An index theorem for Toeplitz operators on partitioned manifolds
(分割された多様体における Toeplitz 作用素の指数定理)

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A thesis submitted for the degree of
Doctor of Philosophy (Mathematical Science)
in Graduate School of Mathematics, Nagoya University

January 12, 2016

Contents

CHAPTER 1

Introduction

1.1. Summary of the main result

Let M be a complete Riemannian manifold and $S \to M$ a Hermitian vector bundle on M. J. Roe [**34**] introduced a non-unital C^*-algebra $C^*(M) = \overline{\mathscr{X}}$, where $\mathscr{X}$ denotes the algebra of bounded integral operators on $L^2(M, S)$ which have smooth kernel functions supported within a bounded neighborhood of the diagonal of $M \times M$. It is called the Roe algebra. One has $K_1(C^*(M)) = \pi_0(GL_\infty(C^*(M)))$ by definition, where π_0 stands for the set of connected components. It turns out to be an abelian group. Let D be the Dirac operator acting on a Clifford bundle S. Roe also defined an odd index for D, ind(D), as an element in $K_1(C^*(M))$. It is known that the odd index is represented by the Cayley transformation of D:

$$\mathrm{ind}(D) = \left[\frac{D-i}{D+i}\right] \in K_1(C^*(M)).$$

Assume that there exist two submanifolds with boundary, M^+ and M^-, of the same dimension as M that satisfy the conditions $M = M^+ \cup M^-$ and $M^+ \cap M^- = \partial M^+ = \partial M^-$. Set $N = M^+ \cap M^-$, which is a submanifold of M of codimension one. We call such M a partitioned manifold if such N is a closed manifold, that is, N is a compact, oriented manifold without boundary; see below Figure 1.1.1. Denote by Π the characteristic function of M^+ and set $\Lambda = 2\Pi - 1$. Π and Λ act on $L^2(M, S)$ as a multiplication operator, respectively.

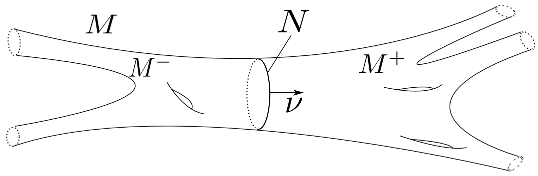

FIGURE 1.1.1. Partitioned manifold

In this setting, Roe also defined a cyclic 1-cocycle ζ:

$$\zeta(A,B) = \frac{1}{4}\mathrm{Tr}(\Lambda[\Lambda, A][\Lambda, B])$$

for $A, B \in \mathscr{A}$, where $\mathscr{A}$ is a certain Banach algebra, which is dense in $C^*(M)$ and the inclusion map induces an isomorphism of K_1 groups: $K_1(\mathscr{A}) \cong K_1(C^*(M))$. Then ζ defines a linear map

$$\zeta_* : K_1(C^*(M)) \to \mathbb{C},$$

which is essentially given by substituting an element in $\mathscr{A}$ in ζ. This coincides with Connes' pairing of a cyclic cohomology with K theory; see Subsection 3.3.2.

On the other hand, it is known that there is a $\mathbb{Z}_2$-graded Dirac operator D_N on $S|_N$ with the grading the Clifford multiplication of ν. Here, ν is the unit normal vector field; see Figure 1.1.1. D_N is an odd operator with respect to the grading, so D_N splits the positive part D_N^+ and the negative part D_N^-. Roe [**34**] proved the following formula in 1988:

$$\zeta_*(\mathrm{ind}(D)) = -\frac{1}{8\pi i}\mathrm{index}(D_N^+).$$

N. Higson [**26**] proved this formula with a simplified proof in 1991, after Roe's work. We shall call the formula the Roe-Higson index theorem from now on.

It is known that the Fredholm index of an elliptic differential operator on an odd-dimensional closed manifold is 0; see [**4**, Proposition 9.2]. Thus we have $\mathrm{index}(D_N^+) = 0$ when N is of odd dimension. Therefore, the Roe-Higson index $\zeta_*(\mathrm{ind}(D))$ is trivial when M is of even dimension. However, the value $\zeta_*(x)$ is non trivial for a general $x \in K_1(C^*(M))$.

In this thesis, we shall study such values on even-dimensional manifolds M replacing $\mathrm{index}(D_N^+)$ by the Toeplitz index, which is a counterpart of the Roe-Higson index theorem on odd-dimensional manifolds. To be more precise, we replace two parts, $\mathrm{ind}(D)$ and the Dirac operator D_N^+ by an index class $\mathrm{Ind}(\phi, D) = [\phi]\hat{\otimes}[D]$ and the Toeplitz operator on N, respectively.

	Main theorem	Roe-Higson index theorem
$\dim M$	even	odd
odd index	$\mathrm{Ind}(\phi, D)$	$\mathrm{ind}(D)$
operator on N	Toeplitz operator $T_{\phi\|_N}$	Dirac operator D_N^+

The odd index $\mathrm{Ind}(\phi, D)$ and the associated Toeplitz operator is defined as follows. Let D be the graded Dirac operator on $S = S^+ \oplus S^-$ and ϵ the grading operator. Denote by $C_w(M)$ the C^*-algebra generated by bounded smooth functions on M with bounded gradient. Then we can define $[D] \in KK^0(C_w(M), C^*(M))$ and $[\phi] \in K_1(C_w(M))$ for $\phi \in GL_l(C_w(M))$. By using the Kasparov product

$$\hat{\otimes} : K_1(C_w(M)) \times KK^0(C_w(M), C^*(M)) \to K_1(C^*(M)),$$

we obtain $\mathrm{Ind}(\phi, D) = [\phi]\hat{\otimes}[D]$ by definition. More explicitly, it is given by

$$\mathrm{Ind}(\phi, D) = \left[\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}\begin{bmatrix}1 & 0\\ 0 & \phi^{-1}\end{bmatrix}\right] \in K_1(C^*(M)),$$

where the bounded operator $\mathcal{D}$ on $L^2(M, S)$ is defined by $\mathcal{D} = (D + \epsilon)(D^2 + 1)^{-1/2}$. Next we construct the Toeplitz operator. Index theory for Toeplitz operators on closed manifolds is developed by P. Baum and R. G. Douglas [**7**], [**8**]. Set $S_N = S^+|_N$ and let $D_N : L^2(N, S_N) \to L^2(N, S_N)$ be the Dirac operator on S_N. It is well known that the spectrum of D_N consists of only real eigenvalues with finite multiplicity. Let $H_+ \subset L^2(N, S_N)$ be the closed subspace generated by all eigenvectors for D_N corresponding to non-negative eigenvalues. Denote by P the projection onto H_+. Let $\psi \in GL_l(C(N))$, which can be considered as a continuous mapping $\psi : N \to GL_l(\mathbb{C})$ at the same time. The Toeplitz operator T_ψ with symbol ψ is defined to be a compression of the multiplication operator on H_+. Namely, we define $T_\psi : H_+^l \to H_+^l$ to be $T_\psi s = P\psi s$. We note that T_ψ is a Fredholm operator since ψ takes the values in $GL_l(\mathbb{C})$.

We shall study $\zeta_*(\mathrm{Ind}(\phi, D))$ and prove that it is equal to the Fredholm index of a Toeplitz operator on N. Recall that $C_w(M)$ is the C^*-algebra generated by bounded smooth functions on M with bounded gradient. Then an element $\phi \in M_l(C_w(M))$ can be considered as a continuous mapping $\phi = [\phi_{ij}] : M \to M_l(\mathbb{C})$ such that $\phi_{ij} \in C_w(M)$ at the same time. Thus an element $\phi \in M_l(C_w(M))$ belongs to $GL_l(C_w(M))$ if and only if ϕ takes the values in $GL_l(\mathbb{C})$ and one has $\phi^{-1} \in M_l(C_w(M))$, where ϕ^{-1} is defined by $\phi^{-1}(x) = \phi(x)^{-1}$. The precise statement of the main theorem is as follows:

Main Theorem (see Theorem 5.2.1). *Let M be a complete Riemannian manifold partitioned by N as previously. Let $S \to M$ be a graded Clifford bundle with grading ϵ and denote by D the graded Dirac operator of S. Take $\phi \in GL_l(C_w(M))$. Then the following formula holds:*

$$\zeta_*(\mathrm{Ind}(\phi, D)) = -\frac{1}{8\pi i}\mathrm{index}(T_{\phi|_N}),$$

where $T_{\phi|_N}$ is the Toeplitz operator with symbol $\phi|_N : N \to GL_l(\mathbb{C})$.

In particular, if $\phi : M \to GL_l(\mathbb{C})$ is a bounded smooth mapping with bounded gradient and $\phi^{-1} : M \to GL_l(\mathbb{C})$ is also bounded, then we have $\phi \in GL_l(C_w(M))$. Here, bounded means the supremum on M of the norm on $M_l(\mathbb{C})$ is finite. Applying the topological formula for Toeplitz operators proved by Baum-Douglas [**7**], [**8**], we obtain the following:

Corollary (see Corollary 5.2.2). *Let M be a complete Riemannian manifold partitioned by N as previously. Denote by Π the characteristic function of M^+. Let $S \to M$ be a graded Clifford bundle with grading ϵ and denote by D the graded Dirac operator of S. Assume that $\phi \in C^\infty(M; GL_l(\mathbb{C}))$ is bounded with bounded gradient and ϕ^{-1} is also bounded. Then one has*

$$\operatorname{index}\left(\Pi(D+\epsilon)^{-1}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}(D+\epsilon)\Pi : \Pi(L^2(M,S))^l \to \Pi(L^2(M,S))^l\right)$$
$$= \int_{S^*N} \pi^*\mathrm{Td}(TN\otimes\mathbb{C})ch(S^+)\pi^*ch(\phi).$$

The idea of the proof is as follows. Firstly, we calculate the Kasparov product $[\phi]\hat{\otimes}[D]$ by using the Cuntz picture of $[D]$. Secondly, we calculate $\zeta_*(\mathrm{Ind}(\phi, D))$ explicitly by using the Hilbert transformation and a homotopy of Fredholm operators in the case for $M = \mathbb{R}\times N$ and $\phi = 1\otimes\psi$ with $\psi \in C^\infty(N; GL_l(\mathbb{C}))$. Finally, we reduce the general case to $\mathbb{R}\times N$ by applying a similar argument in Higson [**26**].

Set $M = \mathbb{R}\times N$ and assume that N is of odd dimension. In this case, the main theorem is derived from the Roe-Higson index theorem by applying the suspension homomorphism *formally*. Let D be the Dirac operator on M and take a mapping $\phi : M \to GL_l(\mathbb{C})$. Then they determine elements $[[D]] \in KK^0(M, \mathrm{pt})$ and $[[\phi]] \in KK^1(M, M)$, respectively. By using the Kasparov product $\hat{\otimes}$, with σ the suspension homomorphism and e an induced mapping by the map to one point, we have the following commutative diagram:

$$\begin{array}{ccc}
KK^1(M,M)\times KK^0(M,\mathrm{pt}) & \longrightarrow & KK^0(M\times S^1, M\times S^1)\times KK^1(M\times S^1,\mathrm{pt})\\
\downarrow{\scriptstyle\hat{\otimes}} & & \downarrow{\scriptstyle\hat{\otimes}}\\
KK^1(M,\mathrm{pt}) & & KK^1(M\times S^1,\mathrm{pt})\\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma}\\
KK^0(N,\mathrm{pt}) & & KK^0(N\times S^1,\mathrm{pt})\\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e}\\
\mathbb{Z} & = & \mathbb{Z}
\end{array}$$

$$\begin{array}{ccc}
([[\phi]],[[D]]) & \longmapsto & ([[E_\phi]],[[D_{M\times S^1}]]) \\
\downarrow \hat{\otimes} & & \downarrow \hat{\otimes} \\
[[\phi]]\hat{\otimes}[[D]] & & [[E_\phi]\hat{\otimes}[[D_{M\times S^1}]] \\
\downarrow \sigma & & \downarrow \sigma \\
[[\phi|_N]]\hat{\otimes}[[D_N]] & & [[D_{\phi|_N}]] \\
\downarrow e & & \downarrow e \\
\mathrm{index}(T_{\phi|_N}) & = & \mathrm{index}(D_{\phi|_N})
\end{array}$$

Here, E_ϕ is the vector bundle clutched by ϕ and $D_{\phi|_N}$ the Dirac operator on $N \times S^1$ twisted by $E_{\phi|_N}$. To be precise, we set $E_\phi = (M \times [0,1] \times \mathbb{C}^l)/\sim$ with $(x,0,v) \sim (x,1,\phi(x)v)$. By the second column and the Roe-Higson index theorem, we have $e \circ \sigma = \zeta_* \circ A$, where $A : KK^1(M,\mathrm{pt}) \to K_1(C^*(M))$ is the map defining the odd index called the assembly map. Thus, we have $\zeta_*(A([[\phi]]\hat{\otimes}[[D]])) = e(\sigma([[\phi]]\hat{\otimes}[[D]])) = \mathrm{index}(T_{\phi|_N})$, which is a statement of the main theorem for $M = \mathbb{R} \times N$.

This formal argument is correct if ϕ is an element in $GL_l(C_0(M))$ since the above KK groups are defined as $KK^1(M,\mathrm{pt}) = KK^1(C_0(M),\mathbb{C})$, for instance. However, if ϕ were chosen as an element in $GL_l(C_0(M))$, ϕ should take a constant value outside a compact set of M. This implies that $\phi|_N$ is homotopic to a constant function in $GL_l(C(N))$ and thus $\mathrm{index}(T_{\phi|_N})$ should vanish. Therefore, in order to obtain non-trivial index, we have to employ a larger algebra than $C_0(M)$.

Higson [**25**] introduced such a C^*-algebra $C_h(M)$ that contains $C_0(M)$, which is now called the Higson algebra. It plays an important role in a K-homological proof of the Roe-Higson index theorem. The Higson algebra is defined as follows: $C_h(M)$ is the C^*-algebra generated by all smooth and bounded functions defined on M of which gradient is vanishing at infinity [**25**, p.26]. $C_h(M)$ contains $C_0(M)$ as an ideal and is contained in $C_w(M)$ by definition. Given $\psi \in C^\infty(N)$, we note that $\phi = 1 \otimes \psi$ does not belong to $C_h(M)$ in general. Thus the Higson algebra is not large enough to prove our main theorem. On the other hand, we have $\phi \in C_w(M)$. Moreover, $C_w(M)$ is the largest C^*-algebra A for which we can define $[D]$ as an element in $KK^0(A, C^*(M))$. They are reasons why we introduced the C^*-algebra $C_w(M)$ in our main theorem.

Last but not least, the Roe-Higson index theorem is generalized by M. E. Zadeh [**37**] and T. Schick and Zadeh [**44**], for instance. However, they treat N of even dimension, essentially.

1.2. Background

Let Σ be a closed Riemannian surface, where closed means compact, oriented and without boundary. By the classification theorem of closed surfaces, Σ is classified by the number of holes. The number $g(\Sigma)$ is called the genus of Σ. Denote by $K : \Sigma \to \mathbb{R}$ the Gaussian curvature of Σ. The Gauss-Bonnet theorem gives a relationship of $g(\Sigma)$ and K:

$$2 - 2g(\Sigma) = \frac{1}{2\pi} \int_\Sigma K * 1.$$

Here, $*$ is the Hodge star operator. The left hand side $2 - 2g(\Sigma)$ is called the Euler number $\chi(\Sigma)$ of Σ. We can see this formula connects a global invariant and a local invariant. Moreover, we can also see this formula connects an analytic invariant and a geometric invariant as described later.

The Gauss-Bonnet theorem is generalized by C. B. Allendoerfer and A. Weil [**1**] in 1943 to closed Riemannian manifolds N of even dimension. Their proof is using the embedding into a Euclidean space of a sufficiently high dimension. After that, S. Chern [**13**] proved the generalized Gauss-Bonnet theorem by using differential forms without embedding into a Euclidean space in 1944. By using the Chern-Weil theory [**14**], its generalization is formulated as follows:

$$\chi(N) = \sum_{j=0}^{\dim N} (-1)^j H^j(N;\mathbb{C}) = \int_N e(TN),$$

where $e(TN)$ is the Euler class of the tangent bundle TN. We can see this formula also connects a global invariant and a local invariant or an analytic invariant and a geometric invariant.

As same as the above formulas, the Riemann-Roch-Hirzebruch theorem and the Hirzebruch signature theorem is also connecting such invariants. The Riemann-Roch-Hirzebruch theorem connects the Euler number of a Dolbeault complex and the Todd class (we can expand it a polynomial of Chern classes). The Hirzebruch signature theorem connects the signature of the intersection form and the $\mathcal{L}$ class (we can expand it a polynomial of Pontrjagin classes).

In 1963, M. F. Atiyah and I. M. Singer [**2**] presented the index theorem for elliptic differential operators on closed manifolds N. In particular, above Euler numbers or signatures are equal to the Fredholm index of Dirac operators D defined by the following:

$$\mathrm{index}(D) = \dim \mathrm{Ker}(D) - \dim \mathrm{Ker}(D^*) \in \mathbb{Z}.$$

This quantity is calculated by the dimension of the solution space of differential equations $Ds = 0$ and $D^*s = 0$. So index(D) is a global and analytic number. For example, by using the Hodge theory, we can see the Euler number $\chi(N)$ in the Gauss-Bonnet-Chern theorem is equal to index$((d + d^*)^+)$, where d is the exterior differential for differential forms and upper $^+$ means the restriction to the set of differential forms of even degree. On the other hand, the notion of Dirac operators is a generalization of the canonical Dirac operator for a spin manifold. In this case, the Atiyah-Singer formula is as follows:

$$\mathrm{index}(D^+) = \int_N \hat{\mathcal{A}}(TN),$$

where upper $^+$ means the restriction of the canonical Dirac operator to the set of positive spinors and $\hat{\mathcal{A}}(TN)$ is a characteristic class of TN called $\hat{\mathcal{A}}$ class (we can expand it a polynomial of Pontrjagin classes). We often assume the Dirac operator D acting on S is graded, that is, D is an odd operator with respect to a $\mathbb{Z}_2$-grading of S like above examples.

Atiyah-Singer generalized their index theorem for elliptic *pseudo*-differential operators by using topological K-theory [**3**]. There are a lot of advantages of this generalization. For example, we can get a nontrivial index for odd dimensional manifolds since every elliptic differential operator has trivial index, that is, index $= 0$; see [**4**, Proposition 9.2]. The most typical example of an elliptic pseudo-differential (not differential) operator is given by the Toeplitz operator T_ϕ. As an example, we assume $N = S^1$ is a unit circle, the simplest closed manifold. Let $\phi \in C^\infty(S^1)$ be a smooth function with $\phi(x) \neq 0$ for $x \in S^1$ and H_+ the Hardy space, that is, the set of L^2-boundary value functions on S^1 of holomorphic functions on the unit disk. Denote by P the projection onto H_+. Then we define the Toeplitz operator $T_\phi : H_+ \to H_+$ to be $T_\phi f = P\phi f$. In this case, $D_\phi = T_\phi \oplus 1$ on $L^2(S^1) = H_+ \oplus H_+^\perp$ is an elliptic pseudo-differential operator of order 0 and we have index$(D_\phi) =$ index(T_ϕ). Then the Atiyah-Singer index formula gives the Gohberg-Krein formula [**23**]:

$$\mathrm{index}(T_\phi) = -\deg(\phi),$$

where deg means the winding number. More generally, index theory for Toeplitz operators on the general N is developed by P. Baum and R. G. Douglas [**7**], [**8**]; see also Subsection 2.3.

Thanks to the K-theoretical approach, we can generalize the notion of the Fredholm index. Indeed, Atiyah-Singer defined the index for equivariant elliptic operators with the action of a group in [**3**] and the index for families of elliptic operators in [**5**]. In particular, the index of a family of elliptic operators $\{D_x\}_{x\in X}$ parametrized by a compact Hausdorff space X is defined as an element in $K(X)$, that is, it is a formal difference of vector bundles over X. If the dimension of $\mathrm{Ker}(D_x)$ is independent of $x \in X$, then it is represented by the following:

$$\mathrm{index}(\{D_x\}) = [\cup \mathrm{Ker}(D_x)] - [\cup \mathrm{Ker}(D_x^*)] \in K(X).$$

Since the category of vector bundles on X and that of finitely generated projective $C(X)$-modules are categorical equivalence by the Swan theorem, we have $K(X) \cong K_0(C(X))$. Here, $K_0(C(X))$ is a K_0 group of $C(X)$, that is, $K_0(C(X))$ is the group generated by the stable homotopy class of idempotents in $M_\infty(C(X))$. Thus we have $\mathrm{index}(\{D_x\}) \in K_0(C(X))$.

By using the above point of view, we assume that a generalized index is an element in K groups for algebras. We use K groups for operator algebras since it has several properties compatible with geometry, for example, it satisfies $K_1(C(X)) \cong K^{-1}(X)$. Its generalization is called an index class.

We can also assume that the ordinary index is an element in $K_0(\mathcal{K})$, where $\mathcal{K}$ is the C^*-algebra of all compact operators on a fixed countably infinite dimensional Hilbert space. It is provided by the Atkinson theorem; see Section 3.1.

By the way, there is another (but related) approach for the identification of the Fredholm index with an element in $K_0(\mathcal{K})$. Let D be the graded Dirac operator on a closed manifold. Then we have $[e_D] - [p] \in K_0(\mathcal{K})$, where we set

$$e_D = (D^2+1)^{-1}\begin{bmatrix} 1 & D^- \\ D^+ & D^+D^- \end{bmatrix} \text{ and } p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

e_D is called the graph projection of D. Moreover, by using an isomorphism $\mathrm{Tr}_* : K_0(\mathcal{K}) \cong \mathbb{Z}$ which is defined essentially by substituting an operator in the trace Tr, we have $\mathrm{index}(D^+) = \mathrm{Tr}_*([e_D]-[p])$. We note that we can see the isomorphism Tr_* is defined by Connes' pairing [**18**] of the cyclic 0-cocycle Tr with $K_0(\mathcal{K})$.

By using an index class, we can study an index problem for more general spaces. For example, non-compact manifolds, foliated manifolds and Hilbert module bundles.

We are interested in non-compact, complete Riemannian manifolds in this thesis. In this case, we can not define the ordinary Fredholm index in general. For example, the Dirac operator $-id/dt$ on $\mathbb{R}$ is not Fredholm. Thus we should define a generalized index.

Let M be a complete Riemannian manifold and D the Dirac operator on M. J. Roe [**34**] defined the C^*-algebra $C^*(M)$, which is called the Roe algebra, and the odd index $\mathrm{ind}(D) \in K_1(C^*(M))$. By using this odd index, Roe [**34**] proved an index theorem for a complete manifold partitioned by a closed hypersurface N like Figure 1.2.1: $M = M^- \cup_N M^+$.

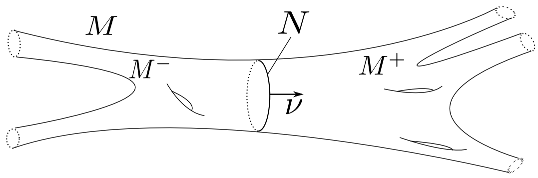

FIGURE 1.2.1. Partitioned manifold

The statement of Roe's theorem is as follows. By using the partition of M, Roe defined the cyclic 1-cocycle ζ, which is called the Roe cocycle. Recall that A. Connes [**18**] defined the pairing of a cyclic cohomology with a K group. Roe proved Connes' pairing of ζ with $\mathrm{ind}(D)$ is equal to the Fredholm index of the restricted Dirac operator D_N^+ on N up to a certain constant multiple:

$$(\spadesuit) \qquad \langle \mathrm{ind}(D), \zeta \rangle = \mathrm{index}(D_N^+),$$

where the grading of D_N is defined by the Clifford action of the unit normal vector field ν.

Higson [**26**] gave a short proof of a variation of Roe's theorem after Roe's work. Thus we call the formula ($\spadesuit$) the Roe-Higson index theorem in this thesis. In fact, let Π be the characteristic function of M^+ and $\varphi \in C^\infty(M)$ a smooth function such that we have $\varphi = \Pi$ on the complement of a compact set in M. We identify φ with a multiplication operator of it. Higson proved

$$\mathrm{index}(1 - \varphi + \varphi(D - i)(D + i)^{-1}) = \mathrm{index}(D_N^+).$$

Roe's original proof is using K-theoretical argument. On the other hand, Higson's proof is based on a calculation of the dimension of the solution space of a first order ordinary linear differential equation. Firstly, he proved the case when $M = \mathbb{R} \times N$ by calculation of its dimension. Secondly, he reduced the proof for a general partitioned manifold M to the case when $M = \mathbb{R} \times N$.

Since the Fredholm index of the Dirac operator on a closed manifold of odd dimension is 0, the right hand side of the Roe-Higson index theorem is trivial when M is of even dimension. Thus, this theorem is a theorem when M is of odd dimension, essentially.

The Roe-Higson index theorem is generalized by some researchers. U. Bunke's Callias-type index theorem [**12**] covers the Roe-Higson index theorem by using the point of view of Higson. P. Siegel [**39**] generalized when the Dirac operator is equivariant with an action of a discrete group and a manifold has a certain geometric condition. M. E. Zadeh [**43, 44**] generalized for Hilbert module bundles on a partitioned manifold of odd dimension. S. Kamimura [**28**] generalized for a product manifold $\mathbb{R}^p \times N^{\mathrm{even}}$ ($\mathbb{R} \times N$ is a partitioned manifold). T. Shick and Zadeh [**37**] generalized for a p-multi-partitioned manifold M with $\dim M - p = \text{even}$ ("$\dim M - p = \text{odd}$" implies trivial index). Siegel [**39**] also treats the case when a multi-partitioned manifold has a certain geometric condition. Here, $\mathbb{R}^p \times N$ is a typical example of a p-multi-partitioned manifold.

1.3. Organization of the thesis

In Section 2, we recall basic properties of the Dirac operator. We also recall the index theorem for Dirac operators and Toeplitz operators. In Section 3, we review a generalization of the notion of the analytic index in terms of noncommutative geometry in the required range. In Section 4, we review the Roe-Higson index theorem. Definitions of the Roe algebra, the Roe cocycle and its pairing are contained in this section. In Section 5, we discuss the main theorem. As the appendix, we review a part of properties of the Hilbert transformation. These properties are used in Sections 4 and 5.

Acknowledgments

I deeply grateful to my advisor, Professor Hitoshi Moriyoshi, for his valuable mathematical and non-mathematical suggestions and teachings. I would also grateful to Professor Toshikazu Natsume for his valuable comments on the thesis.

I also would like to thank my parents for their financial support for my study.

1.4. Notations

Numbers.

$\mathbb{N}$	The set of positive integers.
$\mathbb{Z}_+$	The set of non-negative integers.
$\mathbb{Z}$	The set of integers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{Z}_2$	$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Classes of maps.

C^r	It means C^r-class. We use also $C = C^0$.
C_c^∞	It means compactly supported C^∞-class.
C_0	It means continuous and vanishing at infinity.
C_b	It means continuous and bounded.
L^2	It means square integrable.
$\mathscr{F}(X;A)$	The set of A-valued functions defined on X of class $\mathscr{F}$. We use also $\mathscr{F}(X) = \mathscr{F}(X;\mathbb{C})$.
$\mathscr{F}(X,E)$	The set of sections of a bundle $E \to X$ of class $\mathscr{F}$.
$\mathscr{S}(\mathbb{R}^n)$	The set of rapidly decreasing functions on $\mathbb{R}^n$.
$\mathscr{S}'(\mathbb{R}^n)$	The topological dual of $\mathscr{S}(\mathbb{R}^n)$.
$\mathcal{K}(X,Y)$	The set of compact operators from X to Y. We use also $\mathcal{K}(X) = \mathcal{K}(X,X)$.
$\mathcal{K}$	The C^*-algebra of compact operators on a Hilbert space of countably infinite dimension.
$\mathcal{L}(X,Y)$	The set of bounded operators from X to Y. We use also $\mathcal{L}(X) = \mathcal{L}(X,X)$.

Not often used constructions.

