P(t)$ for large time $t \geq 1$, □

4.5 Proof for the existence of a solution to inhomogeneous type of NLHS

First of all, we state the existence of a unique global solution to the Cauchy problem (4.5) on modulation spaces. Since the proofs of Theorem 4.4 are almost the same as the proof of Theorem 4.1, however, we only state proof of (a–1) in Theorem 4.4 briefly.

Proof of (a-i) in Theorem 4.4. From Proposition 4.3, we obtain for $n \geq 2$

$$\left\| e^{it\phi(\sqrt{-\Delta})} u_0 \right\|_{M_{p,q}} \lesssim (1+|t|)^{-(n-1)\delta(p)} \left\| u_0 \right\|_{M_{p',q}}.$$

So, by the assumption $\kappa_0 < \kappa$, we have

$$(n-1) \times \delta(2+\kappa) \times (1+\kappa) = (n-1)\frac{\kappa(1+\kappa)}{2(2+\kappa)} > 1,$$

and

$$(1+|t|)^{-\theta\delta(2+\kappa)} \in L^{1+\kappa}(\mathbf{R}).$$

Thus, we get the desired statement. The rest parts are the repetitions of the above proof. Therefore, we omit their proofs. □

Next, we consider the similar statement on L^p spaces. Since we have embedding theorems:

$$L^s_p \subset B^s_{p,2}, \text{ for } 1 < p \leq 2 \text{ and } s \in \mathbf{R},$$
$$B^s_{p,2} \subset L^s_p, \text{ for } 2 \leq p < \infty \text{ and } s \in \mathbf{R},$$

we can establish time decay estimates on L^p–Sobolev spaces.

Corollary 4.11. *Let $2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have*

$$\left\| e^{it\phi(\sqrt{-\Delta})} u_0 \right\|_{L^s_p} \lesssim P(t) \left\| u_0 \right\|_{L^{s'}_{p'}},$$

where $P(t)$ and $s, s' \in \mathbf{R}$ are the same as in Proposition 4.10.

By using Corollary 4.11, we have the existence of a global solution as follows.

Corollary 4.12. *Let $f(u) = |u|^\kappa u$, $\kappa \in \mathbf{R}$ and ϕ satisfy the following each condition. Assume that $u_0 \in L^{(2+\kappa)/(1+\kappa)}$ and there exists a small $\rho > 0$ such that $\|u_0\|_{L^{(2+\kappa)/(1+\kappa)}} \leq \rho$. Then, the Cauchy problem (4.5) has a unique global solution*

$$u \in L^{1+\kappa}(\mathbf{R}, L^{2+\kappa}).$$

(a-i) *Let ϕ satisfy (A) and $2n/M < n-1$. For $m = 1$ and $n \geq 2$, $\kappa \in \mathbf{R}$ satisfies $\kappa_0 < \kappa < \kappa_1$. Here,*

$$\kappa_0 \text{ is the positive root of } (n-1)\kappa^2 + (n-3)\kappa - 4 = 0,$$
$$\kappa_1 \text{ is the positive root of } n\kappa^2 + (n-M)\kappa - 2M = 0.$$

(a-ii) *Let ϕ satisfy (A). For $m \geq 2$ and $n \geq 1$, $\kappa \in \mathbf{R}$ satisfies $\kappa_0 < \kappa < \kappa_1$. Here,*

$$\kappa_0 \text{ is the positive root of } n\kappa^2 + (n-m)\kappa - 2m = 0,$$
$$\kappa_1 \text{ is the positive root of } n\kappa^2 + (n-M)\kappa - 2M = 0.$$

(b-i) *Let ϕ satisfy (B) and $2n/M < n-1$. For $m = 1$ and $n \geq 2$, $\kappa \in \mathbf{R}$ satisfies $\kappa_0 < \kappa < \kappa_1$. Here,*

$$\kappa_0 \text{ is the positive root of } (n-1)\kappa^2 + (n-3)\kappa - 4 = 0,$$
$$\kappa_1 \text{ is the positive root of } n\kappa^2 + (n-M)\kappa - 2M = 0.$$

(b-ii) *Let ϕ satisfy (B) and $2n/M < \theta$. For $m \geq 2$ and $n \geq 1$, $\kappa \in \mathbf{R}$ satisfies $\kappa_0 < \kappa < \kappa_1$. Here,*

$$\kappa_0 \text{ is the positive root of } \theta\kappa^2 + (\theta-2)\kappa - 4 = 0,$$
$$\kappa_1 \text{ is the positive root of } n\kappa^2 + (n-M)\kappa - 2M = 0,$$

$\theta = \min\left\{\frac{2n}{m}, 2\Theta_2\right\}$, $\Theta_2 = \min_k \left\{\frac{n-1}{2} + \frac{1}{\ell_{2,k}+2}\right\}$, *and $\{\ell_{2,k}\}$ are the multiplicities of inflection points of ϕ.*

(c-i) *Let ϕ satisfy (C) and $2n/M < 1$. For $m = 1$ and $n \geq 2$, $\kappa \in \mathbf{R}$ satisfies $\kappa_0 < \kappa < \kappa_1$. Here,*

$$\kappa_0 \text{ is the positive root of } \kappa^2 - \kappa - 4 = 0,$$
$$\kappa_1 \text{ is the positive root of } n\kappa^2 + (n-M)\kappa - 2M = 0.$$

(c-ii) *Let ϕ satisfy (C) and $2n/M < 1$. For $m \geq 2$ and $n \geq 1$, $\kappa \in \mathbf{R}$ satisfies $\kappa_0 < \kappa < \kappa_1$. Here,*

$$\kappa_0 \text{ is the positive root of } \theta\kappa^2 + (\theta-2)\kappa - 4 = 0,$$
$$\kappa_1 \text{ is the positive root of } n\kappa^2 + (n-M)\kappa - 2M = 0,$$

and $\theta = \min\left\{\frac{2n}{m}, 1\right\}$.

(d-i) *Let ϕ satisfy (D) and $2n/M < 2\Theta_1$. For $m = 1$ and $n \geq 2$, $\kappa \in \mathbf{R}$ satisfies $\kappa_0 < \kappa < \kappa_1$. Here,*

$$\kappa_0 \text{ is the positive root of } \Theta_1\kappa^2 + (\Theta_1 - 1)\kappa - 2 = 0,$$
$$\kappa_1 \text{ is the positive root of } n\kappa^2 + (n - M)\kappa - 2M = 0,$$

$\Theta_1 = \min_k \left\{ \frac{1}{\ell_{1,k}+1} \right\}$, *and $\{\ell_{1,k}\}$ are the multiplicities of critical points of ϕ.*
(d-ii) *Let ϕ satisfy (D) and $2n/M < \theta$. For $m \geq 2$ and $n \geq 1$, $\kappa \in \mathbf{R}$ satisfies $\kappa_0 < \kappa < \kappa_1$. Here,*

$$\kappa_0 \text{ is the positive root of } \theta\kappa^2 + (\theta - 2)\kappa - 4 = 0,$$
$$\kappa_1 \text{ is the positive root of } n\kappa^2 + (n - M)\kappa - 2M = 0,$$

and $\theta = \min\left\{\frac{2n}{m}, 2\Theta_1, 2\Theta_2\right\}$.

Before we prove Corollary 4.12, we give a lemma.

Lemma 4.13. *Let $0 < \alpha \leq \beta$. Let κ_α be a positive root of $\alpha\kappa^2 + (\alpha - 2)\kappa - 4 = 0$ and let κ_β be a positive root of $\beta\kappa^2 + (\beta - 2)\kappa - 4 = 0$. Then we have*

$$\kappa_\alpha \geq \kappa_\beta.$$

Proof. κ_α and κ_β follow that

$$\kappa_\alpha = \frac{(2-\alpha) + \sqrt{\alpha^2 + 12\alpha + 4}}{2\alpha} \text{ and } \kappa_\beta = \frac{(2-\beta) + \sqrt{\beta^2 + 12\beta + 4}}{2\beta},$$

respectively. From the assumption $\alpha \leq \beta$, we obtain

$$\begin{aligned} \kappa_\alpha &= \frac{1}{2}\left\{ (\frac{2}{\alpha} - 1) + \sqrt{1 + \frac{12}{\alpha} + \frac{4}{\alpha^2}} \right\} \\ &\geq \frac{1}{2}\left\{ (\frac{2}{\beta} - 1) + \sqrt{1 + \frac{12}{\beta} + \frac{4}{\beta^2}} \right\} = \kappa_\beta. \end{aligned}$$

Thus, we have the desired inequality. □

Proof of (a-i) in Corollary 4.12. In this case, we have

$$\left\| e^{it\phi(\sqrt{-\Delta})} u_0 \right\|_{L^p} \lesssim P(t) \left\| u_0 \right\|_{L^{p'}},$$

where

$$P(t) = \begin{cases} |t|^{-\frac{2n}{M}\delta}, & |t| \leq 1, \\ |t|^{-(n-1)\delta}, & |t| \geq 1. \end{cases}$$

So, by the assumptions, we have

$$\frac{2n}{M}\frac{\kappa}{2(2+\kappa)}(1+\kappa) < 1 \text{ and } (n-1)\frac{\kappa}{2(2+\kappa)}(1+\kappa) > 1.$$

Then we have

$$P(t) \in L^{1+\kappa}(\mathbf{R}).$$

From the assumption $2n/M < n-1$, we have $\kappa_1 > \kappa_0$ by Lemma 4.13. Thus, the above two inequalities make sense. Since the rest of proofs are the same as above, we omit the detail proofs. $\square$

Remark 4.14. In this remark, we actually consider the fourth order Schrödinger equations $\phi(r) = r^4 - r^2$, which are applied to (d-ii) in Theorem 4.4 and Corollary 4.12. We have the following dispersive estimates:

$$\left\| e^{it\phi(\sqrt{-\Delta})} u_0 \right\|_{L^p} \lesssim \begin{cases} |t|^{-\frac{n}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{L^{p'}}, & |t| \le 1, \\ |t|^{-\theta\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{L^{p'}}, & |t| \ge 1, \end{cases}$$

where $\theta = 2/3$ if $n = 1$, 1 if $n \ge 2$. Since we need to assume that $n/2 < \theta$, we can prove the existence of a global solution on L^p spaces for only $n = 1$. Furthermore, since we have $1 + \sqrt{7} < \kappa < (3 + \sqrt{41})/2$, we can prove it for only $\kappa = 4$.

On the other hands, if we establish the dispersive estimates on modulation spaces, we obtain

$$\left\| e^{it\phi(\sqrt{-\Delta})} u_0 \right\|_{M_{p,q}} \lesssim \begin{cases} (1+|t|)^{-\frac{2}{3}\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{M_{p',q}}, & n = 1, \\ (1+|t|)^{-\left(\frac{1}{2}-\frac{1}{p}\right)} \|u_0\|_{M_{p',q}}, & n \ge 2, \end{cases}$$

from

$$\min\left\{\frac{2n}{2}, 2\left(\frac{n-1}{2} + \frac{1}{3}\right), \frac{2}{2}\right\} = \begin{cases} 2/3, & n = 1, \\ 1, & n \ge 2. \end{cases}$$

So, for the positive root κ_0 satisfying

$$\begin{aligned} &\kappa^2 - 2\kappa - 6 > 0, \quad n = 1, \\ &\kappa^2 - \kappa - 4 > 0, \quad n \ge 2, \end{aligned}$$

there exists a unique global solution as given in Theorem 4.4. More precisely, if $\kappa \ge 4$ for $n = 1$ or $\kappa \ge 3$ for $n \ge 2$, then there exists a unique global solution (For global well–posedness for fourth order Schrödinger equations, [74] is written in detail).

5 Global well–posedness for the 2D generalized Zakharov–Kuznetsov equations with small initial data on modulation spaces

5.1 Introduction and main theorems

In this paper, we consider the Cauchy problem for the generalized Zakharov–Kuznetsov (gZK) equations:

$$\begin{cases} \partial_t u + \partial_x \Delta u = \partial_x (u^{p+1}) \\ u(0) = u_0, \end{cases} \tag{5.1}$$

where $(x, y) \in \mathbf{R} \times \mathbf{R}^{n-1}$, $n = 2, 3$, $t > 0$, $\Delta = \partial_x^2 + \sum_{i=1}^{n-1} \partial_{y_i}^2$. The cases $p = 1$ and $p = 2$ correspond as Zakharov–Kuznetsov (ZK) equation and modified Zakharov–Kuznetsov (mZK) equation, respectively. The Zakharov–Kuznetsov equation was introduced by Zakharov and Kuznetsov [101] as the extension of the Koreteweg–de Vries (KdV) equation on one dimension to multi–dimensions. This equation describes the propagation of ion–sound waves in the magnetic field.

In this paper, we mainly study the well–posedness for the Cauchy problem (5.1) in the frame of modulation spaces. Modulation spaces $M^s_{p,q}$ are firstly introduced by Feichtinger [22]. Let $\sigma : \mathbf{R}^n \to [0, 1]$ be a window function such that

$$\sigma \in \mathcal{S}, \quad \mathrm{supp}\sigma \subset [-1, 1]^n, \quad \text{and} \quad \sum_{k \in \mathbf{Z}^n} \sigma(x - k) \equiv 1.$$

Then, we denote the modulation spaces $M^s_{p,q}$ by all tempered distribution f such that the norm, for any $s \in \mathbf{R}$ and $1 \le p, q \le \infty$,

$$\|f\|_{M^s_{p,q}} = \left(\sum_{k \in \mathbf{Z}^n} \langle k \rangle^{sq} \|\Box_k f\|^q_{L^p} \right)^{1/q},$$

where the frequency-uniform decomposition operator $\Box_k := \mathcal{F}^{-1} \sigma(\cdot - k) \mathcal{F}$and $\langle k \rangle = 1 + |k| = 1 + |k_1| + \cdots + |k_n|$. We will write $M_{p,q} = M^0_{p,q}$ for simplicity.

In the two dimensional case, scaling critical regularity is $s_c := 1 - 2/p$ (more generally, in multi–dimension, $s_c(n, p) = n/2 - 2/p$). Linares–Pastor [62] studied the local well–posedness in the classical Sobolev spaces $H^s(\mathbf{R}^2)$ for

$$\begin{cases} s > 3/4 & \text{if } 2 \le p \le 7, \\ s > 1 - 3/(2p - 4) & \text{if } p \ge 8. \end{cases}$$

Moreover, for $p \ge 3$, they also studied the global well–posedness in H^1 with small initial data. Ribaud–Vento [77] proved that the local well–posedness in $H^s(\mathbf{R}^2)$ for

$$\begin{cases} s > 1/4 & \text{if } p = 2, \\ s > 5/12 & \text{if } p = 3, \\ s > 1 - 2/p & \text{if } p \ge 4. \end{cases}$$

Farah–Linares–Pastor [21] also showed that local well–posedness in $H^s(\mathbf{R}^2)$ for $s > 1 - 2/p$ and $p > 8$. By symmetrization of the gZK equation, Grünrock [34] proved the global well–posedness in the homogeneous Sobolev spaces $\dot{H}^{s_c}(\mathbf{R}^2)$ with small initial data, where $s_c := 1 - 2/p$ and $p \geq 3$ (more precisely, in homogeneous Besov spaces). Symmetrizing the gZK equation, the equation come down to the KdV–like equation: $\partial_t v + (\partial_x^3 + \partial_y^3)v = C(\partial_x + \partial_y)v^{p+1}$ (see also Grünrock–Herr [35]). Since spacial derivatives in the linear part are separated to each spacial derivative, gZK equation are easier to be handled. Thus, symmetrization is quite useful and powerful method for the gZK equation. However, the symmetrizing transformation doesn't define an isomorphism on modulation spaces because of non–symmetric decomposition σ_k. Thus we cannot use their method in this paper, unfortunately.

In order to prove the well–posedness for the gZK equation, the Kato type smoothing, Strichartz, and maximal function estimates play an important role. For the Kato type smoothing and Strichartz estimates, sufficiently good estimates are already given by Ribaud–Vento [76, Proposition 3.1] and Linares–Pastor [61, Lemma 2.1 or Proposition 2.4], respectively. Therefore, in this paper, we mainly improve the maximal function estimate. Linares–Pastor [61, Proposition 1.5] got the following maximal function estimate: for $s > 3/4$ and $0 \leq T \leq 1$

$$\|U(t)u_0\|_{L_x^4 L_{y,T}^\infty} \lesssim \|u_0\|_{H_{x,y}^s}, \tag{5.2}$$

where the unitary group $\{U(t)\}_{t=-\infty}^{t=\infty}$ denoted by $U(t) = \mathcal{F}^{-1} e^{it(\xi^3 + \xi\eta^2)} \mathcal{F}$. On the other hand, considering the frequency spaces carefully by frequency–uniform decomposition, we have for any $k \in \mathbf{Z}^2$

$$\|\Box_k U(t)u_0\|_{L_x^4 L_{y,t}^\infty} \lesssim \langle k \rangle^{1/4} \|\Box_k u_0\|_{L_{x,y}^2} \tag{5.3}$$

(see Proposition 5.11). Comparing these two maximal function estimates, there is two advantages of regularity and time–variable. The estimate (5.2) has a larger regularity $s > 3/4$ and holds on local time, though, the estimate (5.3) has a smaller regularity $s = 1/4$ and holds on global time (Grünrock has proved global maximal function estimates for the symmetrized gZK equation in [34, Proposition 1 in Section 3.1]). Moreover, the regularity $1/4$ is the same as that one for the KdV equation on one dimension given by Kenig–Ponce–Vega [48]. Using this maximal function estimates (5.3), we prove the global well–posedness in modulation spaces with small initial data. Let auxiliary function spaces $\mathcal{X}^\theta$ satisfy that

$$\mathcal{X}^\theta := \{u \in \mathcal{S}' : \|u\|_{X \cap Y \cap Z} \leq \rho\},$$

with

$$\begin{aligned} \|u\|_X &= \sum_{|k| \gg 1} \langle k \rangle^\theta \|\Box_k u\|_{L_x^\infty L_{y,t}^2}, \\ \|u\|_Y &= \sum_{k \in \mathbf{Z}^n} \|\Box_k u\|_{L_x^p L_{y,t}^\infty}, \\ \|u\|_Z &= \sum_{k \in \mathbf{Z}^n} \langle k \rangle^\theta \|\Box_k u\|_{L_t^\infty L_{x,y}^2 \cap L_{x,y,t}^{2+p}}. \end{aligned}$$

Then, we have the following main theorem.

Theorem 5.1. *Let integers $p \geq 4$ and $n = 2$. There exists $\rho > 0$ such that if $u_0 \in M_{2,1}^{1/p}$ satisfies that $\|u_0\|_{M_{2,1}^{1/p}} \leq \rho$, then the Cauchy problem (5.1) has a unique global solution*

$$u \in C([0,\infty), M_{2,1}^{1/p}) \cap \mathcal{X}^{1/p}.$$

From Wang–Hudzik [97] (see also Kobayashi–Sugimoto [54]), we have the sharp inclusion relations between the modulation and Sobolev or Besov spaces: for any $\varepsilon > 0$.

$$\begin{aligned} M_{2,1}^{s} &\not\subset H^{s+\varepsilon} \cup B_{\infty,\infty}^{s+\varepsilon}, \\ H^{s+n/2+\varepsilon} \subset B_{2,1}^{s+n/2} \subset M_{2,1}^{s} &\subset H^{s}. \end{aligned}$$

If the regularity in the right hand side are made larger, these embedding relations doesn't hold (see Wang [94, Appendix B]). Moreover, we also have the inclusions: $H^{s+n/2} \not\subset M_{2,1}^s$ by Wang–Zhao–Guo [99]. Thus, we see that we deal with the global well–posedness for gZK equation in the new class of functions.

We remark that the sharpness of the regularity $1/p$ is not completely understood. For the KdV equation on one dimension, the global well–posed in $M_{2,1}$ with small initial data has been proved by Wang–Huang [96]. On the other hand, in the y–axis space, we can regard the semi–group of Zakharov–Kuznetsov equation as that of the Schrödinger equation. For the Schrödinger equation with derivative nonlinear term: $i\partial_t u + \Delta u = \sum_{i=1}^n \partial_{x_i}(u^{p+1})$, the global well–posedness in $M_{2,1}^{1/p}$ with small initial data has been proved by Wang [94] (see also [95]). Moreover, Wang also studied the ill–posedness in $M_{2,1}^s$ for $s < 1/p$ in [94]. Thus, we can understand $1/p$ as the valid regularity.

Next, we mention the well-posedness on three dimensions. For the global well–posedness for the gZK equation on three dimensions, there is only one work by Grünrock [34]. He proved the global well–posedness in $\dot{H}^{s_c}(\mathbf{R}^3)$ with small initial data by using U^p and V^p type function spaces (see [36]). Here $s_c := 3/2 - 2/p$ and $p \geq 3$. On the other hand, for the ZK equation ($p = 1$), the local well–posedness in H^s for $s > 9/8$ was proved by Linares–Saut [63]. Ribaud–Vento [76] studied the local one in H^s for $s > 1$, and then Grünrock–Herr [35] extended to $s > 1/2$. Moreover, for the mZK equation ($p = 2$), Grünrock [33] showed the local well–posedness in H^s for $s > 1/2$ and the global well–posedness in H^s for $s \geq 1$ with small initial data.

On three dimensions, the three same estimates are also needed to prove the well-posedness. In order to construct the Strichartz and maximal function estimate, we need the the estimate for the oscillatory integral associated to $U(t)$. Linares–Saut [63] have proved that for $0 < \varepsilon < 1$

$$\left| \int_{\mathbf{R}^3} |\xi|^{\varepsilon + i\beta} e^{it(\xi^3 + \xi|\eta|^2) + ix\xi + iy\cdot\eta} d\xi d\eta \right| \leq C|t|^{-1-\varepsilon/3}, \tag{5.4}$$

which time decay is sharp. Here $y, \eta \in \mathbf{R}^2$. On the other hand, if we consider the estimate with frequency–uniform decomposition, which volume is uniformly constant, then it is came down to that one on two dimensions (see Proposition 5.13). Since the sharp estimate on two dimensions is already given by Linares–Pastor [61], we have for any $k \in \mathbf{Z}^3$

$$\left| \int_{\mathbf{R}^3} e^{it(\xi^3+\xi|\eta|^2)+ix\xi+iy\cdot\eta} \sigma_k(\xi,\eta) d\xi d\eta \right| \lesssim |t|^{-2/3}. \tag{5.5}$$

Comparing these two estimates, the estimate (5.4) has a better time decay term than the estimate (5.5). However, since we can't take $\varepsilon = 0$ in (5.4), the derivative D_x appears to the Strichartz estimates (see [61, Proposition 3.1]). Thus, we can't use the sharp estimate (5.4) and need to use the estimate (5.5), unfortunately. Using the estimate (5.5), we have the global maximal function estimate: for $p > 3$

$$\|\Box_k U(t) u_0\|_{L^p_x L^\infty_{y,t}} \lesssim \langle k \rangle^{2/p} \|\Box_k u_0\|_{L^2_{x,y}},$$

and have the following global well–posedness.

Theorem 5.2. *Let integers $p \geq 4$ and $n = 3$. There exists $\rho > 0$ such that if $u_0 \in M^{2/p}_{2,1}$ satisfies that $\|u_0\|_{M^{2/p}_{2,1}} \leq \rho$, then the Cauchy problem (5.1) has a unique global solution*

$$u \in C([0,\infty), M^{2/p}_{2,1}) \cap \mathcal{X}^{2/p}.$$

From the sharp embeddings as stated above, $M^{2/p}_{2,1}$ is also new class of functions. We finally remark that Theorem 5.2 can be applied to higher dimensions $n \geq 4$. Since only two and three dimensional cases are justified by Lannes–Linares–Saut [57], however, we don't mention higher dimensional cases.

5.2 Known results for oscillatory integrals

In this section, we display two powerful tools to prove the following statements, especially the maximal function estimate. These are given by Kenig–Ponce–Vega [47].

Lemma 5.3. *(See [47, Lemma 3.4]) Let $\Omega \subset \mathbf{R}^n$ be an open ball. Let $\phi \in C^{n+1}$ satisfy that for some $m \geq 2$ there are constants c_1 and c_2*

$$\begin{aligned} c_1|\xi|^{m-1} &\leq |\nabla\phi(\xi)| \leq c_2|\xi|^{m-1}, \\ c_1|\xi|^{(m-2)n} &\leq |H\phi(\xi)| \leq c_2|\xi|^{(m-2)n}, \\ |\partial^\alpha\phi(\xi)| &\leq c_2|\xi|^{m-|\alpha|}, \end{aligned}$$

for any $\alpha \in \mathbf{Z}^n_+$ with $|\alpha| \leq n+1$. Here, $H\phi$ is the Hessian of ϕ. Then we have

$$\left| \int_\Omega e^{it\phi(\xi)+ix\cdot\xi} \psi(\xi) |H\phi(\xi)|^{1/2+i\beta} d\xi \right| \leq C(1+|\beta|)^n |t|^{-n/2}$$

where $(\beta,t,x) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^n$ and $\psi \in C_0^\infty$ with $supp\psi \subset \Omega$.

Remark 5.4. We remark that the constant C doesn't depend on β, t, x. Moreover, if we carefully follows this proof in [47, Lemma 3.4], we see that this constant depends on only $m, c_1, c_2, n, |\Omega|$ and $\sup_{\xi\in\Omega}|\psi|$ in the assumptions. Here $|\Omega|$ is the Lebesgue measure of Ω.

Before we show the next lemma, we denote the class $\mathcal{A}$ of functions as follows: $\phi \in \mathcal{A}$ if

- $\phi : \Omega \to \mathbf{R}$ for some $\Omega \subseteq \mathbf{R}$, $\phi \in C^3(\Omega)$ and Ω is a finite union of intervals,
- the set $S_\phi = \{\xi \in \overline{\Omega} \cup \mathbf{R} : \phi''(\xi) = 0 \text{ or } \lim_{\xi_1\to\xi}\phi''(\xi_1) = \phi''(\xi) = \pm\infty\}$ is finite,
- if $\xi_0 \in S_\phi$ with $\xi_0 \neq \pm\infty$, then there exists constants ε, c_1, c_2 and $\alpha \neq 0$ such that for $|\xi - \xi_0| < \varepsilon$,

$$c_1|\xi-\xi_0|^{\alpha-2} \le |\phi''(\xi)| \le c_2|\xi-\xi_0|^{\alpha-2},$$

- if $\xi_0 = \pm\infty \in S_\phi$, then there exists constants ε, c_1, c_2 and $\alpha \neq 0$ such that for $|\xi| > 1/\varepsilon$,

$$c_1|\xi|^{\alpha-2} \le |\phi''(\xi)| \le c_2|\xi|^{\alpha-2},$$

- ϕ'' has a finite number of changes of monotonicity.

One can find this class in Kenig–Ponce–Vega [47, page 38]. Now, we display the following lemma and corollary.

Lemma 5.5. *(See [47, Lemma 2.7]) Let $\phi \in \mathcal{A}$. Then we have for any $x, t, \beta \in \mathbf{R}$*

$$\left|\int_\Omega e^{it\phi(\xi)+ix\xi}|\phi''(\xi)|^{1/2+i\beta}d\xi\right| \le C(1+|\beta|)|t|^{-1/2}.$$

Here, the constant C depends only on the constant in the above assumptions.

Corollary 5.6. *(See [47, Corollary 2.9]) Let $\phi \in \mathcal{A}$. Then we have*

$$\left|\int_b^a e^{it\phi(\xi)+ix\cdot\xi}\psi(\xi)|\phi''(\xi)|^{1/2+i\beta}d\xi\right| \le C(1+|\beta|)|t|^{-1/2}\left\{\psi(b) + \int_a^b |\psi'(\xi)|d\xi\right\}$$

for any $x, t, \beta \in \mathbf{R}$.

5.3 Linear estimates

In this section, we consider the linear Cauchy problem

$$\begin{cases} \partial_t u + \partial_x \Delta u = 0, \\ u(0) = u_0, \end{cases}$$

for $t \in \mathbf{R}$ and $(x,y) \in \mathbf{R} \times \mathbf{R}^{n-1}$, where $n = 2$ or 3. The solution to the above problem is given by

$$\begin{aligned} u(t) = U(t)u_0(x,y) &= \mathcal{F}^{-1} e^{it(\xi^3+\xi|\eta|^2)} \mathcal{F} u_0 \\ &= \int_{\mathbf{R}^n} e^{it(\xi^3+\xi|\eta|^2)+ix\xi+iy\cdot\eta} \widehat{u_0}(\xi,\eta) d\xi d\eta, \end{aligned}$$

where $|\eta|^2 = \eta^2$ and $y \cdot \eta = y\eta$ if $n = 2$ and $|\eta|^2 = \eta_1^2 + \eta_2^2$ and $y \cdot \eta = y_1\eta_1 + y_2\eta_2$ if $n = 3$. We set $y = (y_1, y_2)$ and $\eta = (\eta_1, \eta_2)$ for three dimensional case. Moreover, we write $k = (k_1, k_2) \in \mathbf{Z}^2$ and $k = (k_1, k_2, k_3) \in \mathbf{Z}^3$.

In the following subsections, we state some estimates for $U(t)u_0$.

5.3.1 Kato type smoothing effects on 2D and 3D

First of all, we display the smoothing effect of Kato type.

Proposition 5.7. *(See [76, Proposition 3.1].) Let $n = 2, 3$ and ∇ be the Riesz potential operator of order -1; $\nabla = \mathcal{F}^{-1}(\xi^2 + |\eta|^2)^{1/2}\mathcal{F}$. Then*

$$\|\Box_k \nabla U(t)u_0\|_{L_x^\infty L_{y,t}^2} \lesssim \|\Box_k u_0\|_{L_{x,y}^2}. \tag{5.6}$$

Proof. Although only the case $n = 3$ was proved by Ribaud–Vento [76], we get that on $n = 2$ if we follow the same lines as they did. Then we only multiply the frequency–uniform decomposition operators $\Box_k$. □

We can replace the Riesz potential ∇ by ∂_x, and then the following corollary holds.

Corollary 5.8. *Let $n = 2, 3$. Then we have for any $k \in \mathbf{Z}^2$*

$$\|\Box_k \partial_x U(t)u_0\|_{L_x^\infty L_{y,t}^2} \lesssim \|\Box_k u_0\|_{L_{x,y}^2}. \tag{5.7}$$

5.3.2 Strichartz and maximal function estimates on 2D

Next, we prove the Strichartz and maximal function estimates on two dimensions. In order to prove these estimate, we show the following lemma given by Linares–Pastor [61].

Lemma 5.9. *(See [61, Lemma 2.1]) Let $n = 2$, $0 \le \varepsilon < 1/2$ and $\beta \in \mathbf{R}$. Then we have*

$$\left| \int_{\mathbf{R}^2} |\xi|^{\varepsilon+i\beta} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} d\xi d\eta \right| \le C|t|^{-(2+\varepsilon)/3}.$$

Now, we prove the Strichartz estimates. Their proof heavily depends on statements by Wang–Hudzik [97, Proposition 5.1] and time decay estimates deduced from Lemma 5.9.