$E \boxtimes F$	The exterior tensor product of two vector bundles $E \to X_1$ and $F \to X_2$: $E \boxtimes F = p_1^*E \otimes p_2^*F$, where $p_i : X_1 \times X_2 \to X_i$ is the projection.
$A \triangle B$	The symmetric difference of A and B: $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Operator K groups.

Let A be a Banach algebra.

A^+	The unital algebra adjoining a unit of A: $A^+ = A \oplus \mathbb{C}$.
SA	The suspension of A: $SA = C_0(\mathbb{R}) \otimes A$.
$M_n(A)$	The set of $n \times n$ matrices of A.
$M_\infty(A)$	The inductive limit of $M_n(A)$: $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$.
$I_n(A)$	The set of idempotents of $M_n(A)$.
$K_0(A)$	The K_0 group of A: $K_0(A) = \{[e] - [f] \,;\, e, f \in I_n(A^+) \text{ and } e - f \in M_n(A)\}$.
$GL_n(A^+)$	The set of $n \times n$ invertible matrices of A^+.
$GL_n(A)$	$GL_n(A) = \{u \in GL_n(A^+) \,;\, u - 1 \in M_n(A)\}$.
$GL_\infty(A)$	The inductive limit of $GL_n(A)$: $GL_\infty(A) = \bigcup_{n=1}^\infty GL_n(A)$.
$K_1(A)$	The K_1 group of A: $K_1(A) = \pi_0(GL_\infty(A))$.

Let A be a C^*-algebra.

$P_n(A)$	The set of projections of $M_n(A)$.
$U_n(A^+)$	The set of $n \times n$ unitary matrices of A^+.
$U_n(A)$	$U_n(A) = \{u \in U_n(A^+) \,;\, u - 1 \in M_n(A)\}$.
$U_\infty(A)$	The inductive limit of $U_n(A)$: $U_\infty(A) = \bigcup_{n=1}^\infty U_n(A)$.

CHAPTER 2

Preliminaries

In this chapter, we recall basic properties of the Dirac operator on a complete Riemannian manifold and the index theorem for Dirac operators on a closed manifold. Furthermore, we also recall the index theorem for Toeplitz operators since the index of the Dirac operator on odd-dimensional manifolds is always trivial. There are a lot of helpful references for topics in this chapter, for example, [**8**], [**9**], [**27**], [**31**] and [**36**].

2.1. Properties of the Dirac operator

Let M be a complete Riemannian manifold of dimension n without boundary. Let $(S, h) \to M$ be a Clifford bundle, that is, S is a Hermitian vector bundle equipped with a metric connection ∇^S and an action $c \in C^\infty(M, \mathrm{Hom}(\mathbb{C}l(TM), \mathrm{End}(S)))$ of the complex Clifford module bundle $\mathbb{C}l(TM)$ of the tangent bundle TM such that

- $h(c(X)s, t) + h(s, c(X)t) = 0$,
- $\nabla^S_X(c(Y)s) = c(\nabla_X Y)s + c(Y)\nabla^S_X s$,

for any X, $Y \in C^\infty(M, TM)$ and $s, t \in C^\infty(M, S)$; see [**36**, Definition 3.4]. By using ∇^S and c, we can define the Dirac operator of S as a composition of these maps:

$$D : C^\infty(M, S) \xrightarrow{\nabla^S} C^\infty(M, T^*M \otimes S) \xrightarrow{\sharp\otimes\mathrm{id}} C^\infty(M, TM \otimes S) \xrightarrow{c} C^\infty(M, S).$$

Here, $\sharp : T^*M \to TM$ is the isomorphism defined by using the Riemannian metric of M. We often identify T^*M with TM by using $\sharp$. By definition, the Dirac operator is a globally defined first-order elliptic differential operator with the principal symbol $ic(\xi)$. We can write

$$D = \sum_{i=1}^{n} c(e_i)\nabla^S_{e_i},$$

where $e_1, \ldots e_n$ is a local orthonormal frame of TM.

We often assume a Clifford bundle is $\mathbb{Z}_2$-graded, that is, $S = S^+ \oplus S^-$ is a $\mathbb{Z}_2$-graded Hermitian vector bundle, ∇^S is an even operator and $c(v)$ for $v \in TM$ is an odd operator. When a Clifford bundle S is $\mathbb{Z}_2$-graded, its Dirac operator is an odd operator by definition:

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} : C^\infty(M, S^+) \oplus C^\infty(M, S^-) \to C^\infty(M, S^+) \oplus C^\infty(M, S^-)$$

We often say graded means $\mathbb{Z}_2$-graded.

Example 2.1.1.

- *We can consider that the exterior tensor product* $\bigwedge^* T^*M \otimes \mathbb{C}$ *of the cotangent bundle* T^*M *is a Clifford bundle. In this case, the Dirac operator* D *is* $D = d + d^*$*, where* d *is the exterior differential for differential forms on* M*.*
- *Let* $W \to M$ *be a holomorphic vector bundle with the canonical connection on a Hermitian manifold* M*. Then* $\bigwedge^*(T^{0,1}M)^* \otimes W$ *is a Clifford bundle and its Dirac operator* D *is* $D = \sqrt{2}(\bar{\partial}_W + \bar{\partial}_W^*) + A$ *for a certain endomorphism* $A \in \mathrm{End}(S)$*. In particular, if* M *is a Kähler manifold, then we have* $A = 0$*.*
- *Let* M *admits a spin structure. In this case, the spin bundle* Δ *defined by its spin structure is a Clifford bundle. In this case, the canonical spinor Dirac operator of* Δ *is the Dirac operator in our definition. If* M *is of even dimension, then* Δ *is* $\mathbb{Z}_2$*-graded by the decomposition of positive and negative spinors.*
- *Let* $D : C^\infty(M, S) \to C^\infty(M, S)$ *be the Dirac operator and* $E \to M$ *a Hermitian vector bundle with a metric connection. We assume a connection of* $S \otimes E$ *is the tensor product connection. Then* $S \otimes E$ *is a Clifford bundle with the Clifford action* $c \otimes 1$*. We denote by* D_E *the Dirac operator on* $S \otimes E$*.* D_E *is called the Dirac operator twisted by* E*.*

Since the boundary of M is empty and S has the compatibility conditions, the Dirac operator D is a formally self-adjoint operator. Because of the completeness of the Riemannian metric of M, we have the following important property.

Theorem 2.1.2. [**15**] *D is an essentially self-adjoint operator, that is, D has a self-adjoint closed extension $\bar{D}$ on $L^2(M, S)$.*

We often denote $\bar{D}$ by D in the sequel. Denote by $H^1(M, S)$ the domain of D. This space is a Hilbert space by the graph norm of D, that is, the Sobolev first inner product

$$\langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \langle Du, Dv\rangle_{L^2}$$

defines a complete norm on $H^1(M, S)$. $H^1(M, S)$ is called the (first) Sobolev space. More generally, we can define higher order Sobolev spaces for $k \in \mathbb{N}$ as follows:

$$H^k(M,S) = \{u \in H^1(M,S)\,;\, \|u\|_{H^k} < +\infty\}.$$

Here, the Sobolev k-th norm $\|\cdot\|_{H^k}$ is defined by the following Sobolev k-th inner product:

$$\langle u, v\rangle_{H^k} = \langle u, v\rangle_{L^2} + \sum_{r=1}^{k} \langle D^r u, D^r v\rangle_{L^2}.$$

$H^k(M,S)$ is a Hilbert space by the Sobolev k-th norm. Set $H^0(M,S) = L^2(M,S)$. We can define these Sobolev type spaces more general order $s \in \mathbb{R}$. However, we use only the case when $k \in \mathbb{Z}_+$.

The last of this subsection, we collect well-known properties of Sobolev spaces which we use. Denote by $H^k(K,S)$ the subspace of $H^k(M,S)$ with supported by a compact set $K \subset M$. Since the principal symbol of the Dirac operator D is $ic(\xi)$, D is a first order elliptic differential operator. Thus, because of the elliptic estimate, our Sobolev spaces coincide with ordinary Sobolev spaces on a compact set, that is, $H^k(K,S)$ coincides with the set of all k-th derivatives are of L^2-class.

Theorem 2.1.3.

- *The inclusion $H^l(M,S) \to H^k(M,S)$ is continuous for $l \geq k$.*
- *$D : H^{k+1}(M,S) \to H^k(M,S)$ is continuous.*
- *(The Sobolev embedding theorem) Let $r \in \mathbb{Z}_+$ and $k > n/2 + r$ and assume $K \subset M$ is a compact subset. If $u \in H^k(K,S)$, then we have $u \in C^r(M,S)$ and there exists $C = C(n,r,s) > 0$ such that we have $\|u\|_{r,\infty} \leq C\|u\|_{H^s}$. Here, $\|\cdot\|_{r,\infty}$ is the uniform C^r norm.*
- *(The Rellich lemma) Assume $K \subset M$ is a compact subset and $l > k$. Then the inclusion $H^l(K,S) \to H^k(M,S)$ is a compact operator.*
- *(The Maurin theorem) Assume $K \subset M$ is a compact subset and $r > \dim M/2$. Then the inclusion $H^{k+r}(K,S) \to H^k(M,S)$ is of Hilbert-Schmidt class.*

2.2. The index theorem for Dirac operators

Let $S \to N$ be a Clifford bundle on a closed Riemannian manifold N and D the Dirac operator of S, where closed means compact, oriented and without boundary. Since D is self adjoint, the spectrum of D is contained in $\mathbb{R}$. Thus, resolvent operators $(D \pm i)^{-1}$ are bounded on $L^2(N,S)$. Furthermore, we can see $(1+D^2)^{-1} = (D+i)^{-1}(D-i)^{-1}$ is a bounded operator as $H^k(N,S) \to H^{k+2}(N,S)$. Thanks to the Rellich lemma and the compactness of N, $(1 + D^2)^{-1}$ is a compact operator on $H^k(N,S)$. By using the spectral decomposition of the self-adjoint compact operator $(1+D^2)^{-1} \in \mathcal{K}(L^2(N,S))$, we can show that the set of spectra of closed operator D on $L^2(N,S)$ does not have a limit point in $\mathbb{R}$ and contains only real eigenvalues with finite multiplicity. Moreover, the definition of Sobolev spaces and the Sobolev embedding theorem imply all eigensections are smooth. In particular, the dimension of the kernel of $D \in \mathcal{L}(H^{k+1}(N,S), H^k(N,S))$ is independent of k. We note that $D \cdot D(D^2+1)^{-1} = \mathrm{id}_{H^k} - (D^2+1)^{-1}$ and $D(D^2+1)^{-1} \cdot D = \mathrm{id}_{H^{k+1}} - (D^2+1)^{-1}$.

By above observations, D is a Fredholm operator and we can define the Fredholm index of D. We recall the notion of a Fredholm operator.

Theorem 2.2.1 (The Atkinson theorem). *Let H and H' are two Hilbert spaces and $T : H \to H'$ a bounded operator. Then the followings are equivalent:*

- *T is a Fredholm operator, that is, the image of T is closed and we have $\dim \mathrm{Ker}(T) < \infty$ and $\dim \mathrm{Coker}(T) < \infty$.*
- *There exists $S \in \mathcal{L}(H', H)$ such that $ST - 1 \in \mathcal{K}(H)$, $TS - 1 \in \mathcal{K}(H')$.*

Let $T \in \mathcal{L}(H,H')$ be a Fredholm operator. We define the Fredholm index of T as follows:

$$\mathrm{index}(T) = \dim \mathrm{Ker}(T) - \dim \mathrm{Coker}(T) \in \mathbb{Z}.$$

The Fredholm index of $D \in \mathcal{L}(H^{k+1}(N,S), H^k(N,S))$ is always 0 since D is self adjoint. In order to avoid this vanishing, we assume S is a $\mathbb{Z}_2$-graded Clifford bundle. In this setting, $D^+ \in \mathcal{L}(H^{k+1}(N,S^+), H^k(N,S^-))$ is also Fredholm and the Fredholm index of D^+ is independent of the choice of k since we have $\dim \mathrm{Coker}(D^+) = \dim \mathrm{Ker}(D^-)$. Summarizing the above, we can define the Fredholm index of D^+:

$$\mathrm{index}(D^+) = \dim \mathrm{Ker}(D^+) - \dim \mathrm{Ker}(D^-) \in \mathbb{Z}.$$

The Atiyah-Singer index theorem for the Dirac operator calculates this quantity by geometrical information. By the Chern-Weil theory, a differential form

$$\hat{\mathcal{A}}(TN) = \left[\det\left(\frac{R/4\pi i}{\sinh R/4\pi i}\right)\right]^{1/2}$$

defines an element in the de Rham cohomology group $H^{4*}_{\mathrm{dR}}(N;\mathbb{C})$, where R is a Riemannian curvature tensor of N. This is called the $\hat{\mathcal{A}}$ class of TN. In fact, $\hat{\mathcal{A}}$ class defines an element in $H^{4*}(N;\mathbb{Q})$ since we can expand it by a polynomial over $\mathbb{Q}$ of Pontrjagin classes as follows:

$$\hat{\mathcal{A}}(TN) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(-4p_2 + 7p_1^2) - \frac{1}{967680}(16p_3 - 44p_1p_2 + 31p_1^3) + \cdots$$

On the other hand, we recall the spinor representation is defined by the restriction of the irreducible representation of the complex Clifford algebra. So a Clifford bundle S is equivalent locally to a tensor product of the spinor bundle $\Delta|_U$ of U with a coefficient vector bundle $E|_U$ locally: $S|_U \cong \Delta|_U \hat{\otimes} E|_U$. Of course, if M is spin, then this equivalence holds globally. When S is $\mathbb{Z}_2$-graded, $E|_U$ has been often $\mathbb{Z}_2$-graded.

Let K be the curvature of S and set

$$R^S(e_i, e_j) = \frac{1}{4}\sum_{k,l} g(R(e_i, e_j)e_k, e_l)c(e_k)c(e_l).$$

Then R^S is a globally defined operator and $R^S|_U$ defines the curvature of $\Delta|_U$. So $F^S = K - R^S$ is a globally defined operator and $F^S|_U$ defines the curvature of $E|_U$. By using the Chern-Weil theory again, a differential form

$$ch_s(S/\Delta) = \mathrm{tr}_s(\exp(-\frac{1}{2\pi i}F^S))$$

defines an element in $H^{2*}_{\mathrm{dR}}(N;\mathbb{C})$. Here, we define $\mathrm{tr}_s(A_0 + A_1) = \mathrm{tr}(A_0) - \mathrm{tr}(A_1)$ for A_0 is an operator on the positive part and A_1 is an operator on the negative part.

We finished the preparation of the ingredients in the index formula. The actual formula as follows:

THEOREM 2.2.2 (Atiyah-Singer). [**2**] *Let $S \to N$ be a $\mathbb{Z}_2$-graded Clifford bundle on a closed Riemannian manifold N and D the Dirac operator of S. Then the following formula holds:*

$$\mathrm{index}(D^+) = \int_N \hat{\mathcal{A}}(TN)ch_s(S/\Delta).$$

EXAMPLE 2.2.3.

- *Assume that D is equal to $d+d^*$ on a Clifford bundle $\bigwedge^* T^*N \otimes \mathbb{C}$ with the $\mathbb{Z}_2$-grading the parity of differential forms. Then the Atiyah-Singer formula implies the Gauss-Bonnet-Chern formula:*

$$\chi(N) = \mathrm{index}((d+\delta)^+) = \int_N e(TN).$$

- *Assume that D is equal to $\sqrt{2}(\bar{\partial}_W + \bar{\partial}_W^*) + A$ on a Clifford bundle $\bigwedge^*(T^{0,1}N)^* \otimes W$ on a Hermitian manifold N with the $\mathbb{Z}_2$-grading the parity of differential forms. Then we have* $\mathrm{index}(D^+) = \mathrm{index}((\bar{\partial}_W + \bar{\partial}_W^*)^+)$. *Therefore the Atiyah-Singer* *formula implies the Riemann-Roch-Hirzebruch formula:*

$$\sum_{j=0}^{n}(-1)^j \dim H^{0,j}(N,W) = \mathrm{index}((\bar{\partial}_W + \bar{\partial}_W^*)^+) = \int_N \mathrm{Td}(TN)ch(W).$$

It seems that the proof of this formula for a general Hermitian manifold is only known this implication by the Atiyah-Singer formula.

- *Assume that D_E is a spinor Dirac operator twisted by an ungraded Hermitian vector bundle E on spin manifold N of even dimension. Then D_E is $\mathbb{Z}_2$-graded by the decomposition of positive and negative spinors. Then the Atiyah-Singer formula becomes more clear:*

$$\mathrm{index}(D_E^+) = \int_N \hat{\mathcal{A}}(TN)ch(E).$$

2.3. Toeplitz operators

Let $S \to N$ be a Clifford bundle on a closed Riemannian manifold N and D be the Dirac operator of S. Let assume N is of odd dimension. By the Atiyah-Singer formula, we can see $\mathrm{index}(D^+) = 0$. Moreover, the Fredholm index of every elliptic differential operator on odd-dimensional closed manifold is always 0; see, for instance [**4**, Proposition 9.2]. In order to avoid this vanishing, we should use non "elliptic differential" operators. We use elliptic *pseudo*-differential operators. The Toeplitz operator which is defined as follows gives the most typical example of elliptic pseudo-differential operators.

Let H_+ be the subspace of $L^2(N,S)$ generated by all eigensections of D corresponding to a non-negative eigenvalue. Denote by P the orthogonal projection onto H_+. Let $\phi \in C(N; M_l(\mathbb{C}))$ be a matrix valued continuous function on N. By using these data, we define the Toeplitz operator as follows:

Definition 2.3.1. [**8**, p.146] *Define the bounded linear operator $T_\phi : H_+^l \to H_+^l$ by $T_\phi s = P(\phi s)$. We call T_ϕ the Toeplitz operator (with symbol ϕ).*

Example 2.3.2. *We assume $N = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, the unit circle, $S = S^1 \times \mathbb{C}$, the product bundle. Set $D = -id/dx$ and $\phi_k(x) = e^{ikx}$ for $k \in \mathbb{Z}$. Then we obtain $H_+ = \mathrm{Span}_{\mathbb{C}}\{e^{inx}\,;\, n \in \mathbb{Z}_+\}$, which is called the Hardy space. In this case, we can see the Toeplitz operator $T_{\phi_k} : H_+ \to H_+$ is a k-shift operator with respect to this basis: $e^0, e^{ix}, e^{2ix}, e^{3ix}, \ldots$.*

The Toeplitz operator T_ϕ is a Fredholm operator for $\phi \in C(N; GL_l(\mathbb{C}))$ [**8**, Lemma 2.10]. We see the outline of a reason why this property holds. As explained in Section 2.2, the set of spectra of closed operator D on $L^2(N,S)$ does not have a limit point in $\mathbb{R}$ and contains only real eigenvalues with finite multiplicity. Thus we have $\delta = \inf\{|\lambda|\,;\, \lambda \in \mathrm{Spec}(D)\setminus\{0\}\} > 0$. Therefore there exists $f \in C^\infty(\mathbb{R};[0,1])$ such that $f|_{[0,\infty)} = 1$, $f|_{(-\infty,-\delta]} = 0$ and we have $P = f(D)$. Here, the right hand side of the last equality is defined by the functional calculus. Thus P is a pseudo-differential operator of order 0 by [**40**, Theorem XII.1.3]. Therefore, $[P,\phi]$ is a pseudo-differential operator of order -1 when ϕ is smooth. This implies $[P,\phi] : L^2(N,S) \to H^1(N,S)$ is a bounded operator. By the Rellich lemma, we have $[P,\phi] \in \mathcal{K}(L^2(N,S))$. Therefore, we have $[P,\phi] \in \mathcal{K}(L^2(N,S))$ for any $\phi \in C(N;M_l(\mathbb{C}))$ since the set of compact operators is closed set in operator norm topology and $C^\infty(N;M_l(\mathbb{C}))$ is dense in $C(N;M_l(\mathbb{C}))$. Thus we have $T_\phi T_{\phi^{-1}} - 1, T_{\phi^{-1}}T_\phi - 1 \in \mathcal{K}(H_+)$ for $\phi \in C(N;GL_l(\mathbb{C}))$. So T_ϕ is a Fredholm operator. Thus we can deal with the Fredholm index of T_ϕ:

$$\mathrm{index}(T_\phi) = \dim\mathrm{Ker}(T_\phi) - \dim\mathrm{Coker}(T_\phi) \in \mathbb{Z}.$$

REMARK 2.3.3. *Set*

$$D_\phi = \begin{bmatrix} T_\phi & 0 \\ 0 & 1 \end{bmatrix} : L^2(N,S_N)^l = H_+^l \oplus (H_+^l)^\perp \to H_+^l \oplus (H_+^l)^\perp$$

for $\phi \in C(N, GL_l(\mathbb{C}))$. *Then* D_ϕ *is an elliptic pseudo-differential operator of order* 0.

There exists the index theorem of the Toeplitz operator. We can consider that this index theorem is a corollary of the general Atiyah-Singer index theorem. Let $\pi : S^*N \to N$ be the unit sphere bundle of T^*N. We denote by $\sigma(x,\xi) \in \mathrm{End}((\pi^*S)_{(x,\xi)})$ the principal symbol of D for all $(x,\xi) \in S^*N$. Denote by $S^+_{(x,\xi)}$ the 1-eigenspace of $\sigma(x,\xi) = ic(\xi)$. Set $S^+ = \bigcup_{(x,\xi)} S^+_{(x,\xi)}$. Then S^+ is a subbundle of π^*S.

THEOREM 2.3.4. [**7**, Corollary 24.8][**8**, Theorem 4] *The Fredholm index of the Toeplitz operator satisfies the following:*

$$\mathrm{index}(T_\phi) = \int_{S^*N} \pi^*\mathrm{Td}(TN\otimes\mathbb{C})ch(S^+)\pi^*ch(\phi).$$

Here, $ch(\phi) \in H^{2*+1}(N;\mathbb{C})$ is an odd Chern character defined by

$$ch(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(2n+1)!}\frac{1}{(2\pi i)^{n+1}}\mathrm{tr}((\phi^{-1}d\phi)^{2n+1})$$

for $\phi \in C^\infty(N;GL_l(\mathbb{C}))$. In particular, if $S \to N$ is a spin bundle on a spin manifold N and D is a spinor Dirac operator on S, this formula becomes clear. This clear formula has proven independently in [**22**].

Corollary 2.3.5. *We assume D is a spinor Dirac operator on a spin manifold N. Then we have*

$$\mathrm{index}(T_\phi) = -\int_N \hat{\mathcal{A}}(TN)ch(\phi).$$

The last of this section, we give an example and a remark for Toeplitz operators.

Example 2.3.6. [**23**] *We use the setting of Example 2.3.2. In this case, we have* $\mathrm{index}(T_{\phi_k}) = -\deg(\phi_k)$. *Here,* deg *is the winding number of* ϕ_k.

Remark 2.3.7. *The notion of Toeplitz operators appears in several complex variables. Let Ω be a strictly pseudo-convex bounded domain and set $N = \partial\Omega$. Then "$H_+ \subset L^2(N)$" is defined by the L^2-boundary values of holomorphic functions on Ω, and it is called a Hardy space. Then the projection onto a Hardy space is called a Szegő projector. These ingredients are coincide with our definition in the case when $N = S^1$, that is, in the case when* $\dim_{\mathbb{C}} \Omega = 1$. *However, these are different in the case when* $\dim_{\mathbb{C}} \Omega \geq 2$. *In fact, Szegő projectors are Heisenberg pseudo-differential operators, but not ordinary pseudo-differential operators. See, for instance* [**21**].

CHAPTER 3

The Kasparov product and the index

We recall that we can define the Fredhlm index of the Dirac operator D on a closed manifold. However, the Dirac operator on non-compact, complete Riemannian manifold is not Fredholm in general. For example, $D = -id/dt$ on $\mathbb{R}$ is a Dirac operator but not Fredholm. Therefore, we should generalize the notion of the Fredholm index in order to study an index theorem on non-compact manifolds. In this chapter we see its generalizations by using operator K-theory. A comprehensive text for operator K-theory is [**10**].

3.1. The Fredholm index and K theory

Let us recall the Atkinson theorem 2.2.1. Let H be a separable infinite dimensional Hilbert space and $T \in \mathcal{L}(H)$ a Fredholm operator. By the Atkinson theorem, T is a Fredholm operator if and only if there exists $S \in \mathcal{L}(H)$ such that we have $ST - 1, TS - 1 \in \mathcal{K}(H)$. Therefore, T is a Fredholm operator if and only if T is an invertible element in the Calkin algebra $\mathcal{Q}(H) = \mathcal{L}(H)/\mathcal{K}(H)$.

Let $0 \to A \overset{\iota}{\to} B \overset{\pi}{\to} C \to 0$ be a short exact sequence of Banach algebras. There exist connecting maps $\partial : K_1(C) \to K_0(A)$ and $\delta : K_0(C) \to K_1(A)$ defined by as follows: For any $u \in GL_n(C)$, let $w \in GL_{2n}(B)$ satisfies

$$\pi(w) = u \oplus u^{-1} = \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}.$$

Set $\partial([u]) = [wp_n w^{-1}] - [p_n]$, where we denote by

$$p_n = \begin{bmatrix} 1_n & 0 \\ 0 & 0 \end{bmatrix} \in M_{2n}(\mathbb{C}).$$

Then $\partial([u])$ turns out to be an element in $K_0(A)$. On the other hand, for any $e, f \in I_n(C^+)$ with $e - f \in M_n(C)$, let $x, y \in M_n(B^+)$ satisfy $\pi(x) = e$ and $\pi(y) = f$. Set $\delta([e] - [f]) = [\exp(2\pi i x)] - [\exp(2\pi i y)]$. Then $\delta([e] - [f])$ turns out to be an element in $K_1(A)$.

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\iota_*} & K_0(B) & \xrightarrow{\pi_*} & K_0(C) \\ \partial \uparrow & & & & \downarrow \delta \\ K_1(C) & \xleftarrow[\pi_*]{} & K_1(B) & \xleftarrow[\iota_*]{} & K_1(A) \end{array}$$

We apply ∂ for the following short exact sequence for C^*-algebras:

$$0 \to \mathcal{K}(H) \to \mathcal{L}(H) \xrightarrow{\pi} \mathcal{Q}(H) \to 0.$$

Since these three algebras $\mathcal{K}(H)$, $\mathcal{L}(H)$ and $\mathcal{Q}(H)$ are C^*-algebras, we can take a representative element in $K_1(\mathcal{Q}(H))$ by a unitary element. Let $T \in \mathcal{L}(H)$ be a Fredholm operator satisfies $\pi(T)^* = \pi(T)^{-1}$. Then we have $[\pi(T)] \in K_1(\mathcal{Q}(H))$. Let V be the partial isometry part of the polar decomposition of T: $T = V|T|$. Then we have $V - T \in \mathcal{K}(H)$ by $1 - T^*T \in \mathcal{K}(H)$. By using the partial isometry V, a unitary $W \in U_2(\mathcal{L}(H))$ defined by

$$W = \begin{bmatrix} V & 1 - VV^* \\ 1 - V^*V & V^* \end{bmatrix}$$

is a lift of $\pi(T) \oplus \pi(T^*)$. Then we have $\partial([\pi(T)]) = [1 - V^*V] - [1 - VV^*] \in K_0(\mathcal{K}(H))$. On the other hand, it is known that taking the dimension of the image of an operator induces an isomorphism $K_0(\mathcal{K}(H)) \to \mathbb{Z}$. Combining this isomorphism, we have $\partial([\pi(T)]) = \dim \mathrm{Ker}(V) - \dim \mathrm{Ker}(V^*) = \mathrm{index}(V) = \mathrm{index}(T) \in \mathbb{Z}$. By this reason, ∂ is called an index map. On the other hand, δ is called an exponential map.

We call an element in $K_n(A)$ an index class in the general situation.