Proposition 5.10. *Let $3 \le p \le \infty$ and $n=2$. Then we have for any $k \in \mathbf{Z}^2$*

$$\|\Box_k U(t)u_0\|_{L^{p+2}_{x,y,t}} \lesssim \|\Box_k u_0\|_{L^2_{x,y}}. \tag{5.8}$$

Proof. We clearly have for any $k \in \mathbf{Z}^2$

$$\left|\int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta}\sigma_k(\xi,\eta)d\xi d\eta\right| \lesssim 1.$$

Thus, setting $\varepsilon = 0$ and $\beta = 0$ in Lemma 5.9, by interpolation theorem we have

$$\|\Box_k U(t)u_0\|_{L^p} \lesssim (1+|t|)^{-4\delta/3}\|\Box_k u_0\|_{L^{p'}},$$

where $\delta = \delta(p) = 1/2 - 1/p$. From the fact in Wang–Hudzik [97, Proposition 5.1], we obtain

$$\|\Box_k U(t)u_0\|_{L^\gamma_t L^p_{x,y}} \lesssim \|\Box_k u_0\|_{L^2_{x,y}}$$

for any $\gamma \ge \max(2, \frac{2}{\gamma(p)})$ and $\gamma(p) = \frac{4}{3}(\frac{1}{2} - \frac{1}{p})$. Then, if $p \ge 5$, it follows that

$$\|\Box_k U(t)u_0\|_{L^p_{x,y,t}} \lesssim \|\Box_k u_0\|_{L^2_{x,y}}.$$

Next, for $p = \infty$, we have by the Young inequality, Hölder inequality, and the Plancherel theorem

$$\begin{aligned}\|\Box_k U(t)u_0\|_{L^\infty_{x,y,t}} &\lesssim \sum_{|\ell|\le 4}\|\sigma_{k+\ell}\sigma_k\mathcal{F}u_0\|_{L^1_{x,y}} \\ &\lesssim \|\sigma_k\mathcal{F}u_0\|_{L^2_{x,y}} \\ &\sim \|\Box_k u_0\|_{L^2_{x,y}},\end{aligned} \tag{5.9}$$

Thus, the proof is done. □

The following estimates are called the maximal function estimates.

Proposition 5.11. *Let $4 \le p \le \infty$ and $n=2$. Then we have for any $k \in \mathbf{Z}^2$*

$$\|\Box_k U(t)u_0\|_{L^p_x L^\infty_{y,t}} \lesssim \langle k\rangle^{1/p}\|\Box_k u_0\|_{L^2_{x,y}}. \tag{5.10}$$

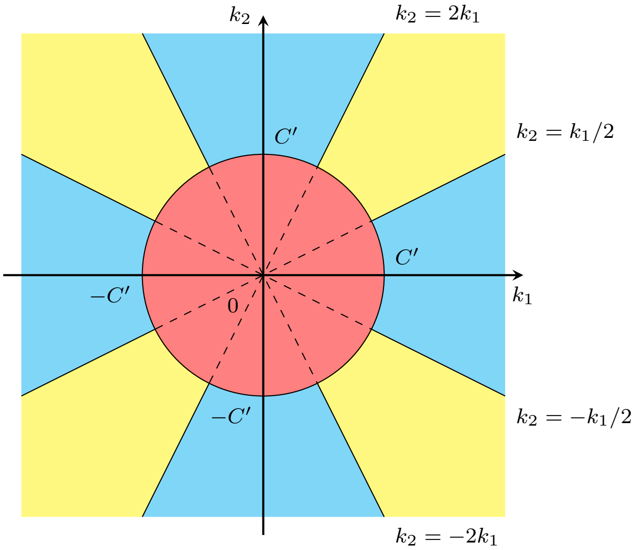
Proof. In order to prove the maximal function estimate (5.10), we divide $k = (k_1, k_2) \in \mathbf{Z}^2$ into the following three cases:

(A) $|k| \le C'$,

(B) $k_2 \ge 2|k_1| \ge 0$ and $|k| \ge C'$; or
$|k_1|/2 \ge k_2 \ge 0$ and $|k| \ge C'$; or
$-|k_1|/2 \le k_2 \le 0$ and $|k| \ge C'$; or
$k_2 \le -2|k_1| \le 0$ and $|k| \ge C'$,

(C) $|k_1|/2 \le k_2 \le 2|k_1|$ and $|k| \ge C'$; or
$-2|k_1| \le k_2 \le -|k_1|/2$ and $|k| \ge C'$

Here, it suffices to take $C' = 50$. We draw a graph for each case as follows. The domains of $(k_1, k_2) \in \mathbf{Z}^2$ colored with red, blue, and yellow correspond to Cases (A), (B), and (C).



Domains for Cases (A), (B), and (C)

Case (A) In this case we use the standard dual estimate method. Thus it suffices to prove that

$$\left\| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \tilde{\sigma}_{k_1}(\xi)\tilde{\sigma}_{k_2}(\eta) d\xi d\eta \right\|_{L_x^{p/2} L_{y,t}^\infty} \lesssim 1,$$

where we used the fact that we can decompose the frequency–uniform decomposition as $\sigma_k(\xi,\eta) = \tilde{\sigma}_{k_1}(\xi)\tilde{\sigma}_{k_2}(\eta)$ for $k = (k_1, k_2) \in \mathbf{Z}^2$. Since the integral on $|x| \le 1$ is obvious, we only consider the case $|x| \ge 1$. In the case $|x| \ge C|t|$ for some suitable constants C, by integration by parts on ξ-variables, we have

$$\left| \int_{\mathbf{R}_\eta} e^{iy\eta} \tilde{\sigma}_{k_2}(\eta) \int_{\mathbf{R}_\xi} e^{it(\xi^3+\xi\eta^2)+ix\xi} \tilde{\sigma}_{k_1}(\xi) d\xi d\eta \right| \lesssim (1+|x|)^{-N} \tag{5.11}$$

for any $N \ge 0$. We used the fact that $|x + t(3\xi^2+\eta^2)| \gtrsim |x|$ in this case. On the other hand, for $|x| \le C|t|$, we have by Lemma 5.9

$$\begin{aligned} \left| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \tilde{\sigma}_{k_1}(\xi)\tilde{\sigma}_{k_2}(\eta) d\xi d\eta \right| &\lesssim (1+|t|)^{-2/3} \\ &\lesssim (1+|x|)^{-2/3}. \end{aligned} \tag{5.12}$$

Combining the estimates (5.11) with (5.12), we have

$$\begin{aligned} &\sup_{y,t} \left| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \tilde{\sigma}_{k_1}(\xi)\tilde{\sigma}_{k_2}(\eta) d\xi d\eta \right| \\ \lesssim\ & (1+|x|)^{-N} + (1+|x|)^{-2/3} \end{aligned}$$

$$\lesssim \quad (1+|x|)^{-2/3}.$$

Now, since we assume that $p \geq 4$, taking $L_x^{p/2}$ norm in the both sides, we have

$$\left\| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \tilde{\sigma}_{k_1}(\xi)\tilde{\sigma}_{k_2}(\eta) d\xi d\eta \right\|_{L_x^{p/2} L_{y,t}^{\infty}} \lesssim 1,$$

Therefore, we have the desired results in this case $|k| \lesssim 1$.

Case (B) From the standard dual estimate method, we prove that

$$\left\| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \tilde{\sigma}_{k_1}(\xi)\tilde{\sigma}_{k_2}(\eta) d\xi d\eta \right\|_{L_x^{p/2} L_{y,t}^{\infty}} \lesssim \langle k \rangle^{2/p}, \tag{5.13}$$

We set $\phi(\xi,\eta) = \xi^3 + \xi\eta^2$. In this case (B), we obtain the following facts; for any $\alpha \in \mathbf{Z}_+^2$ with $|\alpha| \leq 3$

$$\begin{aligned} |\nabla\phi|^2 &= (3\xi^2+\eta^2) + 4\xi^2\eta^2 \sim (\xi^2+\eta^2)^2 > 0, \\ |\partial^\alpha \phi| &\lesssim |\xi|^{3-|\alpha|}, \end{aligned}$$

and the Hessian of ϕ: for $|k| \geq C'$

$$|H\phi| = |12\xi^2 - 4\eta^2| \sim (\xi^2+\eta^2) \sim \langle k \rangle^2 > 0 \tag{5.14}$$

In the calculation of the Hessian of ϕ, we used the fact that ξ and η moves in $|(\xi,\eta)| \gg 1$ and $|\eta|^2 \geq 3.5|\xi|^2$, or $|(\xi,\eta)| \gg 1$ and $|\eta|^2 \leq |\xi|^2/3.5$ since $|k| \gg 1$ and $|k_2|^2 \geq 4|k_1|^2$, or $|k| \gg 1$ and $|k_2|^2 \leq |k_1|^2/4$ holds in this case (B). Therefore, using Lemma 5.3 (also known as stationary phase methods), we have

$$\left| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \sigma_k(\xi,\eta) d\xi d\eta \right| \leq C'' \langle k \rangle^{-1} |t|^{-1}.$$

Here, we see from Remark 5.4 that the constant C'' doesn't depends on $|k|$. Since the estimates (5.13) holds obviously in the case $|x| \leq 1$, we assume that $|x| \geq 1$. For $|x| \leq 4|t|\langle k \rangle^2$, we have

$$\begin{aligned} \left| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \sigma_k(\xi,\eta) d\xi d\eta \right| &\lesssim \frac{1}{1+\langle k \rangle |t|} \\ &\lesssim \frac{\langle k \rangle}{\langle k \rangle + |x|} \end{aligned} \tag{5.15}$$

On the other hand, for $|x| \geq 4|t|\langle k \rangle^2$, we have by integration by parts

$$\left| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \sigma_k(\xi,\eta) d\xi d\eta \right| \lesssim \frac{1}{(1+|x|)^N} \tag{5.16}$$

for any $N \geq 0$. Thus, collecting the estimates (5.15) and (5.16),

$$\sup_{y,t} \left| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \tilde{\sigma}_{k_1}(\xi)\tilde{\sigma}_{k_2}(\eta) d\xi d\eta \right|$$

$$\lesssim \frac{\langle k\rangle}{\langle k\rangle + |x|} + \frac{1}{(1+|x|)^N},$$

Thus, since we assume that $p \geq 4$, taking the $L_x^{p/2}$ norm in the both sides, we have

$$\left\| \int_{\mathbf{R}^2} e^{it(\xi^3+\xi\eta^2)+ix\xi+iy\eta} \tilde{\sigma}_{k_1}(\xi)\tilde{\sigma}_{k_2}(\eta) d\xi d\eta \right\|_{L_x^{p/2} L_{y,t}^\infty}$$
$$\lesssim \langle k\rangle^{2/p}.$$

Case (C) In this case, we prove directly the maximal function estimate;

$$\begin{aligned} \|\Box_k U(t)u_0\|_{L_x^4 L_{y,t}^\infty} &\lesssim \|\Box_k D_x^{-1/4} \nabla^{1/2} u_0\|_{L^2_{x,y}} \\ &\lesssim \langle k\rangle^{1/4} \|\Box_k u_0\|_{L^2_{x,y}}, \end{aligned}$$

where $D_x^{-1/4} = \mathcal{F}^{-1}|\xi|^{-1/4}\mathcal{F}$ and $\nabla^{1/2} = \mathcal{F}^{-1}(\xi^2+\eta^2)^{1/4}\mathcal{F}$. Then, interpolating with the estimate (5.9):

$$\|\Box_k U(t)u_0\|_{L^\infty_{x,y,t}} \lesssim \|\Box_k u_0\|_{L^2_{x,y}},$$

we have the desired result. By the duality, we prove that

$$\left\| \Box_k \int_{\mathbf{R}} P(D_x, D_y)\, U(t) f(t) dt \right\|_{L^2_{x,y}} \lesssim \|\Box_k f\|_{L_x^{4/3} L^1_{y,t}}$$

holds for all $f \in \mathcal{S}(\mathbf{R}^3_{x,y,t})$. Here $P(D_x, D_y) = \mathcal{F}^{-1} P(\xi,\eta)\mathcal{F}$ with

$$P(\xi,\eta) = \left| \frac{6\xi}{3\xi^2+\eta^2} \right|^{1/4}.$$

Thus, by the Hölder inequality, it is equivalent to show that

$$\left\| \Box_k \int_{\mathbf{R}_s} \{P(D_x, D_y)\}^2\, U(t-s) f(s) ds \right\|_{L_x^4 L^\infty_{y,t}} \lesssim \|\Box_k f\|_{L_x^{4/3} L^1_{y,t}}.$$

We set and have

$$\begin{aligned} & |I(x,y,t)| \\ := & \left| \Box_k \int_{\mathbf{R}_s} \{P(D_x, D_y)\}^2\, U(t-s) f(s) ds \right| \\ = & \left| \int_{\mathbf{R}_s} \mathcal{F}^{-1}_{\xi,\eta} \left| \frac{6\xi}{3\xi^2+\eta^2} \right|^{1/2} \sigma_k(\xi,\eta) e^{i(t-s)(\xi^3+\xi\eta^2)} \mathcal{F}_{x,y} f(s) ds \right| \\ \leq & \sum_{|\ell|\leq 3} \left| \int_{\mathbf{R}_s} \mathcal{F}^{-1}_{\xi,\eta} \left| \frac{6\xi}{3\xi^2+\eta^2} \right|^{1/2} \sigma_{k+\ell}(\xi,\eta)\sigma_k(\xi,\eta) e^{i(t-s)(\xi^3+\xi\eta^2)} \mathcal{F}_{x,y} f(s) ds \right| \\ = & \sum_{|\ell|\leq 3} \left| \int_{\mathbf{R}_s} \mathcal{F}^{-1}_{\xi,\eta} \left| \frac{6\xi}{3\xi^2+\eta^2} \right|^{1/2} \sigma_{k+\ell}(\xi,\eta) e^{i(t-s)(\xi^3+\xi\eta^2)} \mathcal{F}_{x,y} \Box_k f(s) ds \right| \end{aligned}$$

$$= \sum_{|\ell|\le 3} \left| \int_{\mathbf{R}_s} \mathcal{F}_\eta^{-1} \left[\tilde{\sigma}_{k_2+\ell_2}(\cdot) I\!I\ (x,\cdot,s,t) \right] ds \right|$$
$$\le \sum_{|\ell|\le 3} \int_{\mathbf{R}_s} \int_{\mathbf{R}_\eta} |\tilde{\sigma}_{k_2+\ell_2}(\eta)| \times |I\!I(x,\eta,s,t)|\, d\eta ds,$$

where $\ell = (\ell_1, \ell_2) \in \mathbf{Z}^2$. Here we set and obtain

$$\begin{aligned} & |I\!I(x,\eta,s,t)| \\ := & \left| \mathcal{F}_\xi^{-1} \left[\left| \frac{6\xi}{3\xi^2+\eta^2} \right|^{1/2} \tilde{\sigma}_{k_1+\ell_1}(\xi) e^{i(t-s)(\xi^3+\xi\eta^2)} \mathcal{F}_{x,y} \Box_k f \right] \right| \\ = & \left| \mathcal{F}_\xi^{-1} \left[\left| \frac{6\xi}{3\xi^2+\eta^2} \right|^{1/2} \tilde{\sigma}_{k_1+\ell_1}(\xi) e^{i(t-s)(\xi^3+\xi\eta^2)} \right] *_x [\mathcal{F}_y \Box_k f] \right| \\ \le & \left| \mathcal{F}_\xi^{-1} \left[\left| \frac{6\xi}{3\xi^2+\eta^2} \right|^{1/2} \tilde{\sigma}_{k_1+\ell_1}(\xi) e^{i(t-s)(\xi^3+\xi\eta^2)} \right] \right| *_x \|\Box_k f(x,\cdot,s)\|_{L^1_y}. \end{aligned}$$

Thus if we could prove that

$$\left| \mathcal{F}_\xi^{-1} \left[\left| \frac{6\xi}{3\xi^2+\eta^2} \right|^{1/2} \tilde{\sigma}_{k_1+\ell_1}(\xi) e^{i\tau(\xi^3+\xi\eta^2)} \right](z) \right| \lesssim |z|^{-1/2} \tag{5.17}$$

for any $z, \tau \in \mathbf{R}$ and $\eta \in \mathrm{supp}\tilde{\sigma}_{k_2+\ell_2}$, then we have

$$\begin{aligned} & |I(x,y,t)| \\ \lesssim & \int_{\mathbf{R}_s} \int_{\mathbf{R}_\eta} |\tilde{\sigma}_{k_2+\ell_2}(\eta)| \left\{ |x|^{-1/2} *_x \|\Box_k f(x,\cdot,s)\|_{L^1_y} \right\} d\eta ds \\ \lesssim & |x|^{-1/2} *_x \|\Box_k f(x,\cdot,\cdot)\|_{L^1_{y,s}}. \end{aligned}$$

Taking $L^4_x L^\infty_{y,t}$ norm to $|I(x,y,t)|$, we have

$$\|I(x,y,t)\|_{L^4_x L^\infty_{y,t}} \lesssim \|\Box_k f\|_{L^{4/3}_x L^1_{y,t}},$$

since $|x|^{-1/2} *_x : L^{4/3}_x \to L^4_x$ (see also Kenig–Ponce–Vega [48]). Then applying the standard dual argument, we have the desired maximal function estimate.

Now, we prove the inequality (5.17). In this case, we don't lose of generality if we change $k_1 + \ell_1$ to k_1 in the estimate (5.17). Thus we consider the following expression;

$$|I\!I\!I(z,\eta,\tau)| := \left| \int_{\mathbf{R}_\xi} e^{i\tau(\xi^3+\xi\eta^2)+iz\xi} \left| \frac{6\xi}{3\xi^2+\eta^2} \right|^{1/2} \tilde{\sigma}_{k_1}(\xi) d\xi \right|.$$

We set $\theta = h(\xi) = \phi(\xi,\eta) = \xi^3 + \xi\eta$. Then, by the change of variables as $\theta = h(\xi)$, we have

$$|I\!I\!I(x,\eta,\tau)| = \left| \int_{\mathbf{R}_\theta} e^{i\tau\theta + izh^{-1}(\theta)} \left| \frac{d^2}{d\theta^2} h^{-1}(\theta) \right|^{1/2} \tilde{\sigma}_{k_1}(h^{-1}(\theta)) d\theta \right| \tag{5.18}$$

Here we used the following argument. For $\theta = h(\xi)$, we have

$$\frac{d}{d\theta}h^{-1}(\theta) = \frac{1}{\frac{d\theta}{d\xi}} = \frac{1}{3\xi^2+\eta^2}$$

and

$$\left|\frac{d^2}{d\theta^2}h^{-1}(\theta)\right| = \left|\frac{\frac{d^2\theta}{d\xi^2}}{\left(\frac{d\theta}{d\xi}\right)^3}\right| = \left|\frac{6\xi}{(3\xi^2+\eta^2)^3}\right|.$$

Thus, we obtain for $\theta = h(\xi)$

$$\begin{aligned}\left|\frac{6\xi}{3\xi^2+\eta^2}\right|^{1/2} \times \frac{d}{d\theta}h^{-1}(\theta) &= \left|\frac{6\xi}{3\xi^2+\eta^2}\right|^{1/2} \times \frac{1}{3\xi^2+\eta^2} \\ &= \left|\frac{6\xi}{(3\xi^2+\eta^2)^3}\right|^{1/2} \\ &= \left|\frac{d^2}{d\theta^2}h^{-1}(\theta)\right|^{1/2}.\end{aligned}$$

Moreover, (ξ, η) moves in $|\eta|/3 \le |\xi| \le 3|\eta|$ and $|(\xi,\eta)| \gg 1$since we assume that $|k_2|/2 \le |k_1| \le 2|k_2|$ and $|k| \ge C' \gg 1$. Thus, we have

$$|\theta| \sim |\xi|^3 \sim |\eta|^3$$

and

$$\left|\frac{d^2}{d\theta^2}h^{-1}(\theta)\right|^{1/2} \sim |\theta|^{-5/3}.$$

Therefore, since $\tilde{\sigma}_{k_1}$ disappear at the end point, we have by using Lemma 5.5 (Corollary 5.6)

$$\begin{aligned}|I\!I\!I(z,\eta,t)| &\le C_3|z|^{-1/2}\int_{\mathbf{R}_\theta}\left|\frac{d}{d\theta}\left[\tilde{\sigma}_{k_1}(h^{-1}(\theta))\right]\right| d\theta \\ &= C_3|z|^{-1/2}\int_{\mathbf{R}_\theta}\left|\frac{d}{d\theta}h^{-1}(\theta)\right| \left|\tilde{\sigma}'_{k_1}(h^{-1}(\theta))\right| d\theta \\ &= C_3|z|^{-1/2}\int_{\mathbf{R}_\xi}\left|\tilde{\sigma}'_{k_1}(\xi)\right| d\xi \\ &\le C_4|z|^{-1/2},\end{aligned}$$

where the constants C_3 and C_4 don't depends on $|k|$ (see Remark 5.12 below). Here, we have changed variables as $\theta = h(\xi) = \xi^3 + \xi\eta^2$ in the third inequality

Collecting the consequences for Cases (A), (B), and (C), we have

$$\|\Box_k U(t)u_0\|_{L^p_x L^\infty_{y,t}} \lesssim \langle k\rangle^{1/p}\|\Box_k u_0\|_{L^2_{x,y}}$$

for any $k \in \mathbf{Z}^n$ and $p \ge 4$. $\Box$

Remark 5.12. In this remark, we actually explain that the constants C_3 in the proof of Proposition 5.11, which appeared when we used Lemma 5.5, are independent of $|k|$. To do this, we use the van der Corput lemma (see Stein [81, Chapter VII I]): Suppose that a real–valued function $\phi \in C^2$ in (a,b) and $|\phi''(\xi)| > 1$ for all $\xi \in (a,b)$. Then

$$\left|\int_a^b e^{it\phi(\xi)}\psi(\xi)d\xi\right| \leq 10|t|^{-1/2}\left\{|\psi(b)| + \int_a^b |\psi'(\xi)|d\xi\right\}.$$

In the Case (C), since we assume that $|k_2|/2 \leq |k_1| \leq 2|k_2|$ and $|k| \geq C' \gg 1$, we have $|\eta|/3 \leq |\xi| \leq 3|\eta|$. Here, $\eta \in [k_2-5, k_2+5]$. Then we have for $\theta = h(\xi) = \xi^3 + \xi\eta^2$

$$\begin{aligned}
\left|\frac{d}{d\theta}h^{-1}(\theta)\right| &= \left|\frac{1}{\frac{d\theta}{d\xi}}\right| = \frac{1}{3\xi^2+\eta^2} \\
\left|\frac{d^2}{d\theta^2}h^{-1}(\theta)\right| &= \left|-\frac{\frac{d^2\theta}{d\xi^2}}{\left(\frac{d\theta}{d\xi}\right)^3}\right| = \left|\frac{6\xi}{(3\xi^2+\eta^2)^3}\right| \leq \left|\frac{5}{(3\xi^2+\eta^2)^{5/2}}\right| \leq C_5\langle k\rangle^{-5} \\
\left|\frac{d^2}{d\theta^2}h^{-1}(\theta)\right| &\geq \left|\frac{\frac{1}{2}}{(3\xi^2+\eta^2)^{5/2}}\right| \geq C_6\langle k\rangle^{-5} \\
\left|\frac{d^3}{d\theta^3}h^{-1}(\theta)\right| &= \left|\frac{3\left(\frac{d^2\theta}{d\xi^2}\right)^2}{\left(\frac{d\theta}{d\xi}\right)^5} - \frac{\frac{d^3\theta}{d\xi^3}}{\left(\frac{d\theta}{d\xi}\right)^4}\right| = \left|\frac{108\xi^2}{(3\xi^2+\eta^2)^5} - \frac{6}{(3\xi^2+\eta^2)^4}\right| \\
&= \left|\frac{90\xi^2-6\eta^2}{(3\xi^2+\eta^2)^5}\right| \leq \left|\frac{30}{(3\xi^2+\eta^2)^4}\right| \leq C_7\langle k\rangle^{-8}
\end{aligned}$$

It is clear that the constants C_5, C_6 and C_7 never depend on $|k|$. Moreover, $\frac{d^3}{d\theta^3}h^{-1}(\theta) > 0$ in Case (C). In Case (C), for any $z, \tau \in \mathbf{R}$ and $\eta \in [k_2-5, k_2+5]$, we have

$$|I\!I\!I(z,\eta,t)| \leq C_8\langle k\rangle^{-1/2} \leq C_8,$$

where the constant C_8 is independent of $|k|$. Thus, for $|z| \leq 1$, it obviously follows that

$$|I\!I\!I(z,\eta,t)| \leq C_8|z|^{-1/2}.$$

Next, we consider the case $|z| > 1$. Since $\tilde{\sigma}_{k_1}$ disappear at the end point and $\tau\theta + zh^{-1}(\theta) = z(\tau\theta/z + h^{-1}(\theta))$ holds, we have by the van der Corput lemma and the above four calculations of h^{-1},

$$\begin{aligned}
&\left|\int_{\mathbf{R}_\theta} e^{i\tau\theta+izh^{-1}(\theta)}\left|\frac{d^2}{d\theta^2}h^{-1}(\theta)\right|^{1/2}\tilde{\sigma}_{k_1}(h^{-1}(\theta))d\theta\right| \\
\leq\ &\frac{10}{(C_6\langle k\rangle^{-5}|x|)^{1/2}}\left\{\int_{\mathbf{R}_\theta}\frac{1}{2}\left|\frac{d^2}{d\theta^2}h^{-1}(\theta)\right|^{-1/2}\left|\frac{d^3}{d\theta^3}h^{-1}(\theta)\right||\tilde{\sigma}_{k_1}(h^{-1}(\theta))|d\theta\right.
\end{aligned}$$

$$
\begin{aligned}
& \quad + \int_{\mathbf{R}_\theta} \left| \frac{d^2}{d\theta^2} h^{-1}(\theta) \right|^{1/2} \left| \frac{d}{d\theta} h^{-1}(\theta) \right| |\tilde{\sigma}'_{k_1}(h^{-1}(\theta))| d\theta \Bigg\} \\
\le & \ \frac{10\langle k\rangle^{5/2}}{(C_6|x|)^{1/2}} \left\{ \int_{\mathbf{R}_\theta} \frac{1}{2} \left(C_6 \langle k\rangle^{-5} \right)^{-1/2} C_7 \langle k\rangle^{-8} \times |\tilde{\sigma}_{k_1}(h^{-1}(\theta))| d\theta \right. \\
& \quad \left. + \int_{\mathbf{R}_\theta} \left(C_5 \langle k\rangle^{-5} \right)^{1/2} \left| \frac{d}{d\theta} h^{-1}(\theta) \right| |\tilde{\sigma}'_{k_1}(h^{-1}(\theta))| d\theta \right\} \\
\le & \ \frac{10}{(C_6|x|)^{1/2}} \left\{ \frac{1}{2} C_6^{-1/2} C_7 \langle k\rangle^{-3} \int_{\mathbf{R}_\theta} |\tilde{\sigma}_{k_1}(h^{-1}(\theta))| d\theta \right. \\
& \quad \left. + C_5^{1/2} \int_{\mathbf{R}_\theta} \left| \frac{d}{d\theta} h^{-1}(\theta) \right| |\tilde{\sigma}'_{k_1}(h^{-1}(\theta))| d\theta \right\} \\
\le & \ \frac{C_9}{|x|^{1/2}} \left\{ \langle k\rangle^{-3} \int_{\mathbf{R}_\xi} |\tilde{\sigma}_{k_1}(\xi)| (3\xi^2 + \eta^2) d\xi + \int_{\mathbf{R}_\xi} |\tilde{\sigma}'_{k_1}(\xi)| d\xi \right\} \\
\le & \ \frac{C_{10}}{|x|^{1/2}} \left\{ \langle k\rangle^{-1} + 1 \right\},
\end{aligned}
$$

where we have changed variables as $\theta = h(\xi) = \xi^3 + \xi\eta^2$ in the fifth inequality. Clearly, the constant C_{10} doesn't depends on $|k|$. Therefore, we could justify that the constant C_3 is independent of $|k|$.

5.3.3 Strichartz and maximal function estimates on 3D

The previous statements concerned with the estimates on two dimension. In this subsection, we prove the same estimates on $n = 3$. Before that, we show the following estimates for the oscillatory integral on 3D. Let us recall that we set $y = (y_1, y_2) \in \mathbf{R}^2$ and $\eta = (\eta_1, \eta_2) \in \mathbf{R}^2$.

Lemma 5.13. *Let $n = 3$, $0 \le \varepsilon < 1/2$ and $\beta \in \mathbf{R}$. Then we have for any $k = (k_1, k_2, k_3) \in \mathbf{Z}^3$*

$$
\left| \int_{\mathbf{R}^3} |\xi|^{\varepsilon + i\beta} e^{it(\xi^3 + \xi|\eta|^2) + ix\xi + iy\cdot\eta} \sigma_k(\xi, \eta) d\xi d\eta \right| \le C|t|^{-(2+\varepsilon)/3}
$$

where the constant C is independent of t, ε, and k.

Proof. By the Young inequality and Lemma 5.9, we have

$$
\begin{aligned}
& \left| \int_{\mathbf{R}^3} |\xi|^{\varepsilon+i\beta} e^{it(\xi^3+\xi|\eta|^2)+ix\xi+iy_1\eta_1+iy_2\eta_2} \sigma_k(\xi, \eta_1, \eta_2) d\xi d\eta_1 d\eta_2 \right| \\
= & \left| \int_{\mathbf{R}^3} |\xi|^{\varepsilon+i\beta} e^{it(\xi^3+\xi\eta_1^2+\xi\eta_2^2)+ix\xi+iy_1\eta_1+iy_2\eta_2} \tilde{\sigma}_{k_1}(\xi) \tilde{\sigma}_{k_2}(\eta_1) \tilde{\sigma}_{k_3}(\eta_2) d\xi d\eta_1 d\eta_2 \right| \\
\lesssim & \int_{\mathbf{R}_{\eta_2}} |\tilde{\sigma}_{k_3}(\eta_2)| \sup_{x, y_1 \in \mathbf{R}} \left| \int_{\mathbf{R}^2_{\xi,\eta_1}} |\xi|^{\varepsilon+i\beta} e^{it(\xi^3+\xi\eta_1^2)+i(x+t\eta_2^2)\xi+iy_1\eta_1} d\xi d\eta_1 \right| d\eta_2 \\
\lesssim & \int_{\mathbf{R}_{\eta_2}} |\tilde{\sigma}_{k_3}(\eta_2)| \, |t|^{-(2+\varepsilon)/3} d\eta_2
\end{aligned}
$$

$$\lesssim \quad |t|^{-(2+\varepsilon)/3}$$

Thus, we have the desired estimates. □

Remark 5.14. Linares–Saut [63] have studied the estimate for this oscillatory integral with the sharp time decay rate: for $0 < \varepsilon < 1$

$$\left| \int_{\mathbf{R}^3} |\xi|^{\varepsilon + i\beta} e^{it(\xi^3 + \xi|\eta|^2) + ix\xi + iy\cdot\eta} d\xi d\eta \right| \leq C|t|^{-1-\varepsilon/3}.$$

However, their estimate doesn't allow us to deal with $\varepsilon = 0$. This means that the derivative D_x appears to the time decay estimates (also known as dispersion estimates) and the Strichartz estimates. For the argument used in this paper, this property has a bad influence when we prove the well–posedness on modulation spaces. Thus, we can't use their sharp estimate and need to use the estimate in Lemma 5.13 with worse decay rate, unfortunately.

The following estimate is the Strichartz estimates in 3D.

Proposition 5.15. *Let* $4 \leq p \leq \infty$ *if* $n = 3$. *Then, we have*

$$\|\Box_k U(t) u_0\|_{L^p_{x,y,t}} \lesssim \|\Box_k u_0\|_{L^2_{x,y}}.$$

Proof. On three dimension, we have by Grünrock [33, Section 2] or [34, Lemma 3]

$$\|\Box_k U(t) u_0\|_{L^4_{x,y,t}} \lesssim \|\Box_k u_0\|_{L^2_{x,y}}.$$

Interpolating with the estimate (5.9), we have the desired estimates on 3D. □

Remark 5.16. We can derive the Strichartz estimates from Lemma 5.13 as we did Proposition 5.10 for two dimensional case. Then, if $p \geq 5$, it follows that

$$\|\Box_k U(t) u_0\|_{L^p_{x,y,t}} \lesssim \|\Box_k u_0\|_{L^2_{x,y}}.$$

However, we see that the Strichartz estimates in Proposition 5.15 are better than the above one. Thus, we chose the better one.