3.2. The Kasparov product

The group $KK^n(A, B)$ for two graded C^*-algebras A, B and the Kasparov product are defined by G. G. Kasparov [**30**]. The notion of KK groups is a generalization of that of K groups, K homology groups and extension groups for C^*-algebras. The Kasparov product is a generalization of an index map ∂ and an exponential map δ. Thus the Kasparov product is a generalization of the Fredholm index. More generally, the Kasparov product gives a bilinear map:

$$\hat{\otimes}_D : KK^n(A_1, B_1 \hat{\otimes} D) \times KK^m(D \hat{\otimes} A_2, B_2) \to KK^{n+m}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

Here, A_1 and A_2 are separable graded C^*-algebras and B_1, B_2 and D are any graded C^*-algebras. In this section, we review the Kasparov product in the required range for our main theorem.

3.2.1. Definition of KK groups. There are a lot of constructions of KK groups. For example, it is made of the Kasparov module due to Kasparov [**30**], the quasihomomorphism due to J. Cuntz [**20**], the unbounded module due to S. Baaj and P. Julg [**6**], and so on. In this subsection, we review a definition of KK groups by Kasparov modules and we see that quasihomomorphisms and unbounded modules give elements in KK groups. In this section, *graded* means $\mathbb{Z}_2$-graded. We assume ungraded C^*-algebra B is also graded with respect to the trivially grading $B^{(0)} = B$ and $B^{(1)} = 0$. We often denote by B^{tri} the C^*-algebra B equipped with the trivially grading.

DEFINITION 3.2.1. *Let B be a C^*-algebra and E a $\mathbb{C}$-linear space. We assume E is a right B-module and the action of B is compatible with $\mathbb{C}$-scalar products: $\lambda(xb) = (\lambda x)b = x(\lambda b)$ for any $\lambda \in \mathbb{C}$, $x \in E$ and $b \in B$. Then E is a Hilbert B-module if there exists B-valued inner product $\langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle : E \times E \to B$ such that*

(i) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$, $\langle x, \lambda y\rangle = \lambda\langle x, y\rangle$,
(ii) $\langle x, yb\rangle = \langle x, y\rangle b$,
(iii) $\langle x, y\rangle = \langle y, x\rangle^*$,
(iv) $\langle x, x\rangle \geq 0$; $\langle x, x\rangle = 0$ *implies* $x = 0$,
(v) *E is complete with respect to the norm* $|x| = \|\langle x, x\rangle\|_B^{1/2}$, *where* $\| \cdot \|_B$ *is the norm of* B,

for any $x, y, z \in E$, $\lambda \in \mathbb{C}$, $b \in B$.

In addition, we assume B is graded. Then a Hilbert B-module E is graded if E is a graded B-module with $\langle E^{(n)}, E^{(m)}\rangle \subset B^{(n+m)}$.

We note that the grading structure of $E_1 \oplus E_2$ for two graded Hilbert B-modules E_1 and E_2 is defined by $(E_1 \oplus E_2)^{(0)} = E_1^{(0)} \oplus E_2^{(0)}$ and $(E_1 \oplus E_2)^{(1)} = E_1^{(1)} \oplus E_2^{(1)}$ unless otherwise noted.

EXAMPLE 3.2.2. *Let B be a graded C^*-algebra. Then B^n is a graded Hilbert B-module with respect to the inner product* $\langle (a_i), (b_i)\rangle = \sum_{i=1}^n a_i^* b_i$. *More generally, let $\mathbb{H}_B$ be the set of the sequence (b_n) for $b_n \in B$ such that $\sum b_n^* b_n$ converges. Then $\mathbb{H}_B$ is a graded Hilbert B-module with respect to the inner product* $\langle (a_n), (b_n)\rangle = \sum a_n^* b_n$. *We call $\mathbb{H}_B$ the Hilbert space over B. In particular, separable Hilbert spaces can be regarded as Hilbert $\mathbb{C}$-modules.*

Set $\hat{\mathbb{H}}_B = \mathbb{H}_B \oplus \mathbb{H}_B^{op}$. *Here, ^{op} means the interchanged grading.*

Kasparov modules are defined by using a graded Hilbert B-module. Let $\mathbb{B}(E_1, E_2)$ be the set of adjointable homomorphisms of right B-modules from E_1 to E_2. We call two Hilbert B-modules E_1 and E_2 are isomorphic if there is a unitary operator $U \in \mathbb{B}(E_1, E_2)$. When the above E_1 and E_2 are graded, E_1 and E_2 are isomorphic if there is a grading preserving unitary operator $U \in \mathbb{B}(E_1, E_2)$. Similar to the theory of Hilbert spaces, $\mathbb{B}(E) = \mathbb{B}(E, E)$ is a C^*-algebra with respect to the operator norm. Let $\mathbb{K}(E)$ be the closure of linear spans of finite rank operators. $\mathbb{K}(E)$ is an ideal in $\mathbb{B}(E)$. We call an element in $\mathbb{K}(E)$ a B-compact operator. For example, we have $\mathbb{K}(B^n) = M_n(B)$ and $\mathbb{K}(\mathbb{H}_B) \cong B \otimes \mathcal{K}$.

Definition 3.2.3. [**30**] *Let A and B are two graded C^*-algebras. The triple (E, ϕ, F) is a Kasparov (A, B)-module if*

- *E is a countably generated graded Hilbert B-module such that $E \oplus \hat{\mathbb{H}}_B \cong \hat{\mathbb{H}}_B$,*
- *$\phi : A \to \mathbb{B}(E)$ is a graded $*$-homomorphism,*
- *$F \in \mathbb{B}(E)$ is an odd operator such that $[F, \phi(a)]_s$, $(F^2 - 1)\phi(a)$ and $(F^* - F)\phi(a)$ are B-compact operators for any $a \in A$. Here, $[\cdot, \cdot]_s$ means a graded commutator.*

Denote by $\mathbb{E}(A, B)$ the set of all Kasparov (A, B)-modules.

Remark 3.2.4. *By using the above definition, we need not to care B is σ-unital or not. Note that any countably generated graded Hilbert B-module E for a σ-unital C^*-algebra B satisfies $E \oplus \hat{\mathbb{H}}_B \cong \hat{\mathbb{H}}_B$ by the Kasparov stabilization theorem* [**29**, §3].

A Kasparov (A, B)-module (E, ϕ, F) is degenerate if we have $[F, \phi(a)]_s = 0$, $(F^2 - 1)\phi(a) = 0$ and $(F^* - F)\phi(a) = 0$ for any $a \in A$. Denote by $\mathbb{D}(A, B)$ the set of all degenerate Kasparov (A, B)-modules.

There are some examples of Kasparov modules.

Example 3.2.5.

- *We have $(E, 0, 0) \in \mathbb{D}(A, B)$. We call it a 0-module.*
- *Let $\phi : A \to B$ be a graded $*$-homomorphism. Then we have $(B, \phi, 0) \in \mathbb{E}(A, B)$.*
- *Let $T : H \to H'$ be a Fredholm operator such that we have $T^*T - 1 \in \mathcal{K}(H)$ and $TT^* - 1 \in \mathcal{K}(H')$. Then we have*
$$\left(H \oplus H', 1, \begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix} \right) \in \mathbb{E}(\mathbb{C}, \mathbb{C}).$$
- *(direct sum) Let (E_1, ϕ_1, F_1) and (E_2, ϕ_2, F_2) are two Kasparov (A, B)-modules. Then $(E_1, \phi_1, F_1) \oplus (E_2, \phi_2, F_2) = (E_1 \oplus E_2, \phi_1 \oplus \phi_2, F_1 \oplus F_2)$ is also a Kasparov (A, B)-module.*

Cuntz defined quasihomomorphisms and proved they determine Kasparov modules for trivially graded C^*-algebras.

EXAMPLE 3.2.6 (The Cuntz picture [**20**]). *We assume A and B are trivially graded. Let $(\phi_0, \phi_1) : A \to \mathbb{B}(\mathbb{H}_B) \triangleright \mathbb{K}(\mathbb{H}_B) = B \otimes \mathcal{K}$ be a quasihomomorphism from A to $B \otimes \mathcal{K}$, that is, $\phi_0, \phi_1 : A \to \mathbb{B}(\mathbb{H}_B)$ are two $*$-homomorphisms such that we have $\phi_0(a) - \phi_1(a) \in \mathbb{K}(\mathbb{H}_B) \cong B \otimes \mathcal{K}$ for any $a \in A$. Then we have*

$$\left(\hat{\mathbb{H}}_B, \begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \in \mathbb{E}(A, B).$$

On the other hand, any homotopy class of Kasparov (A, B)-modules (see below Definition 3.2.10) are represented by a quasihomomorphism.

Baaj and Julg proved "good" unbounded operators determine Kasparov modules.

EXAMPLE 3.2.7 (The Baaj-Julg picture [**6**]). *Let E be a countably generated graded Hilbert B-module such that $E \oplus \hat{\mathbb{H}}_B \cong \hat{\mathbb{H}}_B$, $\phi : A \to \mathbb{B}(E)$ a graded $*$-homomorphism and $D : E \to E$ a self-adjoint odd regular operator. Here, D is regular if D is densely defined operator with densely defined adjoint D^* and $D^*D + 1$ has a dense range. We assume D satisfies the following:*

- *$(1 + D^2)^{-1/2}\phi(a)$ extends as an element in $\mathbb{K}(E)$,*
- *$\{a \in A\,;\, [D, \phi(a)]_s$ is densely defined and extends as an element in $\mathbb{B}(E)\}$ is dense in A.*

Then we have $(E, \phi, D(D^2 + 1)^{-1/2}) \in \mathbb{E}(A, B)$.

Let $\mathbb{C}l_n$ be the complex Clifford algebra with $\mathbb{C}^n$. The grading of $\mathbb{C}l_1 = \mathbb{C} \oplus \mathbb{C}$ is defined by $\mathbb{C}l_1^{(n)} = \{(a, (-1)^n a) \in \mathbb{C} \oplus \mathbb{C}\}$. The grading for higher n is induced by $\mathbb{C}l_{n+1} \cong \mathbb{C}l_n \hat{\otimes} \mathbb{C}l_1$. The Dirac operator on a closed manifold is a self-adjoint regular operator on L^2 sections. Thus it determines a Kasparov module.

EXAMPLE 3.2.8. *Let N be a closed manifold and $D : L^2(N, S) \to L^2(N, S)$ the Dirac operator. Let $\psi : C(N) \to \mathcal{L}(L^2(N, S))$ is defined by the multiplication operator. We assume D is graded. Then we have*

$$[D] = \left(L^2(N, S), \psi, D(D^2 + 1)^{-1/2}\right) \in \mathbb{E}(C(N), \mathbb{C}).$$

On the other hand, if D is ungraded, then we have

$$[D] = \left(L^2(N, S) \oplus L^2(N, S), \psi \oplus \psi, D(D^2 + 1)^{-1/2} \oplus (-D(D^2 + 1)^{-1/2})\right)$$

in $\mathbb{E}(C(N), \mathbb{C}l_1)$, where the grading of $L^2(N, S) \oplus L^2(N, S)$ is defined by $(L^2(N, S) \oplus L^2(N, S))^{(n)} = \{(u, (-1)^n u)\}$. We note that we often assume D is graded if $\dim N$ is even and ungraded if $\dim N$ is odd.

A KK group is the set of homotopy classes of Kasparov modules. The homotopy of Kasparov modules is defined as follows:

DEFINITION 3.2.9. *Let* (E_1, ϕ_1, F_1) *and* (E_2, ϕ_2, F_2) *are two Kasparov* (A, B)*-modules. Then* (E_1, ϕ_1, F_1) *and* (E_2, ϕ_2, F_2) *are unitary equivalent if there exists an even unitary operator* $U \in \mathbb{B}(E_1, E_2)$ *such that we have* $U^*\phi_2(a)U = \phi_1(a)$ *for all* $a \in A$ *and* $U^*F_2U = F_1$*. Denote by* $\sim_u$ *this equivalence relation.*

DEFINITION 3.2.10. *Let* (E_0, ϕ_0, F_0) *and* (E_1, ϕ_1, F_1) *are two Kasparov* (A, B)*-modules. Let* $f_i : C([0,1]; B) \to B$ *be two evaluation maps defined by* $f_0(c) = c(0)$ *and* $f_1(c) = c(1)$*. A homotopy connecting* (E_1, ϕ_1, F_1) *and* (E_2, ϕ_2, F_2) *is a Kasparov* $(A, C([0,1]; B))$*-module* (E, ϕ, F) *such that* $(E\hat{\otimes}_{f_i}B, f_i \circ \phi, F \otimes 1) \sim_u (E_i, \phi_i, F_i)$ *for* $i = 0, 1$*. Here,* $E\hat{\otimes}_{f_i}B$ *is the completion of the algebraic tensor product* $E\hat{\odot}_{C([0,1];B)}B$ *regarded* B *as a left* $C([0,1]; B)$*-module via* f_i *with respect to the following pre-inner product (with its kernels divided out):* $\langle x_1\hat{\otimes}b_1, x_2\hat{\otimes}b_2\rangle = b_1^* f_i(\langle x_1, x_2\rangle_E)b_2$*. If a homotopy exists, then we denote by* $(E_0, \phi_0, F_0) \sim_h (E_1, \phi_1, F_1)$*.*

REMARK 3.2.11. *A homotopy* $\sim_h$ *is an equivalence relation. There are other equivalence relations in* $\mathbb{E}(A, B)$*. Indeed, the Kasparov* (A, B)*-module* (E, ϕ, F_2) *is a compact perturbation of* (E, ϕ, F_1) *if we have* $(F_2 - F_1)\phi(A) \subset \mathbb{K}(E)$*. The equivalence relation* $\sim_{cp}$ *is the equivalence relation generated by* $\sim_u$ *and a compact perturbation. We note that if* (E, ϕ, F_2) *is a compact perturbation of* (E, ϕ, F_1)*, then* (E, ϕ, F_1) *is homotopic to* (E, ϕ, F_2)*.*

A homotopy respects direct sum (see Example 3.2.5) of Kasparov modules. Thus we obtain an abelian group $\mathbb{E}(A, B)/\sim_h$.

DEFINITION 3.2.12. [**30**] *Set* $KK(A, B) = KK^0(A, B) = \mathbb{E}(A, B)/\sim_h$*. More generally, set* $KK^n(A, B) = KK^0(A, B\hat{\otimes}\mathbb{C}l_n)$*.*

REMARK 3.2.13. *Every degenerate element is homotopic to a* 0*-module. Thus, any degenerate elements represent* 0 *in* $KK(A, B)$*.*

KK groups $KK^n(A, B)$ has the following periodicity:

REMARK 3.2.14. [**30**] *It is known that one has* $KK^{n+2} \cong KK^n$ *by formal Bott periodicity:*

$$KK^1(A, B) \cong KK(A\hat{\otimes}\mathbb{C}l_1, B) \text{ and}$$

$$KK(A, B) \cong KK^1(A, B\hat{\otimes}\mathbb{C}l_1) \cong KK^1(A\hat{\otimes}\mathbb{C}l_1, B) \cong KK(A\hat{\otimes}\mathbb{C}l_1, B\hat{\otimes}\mathbb{C}l_1).$$

By this reason, we assume the upper script n *of* KK^n *is an element in* $\mathbb{Z}_2 = \{0, 1\}$*.*

It is also known that one has usual Bott periodicity:

$$KK^1(A, B) \cong KK(SA, B) \cong KK(A, SB) \text{ and}$$

$$KK(A,B) \cong KK^1(A,SB) \cong KK^1(SA,B) \cong KK(SA,SB).$$

Usual Bott periodicity is proved by using the Kasparov product.

A KK group $KK(A,B)$ is depends on gradings of A and B in general. However, if gradings of A and B are even, then $KK(A,B)$ is naturally isomorphic to $KK(A^{\mathrm{tri}}, B^{\mathrm{tri}})$.

DEFINITION 3.2.15. *Let $A = A^{(0)} \oplus A^{(1)}$ be a graded C^*-algebra. Then A is evenly graded if there exists a self-adjoint unitary element $\epsilon \in \mathbb{B}(B)$, which is called the even grading, such that $A^{(n)} = \{a \in A\,;\, \epsilon a = (-1)^n a\epsilon\}$.*

As noted in [**10**, §14.5, 17.8], KK theory for evenly graded C^*-algebra is same as the case for trivially graded. In particular, the natural identification $KK(A,B) \cong KK(A, B^{\mathrm{tri}})$ for evenly graded B is explicitly given by the following. We use it in Subsection 5.5.1 when A is trivially graded.

EXAMPLE 3.2.16. *We assume B is evenly graded. Let $(E \oplus E^{op}, \phi, F) \in \mathbb{E}(A,B)$ be a Kasparov (A,B)-module such that E is a countably generated graded Hilbert B-module with $E \oplus \hat{\mathbb{H}}_B \cong \hat{\mathbb{H}}_B$. We use the stabilization $E \oplus \hat{\mathbb{H}}_B \cong \hat{\mathbb{H}}_B$ and the even grading ϵ of B, then we can induce the even grading $\epsilon_E \in \mathbb{B}(\mathbb{B}(E)) = \mathbb{B}(E)$ of $\mathbb{B}(E)$ such that we have $\langle \epsilon_E e, f\rangle_E = (-1)^{\deg(e)}\epsilon\langle e, f\rangle_E$ for $e, f \in E$. In particular, we have $\epsilon_B = \epsilon$. We denote by $E^{\mathrm{tri}} = E$ as the trivially graded Hilbert B^{tri}-module. Define a map $U : E \oplus E^{op} \to E^{\mathrm{tri}} \oplus (E^{\mathrm{tri}})^{op}$ by*

$$U(e, e_{op}) = \left(\frac{1+\epsilon_E}{2}e + \frac{1-\epsilon_E}{2}e_{op}, \frac{1-\epsilon_E}{2}e + \frac{1+\epsilon_E}{2}e_{op}\right),$$

for $e \in E$ and $e_{op} \in E^{op}$. We can check U is a unitary operator as ungraded Hilbert B^{tri}-modules and $(E^{\mathrm{tri}} \oplus (E^{\mathrm{tri}})^{op}, U\phi U^*, UFU^*)$ is a Kasparov (A, B^{tri})-module. In fact, this construction induces the isomorphism of KK groups:

$$U_A : KK^0(A,B) \to KK^0(A, B^{\mathrm{tri}}).$$

More generally, $KK^n(A,B)$ can be identified with $KK^n(A, B^{\mathrm{tri}})$ by using usual Bott periodicity.

Any Kasparov (A,B)-module can be normalized as follows:

REMARK 3.2.17. *Let $x = [E, \phi, F] \in KK^0(A,B)$. Then there exists a self-adjoint operator $G \in \mathbb{B}(E)$ such that $\|G\| \le 1$ and $x = [E, \phi, G] \in KK^0(A,B)$; see* [**10**, Proposition 17.4.3].

By using the normalization in Remark 3.2.17, the isomorphism U_A in Example 3.2.16 is simplified when A is trivially graded.

REMARK 3.2.18. *We assume A is trivially graded and B is evenly graded. By the normalization in Remark 3.2.17, every element $x \in KK^0(A,B)$ is represented by $x = [E, \phi, F]$, where $F \in \mathbb{B}(E)$ is a self-adjoint operator with $\|F\| \leq 1$. Adding a degenerate module $(E^{op}, 0, F)$, we have*

$$x = [E \oplus E^{op}, \phi \oplus 0, F \oplus F] \in KK^0(A,B).$$

Thus we have $x = [E \oplus E^{op}, \phi \oplus 0, G]$ since $(G - F \oplus F)\phi(a)$ is a B-compact operator, where we set

$$G = \begin{bmatrix} F & \epsilon_E(1-F^2)^{1/2} \\ \epsilon_E(1-F^2)^{1/2} & F \end{bmatrix}.$$

Set $F' = F + \epsilon_E(1-F^2)^{1/2}$. Then we have

$$x = \left[E^{\mathrm{tri}} \oplus (E^{\mathrm{tri}})^{op}, \frac{1+\epsilon_E}{2}\phi \oplus \frac{1-\epsilon_E}{2}\phi, \begin{bmatrix} 0 & F' \\ F' & 0 \end{bmatrix}\right] \in KK^0(A, B^{\mathrm{tri}})$$

under the isomorphism $U_A : KK^0(A,B) \cong KK^0(A, B^{\mathrm{tri}})$ in Example 3.2.16. In particular, every element in $KK^0(A,B)$ can be represented by a quasihomomorphism by conjugating of $F' \oplus 1$ and the stabilization $E^{\mathrm{tri}} \oplus (E^{\mathrm{tri}})^{op} \cong \hat{\mathbb{H}}_{B^{\mathrm{tri}}}$. We often omit ${}^{\mathrm{tri}}$ in the sequel when B is evenly graded.

In the last of this subsection, we see relations to K groups, K homology groups and extensions.

EXAMPLE 3.2.19. *If B is evenly graded, then we have $KK^n(\mathbb{C}, B) \cong KK^n(\mathbb{C}, B^{\mathrm{tri}}) = K_n(B)$. In fact, we assume $a, b \in P_\infty(B^+)$ satisfy $a - b \in M_\infty(B)$. Then we have $[a] - [b] \in K_0(B)$. Two homomorphisms $\phi_0, \phi_1 : \mathbb{C} \to M_\infty(B^+)$ defined by $\phi_0(1) = a$ and $\phi_1(1) = b$ determine a quasihomomorphism $(\phi_0, \phi_1) : \mathbb{C} \to \mathbb{B}(\mathbb{H}_B) \rhd B \otimes \mathcal{K}$.*

REMARK 3.2.20. *We have $KK(\mathbb{C}, \mathbb{C}l_1) = KK^1(\mathbb{C}, \mathbb{C}) = \{0\}$ and $K_0(\mathbb{C}l_1) = K_0(\mathbb{C} \oplus \mathbb{C}) \cong K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus $KK^n(\mathbb{C}, B)$ is not isomorphic to $K_n(B)$ in general when B is not evenly graded.*

EXAMPLE 3.2.21. *If A is evenly graded, then we have $K^n(A, \mathbb{C}) \cong K^n(A)$. In particular, any element in $K^0(A)$ is represented by (H, ϕ, F), where H is a graded Hilbert space, $\phi : A \to \mathcal{L}(H)$ is a graded representation of A and $F : H \to H$ satisfies $(F^2 - 1)\phi(a), (F^* - F)\phi(a), [F, \phi(a)]_s \in \mathcal{K}(H)$. On the other hand, any element in $K^1(A)$ is represented by (H, ψ, T), where H is an ungraded Hilbert space, $\psi : A \to \mathcal{L}(H)$ is a representation of A and T satisfies $(T^2 - 1)\phi(a), (T^* - T)\phi(a), [T, \psi(a)] \in \mathcal{K}(H)$.*

EXAMPLE 3.2.22. *We assume A and B are trivially graded. Then every element in $KK^1(A, B)$ is represented by a pair (ψ, P) satisfies the following:*

- $\psi : A \to \mathbb{B}(\mathbb{H}_B)$ *is a $*$-homomorphism*
- $P \in \mathbb{B}(\mathbb{H}_B)$
- $(P^2 - P)\psi(a) \in \mathbb{K}(\mathbb{H}_B)$, $(P^* - P)\psi(a) \in \mathbb{K}(\mathbb{H}_B)$ *and* $[P, \psi(a)] \in \mathbb{K}(\mathbb{H}_B)$ *for* $a \in A$.

We see this pair defines a Kasparov module. In fact, $(\mathbb{H}_B \oplus \mathbb{H}_B, \psi \oplus \psi, (2P - 1) \oplus (1 - 2P))$ *is a Kasparov* $(A, B\hat{\otimes}\mathbb{C}l_1)$*-module. Here, the grading of* $\mathbb{H}_B \oplus \mathbb{H}_B$ *is defined by* $(\mathbb{H}_B \oplus \mathbb{H}_B)^{(n)} = \{(u, (-1)^n u)\}$. *Thus, it defines an element in* $KK^1(A, B)$.

On the other hand, such pair (ψ, P) *defines an extension of* A *by* $B \otimes \mathcal{K}$:

$$0 \to B \otimes \mathcal{K} \to \mathcal{E} \to A \to 0.$$

Here, we set

$$\mathcal{E} = \{(a, b) \in A \oplus \mathbb{B}(\mathbb{H}_B)\, ;\, P\psi(a)P - b \in \mathbb{K}(\mathbb{H}_B)\}.$$

3.2.2. The Kasparov product. We review the Kasparov product. In general, the Kasparov product defines a bilinear map

$$\hat{\otimes}_D : KK^n(A_1, B_1\hat{\otimes}D) \times KK^m(D\hat{\otimes}A_2, B_2) \to KK^{n+m}(A_1\hat{\otimes}A_2, B_1\hat{\otimes}B_2).$$

However, we do not use this general form. We use it only in the case when $A_1 = A_2 = B_1 = \mathbb{C}$, $D = A$ is trivially graded and $B_2 = B$ is evenly graded in the statement and the proof of the main theorem. We note that the isomorphism in Example 3.2.16 commutes with the Kasparov product, that is, we have

$$U_{\mathbb{C}}(x\hat{\otimes}_A y) = x\hat{\otimes}_A U_A(y) \in KK^{n+m}(\mathbb{C}, B^{\mathrm{tri}}) = K_{n+m}(B)$$

for $x \in KK^n(\mathbb{C}, A) = K_n(A)$ and $y \in KK^m(A, B)$. Thus, the following Kasparov product can be obtained by using $KK(A, B^{\mathrm{tri}})$:

$$\hat{\otimes}_A : K_n(A) \times KK^m(A, B) \to K_{n+m}(B)$$

when A is trivially graded and B is evenly graded.

First, we review the Kasparov product $\hat{\otimes}_A : K_n(A) \times KK^0(A, B) \to K_n(B)$. We use this case in the main theorem. The Cuntz picture gives clear formulation of the Kasparov product [**20**, Remark 1, Theorem 3.3]. See also [**32**, Chapter 3], which contains an explicit formula.

EXAMPLE 3.2.23 (The case for $n = 0$). *Take* $x = [p] \in K_0(A)$ *for any projection* $p \in P_\infty(A)$. *Let* $(\phi_0, \phi_1) : A \to \mathbb{B}(\mathbb{H}_B) \rhd B \otimes \mathcal{K}$ *be a quasihomomorphism. Then we have* $\phi_0(p) - \phi_1(p) \in B \otimes \mathcal{K}$. *This implies we have* $[\phi_0(p)] - [\phi_1(p)] \in K_0(B \otimes \mathcal{K}) \cong K_0(B)$. *Then we have* $x\hat{\otimes}_A[\phi_0, \phi_1] = [\phi_0(p)] - [\phi_0(q)] \in K_0(B)$.

EXAMPLE 3.2.24 (The case for $n = 1$). *Set $x = [u] \in K_1(A)$ for $u \in GL_\infty(A)$. Let $(\phi_0, \phi_1) : A \to \mathbb{B}(\mathbb{H}_B) \triangleright B \otimes \mathcal{K}$ be a quasihomomorphism. We extend ϕ_0 and ϕ_1 as unital $*$-homomorphisms to $M_\infty(A^+)$. We denote this extension by the same letter. Namely, we set $\phi_i([a_{jk} + \lambda_{jk}]) = [\phi_i(a_{jk}) + \lambda_{jk}]$. Therefore, we have $\phi_0(u)\phi_1(u)^{-1} \in GL_\infty(B \otimes \mathcal{K})$. Then we have $x\hat{\otimes}_A[\phi_0, \phi_1] = [\phi_0(u)\phi_1(u)^{-1}] \in K_1(B)$.*

The Kasparov product corresponds to the Fredholm index of the Dirac operator as follows:

EXAMPLE 3.2.25. *Let $D : L^2(N, S) \to L^2(N, S)$ be a graded Dirac operator on a closed manifold N and $i : \mathbb{C} \to C(N)$ an inclusion map. D defines a K-homology element $[D] = [L^2(N, S), \psi, D(D^2 + 1)^{-1/2}] \in KK^0(C(N), \mathbb{C})$ and i defines $1 \in K_0(C(N))$, a class of a constant function. Then we have $1\hat{\otimes}_{C(N)}[D] = \mathrm{index}(D^+)$.*

Next, we review the case when $m = 1$. This case is related to the Fredholm index of the Toeplitz operator. As we can see in Example 3.2.22, $[\psi, P] \in KK^1(A, B)$ defines an extension: $0 \to B \otimes \mathcal{K} \to \mathcal{E} \to A \to 0$. In this case, the Kasparov product is calculated by connecting homomorphisms.