Proposition 5.17. *Let* $3 < p \leq \infty$ *and* $n = 3$. *Then, we have*

$$\|\Box_k U(t) u_0\|_{L^p_x L^\infty_{y,t}} \lesssim \langle k \rangle^{2/p} \|\Box_k u_0\|_{L^2_{x,y}}.$$

Proof. We use the standard dual estimate method. Thus it suffices to prove that

$$\left\| \int_{\mathbf{R}^n} e^{it(\xi^3 + \xi|\eta|^2) + ix\xi + iy\cdot\eta} \sigma_k(\xi, \eta) d\xi d\eta \right\|_{L^{p/2}_x L^\infty_{y,t}} \lesssim \langle k \rangle^{4/p}.$$

Since the case $|k| \lesssim 1$ is proved as in the proof of Proposition 5.11, we only consider the case $|k| \gtrsim 1$. For $|x| \geq 4|t|\langle k \rangle^2$ and $|x| \leq |t|\langle k \rangle^2/4$, by integration by parts we have

$$\left| \int_{\mathbf{R}^2_{\eta_1,\eta_2}} e^{iy_1\eta_1 + iy_2\eta_2} \tilde{\sigma}_{k_2}(\eta_1) \tilde{\sigma}_{k_3}(\eta_2) \int_{\mathbf{R}_\xi} e^{it(\xi^3 + \xi\eta_1^2 + \xi\eta_2^2) + ix\xi} \tilde{\sigma}_{k_1}(\xi) d\xi d\eta_1 d\eta_2 \right|$$

$$\int_{\mathbf{R}^2_{\eta_1,\eta_2}} |\tilde{\sigma}_{k_2}(\eta_1)|\,|\tilde{\sigma}_{k_3}(\eta_2)| \left| \int_{\mathbf{R}_\xi} e^{it(\xi^3+\xi\eta_1^2+\xi\eta_2^2)+ix\xi} \tilde{\sigma}_{k_1}(\xi) d\xi \right| d\eta_1 d\eta_2$$
$$\lesssim \quad (1+|x|)^{-N}$$

for any $N \geq 0$. For the case $|x| \sim |t|\langle k\rangle^2$, we obtain from Lemma 5.13

$$\left| \int_{\mathbf{R}^n} e^{it(\xi^3+\xi|\eta|^2)+ix\xi+iy\cdot\eta} \sigma_k(\xi,\eta) d\xi d\eta \right| \lesssim (1+|t|)^{-2/3} \lesssim \frac{\langle k\rangle^{4/3}}{(\langle k\rangle^2+|x|)^{2/3}}.$$

Therefore, since we assume that $3 < p \leq \infty$, we have

$$\begin{aligned} &\left\| \int_{\mathbf{R}^n} e^{it(\xi^3+\xi|\eta|^2)+ix\xi+iy\cdot\eta} \sigma_k(\xi,\eta) d\xi d\eta \right\|_{L_x^{p/2} L_{y,t}^\infty} \\ \lesssim\ & \left\| \frac{1}{(1+|x|)^N} \right\|_{L_x^{p/2}} + \left\| \frac{\langle k\rangle^{4/3}}{(\langle k\rangle^2+|x|)^{2/3}} \right\|_{L_x^{p/2}} \\ \lesssim\ & \langle k\rangle^{4/p} \end{aligned}$$

Thus we have the desired result. □

5.3.4 Estimates for nonlinear terms

In this section, we give some estimates to deal with the Duhamel terms. First of all, we give the following estimate, which is said to be the smoothing–Strichartz estimate and proved by Ribaud–Vento [76].

Proposition 5.18. *(see Ribaud–Vento [76, Proposition 3.5]) Let $n = 2, 3$. Then we have*

$$\left\| \Box_k \nabla \int_0^t U(t-s) f(s) ds \right\|_{L_t^\infty L_{x,y}^2} \lesssim \|\Box_k f\|_{L_x^1 L_{y,t}^2}. \tag{5.19}$$

Proof. Since we have Proposition 5.7 on $n = 2$, we also have the desired estimates on $n = 2$ if we follow the same line as the proof in the 3D case by Ribaud–Vento [76]. □

As we stated in the previous section, we can replace ∇ by ∂_x.

Corollary 5.19. *Let $n = 2, 3$. Then we have*

$$\left\| \Box_k \partial_x \int_0^t U(t-s) f(s) ds \right\|_{L_t^\infty L_{x,y}^2} \lesssim \|\Box_k f\|_{L_x^1 L_{y,t}^2}.$$

Next, we show the following Strichartz estimate.

Proposition 5.20. *(See Wang–Hudzik [97, Propositions 5.2–5.3]) Let $3 \le p < \infty$ and $n = 2$. Then we have*

$$\left\|\Box_k \int_0^t U(t-s)f(s)ds\right\|_{L_t^\infty L_{x,y}^2} \lesssim \|\Box_k f\|_{L_{x,y,t}^{(2+p)/(1+p)}},$$

$$\left\|\Box_k \int_0^t U(t-s)f(s)ds\right\|_{L_{x,y,t}^{2+p}} \lesssim \|\Box_k f\|_{L_{x,y,t}^{(2+p)/(1+p)}}.$$

Proof. By Lemma 5.9 and [97, Proposition 5.2–5.3], we have for any $\gamma \ge \max(2, \frac{2}{\gamma(p)})$ and $\gamma(p) = \frac{4}{3}(\frac{1}{2} - \frac{1}{p})$

$$\left\|\Box_k \int_0^t U(t-s)f(s)ds\right\|_{L_t^\infty L_{x,y}^2} \lesssim \|\Box_k f\|_{L_t^{\gamma'} L_{x,y}^{p'}},$$

and

$$\left\|\Box_k \int_0^t U(t-s)f(s)ds\right\|_{L_t^\gamma L_{x,y}^p} \lesssim \|\Box_k f\|_{L_t^{\gamma'} L_{x,y}^{p'}}.$$

Then if $p \ge 5$, we have

$$\left\|\Box_k \int_0^t U(t-s)f(s)ds\right\|_{L_t^\infty L_{x,y}^2} \lesssim \|\Box_k f\|_{L_{x,y,t}^{p'}},$$

and

$$\left\|\Box_k \int_0^t U(t-s)f(s)ds\right\|_{L_{x,y,t}^p} \lesssim \|\Box_k f\|_{L_{x,y,t}^{p'}}.$$

Thus, changing p to $p+2$, we have the desired result. $\Box$

Next we present the following estimates called as the smoothing–maximal function estimate and the Strichartz–maximal function estimate.

Proposition 5.21. *Let $4 \le p < \infty$ and $n = 2$. Then we have*

$$\left\|\Box_k \int_0^t U(t-s)f(s)ds\right\|_{L_t^\infty L_{x,y}^2} \lesssim \langle k\rangle^{1/p} \|\Box_k f\|_{L_x^{p'} L_{y,t}^1}, \tag{5.20}$$

$$\left\|\Box_k \nabla \int_0^t U(t-s)f(s)ds\right\|_{L_x^p L_{y,t}^\infty} \lesssim \langle k\rangle^{1/p} \|\Box_k f\|_{L_x^1 L_{y,t}^2}, \tag{5.21}$$

$$\left\|\Box_k \nabla \int_0^t U(t-s)f(s)ds\right\|_{L_x^p L_{y,t}^\infty} \lesssim \langle k\rangle^{1+1/p} \|\Box_k f\|_{L_{x,y,t}^{(2+p)/(1+p)}}, \tag{5.22}$$

Proof. First of all, we prove the estimate (5.20). Using the standard dual argument, we obtain the desired results from the following estimate.

$$\left|\int_{\mathbf{R}_t} \left(\Box_k \int_{\mathbf{R}_s} U(t-s)f(s)ds, g(t)\right)_{L_{x,y}^2} dt\right|$$

$$
\begin{aligned}
&\lesssim \sum_{|\ell|\le 4}\left|\int_{\mathbf{R}_s}\left(\Box_k f(s), \Box_{k+\ell}\int_{\mathbf{R}_t} U(s-t)g(t)dt\right)_{L^2_{x,y}} ds\right| \\
&\lesssim \sum_{|\ell|\le 4}\|\Box_k f\|_{L^{p'}_x L^1_{y,s}}\left\|\Box_{k+\ell}\int_{\mathbf{R}_t} U(s-t)g(t)dt\right\|_{L^p_x L^\infty_{y,s}} \\
&\lesssim \sum_{|\ell|\le 4}\|\Box_k f\|_{L^{p'}_x L^1_{y,t}}\|\Box_{k+\ell} g\|_{L^1_t L^2_{x,y}}\langle k\rangle^{1/p},
\end{aligned}
$$

where we have used Proposition 5.11 in the last step.

Next we show the estimate (5.21). By Proposition 5.18 and the estimate (5.20), we have

$$
\begin{aligned}
&\left|\int_{\mathbf{R}_t}\left(\Box_k\int_{\mathbf{R}_s} U(t-s)\nabla f(s)ds, g(t)\right)_{L^2_{x,y}} dt\right| \\
&\lesssim \sum_{|\ell|\le 4}\left\|\Box_k\int_{\mathbf{R}_s} U(-s)\nabla f(s)ds\right\|_{L^2_{x,y}}\left\|\Box_{k+\ell}\int_{\mathbf{R}_t} U(-t)g(t)dt\right\|_{L^2_{x,y}} \\
&\lesssim \sum_{|\ell|\le 4}\|\Box_k f\|_{L^1_x L^2_{y,t}}\|\Box_{k+\ell} g\|_{L^{p'}_x L^1_{y,t}}\langle k\rangle^{1/p}.
\end{aligned}
$$

Next we consider the estimate (5.22). We have by Propositions 5.20 and the estimate (5.20)

$$
\begin{aligned}
&\left|\int_{\mathbf{R}_t}\left(\Box_k\int_{\mathbf{R}_s} U(t-s)\nabla f(s)ds, g(t)\right)_{L^2_{x,y}} dt\right| \\
&\lesssim \sum_{|\ell|\le 4}\left\|\Box_k\int_{\mathbf{R}_s} U(-s)\nabla f(s)ds\right\|_{L^2_{x,y}}\left\|\Box_{k+\ell}\int_{\mathbf{R}_t} U(-t)g(t)dt\right\|_{L^2_{x,y}} \\
&\lesssim \sum_{|\ell|\le 4}\lesssim\|\Box_k f\|_{L^{(2+p)/(1+p)}_{x,y,t}}\|\Box_{k+\ell} g\|_{L^{p'}_x L^1_{y,t}}\langle k\rangle^{1+1/p}.
\end{aligned}
$$

Finally, from the TT^* argument (see [16, 71, 95]), all of the estimates (5.20), (5.21), and (5.22) hold. $\Box$

The next estimates are said to be the Strichartz–smoothing and smoothing–Strichartz estimates.

Proposition 5.22. *Let $3 \le p < \infty$ and $n = 2$. Then*

$$
\left\|\Box_k\nabla\int_0^t U(t-s)f(s)ds\right\|_{L^\infty_x L^2_{y,t}} \lesssim \|\Box_k f\|_{L^{(2+p)/(1+p)}_{x,y,t}}, \tag{5.23}
$$

$$
\left\|\Box_k\nabla\int_0^t U(t-s)f(s)ds\right\|_{L^{2+p}_{x,y,t}} \lesssim \|\Box_k f\|_{L^1_x L^2_{y,t}}. \tag{5.24}
$$

Proof. We show the estimate (5.23). By Propositions 5.18 and 5.20, we have

$$
\left|\int_{\mathbf{R}_t}\left(\Box_k\int_{\mathbf{R}_s} U(t-s)\nabla f(s)ds, g(t)\right)_{L^2_{x,y}} dt\right|
$$

$$\lesssim \sum_{|\ell|\le 4} \left\| \Box_k \int_{\mathbf{R}_s} U(-s) f(s) ds \right\|_{L^2_{x,y}} \left\| \Box_{k+\ell} \int_{\mathbf{R}_t} U(-t) \nabla g(t) dt \right\|_{L^2_{x,y}}$$
$$\lesssim \sum_{|\ell|\le 4} \|\Box_k f\|_{L^{(2+p)/(1+p)}_{x,y,t}} \|\Box_{k+\ell} g\|_{L^1_x L^2_{y,t}}$$

Next we consider the estimate (5.24). From Propositions 5.18 and 5.20, it follows that

$$\left| \int_{\mathbf{R}_t} \left(\Box_k \int_{\mathbf{R}_s} U(t-s) \nabla f(s) ds, g(t) \right)_{L^2_{x,y}} dt \right|$$
$$\lesssim \sum_{|\ell|\le 4} \left\| \Box_k \int_{\mathbf{R}_s} U(-s) \nabla f(s) ds \right\|_{L^2_{x,y}} \left\| \Box_{k+\ell} \int_{\mathbf{R}_t} U(-t) g(t) dt \right\|_{L^2_{x,y}}$$
$$\lesssim \sum_{|\ell|\le 4} \|\Box_k f\|_{L^1_x L^2_{y,t}} \|\Box_{k+\ell} g\|_{L^{(2+p)/(1+p)}_{x,y,t}} .$$

Finally, by the standard TT^* argument as stated in the previous proof, we have the desired results. □

We show the similar estimates on 3D case.

Proposition 5.23. *We have the following estimates.*

(i) Let $3 < p < \infty$ *and* $n = 3$. *Then*

$$\left\| \Box_k \int_0^t U(t-s) f(s) ds \right\|_{L^\infty_t L^2_{x,y}} \lesssim \langle k \rangle^{2/p} \|\Box_k f\|_{L^{p'}_x L^1_{y,t}},$$
$$\left\| \Box_k \nabla \int_0^t U(t-s) f(s) ds \right\|_{L^p_x L^\infty_{y,t}} \lesssim \langle k \rangle^{2/p} \|\Box_k f\|_{L^1_x L^2_{y,t}},$$
$$\left\| \Box_k \nabla \int_0^t U(t-s) f(s) ds \right\|_{L^p_x L^\infty_{y,t}} \lesssim \langle k \rangle^{1+2/p} \|\Box_k f\|_{L^{(2+p)/(1+p)}_{x,y,t}},$$

(ii) Let $3 \le p < \infty$ *and* $n = 3$. *Then*

$$\left\| \Box_k \int_0^t U(t-s) f(s) ds \right\|_{L^\infty_t L^2_{x,y}} \lesssim \|\Box_k f\|_{L^{(2+p)/(1+p)}_{x,y,t}},$$
$$\left\| \Box_k \int_0^t U(t-s) f(s) ds \right\|_{L^{2+p}_{x,y,t}} \lesssim \|\Box_k f\|_{L^{(2+p)/(1+p)}_{x,y,t}},$$
$$\left\| \Box_k \nabla \int_0^t U(t-s) f(s) ds \right\|_{L^\infty_x L^2_{y,t}} \lesssim \|\Box_k f\|_{L^{(2+p)/(1+p)}_{x,y,t}},$$
$$\left\| \Box_k \nabla \int_0^t U(t-s) f(s) ds \right\|_{L^{2+p}_{x,y,t}} \lesssim \|\Box_k f\|_{L^1_x L^2_{y,t}},$$

Proof. Accordingly to the proof in the 2 dimension, we see that all the estimates in (i) holds true for any $3 < p \le \infty$. On the other hand, since the decay estimate have the decay rate of $-\frac{4}{3}(\frac{1}{2}-\frac{1}{p})$ from the estimate Lemma 5.13 on the dimensions $n = 3$, the Strichartz estimates hold for any $\gamma \ge \max(2, \frac{2}{\gamma(p)})$ and $\gamma(p) = \frac{4}{3}(\frac{1}{2}-\frac{1}{p})$ (see [97, Propositions 5.1 and 5.2]). This fact means that all of the estimate in (ii) hold for $3 \le p \le \infty$ by the same argument as Proposition 5.10. □

Remark 5.24. *We end this section to remark that all of the estimates in Section 5.3 hold if* ∇ *is replaced by* ∂_x.

5.4 Proof of Theorem 5.1

In this section, we actually prove Theorem 5.1. Before that, we give the following lemma to estimate the nonlinear terms.

Lemma 5.25. *(See [97, Lemma 4.2].) Let* $k, k^{(i)} \in \mathbf{Z}^n$ *for* $i = 1, \cdots, p+1$. *Then we have*

$$\Box_k(\Box_{k^{(1)}}u \cdots \Box_{k^{(p+1)}}u) = 0$$

if $|k - k^{(1)} - \cdots - k^{(p+1)}| \ge C_n$.

We also denote some notations will be used in this proof. We set χ_{Ω_k} as a characteristic function on $\Omega_k = \{k \in \mathbf{Z}^n : |k - k^{(1)} - \cdots - k^{(p+1)}| \le C_n\}$. Moreover, in order to divide nonlinear terms into high and low frequencies, we note

$$\begin{aligned} \Lambda_{high} &:= \Big\{ \big(k^{(1)}, \cdots, k^{(p+1)}\big) \in \mathbf{Z}^{(p+1)n} : |k^{max}| \gg 1 \Big\}, \\ \Lambda_{low} &:= \Big\{ \big(k^{(1)}, \cdots, k^{(p+1)}\big) \in \mathbf{Z}^{(p+1)n} : |k^{max}| \lesssim 1 \Big\}, \end{aligned}$$

where $|k^{max}| := \max_{i=1,\cdots p+1} |k^{(i)}|$.

Now, we begin with the proof. It is clear that we obtain from Propositions 5.7, 5.10, and 5.11

$$\|U(t)u_0\|_{\mathcal{X}} \lesssim \|u_0\|_{M^{1/p}_{2,1}}.$$

Next, we consider the Duhamel terms. First of three norms, we estimate on the Z norm. Using the above notations,

$$\begin{aligned} &\left\| \int_0^t U(t-s)\partial_x(u^{p+1})ds \right\|_Z \\ \le\ & \sum_{k\in\mathbf{Z}^n} \langle k\rangle^{1/p} \sum_{k^{(1)},\cdots,k^{(p+1)}\in\mathbf{Z}^n} \chi_{\Omega_k} \left\| \Box_k\partial_x \int_0^t U(t-s)(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)ds \right\|_{L^\infty_t L^2_{x,y}\cap L^{2+p}_{x,y,t}} \\ =\ & \sum_{k\in\mathbf{Z}^n} \langle k\rangle^{1/p} \sum_{\Lambda_{high}} \chi_{\Omega_k} \left\| \Box_k\partial_x \int_0^t U(t-s)(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)ds \right\|_{L^\infty_t L^2_{x,y}\cap L^{2+p}_{x,y,t}} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{1/p}\sum_{\Lambda_{low}}\chi_{\Omega_k}\left\|\Box_k\partial_x\int_0^t U(t-s)(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)ds\right\|_{L^\infty_t L^2_{x,y}\cap L^{2+p}_{x,y,t}}\\
=:\;& I_{high}+I_{low}.
\end{aligned}
$$

By the smoothing–Strichartz estimates (5.19) and (5.24), we have

$$
\begin{aligned}
I_{high} &= \sum_{k\in\mathbf{Z}^n}\langle k\rangle^{1/p}\sum_{\Lambda_{high}}\chi_{\Omega_k}\left\|\Box_k\partial_x\int_0^t U(t-s)(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)ds\right\|_{L^\infty_t L^2_{x,y}\cap L^{2+p}_{x,y,t}}\\
&\lesssim \sum_{k\in\mathbf{Z}^n}\langle k\rangle^{1/p}\sum_{\Lambda_{high}}\chi_{\Omega_k}\left\|\Box_k(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)\right\|_{L^1_x L^2_{y,t}}\\
&\lesssim \sum_{\Lambda_{high}}\langle k^{max}\rangle^{1/p}\|\Box_{k^{max}}u\|_{L^\infty_x L^2_{y,t}}\prod_{i\neq max}\|\Box_{k^{(i)}}u\|_{L^p_x L^\infty_{y,t}}\\
&\le \|u\|_X\|u\|_Y^p,
\end{aligned}
$$

where we have used the fact that $|k|\lesssim|k^{max}|$ on Ω_k in the third inequality. On the other hand, by the Strichartz estimates in Proposition 5.20 we have

$$
\begin{aligned}
I_{low} &= \sum_{k\in\mathbf{Z}^n}\langle k\rangle^{1/p}\sum_{\Lambda_{low}}\chi_{\Omega_k}\left\|\Box_k\partial_x\int_0^t U(t-s)(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)ds\right\|_{L^\infty_t L^2_{x,y}\cap L^{2+p}_{x,y,t}}\\
&\lesssim \sum_{\Lambda_{low}}\left\|\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u\right\|_{L^{(2+p)/(1+p)}_{x,y,t}}\\
&\lesssim (\|u\|_Z)^{1+p}
\end{aligned}
$$

where we used the fact that $|k|\lesssim 1$ on Λ_{low} and Ω_k.

Next, we estimate the Duhamel terms on the Y norm. We divide the Duhamel term into high and low frequencies and set

$$
\begin{aligned}
&\left\|\int_0^t U(t-s)\partial_x(u^{p+1})ds\right\|_Y\\
\le\;& \sum_{k\in\mathbf{Z}^n}\sum_{k^{(1)},\cdots,k^{(p+1)}\in\mathbf{Z}^n}\chi_{\Omega_k}\left\|\Box_k\partial_x\int_0^t U(t-s)(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)ds\right\|_{L^p_x L^\infty_{y,t}}\\
=\;& \sum_{k\in\mathbf{Z}^n}\sum_{\Lambda_{high}}\chi_{\Omega_k}\left\|\Box_k\partial_x\int_0^t U(t-s)(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)ds\right\|_{L^p_x L^\infty_{y,t}}\\
&+\sum_{k\in\mathbf{Z}^n}\sum_{\Lambda_{low}}\chi_{\Omega_k}\left\|\Box_k\partial_x\int_0^t U(t-s)(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)ds\right\|_{L^p_x L^\infty_{y,t}}\\
=:\;& I\!I_{high}+I\!I_{low}.
\end{aligned}
$$

For the high frequency part $I\!I_{high}$, we have by the smoothing–maximal function estimate (5.21)

$$
I\!I_{high} = \sum_{k\in\mathbf{Z}^n}\sum_{\Lambda_{high}}\chi_{\Omega_k}\left\|\Box_k\partial_x\int_0^t U(t-s)(\Box_{k^{(1)}}u\cdots\Box_{k^{(p+1)}}u)ds\right\|_{L^p_x L^\infty_{y,t}}
$$

$$
\begin{aligned}
&\lesssim \sum_{k\in\mathbf{Z}^n} \langle k\rangle^{1/p} \sum_{\Lambda_{high}} \chi_{\Omega_k} \Big\| \Box_k (\Box_{k^{(1)}} u \cdots \Box_{k^{(p+1)}} u) \Big\|_{L^1_x L^2_{y,t}} \\
&\lesssim \sum_{\Lambda_{high}} \langle k^{max}\rangle^{1/p} \|\Box_{k^{max}} u\|_{L^\infty_x L^2_{y,t}} \prod_{i\neq max} \|\Box_{k^{(i)}} u\|_{L^p_x L^\infty_{y,t}} \\
&\leq \|u\|_X \|u\|_Y^p
\end{aligned}
$$

where we have used the fact that $|k| \lesssim |k^{max}|$ on Ω_k in the third inequality. For the low frequency part II_{low}, we have by the Strichartz–maximal function estimate (5.22)

$$
\begin{aligned}
II_{low} &= \sum_{k\in\mathbf{Z}^n} \langle k\rangle^{1/p} \sum_{\Lambda_{low}} \chi_{\Omega_k} \left\| \Box_k \partial_x \int_0^t U(t-s)(\Box_{k^{(1)}} u \cdots \Box_{k^{(p+1)}} u) ds \right\|_{L^p_x L^\infty_{y,t}} \\
&\lesssim \sum_{\Lambda_{low}} \Big\| \Box_{k^{(1)}} u \cdots \Box_{k^{(p+1)}} u \Big\|_{L^{(2+p)/(1+p)}_{x,y,t}} \\
&\lesssim (\|u\|_Z)^{1+p},
\end{aligned}
$$

where we have used the fact that $|k| \lesssim 1$ on Λ_{low} and Ω_k.

Lastly, we state the Duhamel term on the X norm. We have by the Strichartz–smoothing estimate (5.23)

$$
\begin{aligned}
&\left\| \int_0^t U(t-s)\partial_x (u^{p+1}) ds \right\|_X \\
\leq& \sum_{|k|\gg 1} \langle k\rangle^{1/p} \sum_{k^{(1)},\cdots,k^{(p+1)}\in\mathbf{Z}^n} \chi_{\Omega_k} \left\| \Box_k \partial_x \int_0^t U(t-s)(\Box_{k^{(1)}} u \cdots \Box_{k^{(p+1)}} u) ds \right\|_{L^\infty_x L^2_{y,t}} \\
\lesssim& \sum_{|k|\gg 1} \sum_{k^{(1)},\cdots,k^{(p+1)}\in\mathbf{Z}^n} \chi_{\Omega_k} \langle k^{(1)}\rangle^{1/p} \cdots \langle k^{(p+1)}\rangle^{1/p} \Big\| \Box_{k^{(1)}} u \cdots \Box_{k^{(p+1)}} u \Big\|_{L^{(2+p)/(1+p)}_{x,y,t}} \\
\lesssim& \sum_{k^{(1)},\cdots,k^{(p+1)}\in\mathbf{Z}^n} \langle k^{(1)}\rangle^{1/p} \cdots \langle k^{(p+1)}\rangle^{1/p} \|\Box_{k^{(1)}} u\|_{L^{2+p}_{x,y,t}} \cdots \|\Box_{k^{(p+1)}} u\|_{L^{2+p}_{x,y,t}} \\
\lesssim& (\|u\|_Z)^{1+p}.
\end{aligned}
$$

In the second inequality, we used the fact that $\langle k\rangle \leq C_{n,p} \langle k^{(1)}\rangle \cdots \langle k^{(p+1)}\rangle$ on Ω_k.

Remark 5.26. Ribaud–Vento proved the following estimate on the dimension $n = 3$;

$$
\left\| \nabla^2 \int_0^t U(t-s) f(s) ds \right\|_{L^\infty_x L^2_{y,t}} \lesssim \|f\|_{L^1_x L^2_{y,t}}
$$

(see [76, Proposition 3.6]). Following the same lines as they did, we also have the same estimates on the dimension $n = 2$. Thus we see that the estimate on the X norm;

$$
\left\| \int_0^t U(t-s)\partial_x (u^{p+1}) ds \right\|_X \lesssim \|u\|_X \|u\|_Y^p + (\|u\|_Z)^{1+p}
$$

also holds as in the proof for the other two norms Y and Z. However, we chose the above statement for short proof.

Outline of proof of Theorem 5.1. We set

$$\mathcal{N}[u](t) = U(t)u_0 - i\int_0^t U(t-s)\partial_x(u^{p+1})(s)ds,$$

where $U(t) := \mathcal{F}^{-1}e^{it(\xi^3+\xi\eta^2)}\mathcal{F}$. Applying the above three estimates, we have

$$\|\mathcal{N}[u]\|_{\mathcal{X}} \lesssim \|u_0\|_{M_{2,1}^{1/p}} + \|u\|_{\mathcal{X}}^{p+1}$$

and

$$\|\mathcal{N}[u] - \mathcal{N}[u]\|_{\mathcal{X}} \lesssim \left(\|u\|_{\mathcal{X}}^p + \|u\|_{\mathcal{X}}^p\right)\|u-v\|_{\mathcal{X}}.$$

If we assume that $\|u_0\|_{M_{2,1}^{1/p}} \lesssim \rho/2$ and $\rho > 0$ is small enough, then $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ is a strict contraction mapping. Thus $\mathcal{N}$ has a unique fixed-point, and we have the desired result. □

Using Propositions 5.15, 5.17, and 5.23, since the proof of Theorem 5.2 is quite similar to that of Theorem 5.1, we omit the detail proofs.

6 The inclusion relations between $\alpha-$modulation spaces and L^p–Sobolev spaces or local Hardy spaces

6.1 Introduction and main theorems

6.1.1 Introduction

In this section, we determine optimal inclusion relations between $\alpha-$modulation spaces and L^p–Sobolev spaces, and between α–modulation spaces and the local Hardy spaces. The α–modulation space is introduced by Gröbner in his Ph. D thesis [30] to link Besov and modulation spaces by the parameter $0 \leq \alpha \leq 1$. Besov spaces $B^s_{p,q}$ and modulation spaces $M^s_{p,q}$ are constituted by partitioning the frequency components of functions called dyadic decomposition $\{\phi_j\}_{j\in\mathbf{N}\cup\{0\}}$ and frequency uniform decomposition $\{\sigma_k\}_{k\in\mathbf{Z}^n}$, respectively. Their norms are denoted by

$$\|f\|_{B^s_{p,q}} = \left(\sum_{j\geq 0} 2^{jsq} \left\|\mathcal{F}^{-1}\phi_j\mathcal{F}f\right\|^q_{L^p} \right)^{1/q},$$

where supp $\phi_j \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and supp $\phi_0 \subset \{\xi : |\xi| \leq 2\}$, and

$$\|f\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq} \left\|\mathcal{F}^{-1}\sigma_k\mathcal{F}f\right\|^q_p \right)^{1/q},$$

where the supports of σ_k are contained in $k + [-1,1]^n$, $\langle k\rangle = 1 + |k|$ and $|k| = |k_1| + \cdots + |k_n|$ (see Feichtinger [22]). Now, α–modulation spaces $M^{s,\alpha}_{p,q}$ are built from decomposition η^α_k supported in $\{\xi : |\xi - \langle k\rangle^{\alpha/(1-\alpha)}k| \lesssim \langle k\rangle^{\alpha/(1-\alpha)}\}$ $(k \in \mathbf{Z}^n)$ and the norm is denoted by

$$\|f\|_{M^{s,\alpha}_{p,q}} = \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} \left\|\mathcal{F}^{-1}\eta^\alpha_k\mathcal{F}f\right\|^q_p \right)^{1/q},$$

where $\alpha \in [0,1)$. Modulation spaces are the special case when $\alpha = 0$ and Besov spaces can be regarded as the limit case when $\alpha \to 1$. As one can see from the partitions of unity in this norm, volumes of the supports composing α–modulation spaces are $O(\langle k\rangle^{\alpha n/(1-\alpha)})$. On the other hand, those for Besov spaces and modulation spaces are $O(2^{jn})$ and $O(1)$, respectively. So we can say that the way for α–modulation spaces to decompose and analyze the frequency components is in–between the way for Besov spaces and modulation spaces. Moreover, the factor constituting the regularity in the norm is $\langle k\rangle^{1/(1-\alpha)}(= \langle k\rangle^{\alpha/(1-\alpha)+1})$, which depends on distances from the origin and support of decomposition. That ones for Besov and modulation spaces are 2^j and $\langle k\rangle$, which also arise from distances from the origin and supports of decomposition. So, the definition of the α–modulation spaces norm is quite logical and is similar to the other two function spaces. More precise definitions of these function spaces was given in Section 2.2.4. α–modulation spaces have been studied in recent works. Gröbner [30], Han–Wang [37], and Toft–Wahlberg [90] give us the embedding theorems between α_1–modulation spaces and α_2–modulation spaces. In [37] and [90], the inclusions between Besov and α–modulation spaces are also given. As applications, Borup and Nielsen [6] showed that pseudo–differential operators in Hörmander class are bounded on α–modulation spaces. Moreover, Kobayashi, Sugimoto and Tomita [55, 56] studied L^p boundedness of pseudo–differential operators with symbols in α–modulation spaces, where $1 \le p \le 2$. For the other operators, Feichtinger, Huang, and Wang analyze behavior of trace–reactions on α–modulation spaces in [25]. α–modulation spaces are quite recently applied to the field of partial differential equations. In [66], Misiolek and Yoneda proved locally ill-posedness of the Euler equations in the frame of α–modulation spaces. Furthermore, Han and Wang [38] proved a global well-posedness for the nonlinear Schrödinger equations on α–modulation spaces.

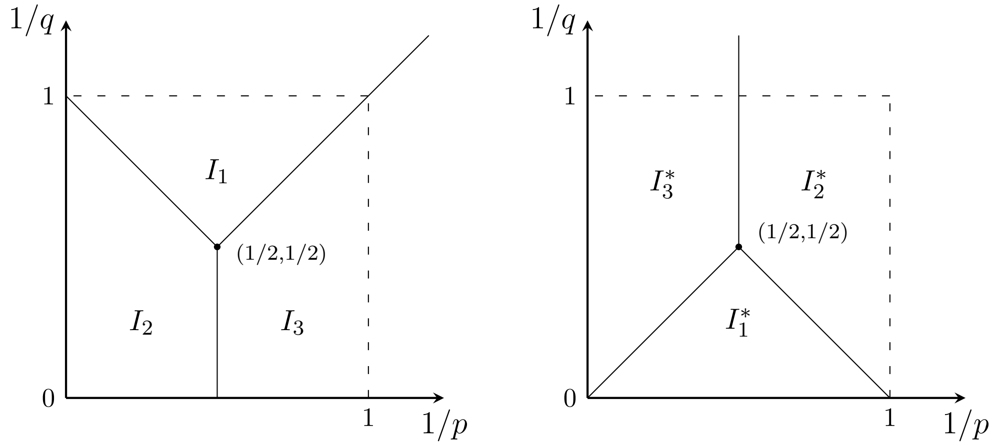
6.1.2 Main theorems

In this subsection, we actually state main results, optimal inclusion relations between α–modulation spaces and L^p–Sobolev spaces, and between α–modulation spaces and the local Hardy spaces. Before we state main theorems, we define the indexes $\nu_1(p,q)$ and $\nu_2(p,q)$ which we will use in the following argument: For $0 < p, q \le \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$,

$$\nu_1(p,q) = \begin{cases} 0 & \text{if } (1/p,1/q) \in I_1^* : \min(1/p,1/p') \ge 1/q, \\ 1/p + 1/q - 1 & \text{if } (1/p,1/q) \in I_2^* : \min(1/q,1/2) \ge 1/p', \\ -1/p + 1/q & \text{if } (1/p,1/q) \in I_3^* : \min(1/q,1/2) \ge 1/p, \end{cases}$$

$$\nu_2(p,q) = \begin{cases} 0 & \text{if } (1/p,1/q) \in I_1 : \max(1/p,1/p') \le 1/q, \\ 1/p + 1/q - 1 & \text{if } (1/p,1/q) \in I_2 : \max(1/q,1/2) \le 1/p', \\ -1/p + 1/q & \text{if } (1/p,1/q) \in I_3 : \max(1/q,1/2) \le 1/p. \end{cases}$$

We note that $\nu_1(p,q) = -\nu_2(p',q')$, and $\nu_1(p,q) \ge 0$, $\nu_2(p,q) \le 0$ for all $0 < p, q \le \infty$.



The index sets of ν_1 and ν_2

Then the following statement is already known:

Theorem 6.1. *(See [37, Theorem 4.2.]) Let* $0 < p, q \leq \infty$, $s_1, s_2 \in \mathbf{R}$, *and* $0 \leq \alpha < 1$. *Then*
(1) $B^{s_1}_{p,q} \subset M^{s_2,\alpha}_{p,q}$ *holds if and only if* $s_1 \geq s_2 + n(1-\alpha)\nu_1(p,q)$ *is satisfied;*
(2) $M^{s_1,\alpha}_{p,q} \subset B^{s_2}_{p,q}$ *holds if and only if* $s_1 \geq s_2 + n(\alpha-1)\nu_2(p,q)$ *is satisfied.*

This theorem coincides with the inclusion relations between Besov and modulation spaces if $\alpha = 0$ (see [30, 86, 89, 96]). Since we have the inclusion relation $L^p_{s+\varepsilon} \hookrightarrow B^s_{p,q} \hookrightarrow L^p_{s-\varepsilon}$ for $\varepsilon > 0$, (see [92, Remark3 in Section 2.3.2]), the following statement is given immediately:

Corollary 6.2. *Let* $1 \leq p, q \leq \infty$, $s_1, s_2 \in \mathbf{R}$, *and* $0 \leq \alpha < 1$. *Then*
(1) $L^p_{s_1} \subset M^{s_2,\alpha}_{p,q}$ *holds if* $s_1 > s_2 + n(1-\alpha)\nu_1(p,q)$ *is satisfied. Conversely, if* $L^p_{s_1} \subset M^{s_2,\alpha}_{p,q}$ *holds, then* $s_1 \geq s_2 + n(1-\alpha)\nu_1(p,q)$ *is satisfied;*
(2) $M^{s_1,\alpha}_{p,q} \subset L^p_{s_2}$ *holds if* $s_1 > s_2 + n(\alpha-1)\nu_2(p,q)$ *is satisfied. Conversely, if* $M^{s_1,\alpha}_{p,q} \subset L^p_{s_2}$ *holds, then* $s_1 \geq s_2 + n(\alpha-1)\nu_2(p,q)$ *is satisfied.*

In Corollary 6.2, there are difference between sufficient conditions and necessary conditions for these inclusion relations holding, that is, the critical case $s_1 = s_2 + n(1-\alpha)\nu_1(p,q)$ and $s_1 = s_2 + n(\alpha-1)\nu_2(p,q)$ are not included in sufficient conditions, though that ones are included in necessary conditions. So, the goal of this paper is to determine whether these critical cases are needed or not. The exact answers are the following theorems:

Theorem 6.3. *Let* $1 \leq p, q \leq \infty$ *and* $s \in \mathbf{R}$. *Then,* $M^{0,\alpha}_{p,q}(\mathbf{R}^n) \hookrightarrow L^p_s(\mathbf{R}^n)$ *holds if and only if one of the following conditions is satisfied.*
(1) $\infty > p \geq q$ *and* $s \leq n(1-\alpha)\nu_2(p,q)$;
(2) $p < q$ *and* $s < n(1-\alpha)\nu_2(p,q)$;
(3) $p = \infty$, $q = 1$, *and* $s \leq n(1-\alpha)\nu_2(\infty, 1)$;
(4) $p = \infty$, $q \neq 1$, *and* $s < n(1-\alpha)\nu_2(\infty, q)$.

Theorem 6.4. *Let* $1 \leq p, q \leq \infty$ *and* $s \in \mathbf{R}$. *Then,* $L^p_s(\mathbf{R}^n) \hookrightarrow M^{0,\alpha}_{p,q}(\mathbf{R}^n)$ *holds if and only if one of the following conditions is satisfied.*
(1) $1 < p \leq q$ *and* $s \geq n(1-\alpha)\nu_1(p,q)$;
(2) $p > q$ *and* $s > n(1-\alpha)\nu_1(p,q)$;
(3) $p = 1$, $q = \infty$, *and* $s \geq n(1-\alpha)\nu_1(1, \infty)$;
(4) $p = 1$, $q \neq \infty$, *and* $s > n(1-\alpha)\nu_1(1, q)$.

In Theorems 6.3 and 6.4, if we set $\alpha = 0$, then these two theorems are exactly the same as inclusion relations between modulation spaces and L^p–Sobolev spaces by Kobayashi and Sugimoto [54]. Moreover, for $0 < p \leq 1$, we have the following inclusion relations between α–modulation spaces and local Hardy spaces:

Theorem 6.5. *Let* $0 < p \leq 1$, $0 < q \leq \infty$, *and* $s \in \mathbf{R}$. *Then,* $M^{s,\alpha}_{p,q}(\mathbf{R}^n) \hookrightarrow h^p(\mathbf{R}^n)$ *holds if and only if either of the following conditions is satisfied.*
(1) $p \geq q$ *and* $s \geq 0$;
(2) $p < q$ *and* $s > n(1-\alpha)(1/p - 1/q)$.

Theorem 6.6. *Let $0 < p \leq 1$, $0 < q \leq \infty$, and $s \in \mathbf{R}$. Then, $h^p(\mathbf{R}^n) \hookrightarrow M^{s,\alpha}_{p,q}(\mathbf{R}^n)$ holds if and only if either of the following conditions is satisfied.*

(1) $p > q$ and $s < -n(1-\alpha)(1/p + 1/q - 1)$;

(2) $p \leq q$ and $s \leq -n(1-\alpha)(1/p + 1/q - 1)$.

In Theorems 6.5 and 6.6, the conditions for regularity $s \in \mathbf{R}$ coincide with $\nu_1(p,q)$ and $\nu_2(p,q)$ for $0 < p \leq 1$ and $0 < q \leq \infty$. If we substitute $\alpha = 0$ into Theorems 6.5 and 6.6, then these theorems are completely the same as with inclusion relations between modulation spaces and local Hardy spaces by Kobayashi, Miyachi, and Tomita [52].

6.2 Key lemmas

In this section, we only display the lemmas to be used in the proofs of the main theorems. Their proofs is given in the next section.

6.2.1 Equivalent (quasi–)norm on α–modulation spaces.

We first give equivalent norms to those on α–modulation spaces with weaker restrictions to decomposition than the α–modulation space norm raised in Subsection 2.2.4. Han and Wang [37] showed the following equivalent norm:

Proposition 6.7. *(See [37, Proposition 6.1.]) Let $0 < p, q \leq \infty$, and $s \in \mathbf{R}$. Let a smooth radial bump function ρ satisfy that*

$$\rho(\xi) = \begin{cases} 1 & on\ |\xi| < 1, \\ 0 & on\ |\xi| \geq 2. \end{cases}$$

Then we have for all $f \in M^{s,\alpha}_{p,q}$

$$\|f\|_{M^{s,\alpha}_{p,q}} \sim \left(\sum_{k \in \mathbf{Z}^n} \langle k \rangle^{sq/(1-\alpha)} \left\| \mathcal{F}^{-1} \rho^\alpha_k \mathcal{F} f \right\|_p^q \right)^{1/q},$$

where

$$\rho^\alpha_k(\xi) := \rho\left(\frac{\xi - \langle k \rangle^{\alpha/(1-\alpha)} k}{C \langle k \rangle^{\alpha/(1-\alpha)}} \right).$$

In the above proposition, the decomposition of frequency spaces ρ^α_k have compact supports. Even if the supports of the decomposition are not compact, however, we are able to have the equivalent norm to that of the α–modulation spaces. The similar statement on modulation spaces is shown in [53, Theorem 2.5], where they have used the specific property of modulation spaces that supports of every decomposition are uniform (see also [31, 28]). On the other hand, those for α–modulation spaces are not so. However, if we set the decomposition as the Schwartz functions, we can ignore overlapping portions of frequency components likewise for modulation spaces:

Lemma 6.8. *Let $0 < p, q \le \infty$ and $s \in \mathbf{R}$. Let a Schwartz function $\Psi \in \mathcal{S}$ be positive and satisfy that $|\Psi(\xi)| \ge c > 0$ on $|\xi| \le 2$. Then we have for all $f \in M^{s,\alpha}_{p,q}$*

$$\|f\|_{M^{s,\alpha}_{p,q}} \sim \left(\sum_{k \in \mathbf{Z}^n} \langle k \rangle^{sq/(1-\alpha)} \left\| \mathcal{F}^{-1} \Psi^\alpha_k \mathcal{F} f \right\|_p^q \right)^{1/q},$$

where

$$\Psi^\alpha_k(\xi) := \Psi \left(\frac{\xi - \langle k \rangle^{\alpha/(1-\alpha)} k}{C \langle k \rangle^{\alpha/(1-\alpha)}} \right).$$

6.2.2 Key lemmas for Theorems 6.5 and 6.6

In this subsection, we show the following four lemmas to prove Theorems 6.5 and 6.6. The first one will be used for the "IF" part of Theorem 6.6 on the critical regularity:

Lemma 6.9. *Let $0 < p \le 1$ and $0 < q \le 2$. Then we have for all h^p–atoms "a"*

$$\|a\|_{M^{s,\alpha}_{p,q}} \le C,$$

where $s = -n(1-\alpha)(1/p + 1/q - 1)$ and the constant C is independent of a and its support.

The remaining three lemmas play roles in the proofs for the "ONLY IF" parts of Theorems 6.5 and 6.6:

Lemma 6.10. *Let $0 < p \le 1$, $0 < q \le \infty$, and $s \in \mathbf{R}$. If $M^{s,\alpha}_{p,q} \hookrightarrow h^p$, then we have*

$$\left(\sum_{k \in \mathbf{Z}^n} |c_k|^p \right)^{1/p} \lesssim \left(\sum_{k \in \mathbf{Z}^n} \langle k \rangle^{sq/(1-\alpha)} |c_k|^q \right)^{1/q},$$

for all finitely supported sequences $\{c_k\}_{k \in \mathbf{Z}^n}$ (that is, $c_k = 0$ except for a finite number of k's).

Lemma 6.11. *Let $0 < p \le 1$, $0 < q \le \infty$, and $s \in \mathbf{R}$. If $h^p \hookrightarrow M^{s,\alpha}_{p,q}$, then we have*

$$\left(\sum_{k \neq 0} |k|^{sq/(1-\alpha) + nq(1/p-1)} \left(\sum_{(1/2)^{1-\alpha} \langle k \rangle \le \langle m \rangle \le 2^{1-\alpha} \langle k \rangle} |c_m|^p \right)^{q/p} \right)^{1/q}$$
$$\lesssim \left(\sum_{k \neq 0} |c_k|^p \right)^{1/p},$$

for all finitely supported sequences $\{c_k\}_{k \in \mathbf{Z}^n \setminus \{0\}}$.

Lemma 6.12. *Let $0 < p \le 1$, $0 < q \le \infty$, and $s \in \mathbf{R}$. If $h^p \hookrightarrow M^{s,\alpha}_{p,q}$, then we have*

$$\left(\sum_{k \neq 0} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \le |m| \le 2\langle k\rangle^{1/(1-\alpha)}} |c_m|^p \right)^{q/p} \right)^{1/q}$$
$$\lesssim \left(\sum_{k \neq 0} |c_k|^p \right)^{1/p},$$

for all finitely supported sequences $\{c_k\}_{k \in \mathbf{Z}^n \setminus \{0\}}$.

The difference between Lemmas 6.11 and 6.12 is only the way to sum $\{c_m\}$ in the each left hand side. The reason why we need these two similar lemmas is written in Remark 6.16.

6.2.3 Key lemmas for Theorems 6.3 and 6.4

Finally, we display the three lemmas for Theorems 6.3 and 6.4. Since we will show the "IF" part by interpolation or dual argument of existing results that pioneers proved, so it suffices to give the following lemmas to show the "ONLY IF" parts:

Lemma 6.13. *Let $1 \le p, q \le \infty$, and $s \in \mathbf{R}$. If $M^{s,\alpha}_{p,q} \hookrightarrow L^p$, then we have*

$$\left(\sum_{k \in \mathbf{Z}^n} |c_k|^p \right)^{1/p} \lesssim \left(\sum_{k \in \mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} |c_k|^q \right)^{1/q},$$

for all finitely supported sequences $\{c_k\}_{k \in \mathbf{Z}^n}$ (that is, $c_k = 0$ except for a finite number of k's).

Lemma 6.14. *Let $1 \le p, q < \infty$, and $s \in \mathbf{R}$. If $L^p \hookrightarrow M^{s,\alpha}_{p,q}$, then we have*

$$\left(\sum_{k \neq 0} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \le |m| \le 2\langle k\rangle^{1/(1-\alpha)}} |c_m|^p \right)^{q/p} \right)^{1/q}$$
$$\lesssim \left(\sum_{k \neq 0} |c_k|^p \right)^{1/p},$$

for all finitely supported sequences $\{c_k\}_{k \in \mathbf{Z}^n \setminus \{0\}}$.

Lemma 6.15. *Let $1 \le q < \infty$ and $s \in \mathbf{R}$. If $L^1 \hookrightarrow M^{s,\alpha}_{1,q}$, then we have*

$$\left(\sum_{k \neq 0} |k|^{sq/(1-\alpha)} \left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \le |m|} |c_m| \right)^{q} \right)^{1/q} \lesssim \sum_{k \neq 0} |c_k|,$$

for all finitely supported sequences $\{c_k\}_{k \in \mathbf{Z}^n \setminus \{0\}}$.

6.3 Proofs of key lemmas

In this section, we show the tools needed to get main theorems. This section is also divided into three subsections. The first one is the proof for the equivalent norm, and the second and third ones are the proof of key lemmas needed to have Theorems 6.5–6.6 and 6.3–6.4, respectively.

6.3.1 Proofs of key lemmas for the equivalent norm

First we prove that it is possible to express the α–modulation space norm by using non–compact supported decomposition:

Proof of Lemma 6.8. We divide the proof into the four steps, that is, the $M^{s,\alpha}_{p,q}$ norm is greater or less than the desired equivalent norm for $0 < p \leq 1$ and $1 < p \leq \infty$. Since the ways to prove them for $0 < p \leq 1$ and $1 < p \leq \infty$ are almost the same, we mainly give the proofs for $0 < p \leq 1$.

Step 1. Let $0 < p \leq 1$. Then, we first prove

$$\|f\|_{M^{s,\alpha}_{p,q}} \gtrsim \left(\sum_{k \in \mathbf{Z}^n} \langle k \rangle^{sq/(1-\alpha)} \left\| \mathcal{F}^{-1} \Psi^\alpha_k \mathcal{F} f \right\|^q_p \right)^{1/q}.$$

Setting an auxiliary function $\kappa \in \mathcal{S}$ satisfying

$$\kappa(\xi) = \begin{cases} 1 & \text{on } |\xi| \leq 1, \\ 0 & \text{on } |\xi| \geq 2, \end{cases} \tag{6.1}$$

and

$$\kappa^\alpha_\ell(\xi) := \kappa\left(\frac{\xi - \langle \ell \rangle^{\alpha/(1-\alpha)} \ell}{C \langle \ell \rangle^{\alpha/(1-\alpha)}} \right),$$

then $\kappa^\alpha_\ell = 1$ on the support of η^α_ℓ. By Proposition 2.1, we have

$$\begin{aligned} \left\| \mathcal{F}^{-1} \Psi^\alpha_k \mathcal{F} f \right\|^p_p &\leq \sum_{\ell \in \mathbf{Z}^n} \left\| \mathcal{F}^{-1} \Psi^\alpha_k \eta^\alpha_\ell \kappa^\alpha_\ell \mathcal{F} f \right\|^p_p \\ &\lesssim \sum_{\ell \in \mathbf{Z}^n} \langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi^\alpha_k \kappa^\alpha_\ell \right\|^p_p \|\Box^\alpha_\ell f\|^p_p. \end{aligned}$$

So, we consider

$$\langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi^\alpha_k \kappa^\alpha_\ell \right\|^p_p = \langle \ell \rangle^{An(1-p)} \left\| \int_{\mathbf{R}^n} e^{ix\cdot\xi} \Psi^\alpha_k(\xi) \kappa^\alpha_\ell(\xi) d\xi \right\|^p_p$$

for the following 6 cases:

(i) $|k| \leq C'$ and $|\ell| \leq |k|/2$;

(ii) $|k| \geq C'$ and $|\ell| \leq |k|/2$;

(iii) $|k| \leq C'$ and $|k|/2 \leq |\ell| \leq 2|k|$;

(iv) $|k| \geq C'$ and $|k|/2 \leq |\ell| \leq 2|k|$;

(iv) $|k| \leq C'$ and $|\ell| \geq 2|k|$;

(vi) $|k| \geq C'$ and $|\ell| \geq 2|k|$,

where $C' = 6C > 0$ and $C > 0$ is a constant written in the definition of the support of η_k^α.

Cases (i) and (iii): Since $|k|, |\ell| \lesssim 1$ for the cases (i) and (iii), we clearly have for sufficiently large $\tilde{N} > 0$

$$\langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi_k^\alpha \kappa_\ell^\alpha \right\|_p^p \lesssim \langle k - \ell \rangle^{-\tilde{N}p}.$$

Cases (ii): ($\Rightarrow \langle \ell \rangle \leq \langle k \rangle$). Changing variables: $\xi \mapsto \langle k \rangle^A \xi$ and $\langle k \rangle^A x \mapsto x$,

$$\begin{aligned} &\langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi_k^\alpha \kappa_\ell^\alpha \right\|_p^p \\ = \ &\left(\frac{\langle \ell \rangle}{\langle k \rangle} \right)^{An(1-p)} \left\| \int_{\mathbf{R}^n} e^{ix\cdot\xi} \Psi\left(\frac{\xi - k}{C} \right) \kappa \left(\frac{\langle k \rangle^A \xi - \langle \ell \rangle^A \ell}{C \langle \ell \rangle^A} \right) d\xi \right\|_p^p . \end{aligned} \tag{6.2}$$

From the setting of the function κ, its support is a subset of the ball:

$$\operatorname{supp} \left[\kappa \left(\frac{\langle k \rangle^A \cdot - \langle \ell \rangle^A}{C \langle \ell \rangle^A} \right) \right] \subset \left\{ \xi \in \mathbf{R}^n : \left| \xi - \frac{\langle \ell \rangle^A}{\langle k \rangle^A} \ell \right| \leq \frac{2C\langle \ell \rangle^A}{\langle k \rangle^A} \right\}.$$

This implies that the domain of integration is included in the set $\{\xi \in \mathbf{R}^n : |\xi| \leq 5|k|/6\}$. Thus, since the function Ψ is a Schwartz function, we have for all $\beta \in \mathbf{Z}_+^n$ and $N \in \mathbf{Z}_+$

$$\begin{aligned} \left| \partial_\xi^\beta \Psi \left(\frac{\xi - k}{C} \right) \right| &\lesssim \langle \xi - k \rangle^{-N} \\ &\lesssim \langle k \rangle^{-N} \\ &\lesssim \langle k \rangle^{-N/2} \langle \ell - k \rangle^{-N/4}. \end{aligned} \tag{6.3}$$

On the other hand, we have

$$\frac{1}{|x|^2} \sum_{j=1}^n x_j \frac{\partial e^{ix\cdot\xi}}{\partial \xi_j} = e^{ix\cdot\xi} \text{ for } |x| \neq 0.$$

So, dividing the L^p–norm into the cases $|x| \leq 1$ and $|x| \geq 1$, and integrating by parts for $|x| \geq 1$ to be bounded in L^p–norm, then it follows that for sufficiently large $\tilde{N} > 0$

$$\langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi_k^\alpha \kappa_\ell^\alpha \right\|_p^p \lesssim \langle k - \ell \rangle^{-\tilde{N}p}.$$

On each time when we use integration by parts, $\langle k \rangle^A$ appears from the function κ in (6.2). However, we can cancel it by $\langle k \rangle^{-N/2}$ in the estimate (6.3).

Cases (iv): Changing variables: $\xi \mapsto \langle \ell \rangle^A \xi$, $\xi - \ell \mapsto \xi$, and $\langle \ell \rangle^A x \mapsto x$,

$$\begin{aligned} &\langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi_k^\alpha \kappa_\ell^\alpha \right\|_p^p \\ = &\left\| \int_{\mathbf{R}^n} e^{ix\cdot\xi} \Psi \left(\frac{\langle \ell \rangle^A \xi + \langle \ell \rangle^A \ell - \langle k \rangle^A k}{C \langle k \rangle^A} \right) \kappa \left(\frac{\xi}{C} \right) d\xi \right\|_p^p . \end{aligned} \tag{6.4}$$

In the same way, we have for all $\beta \in \mathbf{Z}_+^n$ and $N \in \mathbf{Z}_+$

$$\begin{aligned} &\left| \partial_\xi^\beta \Psi \left(\frac{\langle \ell \rangle^A \xi + \langle \ell \rangle^A \ell - \langle k \rangle^A k}{C \langle k \rangle^A} \right) \right| \\ \lesssim &\left(1 + \frac{1}{C} \left| \frac{\langle \ell \rangle^A}{\langle k \rangle^A} \ell - k \right| \right)^{-N} \qquad (6.5) \\ \sim &\left(1 + \frac{1}{C} \left| \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k - \ell \right| \right)^{-N} . \qquad (6.6) \end{aligned}$$

We divide in the two cases when $|k|/2 \le |\ell| \le |k|$ and $|k| < |\ell| \le 2|k|$.

First, we consider $|k|/2 \le |\ell| \le |k|$ ($\Rightarrow \langle k \rangle/2 \le \langle \ell \rangle \le \langle k \rangle$) on 1–dimension. If $k > 0$ and $\ell > 0$, in the estimate (6.6),

$$\ell \le k \le \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k \le 2^A k \quad \text{and} \quad \left| \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k - \ell \right| \ge |k - \ell|.$$

If $k > 0$ and $\ell < 0$,

$$-|\ell| \le k \le \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k \le 2^A k \quad \text{and} \quad \left| \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k - \ell \right| = \left| \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k + |\ell| \right| \ge |k - \ell|.$$

If $k < 0$ and $\ell > 0$,

$$2^A k \le \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k \le k \le -\ell \quad \text{and} \quad \left| \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k - \ell \right| = \left| \frac{\langle k \rangle^A}{\langle \ell \rangle^A} |k| + \ell \right| \ge |k - \ell|.$$

If $k < 0$ and $\ell < 0$,

$$2^A k \le \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k \le k \le \ell \quad \text{and} \quad \left| \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k - \ell \right| \ge |k - \ell|.$$

For n–dimensions, we have

$$\begin{aligned} \left| \frac{\langle k \rangle^A}{\langle \ell \rangle^A} k - \ell \right|^2 &= \sum_{j=1}^n \left(\frac{\langle k \rangle^A}{\langle \ell \rangle^A} k_j - \ell_j \right)^2 \\ &\ge \sum_{j=1}^n (k_j - \ell_j)^2 = |k - \ell|^2 . \end{aligned}$$

Thus, it follows that

$$\left| \partial_\xi^\beta \Psi \left(\frac{\langle \ell \rangle^A \xi + \langle \ell \rangle^A \ell - \langle k \rangle^A k}{C \langle k \rangle^A} \right) \right| \lesssim \langle k - \ell \rangle^{-N}.$$

For the case $|k| < |\ell| \leq 2|k|$, we have the same result by using the similar argument on the estimate (6.5).

Analogously to Case (ii), using integration by parts, then we have for sufficiently large $\tilde{N} > 0$

$$\langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi_k^\alpha \kappa_\ell^\alpha \right\|_p^p \lesssim \langle k - \ell \rangle^{-\tilde{N}p}.$$

Cases (v): ($\Rightarrow \langle k \rangle \leq \langle \ell \rangle$.) The case $2|k| \leq |\ell| \leq 2C'$ is clear, so we only state $2|k| \leq 2C' \leq |\ell|$. By (6.4) and $\Psi \in \mathcal{S}$, we have for all $\beta \in \mathbf{Z}_+^n$ and $N \in \mathbf{Z}_+$

$$\begin{aligned}
\left| \partial_\xi^\beta \Psi \left(\frac{\langle \ell \rangle^A \xi + \langle \ell \rangle^A \ell - \langle k \rangle^A k}{C \langle k \rangle^A} \right) \right| &\lesssim \left(1 + \left| \frac{\langle \ell \rangle^A \xi + \langle \ell \rangle^A \ell - \langle k \rangle^A k}{C \langle k \rangle^A} \right| \right)^{-N} \\
&\leq \left(1 + \frac{1}{C} \left| \frac{\langle \ell \rangle^A}{\langle k \rangle^A} (|\ell| - |\xi|) - |k| \right| \right)^{-N} \\
&\leq \left(1 + \frac{5}{6} |\ell| - |k| \right)^{-N} \\
&\leq \left(1 + \frac{5}{6} |\ell| - \frac{1}{2} |\ell| \right)^{-N} \\
&\lesssim \langle \ell \rangle^{-N} \\
&\lesssim \langle \ell \rangle^{-N/2} \langle k - \ell \rangle^{-N/4} \qquad (6.7)
\end{aligned}$$

In the second inequality, we use $|\xi| \leq 2C = C'/3 \leq |\ell|/6$ and $\langle \ell \rangle / \langle k \rangle \geq 1$. So using the same argument as in the previous cases, we have for sufficiently large $\tilde{N} > 0$

$$\langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi_k^\alpha \kappa_\ell^\alpha \right\|_p^p \lesssim \langle k - \ell \rangle^{-\tilde{N}p}.$$

As mentioned in the last of Case (ii), some $\langle \ell \rangle^A$ appear by integration by parts. But we can cancel them by $\langle \ell \rangle^{-N/2}$ in the inequality (6.24).

Cases (vi): In this case, $|\ell| \geq 2|k| \geq 2C'$. By the same argument as Case (v), we have for sufficiently large $\tilde{N} > 0$

$$\langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi_k^\alpha \kappa_\ell^\alpha \right\|_p^p \lesssim \langle k - \ell \rangle^{-\tilde{N}p}.$$

Taking Cases (i)–(vi) together,

$$\begin{aligned}
&\left(\sum_{k \in \mathbf{Z}^n} \langle k \rangle^{sq/(1-\alpha)} \left\| \mathcal{F}^{-1} \Psi_k^\alpha \mathcal{F} f \right\|_p^q \right)^{1/q} \\
\lesssim &\left(\sum_{k \in \mathbf{Z}^n} \langle k \rangle^{sq/(1-\alpha)} \left(\sum_{\ell \in \mathbf{Z}^n} \langle \ell \rangle^{An(1-p)} \left\| \mathcal{F}^{-1} \Psi_k^\alpha \kappa_\ell^\alpha \right\|_p^p \| \Box_\ell^\alpha f \|_p^p \right)^{q/p} \right)^{1/q} \\
\lesssim &\left(\sum_{k \in \mathbf{Z}^n} \left(\sum_{\ell \in \mathbf{Z}^n} \langle k \rangle^{sp/(1-\alpha)} \langle k - \ell \rangle^{-\tilde{N}p} \| \Box_\ell^\alpha f \|_p^p \right)^{q/p} \right)^{1/q}
\end{aligned}$$

$$\le \left(\sum_{k\in\mathbf{Z}^n}\left(\sum_{\ell\in\mathbf{Z}^n}\langle k-\ell\rangle^{(|s|/(1-\alpha)-\tilde{N})p}\langle \ell\rangle^{sp/(1-\alpha)}\,\|\square_\ell^\alpha f\|_p^p\right)^{q/p}\right)^{1/q}. \tag{6.8}$$

In the last inequality, we use the following inequalities:

$$\begin{cases} \langle k\rangle^x \le \langle k\rangle^x\langle k-\ell\rangle^x & \text{if } x\ge 0,\\ \langle k\rangle^x = (\langle k\rangle^{-|x|}\langle k-\ell\rangle^{-|x|})\langle k-\ell\rangle^{|x|} \le \langle \ell\rangle^{-|x|}\langle k-\ell\rangle^{|x|} & \text{if } x<0.\end{cases}$$

If $0<q/p<1$, by the Fubini–Tonelli theorem and sufficiently large $\tilde{N}>0$, we have

$$\begin{aligned}(6.8) &\le \left(\sum_{k\in\mathbf{Z}^n}\sum_{\ell\in\mathbf{Z}^n}\langle k-\ell\rangle^{(|s|/(1-\alpha)-\tilde{N})q}\langle \ell\rangle^{sq/(1-\alpha)}\,\|\square_\ell^\alpha f\|_p^q\right)^{1/q}\\ &= \left(\sum_{\ell\in\mathbf{Z}^n}\langle \ell\rangle^{sq/(1-\alpha)}\,\|\square_\ell^\alpha f\|_p^q\sum_{k\in\mathbf{Z}^n}\langle k-\ell\rangle^{(|s|/(1-\alpha)-\tilde{N})q}\right)^{1/q}\\ &\lesssim \|f\|_{M_{p,q}^{s,\alpha}}\end{aligned}$$

In the first inequality, we use $(a+b)^p\le a^p+b^p$ for $a,b>0$ and $0<p\le 1$.