EXAMPLE 3.2.26 (Extensions and the Kasparov product). *Let $x \in K_0(A)$ and $y \in K_1(A)$. Then we have $x\hat{\otimes}_A[\psi, P] = \delta(x) \in K_1(B)$ and $y\hat{\otimes}_A[\psi, P] = \partial(y) \in K_0(B)$. Here, δ (resp. ∂) is an exponential map (resp. index map) defined by the short exact sequence as in Example 3.2.22.*

By using this formula, we see a relationship between the Fredholm index of the Toeplitz operator with the Kasparov product.

EXAMPLE 3.2.27 (Toeplitz operators and the Kasparov product). *Let $D : L^2(N, S) \to L^2(N, S)$ be the Dirac operator on a closed manifold N. D defines an element*

$$[D] = [L^2(N, S) \oplus L^2(N, S), \psi \oplus \psi, D(D^2 + 1)^{-1/2} \oplus (-D(D^2 + 1)^{-1/2})]$$

in $KK^1(C(N), \mathbb{C})$; see Example 3.2.8. By the spectral decomposition of D, $[D]$ is equal to $[\psi, P] \in KK^1(C(N), \mathbb{C})$. Here, P is a spectral projection of D onto $[0, \infty)$ as in Section 2.3. This pair (ψ, P) defines the Toeplitz extension

$$0 \to \mathcal{K}(H_+) \to \mathcal{T} \xrightarrow{\pi} C(N) \to 0,$$

where $\mathcal{T}$ is a C^*-algebra generated by Toeplitz operators and $\mathcal{K}(H_+)$.

Let $\phi \in C(N; U_l(\mathbb{C}))$. Then we have $[\phi]\hat{\otimes}_{C(N)}[D] = \partial([\phi]) \in K_0(\mathbb{C})$. Moreover, we have $\partial([\phi]) = \mathrm{index}(T_\phi) \in \mathbb{Z}$ since $\pi(T_\phi) = \phi$. This is proved by a similar argument in Section 3.1.

3.3. The cyclic cohomology and Connes' pairing

As explained above, we can generalize the notion of the Fredholm index as an element in K groups. We call this element an index class. However, it is hard to check that an index class vanishes or not in general. By this reason, we need tools to pick up some numerical information from an index class. We use the cyclic cohomology and Connes' pairing map with K-theory [**18**]. We deal with only the pairing with a K_1 group since we use it only in this case in the main theorem.

3.3.1. The definition of the cyclic cohomology. We recall the definition of the cyclic cohomology.

Proposition 3.3.1. [**18**, p.101] *Let A be an associative algebra over $\mathbb{C}$.*

- *Let $\phi : A^{n+1} \to \mathbb{C}$ be an $(n+1)$-multilinear map, and set*

$$b\phi(a_0, \ldots, a_{n+1}) = \sum_{j=0}^{n} (-1)^j \phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1}\phi(a_{n+1}a_0, a_1, \ldots, a_n).$$

  Then $b\phi$ is an $(n+2)$-multilinear map on A and we have $b^2\phi = bb\phi = 0$.
- *Assume that an $(n+1)$-multilinear map $\phi : A^{n+1} \to \mathbb{C}$ satisfies*

$$\phi(a_0, \ldots, a_n) = (-1)^n \phi(a_n, a_0, \ldots, a_{n-1}). \tag{1}$$

  Then we have

$$b\phi(a_0, \ldots, a_{n+1}) = (-1)^{n+1} b\phi(a_{n+1}, a_0, \ldots, a_n).$$

The condition (1) is called a cyclic condition. By using this proposition, we can define the cyclic cohomology.

Definition 3.3.2. [**18**, p.102] *Let A be an associative algebra over $\mathbb{C}$. Set*

$$C^n_\lambda(A) = \{\phi : A^{n+1} \to \mathbb{C}\,;\, \phi \text{ is } (n+1)\text{-multilinear and satisfies a cyclic condition}\}.$$

By Proposition 3.3.1, b defines a linear map $b : C^n_\lambda(A) \to C^{n+1}_\lambda(A)$. Thus $C^*_\lambda(A) = \{(C^n_\lambda(A), b)\}_{n\geq 0}$ is a cochain complex.

We call an element in $Z^n_\lambda(A) = \mathrm{Ker}(b : C^n_\lambda(A) \to C^{n+1}_\lambda(A))$ a cyclic n-cocycle. Moreover, the cohomology group $H^*_\lambda(A)$ of the cochain complex $C^*_\lambda(A) = \{(C^n_\lambda(A), b)\}_{n\geq 0}$ is called the cyclic cohomology group of A.

3.3.2. Connes' pairing with a K_1 group. In this subsection, we review Connes' pairing with cyclic $(2m-1)$-cocyle with a K_1 group. In the main theorem, we use it in the case when $m = 1$.

Let A be a Banach algebra. For any $\phi \in C^n_\lambda(A)$, we set

$$\tilde{\phi}(a_0 + \lambda_0, a_1 + \lambda_1, \ldots, a_n + \lambda_n) = \phi(a_0, a_1, \ldots, a_n)$$

for $a_j \in A$ and $\lambda_j \in \mathbb{C}$. Then we have $\tilde{\phi} \in C^n_\lambda(A^+)$. We often denote $\tilde{\phi}$ by the same letter ϕ. Under this assumption, we can define Connes' pairing of a cyclic cohomology with a K_1 group.

Definition 3.3.3. [**18**, p.109], [**19**, §3.3, Corollary 4] *Let A be a Banach algebra. The following linear map $\langle \cdot, \cdot \rangle : K_1(A) \times H^{\mathrm{odd}}_\lambda(A) \to \mathbb{C}$ is well defined:*

$$\begin{aligned}&\langle [u], [\phi] \rangle \\ =&\frac{2^{-2m-1}(2\pi i)^{-m}}{(m-1/2)(m-3/2)\ldots 1/2} \sum_{1 \le j_0, \cdots, j_{2m} \le k} \phi(u^{-1}_{j_0 j_1}, u_{j_1 j_2}, \cdots, u^{-1}_{j_{2m-1} j_{2m}}, u_{j_{2m} j_0})\end{aligned}$$

for $[u] \in K_1(A)$ and $[\phi] \in H^{2m-1}_\lambda(A)$. Here, we assume $u = [u_{jk}]_{jk} \in GL_l(A)$.

We use Connes' pairing like the following.

Remark 3.3.4. [**18**, p.92] *Let A and $\mathscr{A}$ be two Banach algebras. We assume $\mathscr{A}$ is a subalgebra in A and closed under holomorphic functional calculus in A. Let $\phi : \mathscr{A}^{2m-1} \to \mathbb{C}$ be a cyclic $(2m-1)$-cocycle on a Banach algebra $\mathscr{A}$. The domain of ϕ may not be extended to A. However, if $\mathscr{A}$ is dense in A, then the inclusion $\mathscr{A} \to A$ induces the isomorphism $K_1(\mathscr{A}) \cong K_1(A)$. Thus Connes' pairing induces the following linear map:*

$$\langle \cdot, [\phi] \rangle : K_1(A) \to \mathbb{C}.$$

For example, the ideal of operators of Schatten p-class is closed under holomorphic functional calculus in the C^*-algebra of compact operators. Similar to this property, we obtain the following:

Example 3.3.5. [**18**, p.92 Proposition 3] *Let A be a Banach algebra and H a countably infinite dimensional Hilbert space. We assume $\phi : A \to \mathcal{L}(H)$ is an action of A on H. Let $F \in \mathcal{L}(H)$ satisfies $F^2 = 1$, $F^* = F$ and $[F, \phi(a)] \in \mathcal{K}(H)$ for any $a \in A$. Such a triple (H, ϕ, F) is called a Fredholm module over A. Set $\mathscr{A}_p = \{a \in A\,;\, [F, \phi(a)]$ is of Schatten p-class $\}$. Then $\mathscr{A}_p$ is a Banach algebra with a norm $\|a\|_{\mathscr{A}_p} = \|\phi(a)\| + \|[F, \phi(a)]\|_p$, where $\|T\|_p = \mathrm{Tr}(|T|^p)^{1/p}$ is a Schatten p-norm. Then $\mathscr{A}_p$ is closed under holomorphic functional calculus in A.*

CHAPTER 4

The Roe-Higson index theorem

In this section, we review the index theorem for partitioned manifolds due to J. Roe and N. Higson. We call this theorem "the Roe-Higson index theorem". For bounded operators T and S, $T \sim S$ means that $T - S$ is a compact operator.

4.1. The Roe algebra

In this section, we review the Roe algebra, which is a C^*-algebra introduced by Roe [**34**]. The definition in [**34**] makes sense for a complete Riemannian manifold. Today, we can extend the notion of the Roe algebra to coarse spaces, for example, proper metric spaces; see [**27**]. We use a definition of the later one. Of course, its definition coincides with Roe's first one; see Remark 4.1.10.

4.1.1. The definition of the Roe algebra. Let (M, g) be a complete Riemannian manifold, d a complete metric defined by g and $S \to M$ a Hermitian vector bundle over M. Firstly, we introduce the notion of finite propagation.

Definition 4.1.1. [**27**, p.148, Definition 6.3.3] *Let $T \in \mathcal{L}(L^2(M, S))$ be a bounded operator and $U, V \subset M$ a non-empty open subset. T is 0 on $U \times V$ if one has $fTg = 0$ for any $f \in C_0(U)$ and $g \in C_0(V)$. Set*

$$\mathrm{Supp}(T) = (M \times M) \setminus \bigcup \Big\{ U \times V\, ;\, U, V \subset M \text{ is open and } T \text{ is } 0 \text{ on } U \times V \Big\}.$$

We call $\mathrm{Supp}(T)$ *the support of T.*

Definition 4.1.2. [**27**, p.152] *For any $T \in \mathcal{L}(L^2(M, S))$, we set*

$$\mathrm{Prop}(T) = \sup\{d(x, y)\, ;\, (x, y) \in \mathrm{Supp}(T)\}.$$

It is called the propagation of T. If we have $\mathrm{Prop}(T) < \infty$*, we call T has finite propagation.*

The propagation of $T \in \mathcal{L}(L^2(M, S))$ measures expansion of the support of a section.

Proposition 4.1.3. *Let $T \in \mathcal{L}(L^2(M, S))$. The followings are equivalent:*

- *We have* $\mathrm{Prop}(T) \le R$*.*
- *We have* $\mathrm{Supp}(Ts) \subset (\mathrm{Supp}(s))_R$ *for any* $s \in C_c^\infty(M,S)$*. Here, we set* $(\mathrm{Supp}(s))_R = \{x \in M\,;\, d(x, \mathrm{Supp}(s)) \le R\}$*.*

EXAMPLE 4.1.4. [**27**, Example 6.3.4] *Let* $T \in \mathcal{L}(L^2(M,S))$ *has a continuous kernel* $k \in C(M \times M, S \boxtimes S^*)$*. Namely,* T *forms as follows:*

$$Ts(x) = \int_M k(x,y)s(y)dy.$$

Then we have $\mathrm{Prop}(T) \le R$ *if and only if we have* $\mathrm{Supp}(k) \subset \Delta(M)_R$*. Here, we set* $\Delta(M)_R = \{(x,y) \in M \times M\,;\, d(x,y) \le R\}$*.*

EXAMPLE 4.1.5. [**27**, Proposition 10.3.1]*,* [**36**, Proposition 7.20] *Let* D *be the Dirac operator on* M*. Then we have* $\mathrm{Prop}(e^{itD}) \le |t|$*.*

Combining finite propagation and the following compactness condition, we define the Roe algebra.

DEFINITION 4.1.6. *Let* $T \in \mathcal{L}(L^2(M,S))$*.*

- T *is pseudolocal if* $[f,T] \sim 0$ *for any* $f \in C_0(M)$*.*
- T *is locally compact if* $fT \sim 0$ *and* $Tf \sim 0$ *for any* $f \in C_0(M)$*.*

Of course, locally compactness implies pseudolocality. Since these properties are closed under the operations of $*$-algebras, we have the following.

DEFINITION 4.1.7. [**27**, Definition 6.3.8] *Set*

$$\mathcal{D}^*(M) = \{T \in \mathcal{L}(L^2(M,S))\,;\, T \textit{ has finite propagation and is pseudolocal}\}$$

and

$$\mathcal{C}^*(M) = \{T \in \mathcal{L}(L^2(M,S))\,;\, T \textit{ has finite propagation and is locally compact}\}.$$

Taking completions of these algebras, we define

$$D^*(M) = \overline{\mathcal{D}^*(M)}^{\|\cdot\|}, \quad C^*(M) = \overline{\mathcal{C}^*(M)}^{\|\cdot\|}.$$

We call $C^*(M)$ *the Roe algebra.*

REMARK 4.1.8. *Pseudolocality and locally compactness are closed conditions, that is, every* $u \in D^*(M)$ *is pseudolocal and every* $u \in C^*(M)$ *is locally compact.* $\mathcal{D}^*(M)$ *is a unital* $*$*-algebra and* $\mathcal{C}^*(M)$ *is an ideal in* $\mathcal{D}^*(M)$*. Thus* $D^*(M)$ *is a unital* C^**-algebra and* $C^*(M)$ *is an ideal in* $D^*(M)$*. In particular,* $C^*(M)$ *is a* C^**-algebra.*

REMARK 4.1.9. *We assume* M *is compact. In this case, we have* $C^*(M) = \mathcal{K}(L^2(M,S))$ *by* $1 \in C_0(M) = C(M)$*.*

The last of this subsection, we remark on Roe's first definition of the Roe algebra.

REMARK 4.1.10. *We denote by* $\mathscr{X}$ *the* $*$*-subalgebra of* $\mathcal{L}(L^2(M,S))$ *with the element has a smooth integral kernel and finite propagation. We denote by* $\overline{\mathscr{X}}$ *the closure of* $\mathscr{X}$ [**34**, Definition 1.2]. *I think the fact that* $C^*(M) = \overline{\mathscr{X}}$ *is well known, but not well documented. So I write an outline of a proof.*

We define $\mathcal{H} \subset \mathcal{L}(L^2(M,S))$ *by the following.* $T \in \mathcal{H}$ *if and only if* T *has finite propagation and is an integral operator with an integral kernel* k *satisfies the following property:* $k|_{K\times M}, k|_{M\times K} \in L^2(M\times M, S\boxtimes S^*)$ *for any compact set* $K \subset M$*. Firstly,* $\mathcal{H}$ *is dense in* $\mathcal{C}^*(M)$*, since the set of Hilbert-Schmidt class operators is dense in the set of compact operators and every Hilbert-Schmidt class operator on* $L^2(M,S)$ *has an* L^2 *kernel. Secondly,* $\mathcal{X}$ *is dense in* $\mathcal{H}$*, since the set of compactly supported smooth sections is dense in the set of* L^2 *sections. These two properties imply* $C^*(M) = \overline{\mathcal{H}} = \overline{\mathscr{X}}$*.*

4.1.2. Functional calculus and the Roe algebra. In this subsection, we obtain elements in $D^*(M)$ and $C^*(M)$ by the functional calculus. Let $D : L^2(M,S) \to L^2(M,S)$ be the Dirac operator over a complete Riemannian manifold M and $A \in \mathrm{End}(S)$ a self-adjoint endomorphism. Then we have $\mathrm{Prop}(e^{it(D+A)}) \leq |t|$. We often denote $D + A$ by the same letter D. Let $\mathscr{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ be the Fourier transformation:

$$\hat{f}(\xi) = \mathscr{F}[f](\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

As well known, the Fourier transformation is extended as a continuous linear map $\mathscr{F} : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$.

PROPOSITION 4.1.11. [**27**, Lemma 10.5.5] *Let* $f \in C_b(\mathbb{R})$ *satisfies* $\mathrm{Supp}(\hat{f}) \subset (-R,R)$*. Then we have* $\mathrm{Prop}(f(D)) \leq R$*.*

PROOF. By the Fourier inversion formula, we have

$$\langle f(D)\sigma, \tau\rangle = \frac{1}{2\pi}\langle \hat{f}(t), \langle e^{itD}\sigma, \tau\rangle_{L^2}\rangle_t$$

for $\sigma, \tau \in C_c^\infty(M,S)$. Take $\sigma \in C_c^\infty(M,S)$ and $\tau \in C_c^\infty(M,S)$ satisfy $\mathrm{Supp}(\tau) \subset ((\mathrm{Supp}(\sigma))_R)^c$. By $\mathrm{Supp}(e^{itD}\sigma) \subset (\mathrm{Supp}(\sigma))_{|t|}$, we have $\langle e^{itD}\sigma, \tau\rangle_{L^2} = 0$ for $|t| \leq R$. Thus the support of the smooth function

$$t \mapsto \langle e^{itD}\sigma, \tau\rangle_{L^2}$$

is contained in $(-R,R)^c$. On the other hand, since the support of $\hat{f}$ is contained in $(-R,R)$, we have

$$\langle f(D)\sigma, \tau\rangle = \frac{1}{2\pi}\langle \hat{f}(t), \langle e^{itD}\sigma, \tau\rangle_{L^2}\rangle_t = 0.$$

This implies $\mathrm{Prop}(f(D)) \leq R$. □

By using Proposition 4.1.11, we get an element in the Roe algebra.

PROPOSITION 4.1.12. [**34**, Proposition 2.3] *We have* $f(D) \in C^*(M)$ *for any* $f \in C_0(\mathbb{R})$.

PROOF. Set $\phi_\pm(x) = 1/(x \pm i)$. Take $g \in C_c^\infty(M)$ and a compact set $K \subset M$ satisfying $\mathrm{Supp}(g) \subset K$. Then we have

$$\begin{aligned} &\|g\phi_\pm(D)s\|_{H^1(K,S)} \\ \leq\ &\|g(D \pm i)^{-1}s\|_{L^2} + \|gD(D \pm i)^{-1}s\|_{L^2} + \|[D,g](D \pm i)^{-1}s\|_{L^2} \\ \leq\ &2(\|g\| + \|\mathrm{grad}(g)\|)\|s\|_{L^2} \end{aligned}$$

for any $s \in L^2(M,S)$. By using the Rellich lemma, this implies $g\phi_\pm(D) \sim 0$. Thus we have $g\phi_\pm(D) \sim 0$ for any $g \in C_0(M)$ since $C_c^\infty(M)$ is dense in $C_0(M)$. On the other hand, we have $\phi_\pm(D)g = (\bar{g}\phi_\mp(D))^* \sim 0$.

We note that $C_0(\mathbb{R})$ is generated by $\phi_+(x)$ as a C^*-algebra by the Stone-Weierstrass theorem for a locally compact Hausdorff space. Thus, we obtain $gf(D) \sim 0$ and $f(D)g \sim 0$. Therefore, $f(D)$ is a locally compact.

We approximate $f(D)$ by an operator of a locally compact with finite propagation. Take $0 < \epsilon < 1/4$. Then, there exists $\phi \in \mathscr{S}(\mathbb{R})$ such that $\|f - \phi\| < \epsilon$ since $\mathscr{S}(\mathbb{R})$ is dense in $C_0(\mathbb{R})$. Moreover, since $C_c^\infty(\mathbb{R})$ is dense in $\mathscr{S}(\mathbb{R})$, there exists $\psi \in \mathscr{S}(\mathbb{R})$ such that we have $\hat{\psi} \in C_c^\infty(\mathbb{R})$ and $d_{\mathscr{S}}(\hat{\phi}, \hat{\psi}) < \epsilon$. Here, $d_{\mathscr{S}}$ is a distance on $\mathscr{S}(\mathbb{R})$ defined by the following system of semi norms:

$$\mathfrak{p}_m(\phi) = \sum_{\substack{r,\alpha \geq 0 \\ 0 \leq \alpha + r \leq m}} \sup_{x \in \mathbb{R}} |(1 + x^2)^r \phi^{(\alpha)}(x)|.$$

Thus, we have

$$\|f - \psi\| \leq \|f - \phi\| + \|\phi - \psi\| \leq \|f - \phi\| + \frac{1}{2}\mathfrak{p}_1(\hat{\phi} - \hat{\psi}).$$

Now, we have $\mathfrak{p}_1(\hat{\phi} - \hat{\psi}) < 4\epsilon$ by

$$\frac{1}{2}\frac{\mathfrak{p}_1(\hat{\phi} - \hat{\psi})}{1 + \mathfrak{p}_1(\hat{\phi} - \hat{\psi})} \leq d_{\mathscr{S}}(\hat{\phi}, \hat{\psi}) < \epsilon.$$

Thus, we have $\|f(D) - \psi(D)\| \leq \|f - \psi\| < 3\epsilon$. This implies $f(D) \in C^*(M)$ since $\psi(D)$ is an element in $\mathcal{C}^*(M)$.

□

We got an element in $C^*(M)$, but we have to use an element in $D^*(M)$. The class of functions which makes an element in $D^*(M)$ is the following:

DEFINITION 4.1.13. *We define* $\chi \in \mathcal{S}$ *if we have* $\chi \in C(\mathbb{R};[-1,1])$ *and* $\lim_{x\to\pm\infty}\chi(x) = \pm 1$.

This class $\mathcal{S}$ contains chopping functions, that is, $[-1,1]$-valued continuous odd functions which satisfy $\lim_{x\to\pm\infty}\chi(x) = \pm 1$. Moreover, it contains good functions as follows:

EXAMPLE 4.1.14. [**27**, Exercises 10.9.3] *Let* $g \in C_c^\infty(\mathbb{R};\mathbb{R})$ *be an odd function which satisfies* $\mathrm{Supp}(g) \subset [-R,R]$ *and* $g \neq 0$. *Set* $f = g * g$ *and we assume* $f(0) = 1/\pi$. *Set*

$$\chi(x) = \int_{-\infty}^{\infty} \frac{e^{itx}-1}{it} f(t)dt.$$

Then we have $\chi \in \mathcal{S}$, χ *is a smooth monotone function, and we have* $\chi(x) > 0$ *for any* $x > 0$. *Moreover, the Fourier transformation* $\hat{\chi} \in \mathscr{S}'(\mathbb{R})$ *satisfies* $\mathrm{Supp}(\hat{\chi}) \subset [-2R,2R]$ *and* $x\hat{\chi} \in C_c^\infty(\mathbb{R})$.

PROOF. By definition, f is an even function and we have $f \in C_c^\infty(\mathbb{R};\mathbb{R})$ and $\mathrm{Supp}(f) \subset [-2R,2R]$. Thus χ is a smooth function. Because of

$$\chi'(x) = \int_{-\infty}^{\infty} e^{itx} f(t)dt = 2\pi\mathscr{F}^{-1}[f](x) = 2\pi(\mathscr{F}^{-1}[g](x))^2 > 0,$$

χ is monotone increasing.

On the other hand, χ is an odd function by

$$\begin{aligned}\chi(-x) &= \int_{-\infty}^{\infty} \frac{e^{-itx}-1}{it} f(t)dt = \int_{-\infty}^{\infty} \frac{e^{-itx}-1}{it} f(-t)dt \\ &= \int_{-\infty}^{\infty} \frac{e^{itx}-1}{-it} f(t)dt = -\chi(x).\end{aligned}$$

Combining $\chi' > 0$, we have $\chi(x) > 0$ for any $x > 0$.

Since χ is an odd function and we have

$$\begin{aligned}2\lim_{x\to\infty}\chi(x) &= \lim_{x\to\infty}\chi(x) - \lim_{x\to-\infty}\chi(x) = \int_{-\infty}^{\infty}\chi'(x)dx \\ &= \int_{-\infty}^{\infty} 2\pi\mathscr{F}^{-1}[f](x)dx \\ &= 2\pi\mathscr{F}[\mathscr{F}^{-1}[f]](0) = 2\pi f(0) = 2,\end{aligned}$$

we have $\lim_{x\to\pm\infty}\chi(x) = \pm 1$.

$x\hat{\chi} \in C_c^\infty(\mathbb{R})$ and $\mathrm{Supp}(x\hat{\chi}) \subset [-2R, 2R]$ is proved by

$$x\hat{\chi}(x) = -i\hat{\chi'}(x) = -i\mathscr{F}[2\pi\mathscr{F}^{-1}[f]](x) = -2\pi i f(x).$$

Let $\phi \in \mathscr{S}(\mathbb{R})$ satisfies $\mathrm{Supp}(\phi) \subset [-2R, 2R]^c$. Then we have $\phi/x \in \mathscr{S}(\mathbb{R})$. Thus, we have $\mathrm{Supp}(\hat{\chi}) \subset [-2R, 2R]$ since $\langle \hat{\chi}, \phi \rangle = \langle x\hat{\chi}, \phi/x \rangle = 0$. $\square$

We prove an element in $\mathcal{S}$ makes an element in $D^*(M)$. For this purpose, we need two lemmas.

LEMMA 4.1.15. [**27**, Lemma 10.6.3] *Take $\chi_1, \chi_2 \in \mathcal{S}$. Then we have $\chi_1(D)g \sim \chi_2(D)g$ and $g\chi_1(D) \sim g\chi_2(D)$ for any $g \in C_0(M)$.*

LEMMA 4.1.16 (The Kasparov lemma). [**27**, Lemma 5.4.7] *Let $T \in \mathcal{L}(L^2(M, S))$. Then, the followings are equivalent.*

- *We have $[T, f] \in \mathcal{K}(L^2(M, S))$ for any $f \in C_0(M)^+$.*
- *We have $fTg \in \mathcal{K}(L^2(M, S))$ for any $f, g \in C_0(M)^+$ with $\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \emptyset$.*

Here, $C_0(M)^+$, the set of continuous functions defined on M constant at infinity, acts on $L^2(M, S)$ as a multiplication operator.

PROPOSITION 4.1.17. [**27**, Lemma 10.6.4] *We have $\chi(D) \in D^*(M)$ for any $\chi \in \mathcal{S}$.*

PROOF. First, we prove $\chi(D)$ is pseudolocal. It suffices to show that $f\chi(D)g \sim 0$ for any $f, g \in C_0(M)^+$ satisfying $\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = \emptyset$ and $\mathrm{Supp}(g) \subset M$ by the Kasparov lemma 4.1.16. Take $R > 0$ satisfying $d(\mathrm{Supp}(f), \mathrm{Supp}(g)) > R$. Then, $|t| \leq R$ implies $fe^{itD}g = 0$. Thanks to Example 4.1.14, there exists $\chi_1 \in \mathcal{S}$ such that $\mathrm{Supp}(\hat{\chi}_1) \subset [-R, R]$. Then we have $f\chi_1(D)g = 0$ by Proposition 4.1.11. By Lemma 4.1.15, we have $0 = f\chi_1(D)g \sim f\chi(D)g$. This implies $\chi(D)$ is pseudolocal.

Combining Proposition 4.1.11, we have $\chi_1(D) \in D^*(M)$. Now, $f = \chi - \chi_1 \in C_0(\mathbb{R})$ and $f(D) \in C^*(M) \subset D^*(M)$ implies $\chi(D) = f(D) + \chi_1(D) \in D^*(M)$. $\square$

4.2. The odd index

In this section, we review the odd index. It is defined by an exponential map of this short exact sequence:

$$0 \to C^*(M) \to D^*(M) \to D^*(M)/C^*(M) \to 0.$$

Take $\chi \in \mathcal{S}$. By $\chi^2 - 1 \in C_0(\mathbb{R})$, we have $\chi(D)^2 \equiv 1 \mod C^*(M)$. Thus we obtain an element $[(\chi(D) + 1)/2] \in K_0(D^*(M)/C^*(M))$.

Now, if we take $\chi_1, \chi_2 \in \mathcal{S}$, then we have $\chi_1(D) - \chi_2(D) \in C^*(M)$ by $\chi_1 - \chi_2 \in C_0(\mathbb{R})$. Therefore, a K-theory class $[(\chi(D) + 1)/2] \in K_0(D^*(M)/C^*(M))$ is independent of the choice of an element in $\mathcal{S}$.

We send an element $[(\chi(D) + 1)/2] \in K_0(D^*(M)/C^*(M))$ by an exponential map $\delta : K_0(D^*(M)/C^*(M)) \to K_1(C^*(M))$ defined by the above short exact sequence. Thus we obtain an element in $K_1(C^*(M))$.

DEFINITION 4.2.1. [**34**, Definition 2.7] *Let* $D : L^2(M, S) \to L^2(M, S)$ *be the Dirac operator over a complete Riemannian manifold* M *and* $\chi \in \mathcal{S}$. *Set* $\mathrm{ind}(D) = \delta([(\chi(D)+1)/2]) \in K_1(C^*(M))$. *We call* $\mathrm{ind}(D)$ *the odd index.*

The odd index vanishes when M is compact.

REMARK 4.2.2. [**34**, Proposition 2.8] *If the spectrum of* D *has a gap, then we have* $\mathrm{ind}(D) = 0$. *In particular, if* M *is compact, then we have* $\mathrm{ind}(D) = 0$.

We use a special function in $\mathcal{S}$. Then the odd index is represented by the Cayley transform of D.

REMARK 4.2.3. *Set*

$$\chi(x) = \frac{1}{\pi}\mathrm{Arg}\left(-\frac{x-i}{x+i}\right) \quad \textit{for } x \in \mathbb{R}.$$

Here, we choose a principal value of the argument of a complex number z *is* $-\pi < \mathrm{Arg}(z) \leq \pi$. *By*

$$-\frac{x-i}{x+i} = -\frac{1-1/x^2}{1+1/x^2} + i\frac{2x}{x^2+1},$$

χ *is monotone increasing and we have*

$$\lim_{x\to\infty} \chi(x) = \frac{1}{\pi}\pi = 1 \quad \textit{and} \quad \lim_{x\to-\infty} \chi(x) = -\frac{1}{\pi}\pi = -1.$$

Therefore, we have $\chi \in \mathcal{S}$.

On the other hand, we have

$$\delta([(\chi(D)+1)/2]) = -[\exp(\pi i \chi(D))] = \left[\frac{D-i}{D+i}\right] \in K_1(C^*(M)).$$

4.3. Roe's cyclic one-cocycle associated to a partition

We want to study the odd index which is defined in Section 4.2. For this purpose, we take Connes' pairing of a certain cyclic cocycle with it. We define such a cyclic cocycle, which is defined by a partition of a manifold. Its cocycle is called the Roe cocycle.

Definition 4.3.1. *Let M be an oriented complete Riemannian manifold. We assume the triple (M^+, M^-, N) satisfies the following conditions:*

- *M^+ and M^- are submanifolds of M of the same dimension as M, $\partial M^+ \neq \emptyset$ and $\partial M^- \neq \emptyset$,*
- *$M = M^+ \cup M^-$,*
- *N is a closed submanifold of M of codimension one,*
- *$N = M^+ \cap M^- = -\partial M^+ = \partial M^-$.*

Then we call (M^+, M^-, N) a partition of M. M is also called a partitioned manifold.

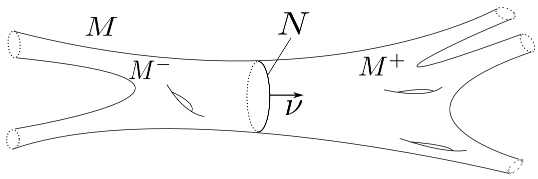

Figure 4.3.1. Partitioned manifold

For example, we can consider $\mathbb{R} \times N$ is partitioned by $(\mathbb{R}_+ \times N, \mathbb{R}_- \times N, \{0\} \times N)$, where we set $\mathbb{R}_+ = \{t \in \mathbb{R}\,;\, t \geq 0\}$ and $\mathbb{R}_- = \{t \in \mathbb{R}\,;\, t \leq 0\}$.

We fix the notation of two functions which are defined by a partition.

Definition 4.3.2. *We assume M is partitioned by (M^+, M^-, N). Then we denote by Π the characteristic function of M^+ and set $\Lambda = 2\Pi - 1$.*

We can prove a commutator condition of these functions with an element in the Roe algebra $C^*(M)$. Recall that we have $C^*(M) = \overline{\mathscr{X}}$. Here, $\mathscr{X}$ is the $*$-subalgebra of $\mathcal{L}(L^2(S))$ with the element has a smooth integral kernel and finite propagation; see Remark 4.1.10.

Proposition 4.3.3. *If M is a partitioned manifold, then the following holds:*

(*i*) *For all $u \in C^*(M)$, one has $[\Pi, u] \sim 0$ and $[\Lambda, u] \sim 0$.*

(*ii*) *For all $u \in C^*(M)$ and $\varphi \in C(M)$ satisfies $\varphi = \Pi$ on the complement of a compact set in M, one has $[\varphi, u] \sim 0$.*

Proof. Due to [**34**, Lemma 1.5], $[\Pi, u]$ is of trace class for all $u \in \mathscr{X}$. So (i) is proved by Remark 4.1.10. Since the support of $\Pi - \varphi$ is compact, there exists $f \in C_0(M)$ such that $f(\Pi - \varphi) = (\Pi - \varphi)f =$

$\Pi - \varphi$. Thus we have $[\varphi, u] \sim [\varphi - \Pi, u] = (\varphi - \Pi)u - u(\varphi - \Pi) = (\varphi - \Pi)fu - uf(\varphi - \Pi) \sim 0$. This proves (ii).

□

We describe the definition of the Roe cocycle, which is a cyclic 1-cocycle defined on $\mathscr{X}$.

DEFINITION 4.3.4. *For any $A, B \in \mathscr{X}$, set*

$$\zeta(A, B) = \frac{1}{4}\mathrm{Tr}(\Lambda[\Lambda, A][\Lambda, B]).$$

We call $\zeta : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ the Roe cocycle.

PROPOSITION 4.3.5. [**34**, Proposition 1.6] *ζ is a cyclic 1-cocycle on $\mathscr{X}$.*

In an index theorem for partitioned manifolds, we take the pairing of ζ with the index class in $K_1(C^*(M))$. For this purpose, we have to extend a domain of ζ.

DEFINITION 4.3.6. *Let M be a partitioned manifold and $S \to M$ a Hermitian vector bundle. Then we define a subalgebra $\mathscr{A}$ in $C^*(M)$ such that one has $u \in \mathscr{A}$ if $[\Lambda, u]$ is of trace class.*

As noted in Example 3.3.5, $\mathscr{A}$ is a Banach algebra with a norm $\|u\|_{\mathscr{A}} = \|u\| + \|[\Lambda, u]\|_1$, where $\|\cdot\|$ is the operator norm and $\|\cdot\|_1$ is the trace norm.

PROPOSITION 4.3.7. *Let M be a partitioned manifold and $S \to M$ a Hermitian vector bundle. Then $\mathscr{A}$ is dense and closed under holomorphic functional calculus in $C^*(M)$.*

PROOF. Because of $\mathscr{X} \subset \mathscr{A} \subset C^*(M)$, $\mathscr{A}$ is dense in $C^*(M)$. By Example 3.3.5 and Proposition 4.3.3 (i), $\mathscr{A}$ is closed under holomorphic functional calculus in $C^*(M)$.

□

Therefore, the inclusion $i : \mathscr{A} \to C^*(M)$ induces an isomorphism $i_* : K_1(\mathscr{A}) \cong K_1(C^*(M))$; see Remark 3.3.4. By using this isomorphism, we can take the pairing of the Roe cocycle with an element in $K_1(C^*(M))$ as follows:

DEFINITION 4.3.8. [**18**, p.109] *Define the map*

$$\langle \cdot, \zeta \rangle : K_1(C^*(M)) \to \mathbb{C}$$

by $\langle [u], \zeta \rangle = \frac{1}{8\pi i} \sum_{i,j} \zeta((u^{-1})_{ji}, u_{ij})$, where we assume $[u]$ is represented by an element $u \in GL_l(\mathscr{A})$ and u_{ij} is the (i, j)-component of u. We note that this is Connes' pairing of cyclic cohomology with K-theory, and $1/8\pi i$ is a constant multiple in Connes' pairing.

We can write its pairing by a Fredholm index.

PROPOSITION 4.3.9. [**18**, p.75] *For any $u \in GL_l(C^*(M))$, one has*

$$\langle [u], \zeta \rangle = -\frac{1}{8\pi i}\operatorname{index}(\Pi u \Pi : \Pi(L^2(M,S))^l \to \Pi(L^2(M,S))^l).$$

PROOF. Since both sides of this equation do not change by homotopy of $u \in GL_l(C^*(M))$, it suffices to show the case when $u \in GL_l(\mathscr{A})$. Then we obtain

$$8\pi i \langle [u], \zeta \rangle = \frac{1}{4}\sum_{i,j} \operatorname{Tr}(\Lambda[\Lambda, (u^{-1})_{ij}][\Lambda, u_{ji}]) = \frac{1}{4}\operatorname{Tr}(\Lambda[\Lambda, u^{-1}][\Lambda, u]).$$

Because of

$$\Pi - \Pi u^{-1}\Pi u \Pi = -\Pi[\Pi, u^{-1}][\Pi, u]\Pi,$$

$\Pi - \Pi u^{-1}\Pi u \Pi$ and $\Pi - \Pi u \Pi u^{-1}\Pi$ are of trace class on $\Pi(L^2(M,S))^l$. Therefore we get

$$\begin{aligned} &\operatorname{index}(\Pi u \Pi : \Pi(L^2(M,S))^l \to \Pi(L^2(M,S))^l) \\ &= \operatorname{Tr}(\Pi - \Pi u^{-1}\Pi u \Pi) - \operatorname{Tr}(\Pi - \Pi u \Pi u^{-1}\Pi) \end{aligned}$$

by [**18**, p.88]. Thus we have

$$\operatorname{index}(\Pi u \Pi : \Pi(L^2(M,S))^l \to \Pi(L^2(M,S))^l) = -\frac{1}{4}\operatorname{Tr}(\Lambda[\Lambda, u^{-1}][\Lambda, u]).$$

This implies

$$\langle [u], \zeta \rangle = -\frac{1}{8\pi i}\operatorname{index}(\Pi u \Pi : \Pi(L^2(M,S))^l \to \Pi(L^2(M,S))^l).$$

□

The last of this section, we see a relationship between Connes' pairing of the Roe cocycle ζ with $K_1(C^*(M))$ and an extension. Set $H = \Pi(L^2(M,S))$. Let $q : \mathcal{L}(H) \to \mathcal{Q}(H)$ be the quotient map to the Calkin algebra. Define $\sigma : C^*(M) \to \mathcal{L}(H)$ by $\sigma(A) = \Pi A \Pi$ and $\tau : C^*(M) \to \mathcal{Q}(H)$ by $\tau = q \circ \sigma$. Set

$$E = \{(A,T) \in C^*(M) \oplus \mathcal{L}(H)\,;\, \tau(A) = q(T)\}.$$

Then we get an extension τ of $C^*(M)$:

$$0 \to \mathcal{K}(H) \hookrightarrow E \to C^*(M) \to 0.$$

This extension is defined by a Fredholm module $(L^2(M,S), \Lambda)$ over $C^*(M)$; see Examples 3.2.21 and 3.2.22. By the definition of an index pairing $\langle \cdot, \cdot \rangle_{\text{ind}} : K_1(C^*(M)) \times \operatorname{Ext}(C^*(M)) \to \mathbb{Z}$ and Proposition 4.3.9, we obtain $\langle [u], \zeta \rangle = \langle [u], [\tau] \rangle_{\text{ind}} = \operatorname{index}(\Pi u \Pi)$ up to a certain constant multiple for any $[u] \in K_1(C^*(M))$.

Moreover, we can prove these are equal to the connecting homomorphism of this extension: $\partial : K_1(C^*(M)) \to K_0(\mathcal{K}(H)) \cong \mathbb{Z}$. In fact, for any unitary $u \in U(C^*(M))$, denote by $v(u)$ the partial isometry part of the polar decomposition of $\sigma(u)$. Then we have $\tau(u) = q(v(u))$ since $\sigma(u)$ is an essential unitary operator on H. Therefore, $(u, v(u)) \in E$ is a partial isometry lift of u. So we obtain $\partial([u]) = [\Pi - v(u)^* v(u)] - [\Pi - v(u)v(u)^*] \in K_0(\mathcal{K}(H))$. By the identification $K_0(\mathcal{K}(H)) \cong \mathbb{Z}$, we have $\partial([u]) = \mathrm{index}(v(u)) = \mathrm{index}(\sigma(u))$. Therefore, we obtain $\langle [u], \zeta\rangle = \langle [u], [\tau]\rangle_{\mathrm{ind}} = [u]\hat{\otimes}_{C^*(M)}[L^2(M, S), \Lambda] = \partial([u]) = \mathrm{index}(\Pi u \Pi)$ up to a certain constant multiple.

4.4. The Roe-Higson index theorem

In this section, we describe the Roe-Higson index theorem. As explained in [**35**, Section 6.1], the Roe cocycle ζ is related to the Poincaré dual $pd(N) \in H^1_c(M)$ of N. In fact, there exists uniquely the element in the coarse cohomology $\alpha \in HX^1(M)$ such that the character map $HX^1(M) \to H^1_c(M)$ sends α to $pd(N)$. Moreover, the character map $HX^1(M) \to HC^1(\mathscr{X})$ sends α to $[\zeta]$. By this relationship with ζ and N, it is expected that Connes' pairing of ζ has some information about N. The Roe-Higson index theorem asserts this expectation is true.

4.4.1. Statement of the Roe-Higson index theorem. Let M be a partitioned manifold, $S \to M$ a Clifford bundle over M and D the Dirac operator on S. Set $S_N = S|_N$. We can induce a structure of a Clifford bundle on S_N by the induced connection ∇^{S_N} and the Clifford action. Moreover, we can induce the $\mathbb{Z}_2$-graded structure on S_N. Let ν be the outward pointing normal unit vector field on $N = \partial M^-$; see Figure 4.3.1. Then $ic(\nu)|_{S_N}$ is $\mathbb{Z}_2$-grading of S_N since we have $(ic(\nu)|_{S_N})^2 = 1$, where c is the Clifford action on S. By definition, S_N is a graded Clifford bundle. Let D_N be the graded Dirac operator on S_N. We note that this formulation makes sense for any dimension of M.

Theorem 4.4.1 (The Roe-Higson index theorem). [**34**, Theorem 3.3] *By using above notations, we have*

$$\langle \mathrm{ind}(D), \zeta\rangle = -\frac{1}{8\pi i}\mathrm{index}(D_N^+).$$

Remark 4.4.2. [**26**] *Let $\varphi \in C^\infty(M)$ be a smooth function such that we have $\varphi = \Pi$ on the complement of a compact set in M. Higson proved a version of Theorem 4.4.1:*

$$\mathrm{index}(1 - \varphi + \varphi u_D) = \mathrm{index}(D_N^+),$$

where $u_D = (D-i)(D+i)^{-1} = 1-2i(D+i)^{-1}$ *is the Cayley transform of* D.

We see $-8\pi i\langle \mathrm{ind}(D), \zeta\rangle = \mathrm{index}(1-\varphi+\varphi u_D)$. *By Remark 4.2.3 and Proposition 4.3.9, we have*

$$-8\pi i\langle \mathrm{ind}(D), \zeta\rangle = \mathrm{index}(\Pi u_D \Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S))).$$

Because of Proposition 4.3.3, the right hand side equals to

$$\mathrm{index}(1-\Pi+\Pi u_D\Pi) = \mathrm{index}(1-\Pi+\Pi u_D) = \mathrm{index}(1-\varphi+\varphi u_D).$$

See also [**24**, §7].

Remark 4.4.3. *As noted in Section 2.3, the Fredholm index of the Dirac operator on an odd-dimensional manifold is always* 0. *Thus the Roe-Higson index* $\langle \mathrm{ind}(D), \zeta\rangle$ *is trivial when* M *is of even dimension. The main theorem in this thesis is motivated by this fact.*

4.4.2. The case for $\mathbb{R}\times N$. In this subsection, we prove Theorem 4.4.1 for $M = \mathbb{R}\times N$. Our proof is similar to that of [**32**, §7.4.2]. We recall that $\mathbb{R}\times N$ is partitioned by $(\mathbb{R}_+\times N, \mathbb{R}_-\times N, \{0\}\times N)$. Let $S_N \to N$ be a graded Clifford bundle. Denote by ϵ the $\mathbb{Z}_2$-grading operator on S_N, c_N the Clifford action on S_N and D_N the Dirac operator of S_N. Let $p : \mathbb{R}\times N \to N$ be a projection map and set $S = p^*S_N$. Then S is a Clifford bundle by the pullback connection and the following Clifford action c:

$$c(\mathrm{d}/\mathrm{d}t) = -i\epsilon, c(X) = c_N(X) \text{ for all } X \in C^\infty(N, TN).$$

Thus we can describe the Dirac operator D on S as follows:

$$D = \begin{bmatrix} -i\frac{\mathrm{d}}{\mathrm{d}t} & D_N^- \\ D_N^+ & i\frac{\mathrm{d}}{\mathrm{d}t} \end{bmatrix}.$$

We note that by the identification $L^2(M,S) = L^2(\mathbb{R})\otimes L^2(N,S_N)$, we identify the characteristic function Π on $M^+ = \mathbb{R}_+\times N$ with that of $\mathbb{R}_+$. Then $\Lambda = 2\Pi - 1$ is identified with the signature function. As we noted in Remark 4.4.2, it suffices to show that

$$\begin{aligned} &\mathrm{index}(\Pi u_D \Pi : \Pi(L^2(\mathbb{R}))\otimes L^2(N,S_N) \to \Pi(L^2(\mathbb{R}))\otimes L^2(N,S_N)) \\ &= \mathrm{index}(D_N^+). \end{aligned}$$

We calculate

$$\mathrm{index}(\Pi u_D \Pi : \Pi(L^2(\mathbb{R}))\otimes L^2(N,S_N) \to \Pi(L^2(\mathbb{R}))\otimes L^2(N,S_N)).$$

For this purpose, we use the Hilbert transformation. Properties about the Hilbert transformation which we use are in Appendix A. Let $\mathscr{F}$:

$L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transformation:

$$\mathscr{F}[f](\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

Let $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Hilbert transformation

$$Hf(t) = -\frac{i}{\pi}\text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{t-y}dy = -\frac{i}{\pi} \lim_{\epsilon \downarrow 0} \int_{|t-y|>\epsilon} \frac{f(y)}{t-y}dy,$$

where p.v. is the Cauchy principal value. Due to $\Lambda\mathscr{F} = -\mathscr{F}H$, we have $\mathscr{F}^{-1}\Pi\mathscr{F} = (1-H)/2$. Set $\hat{P} = (1-H)/2$. On the other hand, by $\mathscr{F}^{-1}\mathrm{d}/\mathrm{d}t\mathscr{F} = -it$, we have

$$\hat{D} = \mathscr{F}^{-1}D\mathscr{F} = \begin{bmatrix} -t & D_N^- \\ D_+ & t \end{bmatrix}.$$

Thus we obtain

$$\begin{aligned} &\text{index}(\Pi u_D \Pi : \Pi(L^2(\mathbb{R})) \otimes L^2(N, S_N) \to \Pi(L^2(\mathbb{R})) \otimes L^2(N, S_N)) \\ &= \text{index}(\hat{P} u_{\hat{D}} \hat{P}^* : \mathscr{H}_- \otimes L^2(N, S_N) \to \mathscr{H}_- \otimes L^2(N, S_N)), \end{aligned}$$

where $u_{\hat{D}}$ is the Cayley transform of $\hat{D}$ and $\mathscr{H}_-$ is the image of $\hat{P}$, that is, the -1-eigenspace of H.

Set $\mathcal{H} = \mathscr{H}_- \otimes \mathrm{Ker}(D_N)$, $v = u_{\hat{D}}|_{\mathcal{H}^\perp}$, $D_0 = (D_N \otimes 1)|_{\mathcal{H}^\perp}$ and $\sigma = i\epsilon D_0 |D_0|^{-1}$. Then we have the following properties by direct computations:

(*i*) $\hat{P}u_{\hat{D}}(\mathcal{H}) \subset \mathcal{H}$,
(*ii*) $\sigma(\mathcal{H}^\perp) \subset \mathcal{H}^\perp$,
(*iii*) $\sigma^2 = 1$,
(*iv*) $\sigma D_0 = -D_0 \sigma$. This implies $\sigma v = v^{-1}\sigma$.

By (*iv*), we obtain

$$\sigma \hat{P} v \hat{P}^* \sigma = \sigma \hat{P} \frac{\hat{D} - i}{\hat{D} + i} \hat{P}^* \sigma = \hat{P} \frac{\hat{D} + i}{\hat{D} - i} \hat{P}^* = \hat{P} v^{-1} \hat{P}^* \text{ on } \mathcal{H}^\perp.$$

Combining this computation and $\text{index}(\sigma) = 0$, we have

$$\begin{aligned} \text{index}(\hat{P} v \hat{P}^* : \mathcal{H}^\perp \to \mathcal{H}^\perp) &= \text{index}(\hat{P} v^{-1} \hat{P}^* : \mathcal{H}^\perp \to \mathcal{H}^\perp) \\ &= -\text{index}(\hat{P} v \hat{P}^* : \mathcal{H}^\perp \to \mathcal{H}^\perp). \end{aligned}$$

This implies $\text{index}(\hat{P} v \hat{P}^* : \mathcal{H}^\perp \to \mathcal{H}^\perp) = 0$.

Thus, we finish a proof as follows:

$$
\begin{aligned}
&\operatorname{index}\Big(\hat{P}u_{\hat{D}}\hat{P}^* : \mathscr{H}_- \otimes L^2(N,S_N) \to \mathscr{H}_- \otimes L^2(N,S_N)\Big)\\
&= \operatorname{index}\left(\hat{P}\begin{bmatrix}(-t-i)/(-t+i) & 0\\ 0 & (t-i)/(t+i)\end{bmatrix}\hat{P}^* : \mathcal{H} \to \mathcal{H}\right)\\
&\quad + \operatorname{index}(\hat{P}v\hat{P}^* : \mathcal{H}^\perp \to \mathcal{H}^\perp)\\
&= \operatorname{index}\left(\hat{P}\frac{t+i}{t-i}\hat{P}^* : \mathscr{H}_- \otimes \operatorname{Ker}(D_N^+) \to \mathscr{H}_- \otimes \operatorname{Ker}(D_N^+)\right)\\
&\quad + \operatorname{index}\left(\hat{P}\frac{t-i}{t+i}\hat{P}^* : \mathscr{H}_- \otimes \operatorname{Ker}(D_N^-) \to \mathscr{H}_- \otimes \operatorname{Ker}(D_N^-)\right)\\
(*)\quad &= \dim\operatorname{Ker}(D_N^+) - \dim\operatorname{Ker}(D_N^-)\\
&= \operatorname{index}(D_N^+).
\end{aligned}
$$

Here, the equality $(*)$ is obtained by Example A.5.

4.4.3. The general case. In this subsection, we complete to prove the Roe-Higson index theorem by a reduction to the case when a product manifold $\mathbb{R} \times N$. For this purpose, we will cover Higson's reduction argument [**26**]. In our main theorem, we use a similar argument to Higson's argument. Firstly, we state a cobordism invariance.

LEMMA 4.4.4. [**26**, Lemma 1.4] *Let (M^+, M^-, N) and $(M^{+\prime}, M^{-\prime}, N')$ be two partitions of M. We assume these two partitions are cobordant, that is, symmetric differences $M^\pm \triangle M^{\mp\prime}$ are compact. Let Π and Π' be the characteristic function of M^+ and $M^{+\prime}$, respectively. Then one has* $\operatorname{index}(\Pi u_D \Pi) = \operatorname{index}(\Pi' u_D \Pi')$.

We use Lemma 4.4.4 in order to construct a Clifford bundle when we change the general manifold to $\mathbb{R} \times N$. Secondly, we state Higson's Lemma.

LEMMA 4.4.5. [**26**, Lemma 3.1] *Let M_1 and M_2 be two partitioned manifolds and $S_j \to M_j$ a Clifford bundle. Denote by D_j the Dirac operator of S_j. Let Π_j be the characteristic function of M_j^+. We assume that there exists an isometry $\gamma : M_2^+ \to M_1^+$ which lifts a Clifford bundle isomorphism $\gamma^* : S_1|_{M_1^+} \to S_2|_{M_2^+}$. Then one has* $\operatorname{index}(\Pi_1 u_{D_1} \Pi_1) = \operatorname{index}(\Pi_2 u_{D_2} \Pi_2)$.

Similarly, if there exists an isometry $\gamma : M_2^- \to M_1^-$ which lifts a Clifford bundle isomorphism $\gamma^* : S_1|_{M_1^-} \to S_2|_{M_2^-}$, then one has $\operatorname{index}(\Pi_1 u_{D_1} \Pi_1) = \operatorname{index}(\Pi_2 u_{D_2} \Pi_2)$.

By using Lemma 4.4.4 and 4.4.5, we can reduce the general case to the case when $\mathbb{R} \times N$ by replacing a manifold without changing index. Let $(-\delta, \delta) \times N$ be diffeomorphic to a tubular neighborhood of N in M. Due to Lemma 4.4.4, we may change a partition of M to $(M^+ \cup ([-\delta, 0] \times N), M^- \setminus ((-\delta, 0] \times N), \{-\delta\} \times N)$ without changing index$(\Pi u_D \Pi)$. Then, due to Lemma 4.4.5 we may change $M^+ \cup ([-\delta, 0] \times N)$ to $[-\delta, \infty) \times N$ without changing index$(\Pi u_D \Pi)$. Here a metric on $[0, \infty) \times N$ is product. We denote this manifold by $M' = ([-\delta, \infty) \times N) \cup (M^- \setminus ((-\delta, 0] \times N))$. M' is partitioned by $([-\delta, \infty) \times N, M^- \setminus ((-\delta, 0] \times N), \{-\delta\} \times N)$. We apply a similar argument to M', we may change M' to a product $\mathbb{R} \times N$ without changing index$(\Pi u_D \Pi)$. Now we have changed M to $\mathbb{R} \times N$.

4.5. Applications

In this section we review two applications of the Roe-Higson index theorem.

4.5.1. The cobordism invariance of the index. Let W be a compact manifold of odd dimension with boundary $\partial W = N$. We assume all geometric structures near ∂W are product. Let D_W be the Dirac operator on W and D_N the induced Dirac operator on N. We also assume D_N is graded by using the unit normal vector field on N. It is well-known fact that we have index$(D_N^+) = 0$; see [**33**, Theorem XVII.3]. This fact is called the cobordism invariance of the index.

We can prove the cobordism invariance of the index independently by using the Roe-Higson index theorem [**26**]. We can construct the Dirac operator D_1 on $(-\infty, 0] \times N$ by using D_N as in Subsection 4.4.2. Let D be the Dirac operator on $M = ((-\infty, 0] \times N) \cup_N W$ defined by using D_W and D_1. M is a partitioned manifold partitioned by $(W, (-\infty, 0] \times N, \{0\} \times N)$. We can construct the Dirac operator on N as in Subsection 4.4.1, but its Dirac operator coincides with D_N.

We use Higson's set up as in Remark 4.4.2. Let $\varphi \in C^\infty(M)$ be a smooth function such that φ is equal to the characteristic function of W on the complement of a compact set in M. By the Roe-Higson index theorem, we have

$$\operatorname{index}(1 - \varphi + \varphi u_D) = \operatorname{index}(D_N^+).$$

By the way, since W is compact, we can use $\varphi = 0$, the constant function on M. Thus the left hand side is equal to index$(\mathrm{id}_{L^2(M,S)}) = 0$. Therefore we have index$(D_N^+) = 0$.

4.5.2. Existence of a Riemannian metric with its scalar curvature is uniformly positive. Let M be a non-compact partitioned manifold of odd dimension. We assume that M can be equipped with a spin structure. We fix a spin structure on M. In this case, we can get necessarily condition for existence of a Riemannian metric with its scalar curvature is uniformly positive.