On the other hand, if $q/p\ge 1$, by the Young inequality with $p/q=p/q+1-1$ and sufficiently large $\tilde{N}>0$, we have

$$\begin{aligned}(6.8) &\le \left\{\left(\sum_{\ell\in\mathbf{Z}^n}\left(\langle \ell\rangle^{sp/(1-\alpha)}\,\|\square_\ell^\alpha f\|_p^p\right)^{q/p}\right)^{p/q}\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{(|s|/(1-\alpha)-\tilde{N})p}\right)\right\}^{1/p}\\ &\lesssim \|f\|_{M_{p,q}^{s,\alpha}}\end{aligned}$$

Hence, we have

$$\|f\|_{M_{p,q}^{s,\alpha}} \gtrsim \left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q}.$$

<u>**Step 2.**</u> Next, we show that for $0<p\le 1$

$$\|f\|_{M_{p,q}^{s,\alpha}} \lesssim \left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q}.$$

Since we know the another equivalent norm in Proposition 6.7, it suffices to show that

$$\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\rho_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q} \lesssim \left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q},$$

where ρ is a smooth radial bump function and satisfies that

$$\rho(\xi) = \begin{cases} 1 & \text{on } |\xi|<1,\\ 0 & \text{on } |\xi|\ge 2.\end{cases}$$

We set $\kappa_k^{\alpha,2}$ and $\kappa_k^{\alpha,M}$ as

$$\kappa_k^{\alpha,2}(\xi) = \kappa\left(\frac{\xi - \langle k\rangle^{\alpha/(1-\alpha)}k}{2C\langle k\rangle^{\alpha/(1-\alpha)}}\right)$$

and

$$\kappa_k^{\alpha,M}(\xi) = \left(\frac{\xi - \langle k\rangle^{\alpha/(1-\alpha)}k}{MC\langle k\rangle^{\alpha/(1-\alpha)}}\right),$$

with sufficiently large $K > 0$. Here κ is an auxiliary function (6.1) used in Step 1. Then $\kappa_k^{\alpha,2} = \kappa_k^{\alpha,M} = 1$ on supp ρ_k^α. Moreover, since $|\Psi| \geq c > 0$ on $|\xi| \leq 2$, Ψ_k^α never vanish on supp ρ_k^α. Thus,

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\rho_k^\alpha\mathcal{F}f\right\|_p^p \\
=\ &\left\|\mathcal{F}^{-1}\rho_k^\alpha\kappa_k^{\alpha,2}\kappa_k^{\alpha,M}\mathcal{F}f\right\|_p^p \\
=\ &\left\|\mathcal{F}^{-1}\rho_k^\alpha\kappa_k^{\alpha,2}\kappa_k^{\alpha,M}\frac{\Psi_k^\alpha}{\Psi_k^\alpha}\mathcal{F}f\right\|_p^p \\
\lesssim\ &\left(M\langle k\rangle^A\right)^{n(1-p)}\left\|\mathcal{F}^{-1}\kappa_k^{\alpha,M}\Psi_k^\alpha\mathcal{F}f\right\|_p^p\left\|\mathcal{F}^{-1}\kappa_k^{\alpha,2}\frac{\rho_k^\alpha}{\Psi_k^\alpha}\right\|_p^p \\
=\ &M^{n(1-p)}\left\|\mathcal{F}^{-1}\kappa_k^{\alpha,M}\Psi_k^\alpha\mathcal{F}f\right\|_p^p\left\|\mathcal{F}^{-1}\kappa(\cdot/2C)\frac{\rho(\cdot/C)}{\Psi(\cdot/C)}\right\|_p^p \\
\sim\ &M^{n(1-p)}\left\|\mathcal{F}^{-1}\kappa_k^{\alpha,M}\Psi_k^\alpha\mathcal{F}f\right\|_p^p. \qquad (6.9)
\end{aligned}$$

In the third inequality, we use Proposition 2.1 with

$$\begin{aligned}
&\text{supp}\ \left[\kappa_k^{\alpha,M}\Psi_k^\alpha\mathcal{F}f\right] \subset \left\{\xi : |\xi| \leq MC\langle k\rangle^A\right\}, \\
&\text{supp}\ \left[\kappa_k^{\alpha,2}\frac{\rho_k^\alpha}{\Psi_k^\alpha}\right] \subset \left\{\xi : |\xi| \leq 2C\langle k\rangle^A\right\} \subset \left\{\xi : |\xi| \leq MC\langle k\rangle^A\right\}.
\end{aligned}$$

In the fourth equality, we have changed variables as $\xi \mapsto \langle k\rangle^A\xi$, $\xi - k \mapsto \xi$, and $\langle k\rangle^A x \mapsto x$. Now, using

$$\mathcal{F}^{-1}\kappa_k^{\alpha,M}\Psi_k^\alpha\mathcal{F}f = \mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f - \mathcal{F}^{-1}\left[\Psi_k^\alpha - \kappa_k^{\alpha,M}\Psi_k^\alpha\right]\mathcal{F}f,$$

we have

$$(6.9) \leq M^{n(1-p)}\left\{\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^p + \left\|\mathcal{F}^{-1}\left[\Psi_k^\alpha\left(1 - \kappa_k^{\alpha,M}\right)\right]\mathcal{F}f\right\|_p^p\right\}.$$

For the second term, by Proposition 2.1,

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[\Psi_k^\alpha\left(1 - \kappa_k^{\alpha,M}\right)\right]\mathcal{F}f\right\|_p^p \\
\leq\ &\sum_{\ell\in\mathbf{Z}^n}\left\|\mathcal{F}^{-1}\left[\Psi_k^\alpha\left(1 - \kappa_k^{\alpha,M}\right)\eta_\ell^\alpha\kappa_\ell^\alpha\right]\mathcal{F}f\right\|_p^p
\end{aligned}$$

$$\lesssim \sum_{\ell\in\mathbf{Z}^n} \langle \ell\rangle^{An(1-p)} \left\| \mathcal{F}^{-1}\left[\Psi_k^\alpha\left(1-\kappa_k^{\alpha,M}\right)\kappa_\ell^\alpha\right]\right\|_p^p \|\Box_\ell^\alpha f\|_p^p,$$

Here, $\kappa_\ell^\alpha = 1$ on the support of η_ℓ^α. In the following, we consider the factor

$$\langle \ell\rangle^{An(1-p)} \left\| \mathcal{F}^{-1}\left[\Psi_k^\alpha\left(1-\kappa_k^{\alpha,M}\right)\kappa_\ell^\alpha\right]\right\|_p^p. \tag{6.10}$$

Since

$$\begin{aligned}
&\text{supp}\ \left[\Psi_k^\alpha\left(1-\kappa_k^{\alpha,M}\right)\right] \subset \left\{\xi : |\xi-\langle k\rangle^A k| \geq MC\langle k\rangle^A\right\},\\
&\text{supp}\ [\kappa_\ell^\alpha] \subset \left\{\xi : |\xi-\langle \ell\rangle^A \ell| \leq 2C\langle \ell\rangle^A\right\},
\end{aligned}$$

the factor (6.10) never vanishes if the condition

$$\left|\langle k\rangle^A k - \langle \ell\rangle^A \ell\right| \geq MC\langle k\rangle^A - 2C\langle \ell\rangle^A \tag{6.11}$$

is satisfied. We divide $k, \ell \in \mathbf{Z}^n$ into the three cases:

(i) $|\ell| \leq |k|/2$;

(ii) $|k|/2 \leq |\ell| \leq 2|k|$;

(iii) $|\ell| \geq 2|k|$.

Cases (i): In this case, since $\langle \ell\rangle \leq \langle k\rangle$, the condition (6.11) is meaningful for sufficient large $M>0$, that is, the right hand side of (6.11) is positive. Changing variables as $\xi \mapsto \langle \ell\rangle^A\xi$, $\xi-\ell\mapsto\xi$, and $\langle \ell\rangle^A x\mapsto x$ in (6.10), we have

$$\begin{aligned}
(6.10) \quad = \quad & \left\| \int_{\mathbf{R}^n} e^{ix\cdot\xi}\kappa(\xi/C)\Psi\left(\frac{\langle \ell\rangle^A\xi + \langle \ell\rangle^A\ell - \langle k\rangle^A k}{C\langle k\rangle^A}\right)\right.\\
&\left.\times\left(1-\kappa\left(\frac{\langle \ell\rangle^A\xi + \langle \ell\rangle^A\ell - \langle k\rangle^A k}{MC\langle k\rangle^A}\right)\right) d\xi\right\|_p^p.
\end{aligned} \tag{6.12}$$

Since $\Psi \in \mathcal{S}$, we have for all $\beta \in \mathbf{Z}_+^n$ and $N \in \mathbf{Z}_+$, and for sufficiently large $M>0$,

$$\begin{aligned}
\left|\partial^\beta\Psi\left(\frac{\langle \ell\rangle^A\xi + \langle \ell\rangle^A\ell - \langle k\rangle^A k}{C\langle k\rangle^A}\right)\right| &\lesssim \left(1+\left|\frac{\langle \ell\rangle^A\xi + \langle \ell\rangle^A\ell - \langle k\rangle^A k}{C\langle k\rangle^A}\right|\right)^{-N}\\
&\lesssim \left(1+\left|\frac{\langle \ell\rangle^A\ell - \langle k\rangle^A k}{C\langle k\rangle^A}\right|\right)^{-N}\\
&\lesssim \left(1+M-2\frac{\langle \ell\rangle^A}{\langle k\rangle^A}\right)^{-N}\\
&\sim M^{-N}
\end{aligned}$$

by the condition (6.11) and $\langle \ell\rangle/\langle k\rangle \leq 1$. On the other hand, we also have

$$\left|\partial^\beta\Psi\left(\frac{\langle \ell\rangle^A\xi + \langle \ell\rangle^A\ell - \langle k\rangle^A k}{C\langle k\rangle^A}\right)\right| \lesssim \left(1+\left|\frac{\langle \ell\rangle^A\ell - \langle k\rangle^A k}{C\langle k\rangle^A}\right|\right)^{-N}$$

$$
\begin{aligned}
&\lesssim \left(1+|k|-\frac{\langle \ell\rangle^A}{\langle k\rangle^A}|\ell|\right)^{-N}\\
&\lesssim (1+|k|-|\ell|)^{-N}\\
&\lesssim (1+|k|-|k|/2)^{-N}\\
&\sim \langle k\rangle^{-N}\\
&\le \langle k-\ell\rangle^{-N/2}.
\end{aligned}
$$

Integrating by parts to be bounded in L^p–norm as we used in Step 1, then we obtain

$$
(6.12) \lesssim M^{-\tilde{N}p}\langle k-\ell\rangle^{-\tilde{N}p}
$$

for sufficiently large $\tilde{N}>0$. Although $\langle \ell\rangle^A/\langle k\rangle^A$ appears on each time when we integrate by parts, we can see it as a constant since $\langle \ell\rangle^A/\langle k\rangle^A \le 1$.

Case (ii): In this case, since $\langle k\rangle/2 \le \langle \ell\rangle \le 2\langle k\rangle$, the condition (6.11) make sense for sufficient large $M>0$. We separate this case into $|k|/2 \le |\ell| \le |k|$ and $|k| < |\ell| \le 2|k|$. For $|k|/2 \le |\ell| \le |k|$ ($\Rightarrow \langle k\rangle/2 \le \langle \ell\rangle \le \langle k\rangle$), we have for Ψ in (6.12) and for $\beta \in \mathbf{Z}_+^n$ and $N \in \mathbf{Z}_+$

$$
\begin{aligned}
\left|\partial^\beta \Psi\left(\frac{\langle \ell\rangle^A\xi+\langle \ell\rangle^A\ell-\langle k\rangle^A k}{C\langle k\rangle^A}\right)\right| &\lesssim \left(1+\left|\frac{\langle \ell\rangle^A\ell-\langle k\rangle^A k}{C\langle k\rangle^A}\right|\right)^{-N}\\
&\sim \left(1+\left|\frac{\langle k\rangle^A}{\langle \ell\rangle^A}k-\ell\right|\right)^{-N}\\
&\lesssim \langle k-\ell\rangle^{-N/2}
\end{aligned}
$$

by the same argument in Case (iv) of Step 1. On the other hand, applying the condition (6.11),

$$
\left|\partial^\beta \Psi\left(\frac{\langle \ell\rangle^A\xi+\langle \ell\rangle^A\ell-\langle k\rangle^A k}{C\langle k\rangle^A}\right)\right| \lesssim M^{-N}.
$$

Thus we obtain

$$
(6.12) \lesssim M^{-\tilde{N}p}\langle k-\ell\rangle^{-\tilde{N}p}
$$

for sufficiently large $\tilde{N}>0$. For $|k| < |\ell| \le 2|k|$, since the proof is same as above, so we omit it.

Case (iii): We divide this case into the following three cases:

(a) $2|k| \le |\ell| \le \frac{1}{2}\left(\frac{M}{2}\right)^{1-\alpha}$;

(b) $2|k| \le \frac{1}{2}\left(\frac{M}{2}\right)^{1-\alpha} \le |\ell|$;

(c) $\frac{1}{2}\left(\frac{M}{2}\right)^{1-\alpha} \le 2|k| \le |\ell|$.

We first consider (6.12) in the case (a). In this case, it follows that $1 \leq \langle \ell \rangle / \langle k \rangle \leq (M/2)^{1-\alpha}$ and from the condition (6.11)

$$\begin{aligned}\left|\frac{\langle \ell \rangle^A}{\langle k \rangle^A}\ell - k\right| &\geq MC - 2C\frac{\langle \ell \rangle^A}{\langle k \rangle^A} \\ &\geq C\left(M - 2\frac{M^\alpha}{2^\alpha}\right).\end{aligned}$$

If we take sufficiently large $M = M_\alpha > 0$ such that $\frac{1}{2}\left(\frac{M}{2}\right)^{1-\alpha} \geq 6C$ where $C > 1$ is written in the definition of η_k^α, then we have $M/24 \geq (M/2)^\alpha$ and

$$\left|\frac{\langle \ell \rangle^A}{\langle k \rangle^A}\ell - k\right| \geq C\left(M - M/2\right) = MC/2.$$

So, we see that for Ψ in (6.12), and for all $\beta \in \mathbf{Z}_+^n$ and $N \in \mathbf{Z}_+$

$$\begin{aligned}\left|\partial^\beta \Psi\left(\frac{\langle \ell \rangle^A \xi + \langle \ell \rangle^A \ell - \langle k \rangle^A k}{C\langle k \rangle^A}\right)\right| &\lesssim \left(1 + \left|\frac{\langle \ell \rangle^A \xi + \langle \ell \rangle^A \ell - \langle k \rangle^A k}{C\langle k \rangle^A}\right|\right)^{-N} \\ &\leq \left(1 + \frac{1}{C}\left|\frac{\langle \ell \rangle^A \ell - \langle k \rangle^A k}{\langle k \rangle^A}\right| - \frac{|\xi|}{C}\frac{\langle \ell \rangle^A}{\langle k \rangle^A}\right)^{-N} \\ &\sim \left(1 + \frac{M}{2} - \frac{M}{12}\right)^{-N} \\ &\lesssim M^{-N} \\ &\lesssim \langle \ell \rangle^{-N} \\ &\lesssim \langle k - \ell \rangle^{-N}\end{aligned} \tag{6.13}$$

In the third inequality, we use $|\xi| \leq 2C$ and $(\langle \ell \rangle / \langle k \rangle)^A \leq (M/2)^\alpha \leq M/24$. In the fifth inequality, we use $\langle \ell \rangle \leq (M/2)^{1-\alpha} \leq M$. Thus, by integration by parts, we have for sufficiently large $\tilde{N} > 0$

$$(6.12) \lesssim M^{-\tilde{N}p}\langle k - \ell \rangle^{-\tilde{N}p}.$$

Although $\langle \ell \rangle^A$ appears on each time when we use integration by parts, we can cancel them by $\langle \ell \rangle^{-N}$ in the inequality (6.13).

Next, we consider (6.12) in the case (b). In this case, it follows that

$$|\ell| \geq \frac{1}{2}\left(\frac{M}{2}\right)^{1-\alpha} \geq 6C \geq 3|\xi|.$$

So, we have for all $\beta \in \mathbf{Z}_+^n$ and $N \in \mathbf{Z}_+$

$$\begin{aligned}\left|\partial^\beta \Psi\left(\frac{\langle \ell \rangle^A \xi + \langle \ell \rangle^A \ell - \langle k \rangle^A k}{C\langle k \rangle^A}\right)\right| &\lesssim \left(1 + \left|\frac{\langle \ell \rangle^A \xi + \langle \ell \rangle^A \ell - \langle k \rangle^A k}{C\langle k \rangle^A}\right|\right)^{-N} \\ &\leq \left(1 + \frac{1}{C}\frac{\langle \ell \rangle^A}{\langle k \rangle^A}\left(|\ell| - |\xi|\right) - \frac{|k|}{C}\right)^{-N}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(1+\frac{2}{3}|\ell|-\frac{1}{2}|\ell|\right)^{-N} \\
&\lesssim \langle \ell\rangle^{-N} \\
&\lesssim M^{-(1-\alpha)N/2}\langle \ell\rangle^{-N/4}\langle k-\ell\rangle^{-N/8}.
\end{aligned}
$$

Thus, by integration by parts, we have for sufficiently large $\tilde{N} > 0$

$$
(6.12) \lesssim M^{-\tilde{N}p}\langle k-\ell\rangle^{-\tilde{N}p}.
$$

For the case (c), we can apply the same argument as is used the case (b). So we omit it.

Taking all cases together,

$$
\begin{aligned}
&\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\rho_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q} \\
\lesssim\ &\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}M^{n(1/p-1)q}\left\{\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^p\right.\right. \\
&\qquad\qquad\left.\left.+\left\|\mathcal{F}^{-1}\left[\Psi_k^\alpha\left(1-\kappa_k^{\alpha,M}\right)\right]\mathcal{F}f\right\|_p^p\right\}^{q/p}\right)^{1/q} \\
\lesssim\ &\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}M^{n(1/p-1)q}\left\{\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^p\right.\right. \\
&\qquad\left.\left.+\sum_{\ell\in\mathbf{Z}^n}\langle \ell\rangle^{An(1-p)}\left\|\mathcal{F}^{-1}\left[\Psi_k^\alpha\left(1-\kappa_k^{\alpha,M}\right)\kappa_\ell^\alpha\right]\right\|_p^p\|\square_\ell^\alpha f\|_p^p\right\}^{q/p}\right)^{1/q} \\
\lesssim\ & M^{n(1/p-1)}\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q} \\
+\ & M^{n(1/p-1)}\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left(\sum_{\ell\in\mathbf{Z}^n}M^{-\tilde{N}p}\langle k-\ell\rangle^{-\tilde{N}p}\|\square_\ell^\alpha f\|_p^p\right)^{q/p}\right)^{1/q} \\
\lesssim\ & M^{n(1/p-1)}\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q} \\
+\ & M^{n(1/p-1)-\tilde{N}}\left(\sum_{k\in\mathbf{Z}^n}\left(\sum_{\ell\in\mathbf{Z}^n}\langle \ell\rangle^{sp/(1-\alpha)}\langle k-\ell\rangle^{|s|p/(1-\alpha)-\tilde{N}p}\|\square_\ell^\alpha f\|_p^p\right)^{q/p}\right)^{1/q}.
\end{aligned} \tag{6.14}
$$

Using the Young inequality if $q/p > 1$ and the Fubini–Tonelli theorem if $0 < q/p \le 1$, the second term in (6.14) is estimated as done in the proof of Step 1. Thus, we have

$$
\|f\|_{M_{p,q}^{s,\alpha}}
$$

$$\sim \left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\rho_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q}$$
$$\lesssim M^{n(1/p-1)}\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q}+M^{n(1/p-1)-\tilde{N}}\|f\|_{M_{p,q}^{s,\alpha}}$$

Since we take $M>0$ and $\tilde{N}>0$ as a sufficiently large number, we see that

$$\|f\|_{M_{p,q}^{s,\alpha}} \lesssim \left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q}.$$

Hence, combining Step 1 and Step 2, we obtain the desired results for $0<p\leq 1$.

Step 3. Thirdly, we prove that for $1<p\leq\infty$

$$\|f\|_{M_{p,q}^{s,\alpha}} \gtrsim \left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q}.$$

Under the same settings of κ as Step 1, it follows that from the Young inequality

$$\begin{aligned}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p &\leq \sum_{\ell\in\mathbf{Z}^n}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\eta_\ell^\alpha\kappa_\ell^\alpha\mathcal{F}f\right\|_p\\ &\lesssim \sum_{\ell\in\mathbf{Z}^n}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\kappa_\ell^\alpha\right\|_1\|\Box_k^\alpha f\|_p.\end{aligned}$$

Then, applying Step 1 for $p=1$ to the first factor in the above inequality, we have for sufficiently large $\tilde{N}>0$

$$\begin{aligned}
&\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q}\\
&\leq \left(\sum_{k\in\mathbf{Z}^n}\left\{\sum_{\ell\in\mathbf{Z}^n}\langle k-\ell\rangle^{|s|/(1-\alpha)-\tilde{N}}\langle\ell\rangle^{s/(1-\alpha)}\|\Box_\ell^\alpha f\|_p\right\}^q\right)^{1/q}\\
&\leq \begin{cases}\left(\sum_{k\in\mathbf{Z}^n}\sum_{\ell\in\mathbf{Z}^n}\langle k-\ell\rangle^{|s|q/(1-\alpha)-\tilde{N}q}\langle\ell\rangle^{sq/(1-\alpha)}\|\Box_\ell^\alpha f\|_p^q\right)^{1/q} & \text{if } 0<q\leq 1,\\ \left(\sum_{k\in\mathbf{Z}^n}\left\{\sum_{\ell\in\mathbf{Z}^n}\langle k-\ell\rangle^{|s|/(1-\alpha)-\tilde{N}}\langle\ell\rangle^{s/(1-\alpha)}\|\Box_\ell^\alpha f\|_p\right\}^q\right)^{1/q} & \text{if } 1<q\leq\infty\end{cases}\\
&\leq \begin{cases}\left(\sum_{\ell\in\mathbf{Z}^n}\langle\ell\rangle^{sq/(1-\alpha)}\|\Box_\ell^\alpha f\|_p^q\sum_{k\in\mathbf{Z}^n}\langle k-\ell\rangle^{|s|q/(1-\alpha)-\tilde{N}q}\right)^{1/q} & \text{if } 0<q\leq 1,\\ \left(\sum_{\ell\in\mathbf{Z}^n}\langle\ell\rangle^{sq/(1-\alpha)}\|\Box_k^\alpha f\|_p^q\right)^{1/q}\times\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{|s|/(1-\alpha)-\tilde{N}}\right) & \text{if } 1<q\leq\infty\end{cases}
\end{aligned}$$

$$\leq \|f\|_{M^{s,\alpha}_{p,q}}.$$

Here, we used the Fubini–Tonelli theorem for $0 < q \leq 1$ and the Young inequality for $1 < q \leq \infty$ in the third and fourth inequality,respectively.

<u>**Step 4.**</u> Finally, we prove that for $1 < p \leq \infty$

$$\|f\|_{M^{s,\alpha}_{p,q}} \lesssim \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} \left\|\mathcal{F}^{-1}\Psi^\alpha_k \mathcal{F}f\right\|_p^q \right)^{1/q}.$$

Using the similar argument to Steps 2 and 3, then we have the above inequality for $1 < p \leq \infty$. So we omit the detail. □

6.3.2 Proofs of key lemmas for Theorems 6.5 and 6.6

Next, we prepare Lemmas 6.9–6.12 for the inclusion relations between α–modulation and local Hardy spaces. The first lemma is to give the "IF" part of Theorem 6.6 for $0 < p \leq 1$ and $0 < q \leq 2$:

Proof of Lemma 6.9. Let $\Psi \in \mathcal{S}$ satisfy that supp $[\mathcal{F}^{-1}\Psi] \subset \{\xi : |\xi| \leq 1\}$ and $|\Psi| \geq c > 0$ on $|\xi| \leq 2$ (the existence of such a function is proven in [52, Lemma 4.3]). We set the Friedrichs mollifier as $\{\varrho_\varepsilon\}_{\varepsilon>0}$, that is, $\varrho_\varepsilon := \varepsilon^{-n}\varrho(x/\varepsilon)$ for $\varrho \in \mathcal{S}$ with $\int_{\mathbf{R}^n} \varrho(x)dx = 1$ and supp $\varrho \subset [-1,1]^n$. Then, $(\varrho_\varepsilon * a)/(2^n\|\varrho\|_{L^1})$ is an h^p–atom of type I (resp. type II) if a is an h^p–atom of type I (resp. type II) and $\varepsilon > 0$ is sufficiently small (see [52]). In the following statement, we denote $a_\varepsilon := (\varrho_\varepsilon * a)/(2^n\|\varrho\|_{L^1})$. So, by the Fatou lemma, we have

$$\begin{aligned}
\|a\|_{M^{s,\alpha}_{p,q}} &= \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} \left\|\mathcal{F}^{-1}\eta^\alpha_k \mathcal{F}a\right\|_p^q \right)^{1/q} \\
&\lesssim \liminf_{\varepsilon\to 0} \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} \left\|\mathcal{F}^{-1}\eta^\alpha_k \mathcal{F}a_\varepsilon\right\|_p^q \right)^{1/q} \\
&\lesssim \liminf_{\varepsilon\to 0} \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} \left\|\mathcal{F}^{-1}\Psi^\alpha_k \mathcal{F}a_\varepsilon\right\|_p^q \right)^{1/q},
\end{aligned}$$

where

$$\Psi^\alpha_k(\xi) := \Psi\left(\frac{\xi - \langle k\rangle^{\alpha/(1-\alpha)}k}{C\langle k\rangle^{\alpha/(1-\alpha)}} \right).$$

Here, we should note that

$$\begin{aligned}
&\varrho_\varepsilon * a \in \mathcal{S} \subset M^{s,\alpha}_{p,q}, \\
&\lim_{\varepsilon\to 0} \varrho_\varepsilon * a = a \text{ in } \mathcal{S}'.
\end{aligned}$$

In the following, we consider

$$\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}a_\varepsilon\right\|_p^q\right)^{1/q}. \tag{6.15}$$

We assume that a is an h^p–atom with the cube $Q=[-r,r]^n$, without loss of generality, since $\|\cdot\|_{L^p}$ is invariant by translation. In the following argument, we divide this proof into two steps for type I and type II atoms.

Step 1. First, we assume that a is an h^p–atom of type I with $Q=[-r,r]^n$ and $r<1/2$. We consider (6.15) in the two cases:

(i) $\langle k\rangle<|Q|^{-(1-\alpha)/n}$;

(ii) $\langle k\rangle\geq|Q|^{-(1-\alpha)/n}$.

Cases (i): We write $N=[n(1/p-1)]$, where $[\cdot]$ is the Gauss symbol. By the Fubini–Tonelli theorem, changes of variables and the Taylor expansion, we have

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}a_\varepsilon\right\|_p^p\\
=&\left\|\int_{\mathbf{R}^n_\xi}e^{ix\cdot\xi}\,\Psi\left(\frac{\xi-\langle k\rangle^Ak}{C\langle k\rangle^A}\right)\int_{\mathbf{R}^n_y}e^{-i\xi\cdot y}\,a_\varepsilon(y)dyd\xi\right\|_p^p\\
=&\ \langle k\rangle^{Anp}C^n\left\|\int_{\mathbf{R}^n_y}a_\varepsilon(y)\exp\left[i\langle k\rangle^Ak\cdot(x-y)\right]\right.\\
&\qquad\left.\times\int_{\mathbf{R}^n_\xi}\exp\left[iC\langle k\rangle^A\xi\cdot(x-y)\right]\ \Psi(\xi)\ d\xi dy\right\|_p^p\\
=&\ \langle k\rangle^{Anp}C^n\left\|\int_{\mathbf{R}^n_y}a_\varepsilon(y)\exp\left[i\langle k\rangle^Ak\cdot(x-y)\right]\check{\Psi}\left(C\langle k\rangle^A(x-y)\right)dy\right\|_p^p\\
=&\ \langle k\rangle^{Anp}C^n\left\|\int_{\mathbf{R}^n_y}a_\varepsilon(y)\left\{\exp\left[i\langle k\rangle^Ak\cdot(x-y)\right]\check{\Psi}\left(C\langle k\rangle^A(x-y)\right)\right.\right.\\
&\qquad\left.\left.-\sum_{|\beta|\leq N}\frac{\left(-C\langle k\rangle^Ay\right)^\beta}{\beta!}\left[\partial^\beta\left(e^{i\frac{k}{C}\cdot\cdot}\,\check{\Psi}(\cdot)\right)\right]\left(C\langle k\rangle^Ax\right)\right\}dy\right\|_p^p\\
=&\ \langle k\rangle^{Anp}C^n\left\|\int_{\mathbf{R}^n_y}a_\varepsilon(y)(N+1)\sum_{|\beta|=N+1}\frac{\left(-C\langle k\rangle^Ay\right)^\beta}{\beta!}\int_0^1(1-\theta)^{N+1}\right.\\
&\qquad\left.\times\left[\partial^\beta\left(e^{i\frac{k}{C}\cdot\cdot}\,\check{\Psi}(\cdot)\right)\right]\left(C\langle k\rangle^A(x-\theta y)\right)d\theta dy\right\|_p^p.
\end{aligned}\tag{6.16}$$

In the fourth equality, we use the property which a_ε is a h^p–atom of type I. We should note the support of the above functions in (6.16):

$$y\ \in\ \operatorname{supp}\,a_\varepsilon(\cdot)\subset[-(r+\varepsilon),r+\varepsilon]^n$$

$$\begin{aligned} x-y &\in \text{supp } \exp\left[i\langle k\rangle^A k\cdot\cdot\right]\check{\Psi}\left(C\langle k\rangle^A\cdot\right) \\ &\subset \left\{\xi : |\xi| \le \frac{1}{C\langle k\rangle^A}\right\} \subset \left[-\frac{1}{\langle k\rangle^A}, \frac{1}{\langle k\rangle^A}\right]^n. \end{aligned}$$

Since we have by the assumption $\langle k\rangle < |Q|^{-(1-\alpha)/n} \sim r^{-(1-\alpha)}$ in this case

$$r < \langle k\rangle^{-1/(1-\alpha)} \le \langle k\rangle^{-A},$$

it follows that

$$\begin{aligned} x &\in \left[-\frac{1}{\langle k\rangle^A}, \frac{1}{\langle k\rangle^A}\right]^n + \left[-\left(\frac{1}{\langle k\rangle^A}+\varepsilon\right), \left(\frac{1}{\langle k\rangle^A}+\varepsilon\right)\right]^n \\ &\subset \left[-\left(\frac{2}{\langle k\rangle^A}+\varepsilon\right), \left(\frac{2}{\langle k\rangle^A}+\varepsilon\right)\right]^n. \end{aligned}$$

Hence,

$$\begin{aligned} &(6.16) \\ \lesssim\; & \langle k\rangle^{Anp}(r+\varepsilon)^{-np/p}\langle k\rangle^{A(N+1)p} \\ & \times\left\| \int_y |y|^{N+1}\int_0^1 \sum_{|\beta|=N+1}\left|\left[\partial^\beta\left(e^{i\frac{k}{C}\cdot\cdot}\check{\Psi}(\cdot)\right)\right]\left(C\langle k\rangle^A(x-\theta y)\right)\right| d\theta dy\right\|_p^p \\ \lesssim\; & \langle k\rangle^{Anp}(r+\varepsilon)^{-np/p}\langle k\rangle^{A(N+1)p}|Q|^{(N+1)p/n}\langle k\rangle^{(N+1)p} \\ & \qquad \times \left\|\int_y\int_0^1 \sum_{|\gamma|\le N+1}\left|\left[\partial^\gamma\check{\Psi}(\cdot)\right]\left(C\langle k\rangle^A(x-\theta y)\right)\right| d\theta dy\right\|_p^p \\ \lesssim\; & \langle k\rangle^{Anp}(r+\varepsilon)^{-np/p}\langle k\rangle^{A(N+1)p}|Q|^{(N+1)p/n}\langle k\rangle^{(N+1)p}\left\|\int_y 1dy\right\|_p^p \\ \lesssim\; & \langle k\rangle^{Anp}(r+\varepsilon)^{-np/p}\langle k\rangle^{A(N+1)p}|Q|^{(N+1)p/n}\langle k\rangle^{(N+1)p}(r+\varepsilon)^{np}\left(2\langle k\rangle^{-A}+\varepsilon\right)^n \\ \underset{\varepsilon\to 0}{\longrightarrow}\; & C'\langle k\rangle^{Ap\{(N+1)-n(1/p-1)\}}\langle k\rangle^{(N+1)p}|Q|^{p\{(N+1)/n-(1/p-1)\}}. \end{aligned}$$

Summing (6.16) on $\langle k\rangle < |Q|^{-(1-\alpha)/n}$,

$$\begin{aligned} &\left(\liminf_{\varepsilon\to 0}\sum_{\langle k\rangle<|Q|^{-(1-\alpha)/n}}\langle k\rangle^{-n(1/p-1)q-n}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}a_\varepsilon\right\|_p^q\right)^{1/q} \\ \lesssim\; & |Q|^{\{(N+1)/n-(1/p-1)\}} \\ & \times\left(\sum_{\langle k\rangle<|Q|^{-(1-\alpha)/n}}\langle k\rangle^{-n(1/p-1)q-n}\langle k\rangle^{Aq\{(N+1)-n(1/p-1)\}}\langle k\rangle^{(N+1)q}\right)^{1/q} \\ =\; & |Q|^{\{(N+1)/n-(1/p-1)\}}\left(\sum_{\langle k\rangle<|Q|^{-(1-\alpha)/n}}\langle k\rangle^{-n}\langle k\rangle^{\{(N+1)-n(1/p-1)\}q/(1-\alpha)}\right)^{1/q} \\ \lesssim\; & |Q|^{\{(N+1)/n-(1/p-1)\}}\times|Q|^{-\{(N+1)-n(1/p-1)\}/n} \end{aligned}$$

$$= \quad 1,$$

where we used the fact $\{(N+1)-n(1/p-1)\}\,q/(1-\alpha)>0$ in the fourth inequality.

Cases (ii): As we did in Case (i), we have

$$\begin{aligned} & \left\|\mathcal{F}^{-1}\Psi_k^{\alpha}\mathcal{F}a_{\varepsilon}\right\|_p \\ = \; & \langle k\rangle^{An}C^n \left\|\exp\left[i\langle k\rangle^{A}k\cdot\cdot\right]\check{\Psi}\left(C\langle k\rangle^{A}\cdot\right) * a_{\varepsilon}(\cdot)\right\|_p \end{aligned} \tag{6.17}$$

and

$$\begin{aligned} y &\in \operatorname{supp}\, a_{\varepsilon}(\cdot)\subset[-(r+\varepsilon),r+\varepsilon]^n \\ x-y &\in \operatorname{supp}\;\exp\left[i\langle k\rangle^{A}k\cdot\cdot\right]\check{\Psi}\left(C\langle k\rangle^{A}\cdot\right) \\ &\subset \left\{\xi : |\xi|\le\frac{1}{C\langle k\rangle^{A}}\right\} \\ &\subset \left[-\frac{1}{C\langle k\rangle^{A}},\frac{1}{C\langle k\rangle^{A}}\right]^n. \end{aligned}$$

Since we assume that $\langle k\rangle \gtrsim r^{-(1-\alpha)}$, we have $\langle k\rangle^{-A}\lesssim r^{\alpha}$ and

$$\begin{aligned} x &\in [-r^{\alpha},r^{\alpha}]^n+[-(r+\varepsilon),r+\varepsilon]^n \\ &\subset [-(2r^{\alpha}+\varepsilon),2r^{\alpha}+\varepsilon]^n, \end{aligned}$$

where we used the fact that $r<r^{\alpha}$ for $0<r<1$ and $0\le\alpha<1$. Thus, using the Hölder inequality, we have

$$\begin{aligned} (6.17) \;&\lesssim\; C^n\langle k\rangle^{An}(2r^{\alpha}+\varepsilon)^{n(1/p-1/2)} \\ &\qquad\qquad \times\left\|\exp\left[i\langle k\rangle^{A}k\cdot\cdot\right]\check{\Psi}\left(C\langle k\rangle^{A}\cdot\right) * a_{\varepsilon}(\cdot)\right\|_2 \\ &= \; (2r^{\alpha}+\varepsilon)^{n(1/p-1/2)}\left\|\mathcal{F}^{-1}\Psi_k^{\alpha}\mathcal{F}a_{\varepsilon}\right\|_2 \end{aligned} \tag{6.18}$$

Summing (6.17) on $\langle k\rangle\ge|Q|^{-(1-\alpha)/n}$,

$$\begin{aligned} & \left(\sum_{\langle k\rangle\ge|Q|^{-(1-\alpha)/n}}\langle k\rangle^{-n(1/p+1/q-1)q}\left\|\mathcal{F}^{-1}\Psi_k^{\alpha}\mathcal{F}a_{\varepsilon}\right\|_p^q\right)^{1/q} \\ \le\; & \left(\sum_{\langle k\rangle\ge|Q|^{-(1-\alpha)/n}}\langle k\rangle^{-n(1/p+1/q-1)qu}\right)^{1/u} \\ & \qquad \times\left(\sum_{\langle k\rangle\ge|Q|^{-(1-\alpha)/n}}\left\|\mathcal{F}^{-1}\Psi_k^{\alpha}\mathcal{F}a_{\varepsilon}\right\|_p^2\right)^{q/2} \end{aligned} \tag{6.19}$$

In this inequality, we used Hölder inequality with $1/u+q/2=1$, i.e., $1<u\le\infty$ from $0<q\le2$. This implies that $n(1/p+1/q-1)qu>n$. So, by (6.18),

(6.19)

$$
\begin{aligned}
&\lesssim \left(\left\{|Q|^{-(1-\alpha)/n}\right\}^{-n(1/p+1/q-1)qu+n}\right)^{1/u} \\
&\qquad \times \left(\sum_{\langle k\rangle \geq |Q|^{-(1-\alpha)/n}} \left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} a_\varepsilon\right\|_2^2\right)^{q/2} \times (2r^\alpha+\varepsilon)^{nq(1/p-1/2)} \\
&\leq |Q|^{(1-\alpha)\{(1/p+1/q-1)q-1/u\}} \times \|a_\varepsilon\|_2^q \times (2r^\alpha+\varepsilon)^{nq(1/p-1/2)} \\
&\leq |Q|^{(1-\alpha)\{(1/p+1/q-1)q-1+q/2\}} \times |Q|^{(1/2-1/p)q} \times (2r^\alpha+\varepsilon)^{nq(1/p-1/2)} \\
&\xrightarrow[\varepsilon\to 0]{} C'|Q|^{(1-\alpha)(1/p-1/2)q} \times |Q|^{-(1/p-1/2)q} \times |Q|^{\alpha q(1/p-1/2)} \\
&\sim 1.
\end{aligned}
$$

Hence, if a is an h^p–atom of type I, then we have

$$\|a\|_{M^{s,\alpha}_{p,q}} < \infty$$

for $s = -n(1-\alpha)(1/p+1/q-1)$, $0 < p \leq 1$, and $0 < q \leq 2$.

Step 2. Next, we assume that a is an h^p–atom of type II with $Q = [-r, r]$ for $r \geq 1/2$. In this case, we have

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} a_\varepsilon\right\|_p \\
&= \langle k\rangle^{An} C^n \left\|\exp\left[i\langle k\rangle^A k\cdot\cdot\right]\check{\Psi}\left(C\langle k\rangle^A\cdot\right) * a_\varepsilon(\cdot)\right\|_p \qquad (6.20)
\end{aligned}
$$

and

$$
\begin{aligned}
y &\in \text{supp } a_\varepsilon(\cdot) \subset [-(r+\varepsilon), r+\varepsilon]^n \\
x &\in \text{supp } \exp\left[i\langle k\rangle^A k\cdot\cdot\right]\check{\Psi}\left(C\langle k\rangle^A\cdot\right) + \text{supp } a_\varepsilon(\cdot) \\
&\subset \left[-\frac{1}{C\langle k\rangle^A}, \frac{1}{C\langle k\rangle^A}\right]^n + [-(r+\varepsilon), r+\varepsilon]^n \\
&\subset [-(2r+\varepsilon), 2r+\varepsilon]^n.
\end{aligned}
$$

Thus, using the Hölder inequality, we have

$$
\begin{aligned}
(6.20) &\lesssim C^n\langle k\rangle^{An}(2r+\varepsilon)^{n(1/p-1/2)} \\
&\qquad \times \left\|\exp\left[i\langle k\rangle^A k\cdot\cdot\right]\check{\Psi}\left(C\langle k\rangle^A\cdot\right) * a_\varepsilon(\cdot)\right\|_2 \\
&= (2r+\varepsilon)^{n(1/p-1/2)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} a_\varepsilon\right\|_2. \qquad (6.21)
\end{aligned}
$$

Thus, by (6.21),

$$
\begin{aligned}
&\left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{-n(1/p+1/q-1)q}\left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} a_\varepsilon\right\|_p^q\right)^{1/q} \\
&\leq \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{-n(1/p+1/q-1)qu}\right)^{1/u} \times \left(\sum_{k\in\mathbf{Z}^n} \left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} a_\varepsilon\right\|_p^2\right)^{q/2} \\
&\lesssim \left(\sum_{k\in\mathbf{Z}^n} \left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} a_\varepsilon\right\|_2^2\right)^{q/2} \times (2r+\varepsilon)^{n(1/p-1/2)q}
\end{aligned}
$$

$$
\begin{aligned}
&\le \quad \|a_\varepsilon\|_2^q \times (2r+\varepsilon)^{n(1/p-1/2)q} \\
&\lesssim \quad |Q|^{(1/2-1/p)q} \times (2r+\varepsilon)^{nq(1/p-1/2)} \\
&\underset{\varepsilon\to 0}{\longrightarrow} \quad C'|Q|^{-(1/p-1/2)q} \times |Q|^{(1/p-1/2)q} \\
&\sim \quad 1,
\end{aligned}
$$

where u is the same index as used in the proof of h^p–atom type I. Hence, if a is an h^p–atom of type II, then we have

$$\|a\|_{M^{s,\alpha}_{p,q}} < \infty$$

for $s = -n(1-\alpha)(1/p+1/q-1)$, $0 < p \le 1$, and $0 < q \le 2$. □

Next we prove the lemma needed to show the "ONLY IF" part of Theorem 6.5.

Proof of Lemma 6.10. We set $\varphi \in \mathcal{S} \setminus \{0\}$ satisfying that $\operatorname{supp} \varphi \subset \{\xi : |\xi| \le 1/2\}$ and

$$f(x) = \sum_{\ell\in\mathbf{Z}^n} c_\ell \langle \ell\rangle^{An/p} \exp[i\langle\ell\rangle^A \ell\cdot x]\, \varphi\left(C\langle\ell\rangle^A (x-\ell)\right)$$

for finitely supported sequences $\{c_\ell\}_{\ell\in\mathbf{Z}^n}$. Then, by changes of variables: $x-\ell \mapsto x$ and $C\langle\ell\rangle^A x \mapsto x$, we have

$$
\begin{aligned}
\widehat{f}(\xi) &= \sum_{\ell\in\mathbf{Z}^n} c_\ell\langle\ell\rangle^{An/p} \int_{\mathbf{R}^n} \exp[-i\xi\cdot x + i\langle\ell\rangle^A\ell\cdot x]\,\varphi\left(C\langle\ell\rangle^A(x-\ell)\right) dx \\
&= C^{-n}\sum_{\ell\in\mathbf{Z}^n} c_\ell\langle\ell\rangle^{An(1/p-1)} \exp\left[-i\xi\cdot\ell + i\langle\ell\rangle^A|\ell|^2\right]\, \widehat{\varphi}\left(\frac{\xi-\langle\ell\rangle^A\ell}{C\langle\ell\rangle^A}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\rho_k^\alpha\mathcal{F}f\right\|_p^p \\
&\lesssim \sum_{\ell\in\mathbf{Z}^n} |c_\ell|^p\langle\ell\rangle^{An(1-p)} \left\|\int_{\mathbf{R}^n} \exp\left[ix\cdot\xi - i\xi\cdot\ell\right]\rho_k^\alpha(\xi)\,\widehat{\varphi}\left(\frac{\xi-\langle\ell\rangle^A\ell}{C\langle\ell\rangle^A}\right) d\xi\right\|_p^p \\
&= \sum_{\ell\in\mathbf{Z}^n} |c_\ell|^p\langle\ell\rangle^{An(1-p)} \left\|\int_{\mathbf{R}^n} e^{ix\cdot\xi}\rho\left(\frac{\xi-\langle k\rangle^A k}{C\langle k\rangle^A}\right)\,\widehat{\varphi}\left(\frac{\xi-\langle\ell\rangle^A\ell}{C\langle\ell\rangle^A}\right) d\xi\right\|_p^p \\
&= \sum_{\ell\in\mathbf{Z}^n} |c_\ell|^p\left(\frac{\langle\ell\rangle}{\langle k\rangle}\right)^{An(1-p)} \\
&\qquad\qquad \times \left\|\int_{\mathbf{R}^n} e^{ix\cdot\xi}\rho\left(\frac{\xi}{C}\right)\,\widehat{\varphi}\left(\frac{\langle k\rangle^A\xi + \langle k\rangle^A k - \langle\ell\rangle^A\ell}{C\langle\ell\rangle^A}\right) d\xi\right\|_p^p \\
&= \sum_{\ell\in\mathbf{Z}^n} |c_\ell|^p \left\|\int_{\mathbf{R}^n} e^{ix\cdot\xi}\rho\left(\frac{\langle\ell\rangle^A\xi - \langle k\rangle^A k}{C\langle k\rangle^A}\right)\,\widehat{\varphi}\left(\frac{\xi-\ell}{C}\right) d\xi\right\|_p^p
\end{aligned}
$$

where ρ_k^α is the function in Proposition 6.7. We estimate the two expressions in the above calculation, namely,

$$\left(\frac{\langle\ell\rangle}{\langle k\rangle}\right)^{An(1-p)} \left\|\int_{\mathbf{R}^n} e^{ix\cdot\xi}\rho\left(\frac{\xi}{C}\right)\,\widehat{\varphi}\left(\frac{\langle k\rangle^A\xi + \langle k\rangle^A k - \langle\ell\rangle^A\ell}{C\langle\ell\rangle^A}\right) d\xi\right\|_p^p \tag{6.22}$$

or

$$\left\| \int_{\mathbf{R}^n} e^{ix\cdot\xi} \rho\left(\frac{\langle \ell\rangle^A \xi - \langle k\rangle^A k}{C\langle k\rangle^A}\right) \widehat{\varphi}\left(\frac{\xi-\ell}{C}\right) d\xi \right\|_p^p \tag{6.23}$$

by dividing $k, \ell \in \mathbf{Z}^n$ into the following six cases as we did in the proof of Lemma 6.8 as follows:

(i) $|k| \le C'$ and $|\ell| \le |k|/2$;

(ii) $|k| \ge C'$ and $|\ell| \le |k|/2$;

(iii) $|k| \le C'$ and $|k|/2 \le |\ell| \le 2|k|$;

(iv) $|k| \ge C'$ and $|k|/2 \le |\ell| \le 2|k|$;

(v) $|k| \le C'$ and $|\ell| \ge 2|k|$;

(vi) $|k| \ge C'$ and $|\ell| \ge 2|k|$,

where $C' = 6C > 0$ and $C > 0$ is a constant written in the definition of the support of η_k^α.

Cases (i) and (iii): We clearly have for sufficiently large $\tilde{N} > 0$

$$(6.22) \lesssim \sum_{\ell\in\mathbf{Z}^n} |c_\ell|^p \langle k-\ell\rangle^{-\tilde{N}p}.$$

Case (ii): ($\Rightarrow \langle \ell\rangle \le \langle k\rangle$.) By (6.22) and $\widehat{\varphi} \in \mathcal{S}$, we have for all $\beta \in \mathbf{Z}_+^n$ and $N \in \mathbf{Z}_+$

$$\begin{aligned} &\left|\partial_\xi^\beta \widehat{\varphi}\left(\frac{\langle k\rangle^A\xi + \langle k\rangle^A k - \langle \ell\rangle^A \ell}{C\langle \ell\rangle^A}\right)\right| \\ \lesssim\ & \left(1 + \left|\frac{\langle k\rangle^A\xi + \langle k\rangle^A k - \langle \ell\rangle^A \ell}{C\langle \ell\rangle^A}\right|\right)^{-N} \\ \le\ & \left(1 + \frac{1}{C}\left|\frac{\langle k\rangle^A}{\langle \ell\rangle^A}(|k| - |\xi|) - |\ell|\right|\right)^{-N} \\ \lesssim\ & \langle k\rangle^{-N} \\ \lesssim\ & \langle k\rangle^{-N/2}\langle k-\ell\rangle^{-N/4} \end{aligned}$$

Case (iv): ($\Rightarrow \langle \ell\rangle \sim \langle k\rangle$.) By (6.22) and $\widehat{\varphi} \in \mathcal{S}$, we have for all $\beta \in \mathbf{Z}_+^n$ and $N \in \mathbf{Z}_+$

$$\begin{aligned} &\left|\partial_\xi^\beta \widehat{\varphi}\left(\frac{\langle k\rangle^A\xi + \langle k\rangle^A k - \langle \ell\rangle^A \ell}{C\langle \ell\rangle^A}\right)\right| \\ \lesssim\ & \left(1 + \frac{1}{C}\left|\frac{\langle k\rangle^A}{\langle \ell\rangle^A}k - \ell\right|\right)^{-N} \\ \sim\ & \left(1 + \frac{1}{C}\left|\frac{\langle \ell\rangle^A}{\langle k\rangle^A}\ell - k\right|\right)^{-N} \end{aligned}$$

We divide in the two cases when $|k|/2 \le |\ell| \le |k|$ and $|k| < |\ell| \le 2|k|$, and have

$$\begin{cases} \left(1 + \frac{1}{C}\left|\frac{\langle k\rangle^A}{\langle \ell\rangle^A}k - \ell\right|\right)^{-N} \lesssim \langle k-\ell\rangle^{-N} & \text{for } |k|/2 \le |\ell| \le |k|, \\ \left(1 + \frac{1}{C}\left|\frac{\langle \ell\rangle^A}{\langle k\rangle^A}\ell - k\right|\right)^{-N} \lesssim \langle k-\ell\rangle^{-N} & \text{for } |k| < |\ell| \le 2|k| \end{cases}$$

as we showed in Step 1 in the proof of Lemma 6.8.

Case (v): ($\Rightarrow \langle k\rangle \le \langle \ell\rangle$.) In this case, we estimate (6.23) and only consider $2|k| \le 2C' \le |\ell|$. From the definition of the function ρ in Proposition 6.7,

$$\operatorname{supp} \rho\left(\frac{\langle \ell\rangle^A \cdot - \langle k\rangle^A k}{C\langle k\rangle^A}\right) \subset \left\{\xi \in \mathbf{R}^n : \left|\, \xi - \frac{\langle k\rangle^A}{\langle \ell\rangle^A}k \,\right| \le \frac{2C\langle k\rangle^A}{\langle \ell\rangle^A}\right\}.$$

This yields that the domain of integration is included in $\{\xi \in \mathbf{R}^n : |\xi| \le 2|\ell|/3\}$. Thus, by $\widehat{\varphi} \in \mathcal{S}$, we have for all $\beta \in \mathbf{Z}^n_+$ and $N \in \mathbf{Z}_+$

$$\begin{aligned} \left|\partial_\xi^\beta \Psi\left(\frac{\xi - k}{C}\right)\right| &\lesssim \langle \xi - k\rangle^{-N} \\ &\lesssim \langle k\rangle^{-N} \\ &\lesssim \langle k\rangle^{-N/2}\langle \ell - k\rangle^{-N/4}. \end{aligned}$$

Case (iv): The proof of this case is the same as Case (v). So we omit it.

Combining Cases (i)–(vi), we have for sufficiently large $\tilde{N} > 0$

$$\begin{aligned} &\left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} \left\|\mathcal{F}^{-1}\rho_k^\alpha \mathcal{F} f\right\|_p^q\right)^{1/q} \\ \lesssim\ &\left(\sum_{k\in\mathbf{Z}^n}\left(\sum_{\ell\in\mathbf{Z}^n} \langle k-\ell\rangle^{(|s|/(1-\alpha)-\tilde{N})p}\langle \ell\rangle^{sp/(1-\alpha)}|c_\ell|^p\right)^{q/p}\right)^{1/q}. \end{aligned}$$

Using the Young inequality if $q/p \ge 1$ and the Fubini–Tonelli theorem if $0 < q/p \le 1$, we obtain

$$\begin{aligned} \|f\|_{M^{s,\alpha}_{p,q}} &\sim \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} \left\|\mathcal{F}^{-1}\rho_k^\alpha \mathcal{F} f\right\|_p^q\right)^{1/q} \\ &\lesssim \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)}|c_k|^q\right)^{1/q}. \end{aligned}$$

Next, we mention the estimate of h^p norm of f. Since $f \in \mathcal{S}$, it follows that $f(x) = \lim_{t\to 0}\psi(tD)f(x)$ for any $x \in \mathbf{R}^n$, where $\psi \in \mathcal{S}$ and $\psi(0) = 1$. So, we have

$$\|f\|_{h^p} = \left\|\sup_{0<t<1}|\psi(tD)f|\,\right\|_{L^p} \ge \|f\|_{L^p}.$$

By the setting of φ which satisfies that supp $\varphi \subset \{\xi : |\xi| \le 1/2\}$,

$$\begin{array}{ll} \text{supp } \varphi\left(C\langle \ell\rangle^A(\cdot-\ell)\right)\cap \text{supp } \varphi\left(C\langle k\rangle^A(\cdot-k)\right)=\emptyset, & \text{if } \ell\neq k. \\ \varphi\left(C\langle \ell\rangle^A(\cdot-\ell)\right)\times \varphi\left(C\langle k\rangle^A(\cdot-k)\right)=0, & \end{array}$$

Thus,

$$\begin{aligned} \|f\|_{L^p}^p &= \int_{\mathbf{R}^n}\left|\sum_{\ell\in\mathbf{Z}^n} c_\ell\langle\ell\rangle^{An/p}\exp[i\langle\ell\rangle^A\ell\cdot x]\,\varphi\left(C\langle\ell\rangle^A(x-\ell)\right)\right|^p dx \\ &= \int_{\mathbf{R}^n}\sum_{\ell\in\mathbf{Z}^n}|c_\ell|^p\langle\ell\rangle^{An}\,\left|\varphi\left(C\langle\ell\rangle^A(x-\ell)\right)\right|^p dx \\ &\sim \sum_{\ell\in\mathbf{Z}^n}|c_\ell|^p\int_{\mathbf{R}^n}|\varphi(x)|^p\,dx \\ &\sim \sum_{\ell\in\mathbf{Z}^n}|c_\ell|^p \end{aligned}$$

Hence, from the assumption $M_{p,q}^{s,\alpha}\hookrightarrow h^p$, we obtain

$$\left(\sum_{k\in\mathbf{Z}^n}|c_k|^p\right)^{1/p}\lesssim\left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}|c_k|^q\right)^{1/q}.$$

□

Lastly, we prove the following two lemmas used for the proof of the "ONLY IF" part of Theorem 6.6.

Proof of Lemma 6.11. We set

$$f(x)=\sum_{\ell\neq 0}c_\ell\langle\ell\rangle^{\frac{n}{(1-\alpha)p}}a\left(\langle\ell\rangle^{1/(1-\alpha)}(x-\ell)\right),$$

where $a\in\mathcal{S}$ satisfies that

$$\begin{aligned} &\text{supp } a\subset[-\delta/8,\delta/8]^n, \\ &\|a\|_\infty\le 1, \\ &\int_{\mathbf{R}^n}x^\beta a(x)dx=0 \text{ for all } |\beta|\le[n(1/p-1)], \\ &|\widehat{a}(\xi)|\ge c>0 \text{ for any } 1/(4C)\le|\xi|\le 2. \end{aligned}\tag{6.24}$$

The existence of this function "a" is shown by Kobayashi–Miyachi–Tomita [52, Lemma 4.3]. Then

$$(\delta/4)^{-n/p}\langle\ell\rangle^{\frac{n}{(1-\alpha)p}}a\left(\langle\ell\rangle^{1/(1-\alpha)}(\cdot-\ell)\right)$$

is an h^p–atom of type I. In fact, these conditions are identified from the following calculations:

$$\text{supp } a\left(\langle\ell\rangle^{1/(1-\alpha)}(\cdot-\ell)\right)\subset \ell+[-\frac{\delta}{8\langle\ell\rangle^{1/(1-\alpha)}},\frac{\delta}{8\langle\ell\rangle^{1/(1-\alpha)}}]^n;$$

$$\begin{aligned}
& \|(\delta/4)^{-n/p}\langle\ell\rangle^{\frac{n}{(1-\alpha)p}}a\left(\langle\ell\rangle^{1/(1-\alpha)}(\cdot-\ell)\right)\|_\infty \\
\leq\ & (\delta/4)^{-n/p}\langle\ell\rangle^{\frac{n}{(1-\alpha)p}} \\
=\ & \left|\ell+[-\delta/(8\langle\ell\rangle^{1/(1-\alpha)}),\delta/(8\langle\ell\rangle^{1/(1-\alpha)})]^n\right|^{-1/p};
\end{aligned}$$

$$\begin{aligned}
& \int_{\mathbf{R}^n} x^\beta a\left(\langle\ell\rangle^{1/(1-\alpha)}(x-\ell)\right)dx \\
=\ & \int_{\mathbf{R}^n}\left(\langle\ell\rangle^{-1/(1-\alpha)}x+\ell\right)^\beta a(x)dx \\
=\ & 0,
\end{aligned}$$

for all $|\beta|\leq[n(1/p-1)]$. Thus, we have

$$\begin{aligned}
\|f\|_{h^p}^p &\leq \sum_{\ell\neq 0}|c_\ell|^p\left\|\langle\ell\rangle^{\frac{n}{(1-\alpha)p}}a\left(\langle\ell\rangle^{1/(1-\alpha)}(x-\ell)\right)\right\|_{h^p}^p \\
&\lesssim \sum_{\ell\neq 0}|c_\ell|^p. \qquad (6.25)
\end{aligned}$$

Next, we estimate f on α–modulation norm. Let a Schwartz function Ψ satisfy that

$$\begin{aligned}
& \text{supp } \check{\Psi}\subset[-3\delta/8,3\delta/8]^n, \\
& \Psi=1 \text{ on } [-\delta/4,\delta/4]^n, \qquad (6.26) \\
& |\Psi(\xi)|\geq c>0 \text{ for any } |\xi|\leq 2. \qquad (6.27)
\end{aligned}$$

The existence of this function Ψ is proved by Kobayashi, et al. [52, Lemma 4.3]. Since $f\in\mathcal{S}$, by the equivalent norm to α–modulation norm in Lemma 6.8, we have

$$\begin{aligned}
\|f\|_{M_{p,q}^{s,\alpha}} &\sim \left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q} \\
&\geq \left(\sum_{k\neq 0}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p^q\right)^{1/q},
\end{aligned}$$

where

$$\Psi_k^\alpha(\xi):=\Psi\left(\frac{\xi-\langle k\rangle^{\alpha/(1-\alpha)}k}{C\langle k\rangle^{\alpha/(1-\alpha)}}\right).$$

By the Fubini–Tonelli theorem, changes of variables: $\xi \mapsto \langle k\rangle^A$; $\xi - k \mapsto \xi$; $\xi \mapsto C\xi$; $C\langle k\rangle^A x \mapsto x$; $Cy \mapsto y$, we have

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} f\right\|_p \\
=\ & \left\| \int_{\mathbf{R}^n_\xi} e^{ix\cdot\xi}\Psi\left(\frac{\xi-\langle k\rangle^A k}{C\langle k\rangle^A}\right)\int_{\mathbf{R}^n_y} e^{-i\xi\cdot y} f(y)dyd\xi\right\|_p \\
=\ & C^n\langle k\rangle^{An}\left\|\int_{\mathbf{R}^n_y} f(y)\exp\left[i\langle k\rangle^A k\cdot(x-y)\right]\check{\Psi}\left(C\langle k\rangle^A(x-y)\right)dy\right\|_p \\
=\ & C^{-n/p}\langle k\rangle^{An(1-1/p)} \\
& \times\left\|\int_{\mathbf{R}^n_y} f(y)\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\check{\Psi}\left(x-\langle k\rangle^A y\right)dy\right\|_p . \qquad (6.28)
\end{aligned}$$