THEOREM 4.5.1. [**34**, Theorem 9.1] *If there exists a Riemannian metric on M with its scalar curvature κ is uniformly positive, then we have $\int_N \hat{\mathcal{A}}(TN) = 0$.*

We review the sketch of proof. Denote by D the canonical spinor Dirac operator on M. N is also spin by the induced spin structure and then D_N is also the canonical spinor Dirac operator. By the Lichnerowicz-Weitzenbock formula, we have $D^2 = \nabla^{S*}\nabla^S + \frac{1}{4}\kappa > 0$. This implies the spectrum of D has a gap near 0. Therefore, we have $\mathrm{ind}(D) = 0$ by Remark 4.2.2. Combine the Atiyah-Singer index theorem and the Roe-Higson index theorem, and we have

$$\int_N \hat{\mathcal{A}}(TN) = \mathrm{index}(D_N^+) = 0.$$

CHAPTER 5

Main theorem

In this chapter, we discuss the main theorem. Let (M, g) be a complete Riemannian manifold, $S \to M$ a graded Clifford bundle with the Clifford action c and the grading ϵ. Denote by D the graded Dirac operator of S. As noted in Remark 4.4.3, the Roe-Higosn index is trivial when M is of even dimension. An index class which is used in the main theorem gives a non-trivial index in this case.

5.1. Definition of the index class

In this section, we define an index class in $K_1(C^*(M))$. Set $\|f\| = \sup_{x \in M} |f(x)|$ for $f \in C(M)$ and $\|X\| = \sup_{x \in M} \sqrt{g_x(X, X)}$ for $X \in C^\infty(M, TM)$.

Definition 5.1.1. *Define $\mathscr{W}(M)$ by the subset in $C^\infty(M)$ such that one has $f \in \mathscr{W}(M)$ if $\|f\| < +\infty$, $\|\mathrm{grad}(f)\| < +\infty$. Define $C_w(M)$ by the closure of $\mathscr{W}(M)$ by the uniform norm on M.*

Proposition 5.1.2. *$\mathscr{W}(M)$ is a unital $*$-subalgebra of $C_b(M)$. Therefore, $C_w(M)$ is a unital C^*-algebra.*

Proof. The proof of this proposition is a routine work. $\square$

We will see a reason why we shall need this C^*-algebra $C_w(M)$ in Section 5.3 and Subsection 5.4.1. We define a Kasparov $(C_w(M), C^*(M))$-module which is made of the Dirac operator D. $C^*(M)$ is an evenly graded C^*-algebra, where the grading is induced by ϵ. Since $\chi_0(x) = x(1 + x^2)^{-1/2}$ is a chopping function, the left composition of $F_D = D(1 + D^2)^{-1/2} \in D^*(M)$ on an element in $C^*(M)$ is an odd operator on $C^*(M)$.

Proposition 5.1.3. *Let $\mu : C_w(M) \to \mathbb{B}(C^*(M))$ be the left composition of the multiplication operator: $\mu(f)u = fu \in C^*(M)$ for $f \in C_w(M)$ and $u \in C^*(M)$. Then one has $[D] = [C^*(M), \mu, F_D] \in KK^0(C_w(M), C^*(M))$.*

Proof. Our proof is similar to the Baaj-Julg picture of Kasparov modules [**6**, Proposition 2.2]. Firstly, we obtain $F_D \in \mathbb{B}(C^*(M))$, since

F_D is a self-adjoint bounded operator on $L^2(M,S)$ and we have $F_D u \in C^*(M)$ for any $u \in C^*(M)$. Now, because of $1 - F_D^2 = 1 - D^2(1+D^2)^{-1} = (1+D^2)^{-1} \in C^*(M) = \mathbb{K}(C^*(M))$ and $F_D^* = F_D$, it suffices to show $[\mu(f), F_D] \in C^*(M)$.

Now, the following integral formula

$$\begin{aligned}
&[\mu(f), F_D]\\
=&\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}[f, D(1+D^2+\lambda)^{-1}]d\lambda\\
=&\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda)(1+D^2+\lambda)^{-1}[f,D](1+D^2+\lambda)^{-1}d\lambda\\
&+\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}D(1+D^2+\lambda)^{-1}[D,f]D(1+D^2+\lambda)^{-1}d\lambda
\end{aligned}$$

is uniformly integrable for any $f \in \mathscr{W}(M)$ since we have $\|(1+D^2+\lambda)^{-1}\| \leq (1+\lambda)^{-1}$ and $\|D(1+D^2+\lambda)^{-1}\| \leq (1+\lambda)^{-1/2}$ for any $\lambda \geq 0$, and $[f,D] = -c(\mathrm{grad}(f)) \in D^*(M)$ for any $f \in \mathscr{W}(M)$. So we obtain $[\mu(f), F_D] \in C^*(M)$ for any $f \in \mathscr{W}(M)$ by $(1+D^2+\lambda)^{-1}, D(1+D^2+\lambda)^{-1} \in C^*(M)$ for any $\lambda \geq 0$. Thus, we obtain $[\mu(f), F_D] \in C^*(M)$ for any $f \in C_w(M)$, since we have $\|[\mu(f), F_D]\| \leq 2\|f\|$ for any $f \in \mathscr{W}(M)$ and $\mathscr{W}(M)$ is dense in $C_w(M)$. This implies $(C^*(M), \mu, F_D)$ is a Kasparov $(C_w(M), C^*(M))$-module. □

REMARK 5.1.4. *Let χ be a chopping function, that is, $\chi \in C(\mathbb{R}; [-1.1])$ is an odd function and one has $\chi(x) \to 1$ as $x \to +\infty$. Then one has $\chi(D) - F_D \in C^*(M)$ by $\chi - \chi_0 \in C_0(M)$. We note that $\chi(D)$ is an odd operator since χ is an odd function. Therefore, we obtain $[D] = [C^*(M), \mu, \chi(D)]$, that is, $[D]$ is independent of the choice of a chopping function χ.*

Any $\phi \in GL_l(C_w(M))$ induces $[\phi] \in K_1(C_w(M))$. By using the Kasparov product

$$\hat{\otimes}_{C_w(M)} : K_1(C_w(M)) \times KK^0(C_w(M), C^*(M)) \to K_1(C^*(M)),$$

we get the index class in $K_1(C^*(M))$ as follows.

DEFINITION 5.1.5. *For any $\phi \in GL_l(C_w(M))$, set*

$$\mathrm{Ind}(\phi, D) = [\phi]\hat{\otimes}_{C_w(M)}[D] \in K_1(C^*(M)).$$

5.2. Main theorem

Roughly speaking, our main theorem is Connes' pairing of the Roe cocycle with $\mathrm{Ind}(\phi, D) \in K_1(C^*(M))$ is calculated by the Fredholm index of the Toeplitz operator on a hypersurface N. In this section, we firstly define its operator.

Let M be a partitioned manifold partitioned by (M^+, M^-, N). Let $\nu \in C^\infty(N, TN)$ be the outward pointing normal unit vector field on $N = \partial M^-$; see Figure 4.3.1 in Section 4.3.

Set $S_N = S^+|_N$. Define $c_N \in C^\infty(N, \mathrm{Hom}(TN, \mathrm{End}(S_N)))$ by $c_N(X) = c(\nu)c(X)$. Then S_N is a Clifford bundle over N with the induced metric and connection and the Clifford action c_N. Denote by D_N the Dirac operator of S_N. We denote the restriction of $\phi \in GL_l(C_w(M))$ to N by the same letter ϕ. Let T_ϕ be the Toeplitz operator with symbol ϕ. This Toeplitz operator T_ϕ is the operator on N in our main theorem.

Now, we can state our main theorem as follows:

THEOREM 5.2.1. *Let M be a partitioned manifold partitioned by (M^+, M^-, N). Let $S \to M$ be a graded Clifford bundle with the grading ϵ and denote by D the graded Dirac operator of S. We denote the restriction of $\phi \in GL_l(C_w(M))$ to N by the same letter ϕ. Then the following formula holds:*

$$\langle \mathrm{Ind}(\phi, D), \zeta \rangle = -\frac{1}{8\pi i}\mathrm{index}(T_\phi).$$

Use the explicit formula of $\mathrm{Ind}(\phi, D)$ and the index theorem for Toeplitz operators, and we obtain the following topological formula:

COROLLARY 5.2.2. *Let M be a partitioned manifold partitioned by (M^+, M^-, N), and Π the characteristic function of M^+. Let $S \to M$ be a graded Clifford bundle with the grading ϵ and denote by D the graded Dirac operator of S. We assume that $\phi \in C^\infty(M; GL_l(\mathbb{C}))$ satisfies $\|\phi\| < \infty$, $\|\mathrm{grad}(\phi)\| < \infty$ and $\|\phi^{-1}\| < \infty$. Then one has*

$$\mathrm{index}\left(\Pi(D+\epsilon)^{-1}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}(D+\epsilon)\Pi : \Pi(L^2(M,S))^l \to \Pi(L^2(M,S))^l\right)$$
$$= \int_{S^*N} \pi^*\mathrm{Td}(TN \otimes \mathbb{C})ch(S^+)\pi^*ch(\phi).$$

The proof of Theorem 5.2.1 and Corollary 5.2.2 is provided in Section 5.6.

5.3. Wrong way functoriality

We see a correspondence between an index theorem for partitioned manifolds with Connes' wrong way functoriality. Let $f : X \to Y$ be a K-oriented smooth map, that is, f is smooth and $TX \oplus f^*TY$ is a spin^c vector bundle. Connes [**16, 17**] defined wrong way functoriality

$f! \in KK^{\dim X + \dim Y}(C_0(X), C_0(Y))$. Roughly speaking, $f!$ is defined by a family of Dirac operators on X parametrized by Y. Note that if $g : Y \to Z$ is a K-oriented smooth map, then $g \circ f$ is also K-oriented and we have $(g \circ f)! = f! \hat{\otimes}_{C_0(Y)} g! \in KK^{\dim X + \dim Z}(C_0(X), C_0(Z))$.

We assume $M = \mathbb{R} \times N$ with N closed. Let $i : \{\mathrm{pt}\} \to \mathbb{R}$ be an inclusion map defined by $i(\mathrm{pt}) = 0$, and $p : \mathbb{R} \to \{\mathrm{pt}\}$ a constant map. Due to Connes, they define wrong way functoriality $i! \in KK^1(\mathbb{C}, C_0(\mathbb{R}))$, $(i \times \mathrm{id}_N)! \in KK^1(C(N), C_0(M))$ and $p! \in KK^1(C_0(\mathbb{R}), \mathbb{C})$, respectively. We note the following:

$$i! \otimes_{C_0(\mathbb{R})} p! = (p \circ i)! = 1_{\mathbb{C}} \in KK^0(\mathbb{C}, \mathbb{C}).$$

Let D_N be the Dirac operator on N and $D_{\mathbb{R}}$ the Dirac operator on $\mathbb{R}$ defined by a spin structure of $\mathbb{R}$. These Dirac operators define elements in K-homology, that is, they define $[D_N] \in KK^*(C(N), \mathbb{C})$ and $[D_{\mathbb{R}}] = p! \in KK^1(C_0(\mathbb{R}), \mathbb{C})$, respectively. Moreover, D_N and $D_{\mathbb{R}}$ determine the Dirac operator D_M on $M = \mathbb{R} \times N$ satisfies $[D_M] = [D_{\mathbb{R}}] \hat{\otimes}_{\mathbb{C}} [D_N] \in KK^{*+1}(C_0(M), \mathbb{C})$.

Firstly, we assume $* = 0$. Let $[[E]] \in KK^0(C_0(M), C_0(M))$ be a KK-element defined by a vector bundle $E \to M$ by using the inclusion map $KK^0(\mathbb{C}, C_0(M)) \to KK^0(C_0(M), C_0(M))$. Then we have

$$\begin{aligned}(i \times \mathrm{id}_N)! \hat{\otimes}_{C_0(M)} ([[E]] \hat{\otimes}_{C_0(M)} [D_M]) &= [[E|_N]] \hat{\otimes}_{C(N)} i! \hat{\otimes}_{C_0(M)} ([D_{\mathbb{R}}] \hat{\otimes}_{\mathbb{C}} [D_N]) \\ &= [[E|_N]] \hat{\otimes}_{C(N)} (i! \hat{\otimes}_{C_0(\mathbb{R})} [D_{\mathbb{R}}]) \hat{\otimes}_{C(N)} [D_N] \\ &= [[E|_N]] \hat{\otimes}_{C(N)} (i! \hat{\otimes}_{C_0(\mathbb{R})} p!) \hat{\otimes}_{C(N)} [D_N] \\ &= [[E|_N]] \hat{\otimes}_{C(N)} [D_N].\end{aligned}$$

Therefore, by using the map $e : KK^0(C(N), \mathbb{C}) \to KK^0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ induced by the map to one point, we have

$$e((i \times \mathrm{id}_N)! \hat{\otimes}_{C_0(M)} ([[E]] \hat{\otimes}_{C_0(M)} [D_M])) = \mathrm{index}(D_{E|_N}),$$

where the right hand side is the Fredholm index of the Dirac operator on N twisted by $E|_N$. This is a similar formula to the Roe-Higson index theorem. Combine the Roe-Higson index theorem, and this implies the composition of the assembly map $A : K^0(C_0(M)) \to K_1(C^*(M))$ with Connes' pairing of ζ is equal to $e(i! \hat{\otimes}_{C_0(M)} -)$:

$$\langle A(x), \zeta \rangle = e(i! \hat{\otimes}_{C_0(M)} x).$$

Note that we have $A([D_M]) = \mathrm{ind}(D_M)$, the odd index of D_M.

On the other hand, we assume $* = 1$. Take $\phi \in GL_l(C_0(M))$, then it defines an element $[[\phi]] \in KK^1(C_0(M), C_0(M))$ by using the inclusion map $KK^1(\mathbb{C}, C_0(M)) \to KK^1(C_0(M), C_0(M))$. The similar argument in $* = 0$ implies

$$\langle A([[\phi]]\hat{\otimes}_{C_0(M)}[D_M]), \zeta\rangle = e((i\times\mathrm{id}_N)!\hat{\otimes}_{C_0(M)}([[\phi]]\hat{\otimes}_{C_0(M)}[D_M])) = \mathrm{index}(T_{\phi|_N}).$$

However, since ϕ is constant ($\neq 0$) at infinity, $\phi|_N$ is homotopic to a constant function in $GL_l(C(N))$. Thus the right hand side is always 0. This vanishing comes from that we take a function ϕ in $C_0(M)^+$. So we have to use a larger algebra than $C_0(M)^+$ in order to get non-trivial index. On the other hand, we take ϕ in $C_w(M)$ in our situation. $C_w(M)$ is a suitable larger algebra in this situation.

5.4. Remarks on the odd index class

5.4.1. A reason why we use $C_w(M)$. In this subsection, we see a reason why we use a C^*-algebra $C_w(M)$. Firstly, we compare the Higson algebra $C_h(M)$ and $C_b(M)$ with $C_w(M)$.

Remark 5.4.1. *Let $C_h(M)$ be the Higson algebra of M, that is, $C_h(M)$ is the C^*-algebra generated by all smooth and bounded functions defined on M of which gradient is vanishing at infinity* [**25**, p.26]. *By definition, one has $C_h(M) \subset C_w(M) \subset C_b(M)$.*

We assume $M = \mathbb{R}$. Then one has $\sin x \notin C_h(\mathbb{R})$ but $\sin x \in C_w(\mathbb{R})$. This implies $C_h(\mathbb{R}) \subsetneq C_w(\mathbb{R})$. On the other hand, any $f \in \mathscr{W}(\mathbb{R})$ is a $\|f'\|$-Lipschitz function. In particular, f is uniformly continuous on $\mathbb{R}$. Thus the uniform limit of Cauchy sequence $\{f_n\} \subset \mathscr{W}(\mathbb{R})$ is also uniformly continuous. Therefore, one has $\sin(x^2) \notin C_w(\mathbb{R})$ but $\sin(x^2) \in C_b(\mathbb{R})$. This implies $C_w(\mathbb{R}) \subsetneq C_b(\mathbb{R})$.

We assume $M = \mathbb{R} \times N$ and $\phi \in C^\infty(N)$. In this case, we have $1\otimes\phi \in C_w(M)$ but $1\otimes\phi \notin C_h(M)$ in general. For example, if $\phi(x) = e^{ix}$ for $x \in S^1$, then we have $1 \otimes \phi \in C_w(\mathbb{R} \times S^1)$ but $1 \otimes \phi \notin C_h(\mathbb{R} \times S^1)$. This is a merit of using $C_w(M)$ (see Subsection 5.6.1).

5.4.2. A relationship with Roe's odd index. In this subsection, we give a formal discussion about a relationship with Roe's odd index. Firstly, we recall the definition of Roe's odd index $\mathrm{ind}(D)$; see Section 4.2. Let M be a complete Riemannian manifold, $S \to M$ a Clifford bundle, D the Dirac operator of S and χ a chopping function. Then we have $\chi(D) \in D^*(M)$ and $q(\chi(D))$ is independent of a choice of χ, where $q : D^*(M) \to D^*(M)/C^*(M)$ is a quotient map. Moreover, we have $[q((\chi(D) + 1)/2)] \in K_0(D^*(M)/C^*(M))$ by $\chi^2 - 1 \in C_0(\mathbb{R})$. Set $\mathrm{ind}(D) = \delta([q((\chi(D) + 1)/2)]) \in K_1(C^*(M))$.

Secondly, we reconstruct this odd index in terms of KK-theory. Define $c_\cdot : \mathbb{C} \to C_w(M)$ by $c_z(x) = z$ for $z \in \mathbb{C}$ and $x \in M$. Then we have $c_\cdot \in KK(\mathbb{C}, C_w(M))$ since this map $c_\cdot$ is a $*$-homomorphism. On the other hand, we have $[C^*(M) \oplus C^*(M), \mu \oplus \mu, \chi(D) \oplus (-\chi(D))] \in KK^1(C_w(M), C^*(M))$ since $\chi_0(x) = x(x^2+1)^{-1/2}$ is a chopping function and we have $\chi - \chi_0 \in C_0(\mathbb{R})$. We denote by $[D]_o$ this KK element. Then we obtain $c.\hat{\otimes}_{C_w(M)}[D]_o = \mathrm{ind}(D)$.

Finally, we see our Kasparov product is a counter part of Roe's odd index. We composite the suspension isomorphism $KK(\mathbb{C}, C_w(M)) \to KK^1(\mathbb{C}, C_w(M) \otimes C_0(\mathbb{R}))$ and a homomorphism induced by the inclusion map $C_w(M) \otimes C_0(\mathbb{R}) \to C_w(M) \otimes C(S^1) \to C_w(M \times S^1)$. Thus we get a homomorphism

$$\sigma : KK(\mathbb{C}, C_w(M)) \to KK^1(\mathbb{C}, C_w(M \times S^1)).$$

On the other hand, there is a homomorphism $KK^1(C_w(M), C^*(M)) \to KK^1(C_w(M), C^*(M \times S^1))$ since KK^1-group is stably isomorphic. Let D_{S^1} be the Dirac operator on S^1. D_{S^1} induces $[D_{S^1}] \in KK^1(C(S^1), \mathbb{C})$. By the composition of the Kasparov product $[D_{S^1}]\hat{\otimes}_{\mathbb{C}}-$ and a map induced by this $*$-homomorphism $C_w(M \times S^1) \ni f \mapsto f|_{M \times \{1\}} \otimes 1 \in C_w(M) \otimes C(S^1)$, we get a homomorphism

$$\tau : KK^1(C_w(M), C^*(M)) \to KK(C_w(M \times S^1), C^*(M \times S^1)).$$

Consequently, by using homomorphisms σ and τ, we may see the Kasparov product which we use is a counterpart of Roe's odd index.

5.4.3. On $KK^n(C_w(M), C^*(M))$. Let N be a closed manifold and D a Dirac operator on N. D defines an element $[D] \in KK^n(C(N), \mathbb{C})$. Since KK groups are stably isomorphic, we have

$$[D] \in KK^n(C(N), \mathbb{C}) \cong KK^n(C(N), \mathcal{K}).$$

Recall that $C^*(N) = \mathcal{K}$; see Remark 4.1.9. Thus the group $KK^n(C_w(M), C^*(M))$ is a variation of $KK^n(C(N), \mathcal{K})$ in the case when M is non compact. Of course, the element $[D] \in KK^0(C_w(M), C^*(M))$ in Proposition 5.1.3 is a variation in the case when M is non compact.

We assume the Dirac operator D is graded. By using the Kasparov product

$$\hat{\otimes}_{C_w(M)} : K_0(C_w(M)) \times KK^0(C_w(M), C^*(M)) \to K_0(C^*(M)),$$

we have $[1]\hat{\otimes}_{C_w(M)}[D] = [e_D] - [p]$, where we set

$$e_D = (1+D^2)^{-1} \begin{bmatrix} 1 & D^- \\ D^+ & D^-D^+ \end{bmatrix} \text{ and } p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This class is studied in [**35**] and [**42**], for instance.

5.5. Calculation of the index class

In this section, we calculate our index class explicitly.

5.5.1. Explicit formula of the index class. In this subsection, we represent the index class by an element in $GL_l(C^*(M))$. For this purpose, we present $[D]$ by the Cuntz picture of $KK(C_w(M), C^*(M))$ and then we calculate Kasparov product $[\phi]\hat{\otimes}_{C_w(M)}[D]$. Set

$$C_b^*(M) = \left\{ u + \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} \; ; \; u \in C^*(M), f, g \in C_b(M) \right\}.$$

Then $C_b^*(M)$ is a unital C^*-subalgebra of $D^*(M)$ and contains $C^*(M)$ as an essential ideal. Let $\chi \in C(\mathbb{R}; [-1,1])$ be a chopping function. Set $\eta(x) = (1 - \chi(x)^2)^{1/2} \in C_0(\mathbb{R})$. Then η is a positive even function and we have $\eta(D) \in C^*(M)$.

Proposition 5.5.1. *Let $\iota : C_b^*(M) \hookrightarrow M_\infty(C_b^*(M))$ be the standard inclusion. We use a standard inclusion $M_\infty(C_b^*(M)) \hookrightarrow \mathbb{B}(\mathbb{H}_{C^*(M)})$. Set $\mathcal{D}_\chi = \chi(D) + \epsilon\eta(D) \in D^*(M)$,*

$$\psi_{\chi,+}(f) = \iota\left(\mathcal{D}_\chi \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} \mathcal{D}_\chi\right) \text{ and } \psi_-(f) = \iota\left(\begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}\right).$$

Then

$$(\psi_+, \psi_-) : C_w(M) \to \mathbb{B}(\mathbb{H}_{C^*(M)}) \rhd C^*(M) \otimes \mathcal{K}$$

is a quasihomomorphism from $C_w(M)$ to $C^*(M) \otimes \mathcal{K}$ and one has $[D] = [\psi_+, \psi_-]$ under the natural identification $KK(C_w(M), C^*(M)) \cong KK(C_w(M), C^*(M)^{\mathrm{tri}})$. We note that we omit the subscript χ for the simplicity.

Proof. We use Remark 3.2.18. Since the even grading of $C^*(M)$ is defined by the decomposition of $S^+ \oplus S^-$ and η is equal to $(1-\chi^2)^{1/2}$ by definition, we have

$$[D] = \left[E = C^*(M) \oplus C^*(M), \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{bmatrix} \right]$$

under the isomorphism:

$$KK^0(C_w(M), C^*(M)) \cong KK^0(C_w(M), C^*(M)^{\mathrm{tri}}).$$

Here, we assume as follows:

$$\begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix} : C_w(M) \to C_b^*(M) \subset \mathbb{B}(C^*(M)).$$

Now, we conjugate by $\mathcal{D} \oplus 1 \in \mathbb{B}(E)$. Then we obtain

$$[D] = \left[C^*(M) \oplus C^*(M), \begin{bmatrix} \mathcal{D}(\mu \oplus 0)\mathcal{D} & 0 \\ 0 & 0 \oplus \mu \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right].$$

Since $\left(\mathbb{H}_{C^*(M)} \oplus \mathbb{H}_{C^*(M)}, 0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ is a degenerate module, we obtain

$$\begin{aligned}[D] &= \left[E, \begin{bmatrix} \mathcal{D}(\mu \oplus 0)\mathcal{D} & 0 \\ 0 & 0 \oplus \mu \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] \oplus \left[\mathbb{H}_{C^*(M)} \oplus \mathbb{H}_{C^*(M)}, 0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] \\ &= \left[(C^*(M) \oplus \mathbb{H}_{C^*(M)})^2, \begin{bmatrix} (\mathcal{D}(\mu \oplus 0)\mathcal{D}) \oplus 0 & 0 \\ 0 & (0 \oplus \mu) \oplus 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right].\end{aligned}$$

Then we define a unitary operator $W : C^*(M) \oplus \mathbb{H}_{C^*(M)} \to \mathbb{H}_{C^*(M)}$ by $W(a_0, (a_i)_{i=1}^\infty) = (a_i)_{i=0}^\infty$ and conjugate by $W \oplus W$. So we obtain

$$[D] = \left[\mathbb{H}_{C^*(M)} \oplus \mathbb{H}_{C^*(M)}, \begin{bmatrix} \psi_+ & 0 \\ 0 & \psi_- \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right].$$

We can show $\psi_+(f) \in M_\infty(C_b^*(M))$ by using

$$\left[\begin{bmatrix} \psi_+ & 0 \\ 0 & \psi_- \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] \in \mathbb{K}(\hat{\mathbb{H}}_{C^*(M)}).$$

Therefore, a pair

$$(\psi_+, \psi_-) : C_w(M) \to \mathbb{B}(\mathbb{H}_{C^*(M)}) \triangleright C^*(M) \otimes \mathcal{K}$$

is a quasihomomorphism from $C_w(M)$ to $C^*(M) \otimes \mathcal{K}$ and we obtain $[D] = [\psi_+, \psi_-]$.

□

REMARK 5.5.2. *By definition, one has*

$$\mathcal{D} \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix} \mathcal{D} - \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} = \mathcal{D} \begin{bmatrix} f\eta(D)^+ & [f, \chi(D)^-] \\ 0 & \eta(D)^- f \end{bmatrix} \in C^*(M)$$

for any $f \in C_w(M)$. This is a direct proof of $\psi_+(f) \in M_\infty(C_b^*(M))$.

We recall that the Cuntz picture of Kasparov modules suits the Kasparov product with an element in K_1-group; see Example 3.2.24.

PROPOSITION 5.5.3. *For any $\phi \in GL_l(C_w(M))$, one has*

$$\mathrm{Ind}(\phi, D) = \left[\mathcal{D} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} \mathcal{D} \begin{bmatrix} 1 & 0 \\ 0 & \phi^{-1} \end{bmatrix}\right] \in K_1(C^*(M)).$$

PROOF. Firstly, we obtain

$$\psi_+(\phi - 1) + 1 = j\left(\mathcal{D} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} \mathcal{D}\right) \text{ and } \psi_-(\phi - 1) + 1 = j\left(\begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix}\right),$$

where $j : GL_l(C^*_b(M)) \to GL_\infty(C^*_b(M))$ is the standard inclusion. Thus we obtain

$$\begin{aligned}\mathrm{Ind}(\phi, D) &= [\{\psi_+(\phi-1)+1\}\{\psi_-(\phi-1)+1\}^{-1}] \\ &= \left[\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}\begin{bmatrix}1 & 0\\ 0 & \phi^{-1}\end{bmatrix}\right] \in K_1(C^*(M)).\end{aligned}$$

□

The last of this subsection, we back to Connes' pairing in our main theorem.