Here, we state the supports of the function in the above equality:

$$\begin{aligned}
\mathrm{supp}\ a\left(\langle \ell\rangle^{1/(1-\alpha)}(\cdot-\ell)\right) &\subset \ell+\left[-\frac{\delta}{8\langle\ell\rangle^{1/(1-\alpha)}},\frac{\delta}{8\langle\ell\rangle^{1/(1-\alpha)}}\right]^n \\
&\subset \ell+[-\delta/8,\delta/8]^n; \\
\mathrm{supp}\ \check{\Psi}\left(x-\langle k\rangle^A\cdot\right) &\subset \frac{x}{\langle k\rangle^A}+\left[-\frac{3\delta}{8\langle\ell\rangle^A},\frac{3\delta}{8\langle\ell\rangle^A}\right]^n \\
&\subset m+[-\delta/2,\delta/2]^n
\end{aligned}$$

for all $x \in \langle k\rangle^A m + [-\delta/8,\delta/8]^n$ and $m \in \mathbf{Z}^n$. Since we see from the above statement that

$$\mathrm{supp}\ a\left(\langle \ell\rangle^{1/(1-\alpha)}(\cdot-\ell)\right)\cap\mathrm{supp}\ \check{\Psi}\left(x-C\langle k\rangle^A\cdot\right)=\emptyset \text{ if } m\neq\ell$$

for all $x \in \langle k\rangle^A m + [-\delta/8,\delta/8]^n$ and $m \in \mathbf{Z}^n$, we have

$$\begin{aligned}
(6.28)\ \gtrsim\ & \langle k\rangle^{An(1-1/p)}\left(\sum_{m\in\mathbf{Z}^n}\int_{x\in\Omega_{k,m}}\left|\int_{\mathbf{R}^n_y}\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\right.\right. \\
& \left.\left.\times\check{\Psi}\left(x-\langle k\rangle^A y\right)\sum_{\ell\neq 0}c_\ell\,\langle\ell\rangle^{\frac{n}{(1-\alpha)p}}a\left(\langle\ell\rangle^{1/(1-\alpha)}(y-\ell)\right)dy\right|^p dx\right)^{1/p} \\
\geq\ & \langle k\rangle^{An(1-1/p)}\left(\sum_{m\neq 0}\int_{x\in\Omega_{k,m}}\left|\int_{\mathbf{R}^n_y}\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\right.\right. \\
& \left.\left.\times\check{\Psi}\left(x-\langle k\rangle^A y\right)\times c_m\,\langle m\rangle^{\frac{n}{(1-\alpha)p}}a\left(\langle m\rangle^{1/(1-\alpha)}(y-m)\right)dy\right|^p dx\right)^{1/p} \\
\geq\ & \langle k\rangle^{An(1-1/p)}\left(\sum_{(1/2)^{1-\alpha}\langle k\rangle\leq\langle m\rangle\leq 2^{1-\alpha}\langle k\rangle}|c_m|^p\langle m\rangle^{n/(1-\alpha)}\right.
\end{aligned}$$

$$\times \int_{x\in\Omega_{k,m}} \left| \int_{\mathbf{R}^n_y} \exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right] \right.$$

$$\left. \times\, \check{\Psi}\left(x-\langle k\rangle^A y\right) a\left(\langle m\rangle^{1/(1-\alpha)}(y-m)\right) dy \right|^p dx \Bigg)^{1/p}, \tag{6.29}$$

where we set $\Omega_{k,m} := \langle k\rangle^A m + [-\delta/8, \delta/8]^n$. Moreover,

$$\begin{aligned}
x-\langle k\rangle^A y &\in \left(\langle k\rangle^A m + [-\delta/8,\delta/8]^n\right) \\
&\qquad -\langle k\rangle^A \left(m + [-\frac{\delta}{8\langle m\rangle^{1/(1-\alpha)}}, \frac{\delta}{8\langle m\rangle^{1/(1-\alpha)}}]^n\right) \\
&\subset \left[-\frac{\delta}{8},\frac{\delta}{8}\right]^n + \left[-\frac{\delta}{8}\frac{\langle k\rangle^A}{\langle m\rangle^{1/(1-\alpha)}}, \frac{\delta}{8}\frac{\langle k\rangle^A}{\langle m\rangle^{1/(1-\alpha)}}\right]^n \\
&\subset \left[-\frac{\delta}{8},\frac{\delta}{8}\right]^n + \left[-\frac{\delta}{4}\frac{\langle k\rangle^A}{\langle k\rangle^{1/(1-\alpha)}}, \frac{\delta}{4}\frac{\langle k\rangle^A}{\langle k\rangle^{1/(1-\alpha)}}\right]^n \\
&\subset \left[-\frac{\delta}{8},\frac{\delta}{8}\right]^n + \left[-\frac{\delta}{4\langle k\rangle}, \frac{\delta}{4\langle k\rangle}\right]^n \\
&\subset \left[-\frac{\delta}{8},\frac{\delta}{8}\right]^n + \left[-\frac{\delta}{8},\frac{\delta}{8}\right]^n \\
&\subset \left[-\frac{\delta}{4},\frac{\delta}{4}\right]^n
\end{aligned} \tag{6.30}$$

where $|k|\geq 1$. This yields that $\check{\Psi}\left(x-\langle k\rangle^A y\right)=1$ by the assumption (6.26). So, by changes of variables in (6.29): $y-m\mapsto y$ and $\langle m\rangle^{1/(1-\alpha)}y\mapsto y$, we have

$$\begin{aligned}
(6.29) &= \langle k\rangle^{An(1-1/p)} \Bigg(\sum_{(1/2)^{1-\alpha}\langle k\rangle\leq\langle m\rangle\leq 2^{1-\alpha}\langle k\rangle} |c_m|^p \langle m\rangle^{n(1-p)/(1-\alpha)} \\
&\qquad \times \int_{x\in\Omega_{k,m}} \left| \int_{\mathbf{R}^n_y} \exp\left[-i\frac{\langle k\rangle^A}{C\langle m\rangle^{1/(1-\alpha)}} k\cdot y\right] a(y)\, dy \right|^p dx \Bigg)^{1/p} \\
&\sim \langle k\rangle^{An(1-1/p)} \Bigg(\sum_{(1/2)^{1-\alpha}\langle k\rangle\leq\langle m\rangle\leq 2^{1-\alpha}\langle k\rangle} |c_m|^p \langle m\rangle^{n(1-p)/(1-\alpha)} \\
&\qquad \times \left| \widehat{a}\left(\frac{\langle k\rangle^A}{C\langle m\rangle^{1/(1-\alpha)}} k\right) \right|^p \Bigg)^{1/p}.
\end{aligned} \tag{6.31}$$

Since we have

$$\begin{aligned}
\frac{\langle k\rangle^A}{C\langle m\rangle^{1/(1-\alpha)}}|k| &\leq \frac{2\langle k\rangle^A}{C\langle k\rangle^{1/(1-\alpha)}}|k| = \frac{2|k|}{C\langle k\rangle} \leq 2; \\
\frac{\langle k\rangle^A}{C\langle m\rangle^{1/(1-\alpha)}}|k| &\geq \frac{\langle k\rangle^A}{2C\langle k\rangle^{1/(1-\alpha)}}|k| = \frac{|k|}{2C\langle k\rangle} \geq \frac{1}{4C},
\end{aligned}$$

and the assumption (6.24), we obtain

$$
\begin{aligned}
(6.31) \quad &\gtrsim \langle k\rangle^{An(1-1/p)}\left(\sum_{(1/2)^{1-\alpha}\langle k\rangle\le\langle m\rangle\le 2^{1-\alpha}\langle k\rangle}|c_m|^p\langle m\rangle^{n(1-p)/(1-\alpha)}\right)^{1/p}\\
&\gtrsim \langle k\rangle^{An(1-1/p)}\langle k\rangle^{n(1/p-1)/(1-\alpha)}\left(\sum_{(1/2)^{1-\alpha}\langle k\rangle\le\langle m\rangle\le 2^{1-\alpha}\langle k\rangle}|c_m|^p\right)^{1/p}\\
&= \langle k\rangle^{n(1/p-1)}\left(\sum_{(1/2)^{1-\alpha}\langle k\rangle\le\langle m\rangle\le 2^{1-\alpha}\langle k\rangle}|c_m|^p\right)^{1/p}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|f\|_{M^{s,\alpha}_{p,q}} \gtrsim &\left(\sum_{k\neq 0}\langle k\rangle^{sq/(1-\alpha)}\langle k\rangle^{n(1/p-1)q}\right.\\
&\left.\times\left(\sum_{(1/2)^{1-\alpha}\langle k\rangle\le\langle m\rangle\le 2^{1-\alpha}\langle k\rangle}|c_m|^p\right)^{q/p}\right)^{1/q}. \qquad (6.32)
\end{aligned}
$$

Therefore, combining (6.25) with (6.32) by using the assumption $h^p \hookrightarrow M^{s,\alpha}_{p,q}$, we obtain the desired result. $\square$

Proof of Lemma 6.12. Let the functions $a \in \mathcal{S}$ and $\Psi \in \mathcal{S}$ be the same ones as is in Proof of Lemma 6.11. We set

$$
f(x) = \sum_{\ell\neq 0} c_\ell |\ell|^{n/p} a\left(|\ell|\,(x-\ell)\right).
$$

Then $(\delta/4)^{-n/p}|\ell|^{n/p}a\left(|\ell|\,(x-\ell)\right)$ is an h^p–atom of type I (see [52, Proof of Lemma 4.4]). So, we have

$$
\begin{aligned}
\|f\|^p_{h^p} &\le \sum_{\ell\neq 0}|c_\ell|^p \left\||\ell|^{n/p}a\left(|\ell|\,(x-\ell)\right)\right\|^p_{h^p}\\
&\lesssim \sum_{\ell\neq 0}|c_\ell|^p. \qquad (6.33)
\end{aligned}
$$

Next, we estimate f on α–modulation norm. Since $f\in\mathcal{S}$, by the equivalent norm to α–modulation norm in Lemma 6.8, we have

$$
\begin{aligned}
\|f\|_{M^{s,\alpha}_{p,q}} &\sim \left(\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi^\alpha_k\mathcal{F}f\right\|^q_p\right)^{1/q}\\
&\ge \left(\sum_{k\neq 0}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi^\alpha_k\mathcal{F}f\right\|^q_p\right)^{1/q},
\end{aligned}
$$

where

$$
\Psi^\alpha_k(\xi) := \Psi\left(\frac{\xi-\langle k\rangle^{\alpha/(1-\alpha)}k}{C\langle k\rangle^{\alpha/(1-\alpha)}}\right).
$$

By the Fubini-Tonelli theorem, a change of variables, we have

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} f\right\|_p \\
\sim\ & \langle k\rangle^{An(1-1/p)} \left\| \int_{\mathbf{R}_y^n} f(y)\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\right. \\
&\left.\times\check{\Psi}\left(x-\langle k\rangle^A y\right)dy\right\|_p . \qquad (6.34)
\end{aligned}$$

Here, we state the supports of the function in the above equality:

$$\begin{aligned}
\operatorname{supp} a\left(|\ell|\left(\cdot-\ell\right)\right) &\subset \ell+[-\frac{\delta}{8|\ell|},\frac{\delta}{8|\ell|}]^n \\
&\subset \ell+[-\delta/8,\delta/8]^n; \\
\operatorname{supp}\check{\Psi}\left(x-\langle k\rangle^A\cdot\right) &\subset \frac{x}{\langle k\rangle^A}+\left[-\frac{3\delta}{8\langle\ell\rangle^A},\frac{3\delta}{8\langle\ell\rangle^A}\right] \\
&\subset m+[-\delta/2,\delta/2]^n
\end{aligned}$$

for all $x\in\langle k\rangle^A m+[-\delta/8,\delta/8]^n$ and $m\in\mathbf{Z}^n$. Since we see that from the above statement

$$\operatorname{supp} a\left(|\ell|\left(\cdot-\ell\right)\right)\cap\operatorname{supp}\check{\Psi}\left(x-C\langle k\rangle^A\cdot\right)=\emptyset \text{ if } m\neq\ell$$

for all $x\in\langle k\rangle^A m+[-\delta/8,\delta/8]^n$ and $m\in\mathbf{Z}^n$, we have

$$\begin{aligned}
(6.34)\ \gtrsim\ & \langle k\rangle^{An(1-1/p)}\left(\sum_{m\in\mathbf{Z}^n}\int_{x\in\Omega_{k,m}}\left|\int_{\mathbf{R}_y^n}\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\right.\right. \\
&\left.\left.\times\check{\Psi}\left(x-\langle k\rangle^A y\right)\sum_{\ell\neq 0}c_\ell\,|\ell|^{n/p}a\left(|\ell|\left(y-\ell\right)\right)dy\right|^p dx\right)^{1/p} \\
\geq\ & \langle k\rangle^{An(1-1/p)}\left(\sum_{m\neq 0}\int_{x\in\Omega_{k,m}}\left|\int_{\mathbf{R}_y^n}\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\right.\right. \\
&\left.\left.\times\check{\Psi}\left(x-\langle k\rangle^A y\right)\ c_m\,|m|^{n/p}a\left(|m|\left(y-m\right)\right)dy\right|^p dx\right)^{1/p} \\
\geq\ & \langle k\rangle^{An(1-1/p)}\left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2\leq|m|\leq 2\langle k\rangle^{1/(1-\alpha)}}|c_m|^p|m|^n\right. \\
&\times\int_{x\in\Omega_{k,m}}\left|\int_{\mathbf{R}_y^n}\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\right. \\
&\left.\left.\times\check{\Psi}\left(x-\langle k\rangle^A y\right)a\left(|m|\left(y-m\right)\right)dy\right|^p dx\right)^{1/p}, \qquad (6.35)
\end{aligned}$$

where we set $\Omega_{k,m} := \langle k\rangle^A m + [-\delta/8, \delta/8]^n$. Moreover,

$$\begin{aligned}
x - \langle k\rangle^A y &\in \left(\langle k\rangle^A m + [-\delta/8, \delta/8]^n\right) - \langle k\rangle^A \left(m + \left[-\frac{\delta}{8|m|}, \frac{\delta}{8|m|}\right]^n\right) \\
&\subset \left[-\frac{\delta}{8}, \frac{\delta}{8}\right]^n + \left[-\frac{\delta}{8}\frac{\langle k\rangle^A}{|m|}, \frac{\delta}{8}\frac{\langle k\rangle^A}{|m|}\right]^n \\
&\subset \left[-\frac{\delta}{8}, \frac{\delta}{8}\right]^n + \left[-\frac{\delta}{4}\frac{\langle k\rangle^A}{\langle k\rangle^{1/(1-\alpha)}}, \frac{\delta}{4}\frac{\langle k\rangle^A}{\langle k\rangle^{1/(1-\alpha)}}\right]^n \\
&\subset \left[-\frac{\delta}{8}, \frac{\delta}{8}\right]^n + \left[-\frac{\delta}{4\langle k\rangle}, \frac{\delta}{4\langle k\rangle}\right]^n \\
&\subset \left[-\frac{\delta}{8}, \frac{\delta}{8}\right]^n + \left[-\frac{\delta}{8}, \frac{\delta}{8}\right]^n \\
&\subset \left[-\frac{\delta}{4}, \frac{\delta}{4}\right]^n \qquad (6.36)
\end{aligned}$$

where $|k| \geq 1$. This implies that $\check{\Psi}\left(x - \langle k\rangle^A y\right) = 1$ by the assumption (6.26). So, by changes of variables in (6.35), we have

$$\begin{aligned}
(6.35) &= \langle k\rangle^{An(1-1/p)} \Bigg(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \leq \langle m\rangle \leq 2\langle k\rangle^{1/(1-\alpha)}} |c_m|^p |m|^{n(1-p)} \\
&\qquad \times \int_{x\in\Omega_{k,m}} \left| \int_{\mathbf{R}^n_y} \exp\left[-i\frac{\langle k\rangle^A}{C\langle m\rangle^{1/(1-\alpha)}} k\cdot y\right] a(y)\, dy \right|^p dx \Bigg)^{1/p} \\
&= \langle k\rangle^{An(1-1/p)} \Bigg(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \leq \langle m\rangle \leq 2\langle k\rangle^{1/(1-\alpha)}} |c_m|^p |m|^{n(1-p)} \\
&\qquad \times \left| \widehat{a}\left(\frac{\langle k\rangle^A}{C|m|} k\right) \right|^p dx \Bigg)^{1/p}. \qquad (6.37)
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\langle k\rangle^A}{C|m|}|k| &\leq \frac{2\langle k\rangle^A}{C\langle k\rangle^{1/(1-\alpha)}}|k| = \frac{2|k|}{C\langle k\rangle} \leq 2; \\
\frac{\langle k\rangle^A}{C|m|}|k| &\geq \frac{\langle k\rangle^A}{2C\langle k\rangle^{1/(1-\alpha)}}|k| = \frac{|k|}{2C\langle k\rangle} \geq \frac{1}{4C},
\end{aligned}$$

we obtain

$$\begin{aligned}
(6.37) &\gtrsim \langle k\rangle^{An(1-1/p)} \left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \leq \langle m\rangle \leq 2\langle k\rangle^{1/(1-\alpha)}} |c_m|^p |m|^{n(1-p)} \right)^{1/p} \\
&\gtrsim \langle k\rangle^{An(1-1/p)} \langle k\rangle^{n(1/p-1)/(1-\alpha)} \left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \leq \langle m\rangle \leq 2\langle k\rangle^{1/(1-\alpha)}} |c_m|^p \right)^{1/p}
\end{aligned}$$

$$= \langle k\rangle^{n(1/p-1)}\left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2\le\langle m\rangle\le 2\langle k\rangle^{1/(1-\alpha)}}|c_m|^p\right)^{1/p}$$

from the assumption (6.24). Hence

$$\|f\|_{M^{s,\alpha}_{p,q}} \gtrsim \left(\sum_{k\neq 0}\langle k\rangle^{sq/(1-\alpha)}\langle k\rangle^{n(1/p-1)q}\right.$$
$$\left.\times\left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2\le\langle m\rangle\le 2\langle k\rangle^{1/(1-\alpha)}}|c_m|^p\right)^{q/p}\right)^{1/q} \tag{6.38}$$

Therefore, combining (6.33) with (6.38) by using the assumption $h^p \hookrightarrow M^{s,\alpha}_{p,q}$, we obtain the desired result. □

6.3.3 Proofs of key lemmas for Theorems 6.3 and 6.4

In this subsection, we prove Lemmas 6.13–6.15. Actually, the argument of the proofs heavily imitates Lemmas 6.10–6.12 in the previous subsection. So we easily mention the proofs. However one can find the precise statement in each related lemma.

Proof of Lemma 6.13. Let $\varphi \in \mathcal{S}\setminus\{0\}$ satisfy that $\operatorname{supp}\varphi \subset \{\xi : |\xi| \le 1/2\}$. We set

$$f(x) = \sum_{\ell\in\mathbf{Z}^n} c_\ell\langle\ell\rangle^{An/p}\exp[i\langle\ell\rangle^A\ell\cdot x]\,\varphi\left(C\langle\ell\rangle^A(x-\ell)\right)$$

for finitely supported sequences $\{c_\ell\}_{k\in\mathbf{Z}^n}$. Then, by changes of variables, we have

$$\widehat{f}(\xi) = C^{-n}\sum_{\ell\in\mathbf{Z}^n} c_\ell\langle\ell\rangle^{An(1/p-1)}\exp\left[-i\xi\cdot\ell + i\langle\ell\rangle^A|\ell|^2\right]\,\widehat{\varphi}\left(\frac{\xi-\langle\ell\rangle^A\ell}{C\langle\ell\rangle^A}\right),$$

and

$$\left\|\mathcal{F}^{-1}\rho_k^\alpha\mathcal{F}f\right\|_p \lesssim \sum_{\ell\in\mathbf{Z}^n}|c_\ell|\left\|\int_{\mathbf{R}^n} e^{ix\cdot\xi}\rho\left(\frac{\langle\ell\rangle^A\xi-\langle k\rangle^A k}{C\langle k\rangle^A}\right)\,\widehat{\varphi}\left(\frac{\xi-\ell}{C}\right)d\xi\right\|_p,$$

where ρ_k^α is the function in Proposition 6.7 as we mentioned in the proof of Lemma 6.10. We divide $k, \ell \in \mathbf{Z}^n$ into the following six cases:

(i) $|k| \le C'$ and $|\ell| \le |k|/2$;

(ii) $|k| \ge C'$ and $|\ell| \le |k|/2$;

(iii) $|k| \le C'$ and $|k|/2 \le |\ell| \le 2|k|$;

(iv) $|k| \ge C'$ and $|k|/2 \le |\ell| \le 2|k|$;

(v) $|k| \le C'$ and $|\ell| \ge 2|k|$;

(vi) $|k| \geq C'$ and $|\ell| \geq 2|k|$,

where $C' = 6C > 0$ and $C > 0$ is a constant written in the definition of the support of η_k^α Then we obtain for sufficiently large $\tilde{N} > 0$

$$\left\|\mathcal{F}^{-1}\rho_k^\alpha \mathcal{F} f\right\|_p \lesssim \sum_{\ell \in \mathbf{Z}^n} |c_\ell| \langle k - \ell\rangle^{-\tilde{N}},$$

by the same argument as used in the proof of Lemma 6.10. Multiplying regularity and summing these inequalities on $k \in \mathbf{Z}^n$,

$$\begin{aligned} \|f\|_{M_{p,q}^{s,\alpha}} &\sim \left(\sum_{k \in \mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} \left\|\mathcal{F}^{-1}\rho_k^\alpha \mathcal{F} f\right\|_p^q\right)^{1/q} \\ &\lesssim \left(\sum_{k \in \mathbf{Z}^n} \left(\sum_{\ell \in \mathbf{Z}^n} \langle k - \ell\rangle^{|s|/(1-\alpha) - \tilde{N}} \langle \ell\rangle^{s/(1-\alpha)} |c_\ell|\right)^q\right)^{1/q}. \end{aligned}$$

By the Young inequality, we have for sufficiently large $\tilde{N} > 0$

$$\begin{aligned} \|f\|_{M_{p,q}^{s,\alpha}} &\sim \left(\sum_{\ell \in \mathbf{Z}^n} \langle \ell\rangle^{sq/(1-\alpha)} |c_\ell|^q\right)^{1/q} \times \left(\sum_{k \in \mathbf{Z}^n} \langle k\rangle^{|s|/(1-\alpha) - \tilde{N}}\right) \\ &\lesssim \left(\sum_{k \in \mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} |c_k|^q\right)^{1/q}. \end{aligned}$$

Next, we consider the L^p norm of f. By the assumption of φ such that supp $\varphi \subset \{\xi : |\xi| \leq 1/2\}$,

$$\left.\begin{aligned} &\operatorname{supp} \varphi\left(C\langle \ell\rangle^A (\cdot - \ell)\right) \cap \operatorname{supp} \varphi\left(C\langle k\rangle^A (\cdot - k)\right) = \emptyset, \\ &\varphi\left(C\langle \ell\rangle^A (\cdot - \ell)\right) \times \varphi\left(C\langle k\rangle^A (\cdot - k)\right) = 0, \end{aligned}\right. \quad \text{if } \ell \neq k.$$

Thus, we have

$$\begin{aligned} \|f\|_{L^p}^p &= \int_{\mathbf{R}^n} \left|\sum_{\ell \in \mathbf{Z}^n} c_\ell \langle \ell\rangle^{An/p} \exp[i\langle \ell\rangle^A \ell \cdot x]\, \varphi\left(C\langle \ell\rangle^A (x - \ell)\right)\right|^p dx \\ &= \sum_{\ell \in \mathbf{Z}^n} \int_{\mathbf{R}^n} \left|c_\ell \langle \ell\rangle^{An/p} \exp[i\langle \ell\rangle^A \ell \cdot x]\, \varphi\left(C\langle \ell\rangle^A (x - \ell)\right)\right|^p dx \\ &\sim \sum_{\ell \in \mathbf{Z}^n} |c_\ell| \int_{\mathbf{R}^n} |\varphi(x)|^p dx \\ &\sim \sum_{\ell \in \mathbf{Z}^n} |c_\ell|^p \end{aligned}$$

as we did in the proof of Lemma 6.10 Hence, from the assumption $M_{p,q}^{s,\alpha} \hookrightarrow L^p$, we obtain

$$\left(\sum_{k \in \mathbf{Z}^n} |c_k|^p\right)^{1/p} \lesssim \left(\sum_{k \in \mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)} |c_k|^q\right)^{1/q}.$$

Therefore this end our proof. □

Proof of Lemma 6.14. Let a function $a \in \mathcal{S}$ satisfy that

$$\begin{aligned}
&\operatorname{supp} a \subset [-\delta/8, \delta/8]^n, \\
&\|a\|_\infty \le 1, \qquad (6.39)\\
&|\widehat{a}(\xi)| \ge c > 0 \text{ on } |\xi| \le 2.
\end{aligned}$$

One can find the existence of this function a in [52, Lemma 4.3]. We set

$$f(x) = \sum_{\ell \neq 0} c_\ell |\ell|^{n/p} a\left(|\ell|\left(x-\ell\right)\right),$$

Since

$$\operatorname{supp} a\left(|\ell|\left(\cdot-\ell\right)\right) \subset \ell + [-\delta/(8|\ell|), \delta/(8|\ell|)]^n,$$

there is no intersection on each support of $a\left(|\ell|\left(\cdot-\ell\right)\right)$. Thus, we have

$$\begin{aligned}
\|f\|_{L^p}^p &= \int_{\mathbf{R}^n} \left|\sum_{\ell\neq 0} c_\ell |\ell|^{n/p} a\left(|\ell|\left(x-\ell\right)\right)\right|^p dx \\
&= \sum_{\ell\neq 0} |c_\ell|^p \, |\ell|^n \int_{\mathbf{R}^n} \left|a\left(|\ell|\left(x-\ell\right)\right)\right|^p dx \\
&= \sum_{\ell\neq 0} |c_\ell|^p \int_{\mathbf{R}^n} |a(x)|^p \, dx \\
&\sim \sum_{\ell\neq 0} |c_\ell|^p. \qquad (6.40)
\end{aligned}$$

Next, we consider f on $M^{s,\alpha}_{p,q}$ norm. Let a Schwartz function Ψ satisfy that

$$\begin{aligned}
&\operatorname{supp} \check{\Psi} \subset [-3\delta/8, 3\delta/8]^n, \\
&\Psi = 1 \text{ on } [-\delta/4, \delta/4]^n, \qquad (6.41)\\
&|\Psi(\xi)| \ge c > 0 \text{ for any } |\xi| \le 2.
\end{aligned}$$

Since $f \in \mathcal{S}$, by Lemma 6.8, we have

$$\|f\|_{M^{s,\alpha}_{p,q}} \gtrsim \left(\sum_{k\neq 0} \langle k\rangle^{sq/(1-\alpha)} \left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} f\right\|_p^q\right)^{1/q}.$$

By the Fubini–Tonelli theorem and changes of variables, we have

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\Psi_k^\alpha \mathcal{F} f\right\|_p \\
\sim\ &\langle k\rangle^{An(1-1/p)} \left\|\int_{\mathbf{R}^n_y} f(y) \exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right] \check{\Psi}\left(x-\langle k\rangle^A y\right) dy\right\|_p,
\end{aligned}$$

as we got (6.28) in the proof of Lemma 6.11. Here, the supports of the functions in the above equality satisfy that

$$\begin{aligned}
\operatorname{supp} a\left(|\ell|\left(\cdot-\ell\right)\right) &\subset \ell+[-\delta/(8|\ell|),\delta/(8|\ell|)]^n \\
&\subset \ell+[-\delta/8,\delta/8]^n; \\
\operatorname{supp} \check{\Psi}\left(x-\langle k\rangle^A\cdot\right) &\subset \frac{x}{\langle k\rangle^A}+\left[-\frac{3\delta}{8\langle\ell\rangle^A},\frac{3\delta}{8\langle\ell\rangle^A}\right] \\
&\subset m+[-\delta/2,\delta/2]^n
\end{aligned}$$

for all $x \in \langle k\rangle^A m+[-\delta/8,\delta/8]^n$ and $m \in \mathbf{Z}^n$. So it follows that

$$\operatorname{supp} a\left(|\ell|\left(\cdot-\ell\right)\right)\cap \operatorname{supp} \check{\Psi}\left(x-C\langle k\rangle^A\cdot\right)=\emptyset \text{ if } m\neq \ell$$

for all $x \in \langle k\rangle^A m+[-\delta/8,\delta/8]^n$ and $m \in \mathbf{Z}^n$. Then, we have

$$\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_p \gtrsim \langle k\rangle^{n(1/p-1)}\left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2\le|m|\le 2\langle k\rangle^{1/(1-\alpha)}}|c_m|^p\right)^{1/p}$$

as we showed in the proof of Lemma 6.12. Hence, multiplying a regularity and summing on $k \in \mathbf{Z}^n\setminus\{0\}$,

$$\begin{aligned}
\|f\|_{M^{s,\alpha}_{p,q}} \gtrsim &\left(\sum_{k\neq0}\langle k\rangle^{sq/(1-\alpha)}\langle k\rangle^{n(1/p-1)q}\right. \\
&\left.\times\left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2\le|m|\le 2\langle k\rangle^{1/(1-\alpha)}}|c_m|^p\right)^{q/p}\right)^{1/q}. \qquad (6.42)
\end{aligned}$$

Therefore, combining (6.40) with (6.42) and using the assumption $h^p \hookrightarrow M^{s,\alpha}_{p,q}$, we obtain

$$\begin{aligned}
&\left(\sum_{k\neq0}|k|^{sq/(1-\alpha)+nq(1/p-1)}\left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2\le|m|\le 2\langle k\rangle^{1/(1-\alpha)}}|c_m|^p\right)^{q/p}\right)^{1/q} \\
\lesssim &\left(\sum_{k\neq0}|c_k|^p\right)^{1/p}.
\end{aligned}$$

Thus we got the desired results. □

Proof of Lemma 6.15. Let functions $a\in\mathcal{S}$ and $\Psi\in\mathcal{S}$ satisfy the same settings in the proof of Lemma 6.11. We set

$$f(x)=\sum_{\ell\neq0}c_\ell|\ell|^n a\left(|\ell|\left(x-\ell\right)\right).$$

for a finitely supported sequence $\{c_\ell\}_{k\in\mathbf{Z}^n\setminus\{0\}}$. Then, we have

$$\|f\|_{L^1} \sim \sum_{\ell\neq 0} |c_\ell|. \tag{6.43}$$

Next, we consider f on $M^{s,\alpha}_{1,q}$ norm. Since $f\in\mathcal{S}$, by Lemma 6.8,

$$\|f\|_{M^{s,\alpha}_{1,q}} \gtrsim \left(\sum_{k\neq 0}\langle k\rangle^{sq/(1-\alpha)}\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_1^q\right)^{1/q}.$$

Analogously to the proofs of Lemma 6.11, we have

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\Psi_k^\alpha\mathcal{F}f\right\|_1\\ \sim\ &\left\|\int_{\mathbf{R}^n_y} f(y)\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\check{\Psi}\left(x-\langle k\rangle^A y\right)dy\right\|_1,\end{aligned} \tag{6.44}$$

and

$$\begin{aligned}\text{supp } a\left(|\ell|\left(\cdot-\ell\right)\right) &\subset \ell+[-\delta/(8|\ell|),\delta/(8|\ell|)]^n\\ &\subset \ell+[-\delta/8,\delta/8]^n;\\ \text{supp } \check{\Psi}\left(x-\langle k\rangle^A\cdot\right) &\subset \frac{x}{\langle k\rangle^A}+\left[-\frac{3\delta}{8\langle \ell\rangle^A},\frac{3\delta}{8\langle \ell\rangle^A}\right]\\ &\subset m+[-\delta/2,\delta/2]^n,\end{aligned}$$

for all $x\in\langle k\rangle^A m+[-\delta/8,\delta/8]^n$ and $m\in\mathbf{Z}^n$. So it follows that

$$\text{supp } a\left(|\ell|\left(\cdot-\ell\right)\right)\cap\text{supp } \check{\Psi}\left(x-C\langle k\rangle^A\cdot\right)=\emptyset \text{ if } m\neq\ell$$

for all $x\in\langle k\rangle^A m+[-\delta/8,\delta/8]^n$ and $m\in\mathbf{Z}^n$. From these properties, we obtain

$$\begin{aligned}(6.44)\ \gtrsim\ &\sum_{m\in\mathbf{Z}^n}\int_{x\in\Omega_{k,m}}\left|\int_{\mathbf{R}^n_y}\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\check{\Psi}\left(x-\langle k\rangle^A y\right)\right.\\ &\qquad\left.\times\sum_{\ell\neq 0}c_\ell\,|\ell|^n a\left(|\ell|\left(y-\ell\right)\right)dy\right|dx\\ \geq\ &\sum_{\langle k\rangle^{1/(1-\alpha)}/2\leq|m|}|c_m||m|^n\int_{x\in\Omega_{k,m}}\left|\int_{\mathbf{R}^n_y}\exp\left[i\frac{k}{C}\cdot\left(x-\langle k\rangle^A y\right)\right]\right.\\ &\qquad\left.\times\check{\Psi}\left(x-\langle k\rangle^A y\right)a\left(|m|\left(y-m\right)\right)dy\right|dx,\end{aligned} \tag{6.45}$$

where we set $\Omega_{k,m}:=\langle k\rangle^A m+[-\delta/8,\delta/8]^n$. Moreover, we knows that (6.30) for $|k|\geq 1$:

$$x-\langle k\rangle^A y\in\left[-\frac{\delta}{4},\frac{\delta}{4}\right]^n.$$

This implies that $\check{\Psi}\left(x-\langle k\rangle^A y\right)=1$. So, by changes of variables, we have

$$\begin{aligned}(6.45) \quad &= \sum_{\langle k\rangle^{1/(1-\alpha)}/2\le|m|} |c_m| \int_{x\in\Omega_{k,m}} \left| \int_{\mathbf{R}^n_y} \exp\left[-i\frac{\langle k\rangle^A}{C|m|}k\cdot y\right] a(y)\,dy \right| dx \\ &\sim \sum_{\langle k\rangle^{1/(1-\alpha)}/2\le|m|} |c_m| \left| \widehat{a}\left(\frac{\langle k\rangle^A}{C|m|}k\right)\right|. \qquad (6.46)\end{aligned}$$

Since

$$\frac{\langle k\rangle^A}{C|m|}|k| \le \frac{2\langle k\rangle^A}{C\langle k\rangle^{1/(1-\alpha)}}|k| = \frac{2|k|}{C\langle k\rangle} \le 2,$$
$$\left|\widehat{a}\left(\frac{\langle k\rangle^A}{C\langle m\rangle^{1/(1-\alpha)}}k\right)\right| \ge c > 0,$$

we obtain

$$(6.46) \quad \gtrsim \sum_{\langle k\rangle^{1/(1-\alpha)}/2\le|m|} |c_m|.$$

Hence

$$\|f\|_{M^{s,\alpha}_{1,q}} \gtrsim \left(\sum_{k\neq 0}\langle k\rangle^{sq/(1-\alpha)}\left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2\le|m|}|c_m|\right)^q\right)^{1/q} \qquad (6.47)$$

Therefore, combining (6.43) with (6.47) by using the assumption $L^1 \hookrightarrow M^{s,\alpha}_{1,q}$, we obtain the desired result. □

6.4 Proof of inclusion relations between α–modulation spaces local Hardy spaces

In the previous sections, we finish the preparations to prove the main theorems. So we begin with the proofs of Theorems 6.5 and 6.6 in the following four subsections.