Remark 5.5.4. *By Proposition 4.3.9, one has*

$$\begin{aligned}&\langle \mathrm{Ind}(\phi, D), \zeta\rangle \\ &= -\frac{1}{8\pi i}\mathrm{index}\left(\Pi\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}\begin{bmatrix}1 & 0\\ 0 & \phi^{-1}\end{bmatrix}\Pi : \Pi(L^2(M,S))^l \to \Pi(L^2(M,S))^l\right).\end{aligned}$$

On the other hand, $\Pi u \Pi$ *is Fredholm operator for any* $u \in GL_l(C^*_b(M))$ *by* $[f, \Pi] = 0$ *for any* $f \in C_b(M)$*. This implies*

$$\begin{aligned}-8\pi i\langle \mathrm{Ind}(\phi, D), \zeta\rangle &= \mathrm{index}\left(\Pi\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}\Pi\right) + \mathrm{index}\left(\Pi\begin{bmatrix}1 & 0\\ 0 & \phi^{-1}\end{bmatrix}\Pi\right) \\ &= \mathrm{index}\left(\Pi\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}\Pi\right).\end{aligned}$$

In order to use bellow sections, we fix notation. Set $u_{\chi,\phi} = \mathcal{D}_\chi\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}_\chi$ and $v_{\chi,\phi} = u_{\chi,\phi} - \begin{bmatrix}1 & 0\\ 0 & \phi\end{bmatrix}$. Then we obtain

$$v_{\chi,\phi} = \mathcal{D}_\chi\begin{bmatrix}(\phi-1)\eta(D)^+ & [\phi, \chi(D)^-]\\ 0 & \eta(D)^-(\phi-1)\end{bmatrix}.$$

5.5.2. Another formula in the special case. By Remark 5.5.4, our main theorem is the coincidence of two Fredholm indices:

$$\mathrm{index}\,(\Pi u_{\chi,\phi}\Pi) = \mathrm{index}(T_\phi).$$

Both sides of this equation do not change a homotopy of ϕ. Therefore, it suffices to show the case when $\phi \in GL_l(\mathscr{W}(M))$. In this case, $\phi : M \to GL_l(\mathbb{C})$ is a smooth function such that $\|\phi\| < \infty$, $\|\mathrm{grad}(\phi)\| < \infty$ and $\|\phi^{-1}\| < \infty$. Moreover, we also assume that ϕ satisfies $[|D|, \phi] \in \mathcal{L}(L^2(M,S))$. This condition is a technical assumption in this subsection. Set $\mathscr{W}_1(M) = \{f \in \mathscr{W}(M); [|D|, f] \in \mathcal{L}(L^2(M,S))\}$. In this subsection, we use $\chi_0(x) = x(1+x^2)^{-1/2}$ as a chopping function, that is, $\mathcal{D} = \mathcal{D}_{\chi_0}$.

In order to prove our main theorem, we perturb the operator $\mathcal{D}\begin{bmatrix}\phi & 0\\ 0 & 1\end{bmatrix}\mathcal{D}$ by a homotopy. Firstly, for any $t \in [0,1]$, set $F_t = t + (1-t)(1+D^2)^{-1/2} \in D^*(M)$. For any $t \in [0,1]$ and $x \in \mathbb{R}$, set

$$f_t(x) = \frac{1}{t + (1-t)(1+x^2)^{-1/2}}.$$

We assume $t \in (0,1]$. Then we obtain $F_t^{-1} = f_t(D) \in D^*(M)$ by $f_t - 1/t \in C_0(M)$.

Secondly, because of

$$(D+\epsilon)^{-1}\begin{bmatrix}f & 0\\ 0 & 0\end{bmatrix}(D+\epsilon)\sigma - \begin{bmatrix}0 & 0\\ 0 & f\end{bmatrix}\sigma = (D+\epsilon)^{-1}\begin{bmatrix}f & -c(\mathrm{grad}(f))^-\\ 0 & f\end{bmatrix}\sigma$$

for any $f \in M_l(\mathscr{W}(M))$ and $\sigma \in C_c^\infty(M,S)$, we obtain

$$\left\|(D+\epsilon)^{-1}\begin{bmatrix}f & 0\\ 0 & 0\end{bmatrix}(D+\epsilon)\sigma\right\|_{L^2} \leq (2\|f\| + \|\mathrm{grad}(f)\|)\|\sigma\|_{L^2}.$$

This implies

$$\rho(f) = (D+\epsilon)^{-1}\begin{bmatrix}f & 0\\ 0 & 0\end{bmatrix}(D+\epsilon) \in \mathcal{L}(L^2(M,S))$$

since $C_c^\infty(M,S)$ is dense in $L^2(M,S)$. Moreover, we obtain $\rho(f) \in C_b^*(M)$ by $(D+\epsilon)^{-1} \in C^*(M)$ and

$$\begin{bmatrix}f & -c(\mathrm{grad}(f))^-\\ 0 & f\end{bmatrix} \in D^*(M).$$

Finally, set $\rho_0(f) = \mathcal{D}\begin{bmatrix}f & 0\\ 0 & 0\end{bmatrix}\mathcal{D}$ and $\rho_t(f) = F_t^{-1}\rho(f)F_t$ for any $t \in (0,1]$ and $f \in \mathscr{W}(M)$. Formally, we set $F_0^{-1} = (1+D^2)^{1/2}$. Then we obtain $\rho_t(f) = F_t^{-1}\rho(f)F_t \in \mathcal{L}(L^2(M,S))$ for any $t \in [0,1]$ and $f \in \mathscr{W}(M)$. Note that we have

$$\rho_0(f) = \mathcal{D}\begin{bmatrix}f & 0\\ 0 & 0\end{bmatrix}\mathcal{D} \text{ and } \rho_1(f) = \rho(f).$$

This family of bounded operator $t \mapsto \rho_t(f)$ is continuous in $C_b^*(M)$ for $f \in \mathscr{W}_1(M)$.

PROPOSITION 5.5.5. *For any $t \in [0,1]$ and $f \in M_l(\mathscr{W}_1(M))$, one has $\rho_t(f) \in M_l(C_b^*(M))$. Moreover, $[0,1] \ni t \mapsto \rho_t(f) \in M_l(C_b^*(M)) \subset M_l(\mathcal{L}(L^2(M,S)))$ is continuous.*

Proof. It suffices to show the case when $l = 1$.

Firstly we show $\rho_t(f) \in C_b^*(M)$. When $t = 0, 1$, we already proved. We assume $t \in (0, 1)$. We have

$$\begin{aligned}
&\rho_t(f) - \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} \\
=&F_t^{-1}(D+\epsilon)^{-1} \begin{bmatrix} tf + (1-t)f(1+D^2)^{-1/2} & tc(\mathrm{grad}(f))^- + (1-t)[f, D^-(1+D^2)^{-1/2}] \\ 0 & tf + (1-t)(1+D^2)^{-1/2}f \end{bmatrix}.
\end{aligned}$$

Because of $F_t^{-1} \in D^*(M)$, $(D+\epsilon)^{-1} \in C^*(M)$ and

$$\begin{bmatrix} tf + (1-t)f(1+D^2)^{-1/2} & tc(\mathrm{grad}(f))^- + (1-t)[f, D^-(1+D^2)^{-1/2}] \\ 0 & tf + (1-t)(1+D^2)^{-1/2}f \end{bmatrix} \in D^*(M),$$

we obtain $\rho_t(f) \in C_b^*(M)$.

Next, we show continuity of $t \mapsto \rho_t(f)$. F_t^{-1}, $\rho(f)$ and F_t are bounded operators for any $t \in (0, 1]$, and $[0, 1] \ni t \mapsto F_t \in \mathcal{L}(L^2(M))$ is continuous. Thus $t \mapsto \rho_t(f)$ is continuous on $(0, 1]$. The rest of proof is continuity at $t = 0$. First, we show $\|(D+\epsilon)^{-1}F_t^{-1}\| \leq 2$ for any $t \in [0, 1]$. Set

$$\begin{aligned}
g_t(x) &= \frac{x}{(1+x^2)(t+(1-t)(1+x^2)^{-1/2})} = \frac{xf_t(x)}{1+x^2} \text{ and} \\
h_t(x) &= \frac{1}{(1+x^2)(t+(1-t)(1+x^2)^{-1/2})} = \frac{f_t(x)}{1+x^2}.
\end{aligned}$$

Then we have

$$|g_t(x)| = \frac{1}{t(|x| + 1/|x|) + (1-t)\sqrt{1+1/x^2}} \leq \frac{1}{2t + 1 - t} \leq 1$$

and $|h_t(x)| \leq 1$. Thus we obtain $\|(D+\epsilon)^{-1}F_t^{-1}\| \leq 2$ by

$$(D+\epsilon)^{-1}F_t^{-1} = D(1+D^2)^{-1}F_t^{-1} + \epsilon(1+D^2)^{-1}F_t^{-1} = g_t(D) + \epsilon h_t(D).$$

By using $\|(D+\epsilon)^{-1}F_t^{-1}\| \leq 2$, we can prove continuity at $t = 0$. For any $t > 0$, we can calculate

$$\begin{aligned}
&\rho_t(f) - \rho_0(f) \\
=&(D+\epsilon)^{-1}F_t^{-1} \begin{bmatrix} tf - tf(1+D^2)^{-1/2} & tc(\mathrm{grad}(f))^- - t[f, D^-(1+D^2)^{-1/2}] \\ 0 & tf - t(1+D^2)^{-1/2}f \end{bmatrix} \\
&+ \{(D+\epsilon)^{-1}F_t^{-1} - \mathcal{D}\} \begin{bmatrix} f(1+D^2)^{-1/2} & [f, D^-(1+D^2)^{-1/2}] \\ 0 & (1+D^2)^{-1/2}f \end{bmatrix}.
\end{aligned}$$

So the first term converges to 0 with the operator norm as $t \to 0$.

We show the second term converges to 0 with the operator norm as $t \to 0$. Because of

$$\mathcal{D} - (D+\epsilon)F_t = t(D+\epsilon)\{1-(1+D^2)^{-1/2}\},$$

the second term is equal to

$$t(D+\epsilon)^{-1}F_t^{-1}\{(1+D^2)^{-1/2}-1\}(1+D^2)^{1/2}\begin{bmatrix} f(1+D^2)^{-1/2} & [f, D^-(1+D^2)^{-1/2}] \\ 0 & (1+D^2)^{-1/2}f \end{bmatrix}.$$

Therefore, if $(1+D^2)^{1/2}f(1+D^2)^{-1/2}$ and $(1+D^2)^{1/2}[f, D(1+D^2)^{-1/2}]$ are bounded, the second term converges to 0 with the operator norm as $t \to 0$. We show that $(1+D^2)^{1/2}f(1+D^2)^{-1/2}$ and $(1+D^2)^{1/2}[f, D(1+D^2)^{-1/2}]$ are bounded. By using following equalities

$$(D^2+1)^{1/2}f(D^2+1)^{-1/2} = [(D^2+1)^{1/2}, f](D^2+1)^{-1/2} + f$$

and

$$(D^2+1)^{1/2}[f, D(D^2+1)^{-1/2}] = [(D^2+1)^{1/2}, f]D(D^2+1)^{-1/2} + [f, D],$$

it suffices to show $[(D^2+1)^{1/2}, f]$ is a bounded operator. Because of $\alpha(x) = \sqrt{x^2+1} - |x| \in C_0(\mathbb{R})$, we have $\alpha(D) \in \mathcal{L}(L^2(M, S))$. This implies $[(D^2+1)^{1/2}, f]$ is bounded if and only if $[|D|, f]$ is bounded. We note that boundness of $[|D|, f]$ is required the definition of the algebra $\mathscr{W}_1(M)$. Hence $(D^2+1)^{1/2}f(D^2+1)^{-1/2}$ and $(D^2+1)^{1/2}[f, D(D^2+1)^{-1/2}]$ are bounded. Thus the second term converges to 0 as $t \to 0$. Therefore, $t \mapsto \rho_t(f)$ is continuous.

□

Due to Proposition 5.5.5, the following maps

$$\Pi\{\rho_t(\phi-1)+1\}\Pi : \Pi(L^2(M,S))^l \to \Pi(L^2(M,S))^l$$

determine a continuous family of Fredholm operators for any $\phi \in GL_l(\mathscr{W}_1(M))$. Therefore, we obtain

$$\langle \mathrm{Ind}(\phi, D), \zeta \rangle = -\frac{1}{8\pi i}\mathrm{index}\left(\Pi(D+\epsilon)^{-1}\begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix}(D+\epsilon)\Pi\right)$$

for any $\phi \in GL_l(\mathscr{W}_1(M))$.

REMARK 5.5.6. *In the definition of* ρ_t*, we don't use the assumption* $[|D|, f] \in \mathcal{L}(L^2(M,S))$. *In particular, one has* $\rho(f) \in C_b^*(M)$ *for* $f \in \mathscr{W}(M)$. *Set* $\varrho(\phi) = \rho(\phi-1)+1$ *for any* $\phi \in GL_l(\mathscr{W}(M))$. *Then the operator* $\Pi\varrho(\phi)\Pi$ *is Fredholm for all* $\phi \in GL_l(\mathscr{W}(M))$.

5.6. Proof of Main theorem

We prove our main theorem. The proof is made of two Steps. The first step is the proof in the case when $\mathbb{R} \times N$. The second step is the reduction to $\mathbb{R} \times N$. This strategy is a common strategy for the proof of an index theorem for partitioned manifolds.

5.6.1. The case for $\mathbb{R} \times N$. Let N be a closed manifold. In this subsection, we prove Theorem 5.2.1 in the case when $M = \mathbb{R} \times N$. Recall that $\mathbb{R} \times N$ is partitioned by $(\mathbb{R}_+ \times N, \mathbb{R}_- \times N, \{0\} \times N)$. Let $S_N \to N$ be a Clifford bundle, c_N the Clifford action on S_N and D_N the Dirac operator on S_N. Given $\phi \in C^\infty(N; GL_l(\mathbb{C}))$, we define the map $\tilde{\phi} : \mathbb{R} \times N \to GL_l(\mathbb{C})$ by $\tilde{\phi}(t, x) = \phi(x)$. We often denote $\tilde{\phi}$ by ϕ in the sequel. Note that we have $\phi \in GL_l(\mathscr{W}_1(\mathbb{R} \times N))$.

Let $p : \mathbb{R} \times N \to N$ be the projection to N. Set $S = p^*S_N \oplus p^*S_N$ and $\epsilon = 1 \oplus (-1)$, where ϵ is the grading operator on S. Then we define a Clifford action c on S by

$$c(\mathrm{d}/\mathrm{d}t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } c(X) = \begin{bmatrix} 0 & c_N(X) \\ c_N(X) & 0 \end{bmatrix}$$

for all $X \in C^\infty(N, TN)$. Here $\mathrm{d}/\mathrm{d}t$ is a coordinate unit vector field on $\mathbb{R}$. Then $S \to M$ is a Clifford bundle and the Dirac operator D of S is given by

$$D = \begin{bmatrix} 0 & \mathrm{d}/\mathrm{d}t + D_N \\ -\mathrm{d}/\mathrm{d}t + D_N & 0 \end{bmatrix}.$$

Denote by H_+ the subspace of $L^2(N, S_N)$ which is generated by non-negative eigenvectors of D_N. Also denote by H_- the orthogonal complement of H_+ in $L^2(M, S)$. Set $F = 2P - 1$, where P is the projection to H_+.

Due to Subsection 5.5.2, it suffices to show

$$\operatorname{index}\left(\Pi(D+\epsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D+\epsilon)\Pi\right) = \operatorname{index}(T_\phi).$$

For this purpose, we perturb the operator $\Pi\varrho(\phi)\Pi$ by a homotopy. We firstly estimate the supuremum of some functions to prove a continuity of the homotopy.

LEMMA 5.6.1. *(i) Set*

$$f_s(x) = \frac{x}{x^2 + (1-s)^2} \text{ and } g_s(x) = \frac{1}{x^2 + (1-s)^2}$$

for all $s \in [0, 1]$ and $x \in \mathbb{R} \backslash (-s, s)$. Then one has $\sup_x |f_s(x)| \leq 2$ and $\sup_x |g_s(x)| \leq 2$ for all $s \in [0, 1]$.

(ii) Set

$$\mu_{\lambda,s}(x) = \frac{1}{x^2 + \{(1-s)\lambda + s\mathrm{sgn}(\lambda)\}^2 + (1-s)^2}$$

and

$$\nu_{\lambda,s}(x) = \frac{x}{x^2 + \{(1-s)\lambda + s\mathrm{sgn}(\lambda)\}^2 + (1-s)^2}$$

for all $\lambda \in \mathbb{R}$, $s \in [0,1)$ *and* $x \in \mathbb{R}$, *where* $\mathrm{sgn}(\lambda)$ *is* 1 *if* $\lambda \geq 0$ *or* -1 *if* $\lambda < 0$. *Then one has*

$$\sup_x |\mu_{\lambda,s}(x)| \leq \frac{1}{(1-s)^2(\lambda^2+1)} \text{ and } \sup_x |\nu_{\lambda,s}(x)| \leq \frac{1}{2(1-s)\sqrt{\lambda^2+1}}$$

for all $\lambda \in \mathbb{R}$, $s \in [0,1)$.

Proof. (i) For $0 \leq s \leq 1/2$, we have $|f_s(x)| \leq f_s(1-s) \leq 1$. For $1/2 \leq s \leq 1$, we have $|f_s(x)| \leq f_s(s) \leq 2$. This implies $\sup_x |f_s(x)| \leq 2$. On the other hand, we have $|g_s(x)| \leq g_s(s) \leq 2$.
(ii) For $\lambda \geq 0$, we have $(1-s)\lambda + s\mathrm{sgn}(\lambda) \geq (1-s)\lambda \geq 0$. On the other hand, for $\lambda < 0$, we have $(1-s)\lambda + s\mathrm{sgn}(\lambda) \leq (1-s)\lambda < 0$. So we obtain $|\mu_{\lambda,s}(x)| \leq h_{\lambda,s}(0) \leq 1/(1-s)^2(\lambda^2+1)$.

On the other hand, we obtain

$$|\nu_{\lambda,s}(x)| \leq \nu_{\lambda,s}\left(\sqrt{\{(1-s)\lambda + s\mathrm{sgn}(\lambda)\}^2 + (1-s)^2}\right) \leq \frac{1}{2(1-s)\sqrt{\lambda^2+1}}.$$

□

Proposition 5.6.2. *Set*

$$D_s = \begin{bmatrix} 0 & \mathrm{d}/\mathrm{d}t + (1-s)D_N + sF \\ -\mathrm{d}/\mathrm{d}t + (1-s)D_N + sF & 0 \end{bmatrix}$$

for all $s \in [0,1]$ *and*

$$u_{\phi,s} = (D_s + (1-s)\epsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D_s + (1-s)\epsilon).$$

Then the map $[0,1] \ni s \mapsto u_{\phi,s} \in \mathcal{L}(L^2(M,S)^l)$ *is continuous.*

Proof. It suffices to show the case when $l = 1$. Since we have $(\mathrm{d}/\mathrm{d}t)^* = -\mathrm{d}/\mathrm{d}t$ and D_N is the Dirac operator on N, D_s is a self-adjoint closed operator densely defined on $\mathrm{domain}(D_s) = \mathrm{domain}(D)$.

Next we show $\sigma(D_s) \cap (-s,s) = \emptyset$ for all $s \in (0,1]$. Set

$$T_s = \begin{bmatrix} 0 & \mathrm{d}/\mathrm{d}t + (1-s)D_N \\ -\mathrm{d}/\mathrm{d}t + (1-s)D_N & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 & F \\ F & 0 \end{bmatrix}.$$

These operators T_s and J are self-adjoint and we have $D_s = T_s + sJ$ and $T_sJ + JT_s = 2(1-s)D_NF \geq 0$ on domain(D). So for any $\sigma \in$ domain(D), we obtain

$$\|D_s\sigma\|_{L^2}^2 = \|T_s\sigma\|_{L^2}^2 + s^2\|J\sigma\|_{L^2}^2 + s\langle (T_sJ+JT_s)\sigma, \sigma\rangle_{L^2} \geq s^2\|J\sigma\|_{L^2}^2 = s^2\|\sigma\|_{L^2}^2.$$

This implies $\sigma(D_s) \cap (-s, s) \neq \emptyset$. In particular, D_1 has a bounded inverse.

On the other hand, when $s \in [0,1)$, we have $(D_s + (1-s)\epsilon)^{-1} \in \mathcal{L}(L^2(M,S))$ since $(D_s + (1-s)\epsilon)^2 = D_s^2 + (1-s)^2$ is invertible. Therefore, $u_{\phi,s}$ is well defined as a closed operator on $L^2(M,S)$ with domain$(u_{\phi,s}) =$ domain(D) for all $s \in [0,1]$. Thus we obtain $u_{\phi,s} \in \mathcal{L}(L^2(M,S))$ by

$$u_{\phi,s} = \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + (D_s + (1-s)\epsilon)^{-1} \begin{bmatrix} (1-s)(\phi - 1) & -(1-s)c_N(\mathrm{grad}(\phi)) + s[\phi, F] \\ 0 & (1-s)(\phi-1) \end{bmatrix}.$$

Next we show continuity of $[0,1] \ni s \mapsto u_{\phi,s} \in \mathcal{L}(L^2(M,S))$. First, because of

$$(D_s + (1-s)\epsilon)^{-1} = f_s(D_s) + (1-s)\epsilon g_s(D_s),$$

we have

$$(*) \qquad \|(D_s + (1-s)\epsilon)^{-1}\| \leq \sup_x |f_s(x)| + (1-s)\sup_x |g_s(x)| \leq 4$$

by Lemma 5.6.1. Therefore, $\{\|(D_s + (1-s)\epsilon)^{-1}\|\}_{s\in[0,1]}$ is a bounded set.

Next, for any $s, s' \in [0,1]$, we obtain

$$\begin{aligned} &u_{\phi,s} - u_{\phi,s'} \\ =&(D_s + (1-s)\epsilon)^{-1} \begin{bmatrix} (s'-s)(\phi-1) & (s-s')c_N(\mathrm{grad}(\phi)) + (s-s')[\phi, F] \\ 0 & (s'-s)(\phi-1) \end{bmatrix} \\ &+ \{(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}\} \\ &\qquad \begin{bmatrix} (1-s')(\phi-1) & -(1-s')c_N(\mathrm{grad}(\phi)) + s'[\phi, F] \\ 0 & (1-s')(\phi-1) \end{bmatrix} \\ =:& \alpha_{s,s'} + \beta_{s,s'}. \end{aligned}$$

The first term $\alpha_{s,s'}$ converges to 0 with the operator norm as $s \to s'$.

The rest of proof is the second term $\beta_{s,s'}$ converges to 0. First, we assume $s' = 1$. Then we obtain

$$\beta_{s,1} = \{(D_s + (1-s)\epsilon)^{-1} - D_1^{-1}\} \begin{bmatrix} 0 & [\phi, F] \\ 0 & 0 \end{bmatrix}$$

and

$$(D_s + (1-s)\epsilon)^{-1} - D_1^{-1}$$
$$= (s-1)(D_s + (1-s)\epsilon)^{-1} D_1^{-1} \begin{bmatrix} 0 & D_N \\ D_N & 0 \end{bmatrix} + (1-s)(D_s + (1-s)\epsilon)^{-1}(J-\epsilon)D_1^{-1}$$

since D_N commutes F and $\mathrm{d}/\mathrm{d}t$ on domain(D), respectively. Therefore, the following operator

$$\beta_{s,1} = (s-1)(D_s + (1-s)\epsilon)^{-1} D_1^{-1} \begin{bmatrix} 0 & 0 \\ 0 & D_N[\phi, F] \end{bmatrix} + (1-s)(D_s + (1-s)\epsilon)^{-1}(J-\epsilon)D_1^{-1} \begin{bmatrix} 0 & [\phi, F] \\ 0 & 0 \end{bmatrix}$$

converges to 0 with the operator norm as $s \to 1$ since $D_N[\phi, F]$ is a pseudodifferential operator of order 0 on N and $\|(D_s + (1-s)\epsilon)^{-1}\|$, $\|J\|$, $\|\epsilon\|$ and $\|D_1^{-1}\|$ are uniformly bounded.

We assume $0 \le s' < 1$. Since an operator

$$\begin{bmatrix} (1-s')(\phi - 1) & -(1-s')c_N(\mathrm{grad}(\phi)) + s'[\phi, F] \\ 0 & (1-s')(\phi-1) \end{bmatrix}$$

is bounded, it suffices to show

$$\|(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}\| \to 0$$

as $s \to s'$. We have

$$(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}$$
$$=(s-s')(D_s + (1-s)\epsilon)^{-1} \begin{bmatrix} 0 & D_N \\ D_N & 0 \end{bmatrix} (D_{s'} + (1-s')\epsilon)^{-1}$$
$$+ (s'-s)(D_s + (1-s)\epsilon)^{-1}(J-\epsilon)(D_{s'} + (1-s')\epsilon)^{-1}$$

and the second term converges to 0 with the operator norm as $s \to s'$ by $(*)$. So it suffices to show

$$U = \begin{bmatrix} 0 & D_N \\ D_N & 0 \end{bmatrix} (D_{s'} + (1-s')\epsilon)^{-1}$$
$$= \begin{bmatrix} D_N A_{s'}^{-1}(-\mathrm{d}/\mathrm{d}t + (1-s')D_N + s'F) & -(1-s')D_N A_{s'}^{-1} \\ (1-s')D_N A_{s'}^{-1} & D_N A_{s'}^{-1}(\mathrm{d}/\mathrm{d}t + (1-s')D_N + s'F) \end{bmatrix}$$

is a bounded operator on $L^2(M, S) = L^2(\mathbb{R})^2 \otimes L^2(N, S_N)$, where set

$$A_{s'} = -\mathrm{d}^2/\mathrm{d}t^2 + \{(1-s')D_N + s'F\}^2 + (1-s')^2.$$

Now, if $D_N A_{s'}^{-1}$, $iD_N A_{s'}^{-1}\mathrm{d}/\mathrm{d}t$ and $D_N A_{s'}^{-1} D_N$ are bounded, then U is also bounded. We show $D_N A_{s'}^{-1} D_N$ is bounded. Denote by E_λ the

λ-eigenspace of D_N. Then $D_N A_{s'}^{-1} D_N$ acts as

$$\lambda^2\{-\mathrm{d}^2/\mathrm{d}t^2 + ((1-s')\lambda + s'\mathrm{sgn}(\lambda))^2 + (1-s')^2\}^{-1}$$

on $L^2(\mathbb{R}) \otimes E_\lambda$. This operator equals to $\lambda^2\mu_{\lambda,s'}(i\mathrm{d}/\mathrm{d}t)$ and we have $\|\lambda^2\mu_{\lambda,s'}(i\mathrm{d}/\mathrm{d}t)\| \leq 1/(1-s')^2$ by Lemma 5.6.1. Therefore, we obtain $\|D_N A_{s'}^{-1} D_N\| \leq 1/(1-s')^2$. Similarly, we can show $\|D_N A_{s'}^{-1}\| \leq 1/(1-s')^2$ (use $\mu_{\lambda,s'}$) and $\|iD_N A_{s'}^{-1}\mathrm{d}/\mathrm{d}t\| \leq 1/2(1-s')$ (use $\nu_{\lambda,s'}$). Thus U is bounded. Therefore, we obtain

$$\|(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}\| \to 0$$

as $s \to s'$ as required.

□

By Proposition 5.6.2, $\Pi u_{\phi,s}\Pi$ is a continuous path in $\mathcal{L}(\Pi(L^2(M,S))^l)$. In fact, this continuous path is a desired homotopy of Fredholm operators.

Proposition 5.6.3. *Set*

$$v_{\phi,s} = u_{\phi,s} - \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix}$$

for all $s \in [0,1]$. *One has* $[\Pi, v_{\phi,s}] \sim 0$. *Therefore* $\Pi u_{\phi,s}\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S))$ *is a Fredholm operator.*

Proof. It suffices to show the case when $l = 1$. Due to Proposition 5.6.2 and closedness of $\mathcal{K}(L^2(M,S))$, we may assume $s \in [0,1)$.

First, we show $g \cdot (D_s + (1-s)\epsilon)^{-1} \sim 0$ for any $g \in C_0(\mathbb{R})$. Since $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, it suffices to show the case when $g \in C_c^\infty(\mathbb{R})$. Because T_s (see in the proof of Proposition 5.6.2) is a first order elliptic differential operator and g commutes with a operator on N, we have

$$\begin{aligned}
&\|g(D_s + (1-s)\epsilon)^{-1}u\|_{H^1} \\
\leq\ & C(\|g(D_s + (1-s)\epsilon)^{-1}u\|_{L^2} + \|T_s g(D_s + (1-s)\epsilon)^{-1}u\|_{L^2}) \\
\leq\ & C'\|u\|_{L^2}
\end{aligned}$$

for any $u \in L^2(M,S)$. Here, $\|\cdot\|_{H^1}$ is the Sobolev first norm on a compact set $\mathrm{Supp}(g) \times N$ and $C' > 0$ depends only on $\|g\|$ and $\|g'\|$. By the Rellich lemma, we have $g(D_s + (1-s)\epsilon)^{-1} \sim 0$. Thus we also have $(D_s + (1-s)\epsilon)^{-1}g = (\bar{g}(D_s + (1-s)\epsilon)^{-1})^* \sim 0$.