6.4.1 Proof of "IF" part in Theorem 6.5

Let $0<p\le 1$, $0<q\le\infty$, $s\in\mathbf{R}$, and $0\le\alpha<1$. Then we have

$$h^p \approx F^0_{p,2} \hookleftarrow B^0_{p,p} \hookleftarrow M^{0,\alpha}_{p,p} \hookleftarrow M^{s,\alpha}_{p,q}$$

if either of the following conditions is satisfied:
(1) $p\ge q$ and $s\ge n\alpha(1/p-1/p)=0$;
(2) $p<q$ and $s>n\alpha(1/p-1/p)+n(1-\alpha)(1/p-1/q)=n(1-\alpha)(1/p-1/q)$.

In the second embedding relation, we used the fact that $B^s_{p,u}\hookrightarrow F^s_{p,q}$ holds if and only if $u\le\min(p,q)$. Theorem 6.1 and Proposition 2.6 yields the third and fourth embeddings. The first and second embedding can be found in [91]. □

6.4.2 Proof of "ONLY IF" part in Theorem 6.5

We assume that $M^{s,\alpha}_{p,q} \hookrightarrow h^p$. By Lemma 6.10, we have for all finitely supported sequences $\{c_k\}_{k\in\mathbf{Z}^n}$,

$$\left(\sum_{k\in\mathbf{Z}^n} |c_k|^p\right)^{1/p} \lesssim \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)}|c_k|^q\right)^{1/q}.$$

We set $c_k = \langle k\rangle^{-s/(1-\alpha)}|d_k|^{1/p}$ for all finitely supported sequences $\{d_k\}_{k\in\mathbf{Z}^n}$. Then

$$\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{-sp/(1-\alpha)}|d_k| \lesssim \left(\sum_{k\in\mathbf{Z}^n} |d_k|^{q/p}\right)^{p/q} \tag{6.48}$$

Here, we take $\{d_k\}_{k\in\mathbf{Z}^n}$ satisfying

$$d_k = \begin{cases} 1 & \text{if } k = (K, 0, \cdots, 0), \\ 0 & \text{if otherwise,} \end{cases}$$

where $K \in \mathbf{N}$. Substituting this $\{d_k\}_{k\in\mathbf{Z}^n}$ for the inequality (6.48), $\langle K\rangle^{-sp/(1-\alpha)} \lesssim 1$. This implied that necessarily of $s \geq 0$.

Next, we assume that $p < q$ and $q \neq \infty$ ($\Rightarrow 1 < q/p < \infty$). Taking the supremum over $\{d_k\}$ such that $\|d_k\|_{\ell^{q/p}} = 1$, then by the inequality (6.48)

$$\left\|\langle k\rangle^{-sp/(1-\alpha)}\right\|_{\ell^{(q/p)'}} = \sup_{\|d_k\|_{\ell^{q/p}}=1} \left|\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{-sp/(1-\alpha)} d_k\right| \lesssim 1.$$

This yields that $(q/p)'sp/(1-\alpha) > n$, namely, $s > n(1-\alpha)(1/p - 1/q)$.

Finally, we assume that $p < q = \infty$. We have $\left\|\langle k\rangle^{-sp/(1-\alpha)}\right\|_{\ell^1} \lesssim 1$. Thus, $s > n(1-\alpha)/p$. □

6.4.3 Proof of "IF" part in Theorem 6.6

Let $0 < p \leq 1$, $0 < q \leq \infty$, $s \in \mathbf{R}$, and $0 \leq \alpha < 1$.

We first assume that $2 \leq q \leq \infty$. Then we have

$$h^p \approx F^0_{p,2} \hookrightarrow B^0_{p,q} \hookrightarrow M^{s,\alpha}_{p,q}$$

holds if $s \leq -n(1-\alpha)(1/p + 1/q - 1)$. In the second embeddings, we used the fact that $F^s_{p,q} \hookrightarrow B^s_{p,v}$ holds if and only if $v \geq \max(p, q)$. One can see this fact in [91]. The third embeddings follows from Theorem 6.1.

Next, we consider the case when $0 < q \leq 2$. We assume that $p \leq q$ ($\Rightarrow p/q \leq 1$). We set

$$f = \sum_{i=1}^{\infty} \lambda_i a_i \in h^p,$$

where a_i are h^p–atoms and λ_i is complex numbers with $\sum |\lambda_i|^p < \infty$. Since $\|f+g\|^p_{M^{s,\alpha}_{p,q}} \le \|f\|^p_{M^{s,\alpha}_{p,q}} + \|g\|^p_{M^{s,\alpha}_{p,q}}$, we have by Lemma 6.9

$$\|f\|^p_{M^{s,\alpha}_{p,q}} \le \sum_{i=1}^{\infty} |\lambda_i|^p \, \|a_i\|^p_{M^{s,\alpha}_{p,q}} \le \sum_{i=1}^{\infty} |\lambda_i|^p < \infty,$$

where $s = -n(1-\alpha)(1/p+1/q-1)$, $0 < p \le 1$, and $0 < q \le 2$. This means that for $0 < p \le 1$, and $0 < q \le 2$

$$h^p \hookrightarrow M^{s,\alpha}_{p,q}$$

holds if $p \le q$ and $s \le -n(1-\alpha)(1/p+1/q-1)$.

Finally, we state the case $p > q$ and $0 < q \le 2$. From the just above relation,

$$h^p \hookrightarrow M^{s,\alpha}_{p,p}$$

holds if $s = -n(1-\alpha)(2/p-1)$. Thus, by Proposition 2.6,

$$h^p \hookrightarrow M^{-n(1-\alpha)(2/p-1),\alpha}_{p,p} \hookrightarrow M^{s,\alpha}_{p,q}$$

holds if $p > q$ and $-n(1-\alpha)(2/p-1) > s + n\alpha(1/p-1/p) + n(1-\alpha)(1/q-1/p)$, that is, $s > n(1-\alpha)(1/p+1/q-1)$. □

6.4.4 Proof of "ONLY IF" part in Theorem 6.6

We assume that $h^p \hookrightarrow M^{s,\alpha}_{p,q}$. Then Lemma 6.11 guide us the fact that

$$\left(\sum_{k\neq 0} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\sum_{(1/2)^{1-\alpha}\langle k\rangle \le \langle m\rangle \le 2^{1-\alpha}\langle k\rangle} |c_m|^p \right)^{q/p} \right)^{1/q} \lesssim \left(\sum_{k\neq 0} |c_k|^p \right)^{1/p} \tag{6.49}$$

for all finitely supported sequence $\{c_k\}_{k\in\mathbf{Z}^n\setminus\{0\}}$. Here, we take $\{c_k\}_{k\in\mathbf{Z}^n\setminus\{0\}}$ satisfying that for a positive integer $K \ge 2$

$$c_k = \begin{cases} 1 & \text{if } |k| \le K, \\ 0 & \text{if } |k| > K. \end{cases}$$

If we substitute this sequence for (6.49), we have

$$\left(\sum_{k\neq 0} |c_k|^p \right)^{1/p} = \left(\sum_{0<k\le K} 1 \right)^{1/p} \lesssim K^{n/p}$$

and

$$\left(\sum_{k\neq 0} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\sum_{(1/2)^{1-\alpha}\langle k\rangle \le \langle m\rangle \le 2^{1-\alpha}\langle k\rangle} |c_m|^p \right)^{q/p} \right)^{1/q}$$

$$\geq \left(\sum_{0<|k|\leq 2^{-(1-\alpha)}K} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\sum_{(1/2)^{1-\alpha}\langle k\rangle \leq \langle m\rangle \leq 2^{1-\alpha}\langle k\rangle} |c_m|^p \right)^{q/p} \right)^{1/q}$$

$$= \left(\sum_{0<|k|\leq 2^{-(1-\alpha)}K} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\sum_{(1/2)^{1-\alpha}\langle k\rangle \leq \langle m\rangle \leq 2^{1-\alpha}\langle k\rangle} 1 \right)^{q/p} \right)^{1/q}$$

$$\gtrsim \left(\sum_{0<|k|\leq 2^{-(1-\alpha)}K} |k|^{sq/(1-\alpha)+nq(2/p-1)} \right)^{1/q}$$

$$\gtrsim K^{s/(1-\alpha)+n(2/p-1)+n/q}.$$

This connotes the inequality $s/(1-\alpha)+n(2/p-1)+n/q \leq n/p$, that is, $s \leq -n(1-\alpha)(1/p+1/q-1)$. The case $p \leq q$ is proven.

Next, we examine $p > q$. We assume that $s \geq -n(1-\alpha)(1/p+1/q-1)$. We can take $\varepsilon > 0$ such that $(1+\varepsilon)q/p < 1$. By the assumption $h^p \hookrightarrow M^{s,\alpha}_{p,q}$ and Lemma 6.12, we have

$$\left(\sum_{k\neq 0} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \leq |m| \leq 2\langle k\rangle^{1/(1-\alpha)}} |c_m|^p \right)^{q/p} \right)^{1/q}$$

$$\lesssim \left(\sum_{k\neq 0} |c_k|^p \right)^{1/p}, \tag{6.50}$$

for all $\{c_k\}_{k\in\mathbf{Z}^n\setminus\{0\}} \in \ell^p$, where we have used the limit argument. We set the sequence $\{c_k\}$ as

$$c_k = \begin{cases} |k|^{-n/p} (\log|k|)^{-(1+\varepsilon)/p} & \text{if } |k| \geq K, \\ 0 & \text{if } |k| < K, \end{cases}$$

where $K > 0$ is a sufficiently large integer. Here, we use the fact that

$$\left\{ |k|^{-n/\gamma} (\log|k|)^{-\beta/\gamma} \right\}_{|k|\geq K} \in \ell^\gamma \text{ if } \beta > 1,$$

$$\left\{ |k|^{-n/\gamma} (\log|k|)^{-\beta/\gamma} \right\}_{|k|\geq K} \notin \ell^\gamma \text{ if } \beta \leq 1.$$

This is written by Sugimoto and Tomita [87, Remark 4.3]. Hence, the right hand side of (6.50) follows that

$$\left(\sum_{k\neq 0} |c_k|^p \right)^{1/p} = \left(\sum_{k\neq 0} \left(|k|^{-n/p} (\log|k|)^{-(1+\varepsilon)/p} \right)^p \right)^{1/p} < \infty.$$

On the other hand, the left hand side of (6.50) follows that

$$
\begin{aligned}
&\left(\sum_{k\neq 0}\langle k\rangle^{sq/(1-\alpha)+nq(1/p-1)}\left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2\leq|m|\leq 2\langle k\rangle^{1/(1-\alpha)}}|c_m|^p\right)^{q/p}\right)^{1/q}\\
=\;&\left(\sum_{k\neq 0}\langle k\rangle^{sq/(1-\alpha)+nq(1/p-1)}\right.\\
&\qquad\left.\times\left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2\leq|m|\leq 2\langle k\rangle^{1/(1-\alpha)}}|m|^{-n}\left(\log|m|\right)^{-(1+\varepsilon)}\right)^{q/p}\right)^{1/q}\\
\gtrsim\;&\left(\sum_{|k|\geq 2^{1-\alpha}K}\langle k\rangle^{-n}\left(\log[\langle k\rangle^{1/(1-\alpha)}]\right)^{-(1+\varepsilon)q/p}\right)^{1/q}\\
\gtrsim\;&\left(\sum_{|k|\geq 2^{1-\alpha}K}\left\{|k|^{-n/q}\left(\log|k|\right)^{-[(1+\varepsilon)q/p]/q}\right\}^{q}\right)^{1/q}\\
=\;&\infty,
\end{aligned}
$$

where we used the fact that $\log\langle k\rangle\sim\log|k|$ if $\langle k\rangle\sim|k|$. However, these two estimates are contradiction to (6.50). Therefore, we obtain $s<-n(1-\alpha)(1/p+1/q-1)$. □

Remark 6.16. We explain the reason why we need these two quite similar Lemmas 6.11 and 6.12. Recall that we set the two functions:

$$
\begin{aligned}
f_1(x) &:= \sum_{\ell\neq 0}c_\ell\langle\ell\rangle^{\frac{n}{(1-\alpha)p}}a\left(\langle\ell\rangle^{1/(1-\alpha)}(x-\ell)\right),\\
f_2(x) &:= \sum_{\ell\neq 0}c_\ell|\ell|^{n/p}a\left(|\ell|(x-\ell)\right)
\end{aligned}
$$

to prove Lemmas 6.11 and 6.12, respectively. Then we could apply the Fubini–Tonelli theorem in the expression (6.28) since we assumed that $\{c_\ell\}_{\ell\in\mathbf{Z}^n\setminus\{0\}}$ is a finitely supported sequence. However, in the latter part of the above proof, i.e. the case $p>q$, we use the limit argument under the setting

$$
c_k=\begin{cases}|k|^{-n/p}\left(\log|k|\right)^{-(1+\varepsilon)/p} & \text{if } |k|\geq K,\\ 0 & \text{if } |k|<K.\end{cases}
$$

So, we need to be careful to treat these functions. Actually, f_2 belongs to L^1, although we don't know whether f_1 belongs to L^1 for all $0\leq\alpha<1$ and $0<p\leq 1$. In fact,

$$
\|f_2\|_{L^1} = \int_{\mathbf{R}^n}\left|\sum_{|\ell|\geq K}|\ell|^{-n/p}\left(\log|\ell|\right)^{-(1+\varepsilon)/p}|\ell|^{n/p}a\left(|\ell|(x-\ell)\right)\right|dx
$$

$$
\begin{aligned}
&\le \sum_{|\ell|\ge K} |\ell|^{-n} \left(\log|\ell|\right)^{-(1+\varepsilon)/p} \int_{\mathbf{R}^n} |a(x)|\,dx \\
&\lesssim \sum_{|\ell|\ge K} |\ell|^{-n} \left(\log|\ell|\right)^{-(1+\varepsilon)} \\
&< +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
\|f_1\|_{L^1} &= \int_{\mathbf{R}^n} \left| \sum_{|\ell|\ge K} |\ell|^{-n/p} \left(\log|\ell|\right)^{-(1+\varepsilon)/p} \langle \ell \rangle^{\frac{n}{(1-\alpha)p}} a\left(\langle \ell \rangle^{1/(1-\alpha)} (x-\ell) \right) \right| dx \\
&\lesssim \sum_{|\ell|\ge K} |\ell|^{-\frac{n}{p}-\frac{n}{1-\alpha}+\frac{n}{(1-\alpha)p}} \left(\log|\ell|\right)^{-(1+\varepsilon)/p} \int_{\mathbf{R}^n} |a(x)|\,dx \\
&\lesssim \sum_{|\ell|\ge K} |\ell|^{\frac{(\alpha-p)n}{(1-\alpha)p}} \left(\log|\ell|\right)^{-(1+\varepsilon)/p} .
\end{aligned}
$$

So, we need to use Lemma 6.12 to show the latter part in the above proof. On the other hand, if we use Lemma 6.12 to prove the former proof, namely, the case $p \le q$, then it doesn't work well. Indeed, if we substitute

$$
c_k = \begin{cases} 1 & \text{if } |k| \le K, \\ 0 & \text{if } |k| > K. \end{cases}
$$

into the inequality (6.50), we have $s \le -n(1-\alpha)(1/q-1) - n/p$, which is a rougher data than desired. Hence we have to use two lemmas properly.

6.5 Proof of inclusion relations between α–modulation spaces and L^p–Sobolev spaces

Finally, we prove Theorems 6.3 and 6.4. As a matter of fact, Theorems 6.3 and 6.4 are closely related with each other by the duality argument. So it suffices to show one theorem. Moreover, in Corollary 6.2, the remaining question is to get the answer whether the critical cases, that is, $s = n(1-\alpha)\nu_1(p,q)$ or $s = n(1-\alpha)\nu_2(p,q)$, are necessary or not for the inclusion relations.

6.5.1 Proof of "IF" part in Theorem 6.3

We begin this subsection by introducing the following lemma:

Lemma 6.17. *Let $1 < p \le 2$, $p \le q \le p'$, and $s \le -(1-\alpha)(1/p+1/q-1)$. Then $L^p \hookrightarrow M^{s,\alpha}_{p,q}$ holds.*

Proof of Lemma 6.17. By Theorem 6.6, $h^1 \hookrightarrow M^{s,\alpha}_{1,q}$, where $s = -n(1-\alpha)/q$. Moreover, the fact that $M^{0,\alpha}_{2,2} \approx L^2$ by Proposition 2.4 and $h^p \approx L^p$ if $1 < p < \infty$ follows. Thus, by interpolation, we have

$$
h^p \hookrightarrow M^{s,\alpha}_{p,q}
$$

if $1 \le p \le 2$, $p \le q \le p'$, and $s = -(1-\alpha)(1/p+1/q-1)$. □

Now, we start the proof. We assume that $p \ge q$. If $q \le \min(p,p')$ and $s \le 0$, then we have by Proposition 2.5

$$M^{0,\alpha}_{p,q} \hookrightarrow M^{s,\alpha}_{p,q} \hookrightarrow M^{s,\alpha}_{p,\min(p,p')} \hookrightarrow L^p_s.$$

Next, by the duality of the inclusion relation in Lemma 6.17, we have

$$M^{s,\alpha}_{p,q} \hookrightarrow L^p$$

if $2 < p < \infty$, $p' \le q \le p$, and $s \le -n(1-\alpha)(1/p+1/q-1)$. Thus, by Proposition 2.3, $M^{0,\alpha}_{p,q} \hookrightarrow L^p_s$ holds. The rest parts of (p,q) are sufficient by Corollary 6.2. □

6.5.2 Proof of "ONLY IF" part in Theorem 6.3

We start this subsection by the preparation to prove the "ONLY IF" part in Theorem 6.3. As stated in the beginning of Subsection 6.3.3, the following statements are almost repeats of the lemmas for the embedding relations between α–modulation and local Hardy spaces. So, we show the simplified proofs, though, one can find the details in each related statement.

Lemma 6.18. *Let $1 \le p,q \le \infty$, $p < q$, and $s \in \mathbf{R}$. Then if $M^{s,\alpha}_{p,q} \hookrightarrow L^p$ holds, then $s > n(1-\alpha)(1/p-1/q)$.*

Proof of Lemma 6.18. We assume that $M^{s,\alpha}_{p,q} \hookrightarrow L^p$ for $p < q$ and $q \ne \infty$ ($\Rightarrow 1 < q/p < \infty$). By Lemma 6.13, we have for all finitely supported sequences $\{c_k\}_{k\in\mathbf{Z}^n}$,

$$\left(\sum_{k\in\mathbf{Z}^n} |c_k|^p\right)^{1/p} \lesssim \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq/(1-\alpha)}|c_k|^q\right)^{1/q}.$$

We set $c_k = \langle k\rangle^{-s/(1-\alpha)}|d_k|^{1/p}$ for all finitely supported sequences $\{d_k\}_{k\in\mathbf{Z}^n}$. Then

$$\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{-sp/(1-\alpha)}|d_k| \lesssim \left(\sum_{k\in\mathbf{Z}^n} |d_k|^{q/p}\right)^{p/q} \tag{6.51}$$

Taking the supremum over $\{d_k\}$ such that $\|d_k\|_{\ell^{q/p}} = 1$, then by the inequality (6.51)

$$\begin{aligned}\left\|\langle k\rangle^{-sp/(1-\alpha)}\right\|_{\ell^{(q/p)'}} &= \sup_{\|d_k\|_{\ell^{q/p}}=1}\left|\sum_{k\in\mathbf{Z}^n}\langle k\rangle^{-sp/(1-\alpha)}d_k\right| \\ &\lesssim 1.\end{aligned}$$

This yields that $(q/p)'sp/(1-\alpha) > n$, namely, $s > n(1-\alpha)(1/p-1/q)$.

Next, we assume that $p < q = \infty$. We have $\left\|\langle k\rangle^{-sp/(1-\alpha)}\right\|_{\ell^1} \lesssim 1$. Thus, $s > n(1-\alpha)/p$. □

Lemma 6.19. *Let $1 \leq q < p < \infty$ and $s \in \mathbf{R}$. Then if $L^p \hookrightarrow M^{s,\alpha}_{p,q}$ holds, then $s < -n(1-\alpha)(1/p+1/q-1)$.*

Proof of Lemma 6.19. We assume that $L^p \hookrightarrow M^{s,\alpha}_{p,q}$. Then Lemma 6.14 give us the inequality

$$\left(\sum_{k \neq 0} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\sum_{\langle k \rangle^{1/(1-\alpha)}/2 \leq |m| \leq 2\langle k \rangle^{1/(1-\alpha)}} |c_m|^p \right)^{q/p} \right)^{1/q}$$
$$\lesssim \left(\sum_{k \neq 0} |c_k|^p \right)^{1/p} \tag{6.52}$$

for all $\{c_k\}_{k \in \mathbf{Z}^n \setminus \{0\}} \in \ell^p$, where we have used the limit argument. We assume that $s \geq -n(1-\alpha)(1/p+1/q-1)$. Since $q<p$, we can take $\varepsilon > 0$ such that $(1+\varepsilon)q/p < 1$. We set the sequence $\{c_k\}$ as

$$c_k = \begin{cases} |k|^{-n/p} (\log |k|)^{-(1+\varepsilon)/p} & \text{if } |k| \geq K, \\ 0 & \text{if } |k| < K, \end{cases}$$

where $K > 0$ is a sufficiently large integer. Here, we use the fact that

$$\left\{ |k|^{-n/\gamma} (\log |k|)^{-\beta/\gamma} \right\}_{|k| \geq K} \in \ell^\gamma \text{ if } \beta > 1,$$
$$\left\{ |k|^{-n/\gamma} (\log |k|)^{-\beta/\gamma} \right\}_{|k| \geq K} \notin \ell^\gamma \text{ if } \beta \leq 1.$$

in [87, Remark 4.3]. Hence, the right hand side of (6.52) follows that

$$\left(\sum_{k \neq 0} |c_k|^p \right)^{1/p} = \left(\sum_{k \neq 0} \left(|k|^{-n/p} (\log |k|)^{-(1+\varepsilon)/p} \right)^p \right)^{1/p} < \infty.$$

On the other hand, the left hand side of (6.52) follows that

$$\left(\sum_{k \neq 0} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\sum_{\langle k \rangle^{1/(1-\alpha)}/2 \leq |m| \leq 2\langle k \rangle^{1/(1-\alpha)}} |c_m|^p \right)^{q/p} \right)^{1/q}$$
$$= \left(\sum_{k \neq 0} |k|^{sq/(1-\alpha)+nq(1/p-1)} \right.$$
$$\left. \times \left(\sum_{\langle k \rangle^{1/(1-\alpha)}/2 \leq |m| \leq 2\langle k \rangle^{1/(1-\alpha)}} |k|^{-n} (\log |k|)^{-(1+\varepsilon)} \right)^{q/p} \right)^{1/q} \tag{6.53}$$

$$
\begin{aligned}
&\gtrsim \left(\sum_{|k|\geq 2K} |k|^{sq/(1-\alpha)+nq(1/p-1)} \left(\log\left[\langle k\rangle^{1/(1-\alpha)}\right]\right)^{-(1+\varepsilon)q/p} \right)^{1/q} \\
&\gtrsim \left(\sum_{|k|\geq 2K} \left\{ |k|^{-n/q} (\log|k|)^{-[(1+\varepsilon)q/p]/q} \right\}^q \right)^{1/q} \\
&= \infty.
\end{aligned}
$$

However, these two estimates are contradiction to (6.52). Therefore, we obtain $s < -n(1-\alpha)(1/p+1/q-1)$. □

Lemma 6.20. *Let $1 \leq q < \infty$. Then $L^1 \hookrightarrow M^{s,\alpha}_{1,q}$ only if $s < -n(1-\alpha)/q$.*

Proof of Lemma 6.20. Let $1 \leq q < \infty$. The assumption $L^1 \hookrightarrow M^{s,\alpha}_{1,q}$ give us from Lemma 6.15

$$
\left(\sum_{k\neq 0} |k|^{sq/(1-\alpha)} \left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \leq |m|} |c_m| \right)^q \right)^{1/q} \lesssim \sum_{k\neq 0} |c_k|, \tag{6.54}
$$

for all $\{c_k\}_{k\in\mathbf{Z}^n\setminus\{0\}} \in \ell^1$, where we have used the limit argument. We assume that $s \geq -n(1-\alpha)/q$. We can take $\varepsilon > 0$ such that $\varepsilon q < 1$. For this $\varepsilon > 0$, we set the sequence $\{c_k\}$ as

$$
c_k = \begin{cases} |k|^{-n} (\log|k|)^{-(1+\varepsilon)} & \text{if } |k| \geq K, \\ 0 & \text{if } |k| < K, \end{cases}
$$

where $K > 0$ is a sufficiently large integer. Here, we use the fact that

$$
\begin{aligned}
\left\{ |k|^{-n/\gamma} (\log|k|)^{-\beta/\gamma} \right\}_{|k|\geq K} &\in \ell^\gamma \text{ if } \beta > 1, \\
\left\{ |k|^{-n/\gamma} (\log|k|)^{-\beta/\gamma} \right\}_{|k|\geq K} &\notin \ell^\gamma \text{ if } \beta \leq 1
\end{aligned}
$$

(see [87, Remark 4.3]). Hence, the right hand side of (6.54) follows that

$$
\sum_{k\neq 0} |c_k| = \sum_{k\neq 0} |k|^{-n} (\log|k|)^{-(1+\varepsilon)} < \infty.
$$

On the other hand, the left hand side of (6.54) follows that

$$
\begin{aligned}
&\left(\sum_{k\neq 0} |k|^{sq/(1-\alpha)} \left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \leq |m|} |c_m| \right)^q \right)^{1/q} \\
&\geq \left(\sum_{|k|\geq 2K} |k|^{sq/(1-\alpha)} \left(\sum_{\langle k\rangle^{1/(1-\alpha)}/2 \leq |m|} |m|^{-n} (\log|m|)^{-(1+\varepsilon)} \right)^q \right)^{1/q} \qquad (6.55)
\end{aligned}
$$

$$
\begin{aligned}
&\sim \left(\sum_{|k|\geq 2K} |k|^{sq/(1-\alpha)} \left(\left(\log\left[\langle k\rangle^{1/(1-\alpha)}\right]\right)^{-\varepsilon}\right)^q \right)^{1/q} \\
&\gtrsim \left(\sum_{|k|\geq 2K} \left\{ |k|^{-n/q} \left(\log|k|\right)^{-[\varepsilon q]/q} \right\}^q \right)^{1/q} \\
&= \infty,
\end{aligned}
$$

where we have used tha fact that $\log\langle x\rangle \sim \log|x|$ for large $|x| > 0$. However, these two estimates are contradiction to (6.54). Therefore, we obtain $s < -n(1-\alpha)/q$. $\square$

Now, we begin with the proof of Theorem "ONLY IF" part in Theorem 6.3. We assume that $M^{0,\alpha}_{p,q} \hookrightarrow L^p_s$. By Corollary 6.2, we have $s \leq n(1-\alpha)\nu_2(p,q)$. So, we only consider the case $p < q$. Lemma 6.18 and Proposition 2.3 imply that $s < -n(1-\alpha)(1/p-1/q)$. This means that $s < n(1-\alpha)\nu_2(p,q)$ for $p \leq 2$. Next, the dual statement of Lemma 6.19 and Proposition 2.3 yield that $s < n(1-\alpha)(1/p+1/q-1)$. This means that $s < n(1-\alpha)\nu_2(p,q)$ for $p \geq 2$. Finally we state the case when $p = \infty$. In this case, we can not use the duality argument. We can regard $M^{0,\alpha}_{\infty,q}$ $(1 < q \leq \infty)$ and L^∞_s as the dual spaces of $M^{0,\alpha}_{1,q'}$ $(1 \leq q' < \infty)$ and L^1_{-s}, that is, $\left(M^{0,\alpha}_{1,q'}\right)^*$ and $\left(L^1_{-s}\right)^*$, respectively. Applying the Hahn–Banach theorem, we see that, for $x_0 \in M^{0,\alpha}_{1,q'}$,

$$
\|x_0\|_{M^{0,\alpha}_{1,q'}} = \sup\left\{ |F(x_0)| : F \in \left(M^{0,\alpha}_{1,q'}\right)^* \text{ and } \|F\|_{\left(M^{0,\alpha}_{1,q'}\right)^*} \leq 1 \right\}.
$$

So, from the assumption, we obtain

$$
\|x_0\|_{M^{0,\alpha}_{1,q'}} \lesssim \sup\left\{ |F(x_0)| : F \in \left(L^1_{-s}\right)^* \text{ and } \|F\|_{\left(L^1_{-s}\right)^*} \lesssim 1 \right\} = \|x_0\|_{L^1_{-s}}.
$$

Thus, we have $L^1 \hookrightarrow M^{s,\alpha}_{1,q'}$ by the lift operator, and $s < -n(1-\alpha)/q' = -n(1-\alpha)(1/q-1)$ for $1 < q \leq \infty$ by Lemma 6.20. $\square$

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