Second, we show $[\varphi, (D_s + (1-s)\epsilon)^{-1}] \sim 0$ for any $\varphi \in C^\infty(\mathbb{R})$ satisfying $\varphi = \Pi$ on the complement of a compact set in M. Since φ commutes with a operator on N, we have

$$[\varphi, (D_s+(1-s)\epsilon)^{-1}] = (D_s+(1-s)\epsilon)^{-1} \begin{bmatrix} 0 & \varphi' \\ -\varphi' & 0 \end{bmatrix} (D_s+(1-s)\epsilon)^{-1} \sim 0.$$

By a similar proof in the proof of Proposition 4.3.3 (ii), we have $[\Pi, (D_s + (1-s)\epsilon)^{-1}] \sim 0$. Therefore, we have

$$\begin{aligned}\Pi v_{\phi,s} &= \Pi(D_s + (1-s)\epsilon)^{-1} \begin{bmatrix}(1-s)(\phi-1) & -(1-s)c_N(\mathrm{grad}(\phi)) + s[\phi, F] \\ 0 & (1-s)(\phi-1)\end{bmatrix} \\ &\sim (D_s + (1-s)\epsilon)^{-1}\Pi \begin{bmatrix}(1-s)(\phi-1) & -(1-s)c_N(\mathrm{grad}(\phi)) + s[\phi, F] \\ 0 & (1-s)(\phi-1)\end{bmatrix} \\ &= v_{\phi,s}\Pi.\end{aligned}$$

This implies $\Pi u_{\phi,s}\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S))$ is a Fredholm operator.

□

Due to Propositions 5.6.2 and 5.6.3,

$$\mathrm{index}(\Pi\varrho(\phi)\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S)))$$

is equal to $\mathrm{index}(\Pi u_{\phi,1}\Pi)$. Let $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Hilbert transformation:

$$Hf(t) = -\frac{i}{\pi}\mathrm{p.v.}\int_{\mathbb{R}} \frac{f(y)}{t-y}dy.$$

Then the eigenvalues of H are only 1 and -1 by $H^2 = 1$ and $H \neq \pm 1$. Let $\mathscr{H}_-$ be the (-1)-eigenspace of H and $\hat{P} : L^2(\mathbb{R}) \to \mathscr{H}_-$ the projection to $\mathscr{H}_-$.

Proposition 5.6.4. *Set $\mathscr{T}_\phi = (-it+F)^{-1}\phi(-it+F)$. Then $\hat{P}\mathscr{T}_\phi\hat{P}^*$ is a Fredholm operator and one has*

$$\mathrm{index}(\Pi\varrho(\phi)\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S))) = \mathrm{index}(\hat{P}\mathscr{T}_\phi\hat{P}^* : X \to X),$$

where we set $X = \mathscr{H}_- \otimes L^2(N, S_N)$.

Proof. Due to Propositions 5.6.2 and 5.6.3, we have

$$\begin{aligned}&\mathrm{index}(\Pi\varrho(\phi)\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S))) \\ =&\mathrm{index}(\Pi u_{\phi,0}\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S))) \\ =&\mathrm{index}(\Pi u_{\phi,1}\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S))).\end{aligned}$$

Now, because of

$$\begin{aligned}u_{\phi,1} &= \begin{bmatrix}0 & (-\mathrm{d}/\mathrm{d}t + F)^{-1} \\ (\mathrm{d}/\mathrm{d}t + F)^{-1} & 0\end{bmatrix}\begin{bmatrix}\phi & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & \mathrm{d}/\mathrm{d}t + F \\ -\mathrm{d}/\mathrm{d}t + F & 0\end{bmatrix} \\ &= \begin{bmatrix}1 & 0 \\ 0 & (\mathrm{d}/\mathrm{d}t + F)^{-1}\phi(\mathrm{d}/\mathrm{d}t + F)\end{bmatrix},\end{aligned}$$

we have

$$\begin{aligned}&\operatorname{index}(\Pi\varrho(\phi)\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S)))\\ =\ &\operatorname{index}(\Pi(\mathrm{d}/\mathrm{d}t+F)^{-1}\phi(\mathrm{d}/\mathrm{d}t+F)\Pi \text{ on } \Pi(L^2(\mathbb{R}))\otimes L^2(N,S_N)).\end{aligned}$$

Let $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transformation:

$$\mathscr{F}[f](\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

Then, we have $\mathscr{F}^{-1}\Pi\mathscr{F} = (1-H)/2 = \hat{P}$ and $\mathscr{F}^{-1}\mathrm{d}/\mathrm{d}t\mathscr{F} = -it$. This implies

$$\begin{aligned}&\operatorname{index}(\Pi\varrho(\phi)\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S)))\\ =\ &\operatorname{index}(\Pi(\mathrm{d}/\mathrm{d}t+F)^{-1}\phi(\mathrm{d}/\mathrm{d}t+F)\Pi \text{ on } \Pi(L^2(\mathbb{R}))\otimes L^2(N,S_N))\\ =\ &\operatorname{index}(\hat{P}\mathscr{T}_\phi\hat{P}^* : X \to X).\end{aligned}$$

□

Thus, it suffices to calculate $\operatorname{index}(\hat{P}\mathscr{T}_\phi\hat{P}^*)$ in order to prove the main theorem. For this purpose, we use eigenfunctions of the Hilbert transformation; see Theorem A.3.

PROPOSITION 5.6.5. *One has* $\operatorname{index}(\hat{P}\mathscr{T}_\phi\hat{P}^*) = \operatorname{index}(T_\phi)$. *Therefore, Theorem 5.2.1 for* $M = \mathbb{R}\times N$ *holds.*

PROOF. Set $X_0 = \mathbb{C}\{a_0\}\otimes H_+$ and $X_1 = (\mathrm{Span}_{\mathbb{C}}\{a_n\}_{n\geq 1}\otimes H_+)\oplus(\mathscr{H}_-\otimes H_-)$. Note that we have $X_0\oplus X_1 = \mathscr{H}_-\otimes L^2(N,S_N) = X$. Let $p : \mathscr{H}_- \to \mathbb{C}\{a_0\}$ be the projection to $\mathbb{C}\{a_0\}$. Then $p_0 = p\otimes P : X \to X_0$ is the projection to X_0 and $p_1 = \mathrm{id}_X - p_0 : X \to X_1$ is the projection to X_1.

By the decomposition of $L^2(N,S_N) = H_+\oplus H_-$, we have

$$\mathscr{T}_\phi = \begin{bmatrix} \mathrm{id}_{L^2(\mathbb{R})}\otimes P\phi P^* & \frac{t-i}{t+i}\otimes P\phi(1-P)^* \\ \frac{t+i}{t-i}\otimes(1-P)\phi P^* & \mathrm{id}_{L^2(\mathbb{R})}\otimes(1-P)\phi(1-P)^* \end{bmatrix}.$$

So we obtain

$$\hat{P}\mathscr{T}_\phi\hat{P}^*p_0^* = p^*\otimes P\phi P^* = \mathrm{id}_{\mathbb{C}\{a_0\}}\otimes T_\phi$$

and

$$\mathscr{T}_\phi\hat{P}^*p_1^* = \begin{bmatrix} (\hat{P}-p)^*\otimes P\phi P^* & \frac{t-i}{t+i}\hat{P}^*\otimes P\phi(1-P)^* \\ \frac{t+i}{t-i}(\hat{P}-p)^*\otimes(1-P)\phi P^* & \hat{P}^*\otimes(1-P)\phi(1-P)^* \end{bmatrix}.$$

This implies $\mathrm{Image}(\hat{P}\mathscr{T}_\phi\hat{P}^*p_0^*)\subset X_0$, $\mathrm{Image}(\mathscr{T}_\phi\hat{P}^*p_1^*)\subset X_1$ and

$$(\hat{P}\mathscr{T}_{\phi^{-1}}\hat{P}^*p_1^*)(\hat{P}\mathscr{T}_\phi\hat{P}^*p_1^*) = \hat{P}\mathscr{T}_{\phi^{-1}}\mathscr{T}_\phi\hat{P}^*p_1^* = \mathrm{id}_{X_1}.$$

So $\hat{P}\mathscr{T}_\phi\hat{P}^*$ forms a direct sum of an invertible part $\hat{P}\mathscr{T}_\phi\hat{P}^*p_1^*$ and another part $\hat{P}\mathscr{T}_\phi\hat{P}^*p_0^*$:

$$\hat{P}\mathscr{T}_\phi\hat{P}^* = \begin{bmatrix} \hat{P}\mathscr{T}_\phi\hat{P}^*p_0^* & 0 \\ 0 & \hat{P}\mathscr{T}_\phi\hat{P}^*p_1^* \end{bmatrix} \text{ on } X_0 \oplus X_1.$$

This proves $\mathrm{index}(\hat{P}\mathscr{T}_\phi\hat{P}^*) = \mathrm{index}(\hat{P}\mathscr{T}_\phi\hat{P}^*p_0^*) = \mathrm{index}(T_\phi)$. □

We note that we also get

$$\mathrm{index}\,(\Pi u_{\chi,\phi}\Pi) = \mathrm{index}(T_\phi).$$

5.6.2. The general case. In this section we reduce the proof for the general partitioned manifold to that of $\mathbb{R} \times N$. Our argument is similar to Higson's argument in Subsection 4.4.3. However, we should rewrite Higson's argument to suit our theorem. By above sections, it suffices to show the case when $\phi \in GL_l(\mathscr{W}(M))$. Firstly, we shall show a cobordism invariance.

LEMMA 5.6.6. *Let* (M^+, M^-, N) *and* $(M^{+\prime}, M^{-\prime}, N')$ *be two partitions of* M. *Assume that these two partitions are cobordant, that is, symmetric differences* $M^\pm \triangle M^{\mp\prime}$ *are compact. Let* Π *and* Π' *be the characteristic function of* M^+ *and* $M^{+\prime}$, *respectively. Take* $\phi \in GL_l(\mathscr{W}(M))$. *Then one has* $\mathrm{index}(\Pi u_{\chi,\phi}\Pi) = \mathrm{index}(\Pi' u_{\chi,\phi}\Pi')$ *and* $\mathrm{index}(\Pi\varrho(\phi)\Pi) = \mathrm{index}(\Pi'\varrho(\phi)\Pi')$.

PROOF. It suffices to show the case when $l = 1$. By $[\phi, \Pi] = 0$ and $[u_{\chi,\phi}, \Pi] \sim 0$, we obtain

$$\begin{aligned} &\mathrm{index}(\Pi u_{\chi,\phi}\Pi : \Pi(L^2(M,S)) \to \Pi(L^2(M,S))) \\ =&\mathrm{index}\left((1-\Pi)\begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \Pi u_{\chi,\phi}\Pi : L^2(M,S) \to L^2(M,S)\right) \\ =&\mathrm{index}\left((1-\Pi)\begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \Pi u_{\chi,\phi} : L^2(M,S) \to L^2(M,S)\right) \\ =&\mathrm{index}\left(\begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \Pi v_{\chi,\phi} : L^2(M,S) \to L^2(M,S)\right). \end{aligned}$$

Therefore, it suffices to show $\Pi v_{\chi,\phi} \sim \Pi' v_{\chi,\phi}$. Now, since $M^\pm \triangle M^{\mp\prime}$ are compact, there exists $f \in C_0(M)$ such that $\Pi - \Pi' = (\Pi - \Pi')f$. So we obtain $\Pi v_{\chi,\phi} - \Pi' v_{\chi,\phi} = (\Pi - \Pi')f v_{\chi,\phi} \sim 0$. By the similar argument, we can prove $\mathrm{index}(\Pi\varrho(\phi)\Pi) = \mathrm{index}(\Pi'\varrho(\phi)\Pi')$.

□

Secondly, we shall prove an analogue of Higson's Lemma.

Lemma 5.6.7. *Let M_1 and M_2 be two partitioned manifolds and $S_j \to M_j$ a Hermitian vector bundle. Let Π_j be the characteristic function of M_j^+. We assume that there exists an isometry $\gamma : M_2^+ \to M_1^+$ which lifts an isomorphism $\gamma^* : S_1|_{M_1^+} \to S_2|_{M_2^+}$. We denote the Hilbert space isometry defined by γ^* by the same letter $\gamma^* : \Pi_1(L^2(M_1, S_1)) \to \Pi_2(L^2(M_2, S_2))$. Take $u_j \in GL_l(C_b^*(M_j))$ such that $\gamma^* u_1 \Pi_1 \sim \Pi_2 u_2 \gamma^*$. Then one has* $\mathrm{index}(\Pi_1 u_1 \Pi_1) = \mathrm{index}(\Pi_2 u_2 \Pi_2)$.

Similarly, if there exists an isometry $\gamma : M_2^- \to M_1^-$ which lifts an isomorphism $\gamma^* : S_1|_{M_1^-} \to S_2|_{M_2^-}$ and $\gamma^* u_1 \Pi_1 \sim \Pi_2 u_2 \gamma^*$, then one has $\mathrm{index}(\Pi_1 u_1 \Pi_1) = \mathrm{index}(\Pi_2 u_2 \Pi_2)$.

Proof. It suffices to show the case when $l = 1$. Let $v : (1 - \Pi_1)(L^2(M_1, S_1)) \to (1 - \Pi_2)(L^2(M_2, S_2))$ be any invertible operator. Then $V = \gamma^* \Pi_1 + v(1 - \Pi_1) : L^2(M_1, S_1) \to L^2(M_2, S_2)$ is also an invertible operator. Hence we obtain

$$
\begin{aligned}
&V((1-\Pi_1) + \Pi_1 u_1 \Pi_1) - ((1-\Pi_2) + \Pi_2 u_2 \Pi_2)V \\
&= \gamma^* \Pi_1 u_1 \Pi_1 - \Pi_2 u_2 \Pi_2 \gamma^* \\
&\sim \gamma^* u_1 \Pi_1 - \Pi_2 u_2 \gamma^* \sim 0.
\end{aligned}
$$

Therefore, we obtain $\mathrm{index}(\Pi_1 u_1 \Pi_1) = \mathrm{index}(\Pi_2 u_2 \Pi_2)$ since V is an invertible operator and one has $\mathrm{index}(\Pi_j u_j \Pi_j) = \mathrm{index}((1 - \Pi_j) + \Pi_j u_j \Pi_j)$ for $j = 1, 2$.

□

Applying Lemma 5.6.7, we prove the following:

Corollary 5.6.8. *Let M_1 and M_2 be two partitioned manifolds. Let $S_j \to M_j$ be a graded Clifford bundle with the grading ϵ_j, and denote by D_j the graded Dirac operator of S_j. We assume that there exists an isometry $\gamma : M_2^+ \to M_1^+$ which lifts isomorphism $\gamma^* : S_1|_{M_1^+} \to S_2|_{M_2^+}$ of graded Clifford structures. Moreover, we assume that $\phi_j \in GL_l(\mathscr{W}(M))$ satisfies $\phi_1(\gamma(x)) = \phi_2(x)$ for all $x \in M_2^+$. Then one has* $\mathrm{index}(\Pi_1 u_{\chi,\phi_1} \Pi_1) = \mathrm{index}(\Pi_2 u_{\chi,\phi_2} \Pi_2)$.

Proof. Fix small $R > 0$. It suffices to show $\gamma^* u_{\chi,\phi_1} \Pi_1 \sim \Pi_2 u_{\chi,\phi_2} \gamma^*$ the case when a chopping function $\chi \in C(\mathbb{R}; [-1, 1])$ satisfies $\mathrm{Supp}(\hat{\chi}) \subset (-R, R)$. Set $N_{2R} = \{x \in M_1^+ ; d(x, N_1) \leq 2R\}$. Let φ_1 be a smooth function on M_1 such that $\mathrm{Supp}(\varphi_1) \subset M_1^+ \setminus N_{2R}$ and assume that there exists a compact set $K \subset M_1$ such that $\varphi_1 = \Pi_1$ on $M_1 \setminus K$. Set $\varphi_2(x) = \varphi_1(\gamma(x))$ for all $x \in M_2^+$ and $\varphi_2 = 0$ on M_2^-. Then we have $\gamma^* v_{\chi,\phi_1} \Pi_1 \sim \gamma^* v_{\chi,\phi_1} \varphi_1$ and $\Pi_2 v_{\chi,\phi_2} \gamma^* \sim \varphi_2 v_{\chi,\phi_2} \gamma^*$. Thus, if one has $\gamma^* v_{\chi,\phi_1}\varphi_1 \sim \varphi_2 v_{\chi,\phi_2}\gamma^*$, then we obtain

$$\gamma^* u_{\chi,\phi_1}\Pi_1 \sim \gamma^* v_{\chi,\phi_1}\varphi_1 + \gamma^* \begin{bmatrix} 1 & 0 \\ 0 & \phi_1 \end{bmatrix} \Pi_1 \sim \varphi_2 v_{\chi,\phi_2}\gamma^* + \Pi_2 \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \gamma^* \sim \Pi_2 u_{\chi,\phi_2}\gamma^*.$$

We shall show $\gamma^* v_{\chi,\phi_1}\varphi_1 \sim \varphi_2 v_{\chi,\phi_2}\gamma^*$. Now, we have $\gamma^* v_{\chi,\phi_1}\varphi_1 = v_{\chi,\phi_2}\gamma^*\varphi_1$ since we have $\gamma^* D = D\gamma^*$ on M^+ and the propagation of $\chi(D)$ and $\eta(D)$ is less than R, respectively. Moreover, we have $[v_{\chi,\phi_2}, \varphi_2] \sim 0$ by $v_{\chi,\phi_2} \in M_l(C^*(M))$. Therefore, we have

$$\gamma^* v_{\chi,\phi_1}\varphi_1 = v_{\chi,\phi_2}\gamma^*\varphi_1 = v_{\chi,\phi_2}\varphi_2\gamma^* \sim \varphi_2 v_{\chi,\phi_2}\gamma^*.$$

□

In order to prove Corollary 5.2.2, we apply Lemma 5.6.7 as follows:

Corollary 5.6.9. *We also assume as in Corollary 5.6.8. Then one has* $\operatorname{index}(\Pi_1\varrho(\phi_1)\Pi_1) = \operatorname{index}(\Pi_2\varrho(\phi_2)\Pi_2)$.

Proof. It suffices to show $\gamma^*\varrho(\phi_1)\Pi_1 \sim \Pi_2\varrho(\phi_2)\gamma^*$. Let φ_1 be a smooth function on M_1 such that $\operatorname{Supp}(\varphi_1) \subset M_1^+$ and assume that there exists a compact set $K \subset M_1$ such that $\varphi_1 = \Pi_1$ on $M_1 \setminus K$. Set $\varphi_2(x) = \varphi_1(\gamma(x))$ for all $x \in M_2^+$ and $\varphi_2 = 0$ on M_2^-. Set $v_{\phi_j} = \varrho(\phi_j) - \begin{bmatrix} 1 & 0 \\ 0 & \phi_j \end{bmatrix}$. Then we have $\gamma^* v_{\phi_1}\Pi_1 \sim \gamma^* v_{\phi_1}\varphi_1$ and $\Pi_2 v_{\phi_2}\gamma^* \sim \varphi_2 v_{\phi_2}\gamma^*$. Thus, if one has $\gamma^* v_{\phi_1}\varphi_1 \sim \varphi_2 v_{\phi_2}\gamma^*$, then we obtain

$$\gamma^*\varrho(\phi_1)\Pi_1 \sim \gamma^* v_{\phi_1}\varphi_1 + \gamma^* \begin{bmatrix} 1 & 0 \\ 0 & \phi_1 \end{bmatrix} \Pi_1 \sim \varphi_2 v_{\phi_2}\gamma^* + \Pi_2 \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \gamma^* \sim \Pi_2\varrho(\phi_2)\gamma^*.$$

We shall show $\gamma^* v_{\phi_1}\varphi_1 \sim \varphi_2 v_{\phi_2}\gamma^*$. In fact, we obtain

$$\begin{aligned}
&\gamma^* v_{\phi_1}\varphi_1 - \varphi_2 v_{\phi_2}\gamma^* \\
&= \gamma^*(D_1+\epsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\operatorname{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \varphi_1 \\
&\quad - \varphi_2(D_2+\epsilon_2)^{-1} \begin{bmatrix} \phi_2 - 1 & -c(\operatorname{grad}(\phi_2))^- \\ 0 & \phi_2 - 1 \end{bmatrix} \gamma^* \\
&= \{\gamma^*(D_1+\epsilon_1)^{-1}\varphi_1 - \varphi_2(D_2+\epsilon_2)^{-1}\gamma^*\} \begin{bmatrix} \phi_1 - 1 & -c(\operatorname{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\
&\sim \{\gamma^*\varphi_1(D_1+\epsilon_1)^{-1} - (D_2+\epsilon_2)^{-1}\gamma^*\varphi_1\} \begin{bmatrix} \phi_1 - 1 & -c(\operatorname{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\
&= (D_2+\epsilon_2)^{-1}\{(D_2+\epsilon_2)\gamma^*\varphi_1 - \gamma^*\varphi_1(D_1+\epsilon_1)\}(D_1+\epsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\operatorname{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\
&\sim (D_2+\epsilon_2)^{-1}\gamma^*[D_1, \varphi_1](D_1+\epsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\operatorname{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix}
\end{aligned}$$

~ 0

since $\mathrm{grad}(\varphi_1)$ has a compact support and $[D_1, \varphi_1] = c(\mathrm{grad}(\varphi_1))$. Thus we get $\gamma^* u_{\phi_1} \Pi_1 \sim \Pi_2 u_{\phi_2} \gamma^*$. Therefore, we obtain $\mathrm{index}(\Pi_1 u_{\phi_1} \Pi_1) = \mathrm{index}(\Pi_2 u_{\phi_2} \Pi_2)$ by Lemma 5.6.7.

□

PROOF OF THEOREM 5.2.1, THE GENERAL CASE. We assume $\phi \in GL_l(\mathscr{W}(M))$. Firstly, let $a \in C^\infty([-1,1];[-1,1])$ satisfies

$$a(t) = \begin{cases} -1 & \text{if } -1 \le t \le -3/4 \\ 0 & \text{if } -2/4 \le t \le 2/4 \\ 1 & \text{if } 3/4 \le t \le 1 \end{cases}.$$

Let $(-4\delta, 4\delta) \times N$ be diffeomorphic to a tubular neighborhood of N in M satisfies

$$\sup_{(t,x),(s,y) \in [-3\delta, 3\delta] \times N} |\phi(t,x) - \phi(s,y)| < \|\phi^{-1}\|^{-1}.$$

Set $\psi(t,x) = \phi(4\delta a(t), x)$ on $(-4\delta, 4\delta) \times N$ and $\psi = \phi$ on $M \setminus (-4\delta, 4\delta) \times N$. Then we obtain $\psi \in GL_l(\mathscr{W}(M))$ and $\|\psi - \phi\| < \|\phi^{-1}\|^{-1}$. Thus a map $[0,1] \ni t \mapsto \psi_t = t\psi + (1-t)\phi \in GL_l(\mathscr{W}(M))$ is continuous with the uniform norm. Therefore, it suffices to show the case when $\phi \in GL_l(\mathscr{W}(M))$ satisfies $\phi(t,x) = \phi(0,x)$ on $(-2\delta, 2\delta) \times N$. Due to Lemma 5.6.6, we may change a partition of M to $(M^+ \cup ([-\delta, 0] \times N), M^- \setminus ((-\delta, 0] \times N), \{-\delta\} \times N)$ without changing $\mathrm{index}(\Pi u_{\chi,\phi} \Pi)$. Due to Corollary 5.6.8, we may change $M^+ \cup ([-\delta, 0] \times N)$ to $[-\delta, \infty) \times N$ without changing $\mathrm{index}(\Pi u_{\chi,\phi} \Pi)$. Here ϕ is equal to $\phi(0,x)$ on $[-\delta, \infty) \times N$ and the metric on $[0, \infty) \times N$ is product. We denote this manifold by $M' = ([-\delta, \infty) \times N) \cup (M^- \setminus ((-\delta, 0] \times N))$. M' is partitioned by $([-\delta, \infty) \times N, M^- \setminus ((-\delta, 0] \times N), \{-\delta\} \times N)$. We apply a similar argument to M', we may change M' to a product $\mathbb{R} \times N$ without changing $\mathrm{index}(\Pi u_{\chi,\phi} \Pi)$. Now we have changed M to $\mathbb{R} \times N$. □

PROOF OF COROLLARY 5.2.2, THE GENERAL CASE. Similar. □

5.7. Example

In this section, we deal with $M = \mathbb{R} \times S^1$ and $\phi_k(x) = e^{ikx}$ as an example of the main theorem. We can calculate independently both sides of our main formula. This calculation is contained in [**38**]. The Dirac operator D on $S = \mathbb{R} \times S^1 \times \mathbb{C}^2$ is given by the following formula:

$$D = \begin{bmatrix} 0 & \partial/\partial t - i\partial/\partial x \\ -\partial/\partial t - i\partial/\partial x & 0 \end{bmatrix},$$

where we use the coordinate $(t,x) \in \mathbb{R} \times S^1$.

Due to Subsection 5.5.2, we have

$$\langle \mathrm{Ind}(\phi_k, D), \zeta\rangle = -\frac{1}{8\pi i}\mathrm{index}\left(\Pi(D+\epsilon)^{-1}\begin{bmatrix}\phi_k & 0\\ 0 & 1\end{bmatrix}(D+\epsilon)\Pi\right).$$

In order to calculate the right hand side of the above, we firstly perturb this operator by a homotopy.

Proposition 5.7.1. *For any* $s \in [0,1]$*, set*

$$D_s = \begin{bmatrix} 0 & \partial/\partial t + s/2 - i\partial/\partial x \\ -\partial/\partial t + s/2 - i\partial/\partial x & 0\end{bmatrix} = D + \begin{bmatrix} 0 & s/2 \\ s/2 & 0\end{bmatrix}$$

and $u_{k,s} = (D_s + (1-s)\epsilon)^{-1}\begin{bmatrix}\phi_k & 0\\ 0 & 1\end{bmatrix}(D_s + (1-s)\epsilon)$.

Then $[0,1] \ni s \mapsto u_{k,s} \in GL_1(C_b^*(M))$ *is continuous.*

Proof. For the simplicity, we omit the subscript k in this proof. We note that $\|D_s f\|_{L^2} \geq s\|f\|_{L^2}/2$ for any $f \in \mathrm{domain}(D_s) = \mathrm{domain}(D)$ and $s \in (0,1]$. Moreover, D_s is self-adjoint. Therefore, the spectrum of D_s and $(-s/2, s/2)$ are disjoint, in particular, we have $D_1^{-1} \in \mathcal{L}(L^2(M,S))$.

Because of $(D_s + (1-s)\epsilon)^2 = D_s^2 + (1-s)^2$, we obtain

$$(D_s + (1-s)\epsilon)^{-1} = \frac{D_s}{D_s^2 + (1-s)^2} + (1-s)\frac{\epsilon}{D_s^2+(1-s)^2} \in C^*(M).$$

Therefore, u_s is well defined as a closed operator densely defined on $\mathrm{domain}(u_s) = \mathrm{domain}(D)$. By simple computation, we obtain

$$u_s = \begin{bmatrix}1 & 0\\ 0 & \phi\end{bmatrix} + (D_s + (1-s)\epsilon)^{-1}\begin{bmatrix}(1-s)(\phi-1) & i\partial\phi/\partial x \\ 0 & (1-s)(\phi-1)\end{bmatrix}$$

and $u_s \in GL_1(C_b^*(M))$.

Next we show $\|u_s - u_{s'}\| \to 0$ as $s \to s'$ for all $s' \in [0,1]$. First, we show $\{\|(D_s + (1-s)\epsilon)^{-1}\|\}_{s\in[0,1]}$ is a bounded set. Set $f_s(x) = \frac{x}{x^2+(1-s)^2}$ and $g_s(x) = \frac{1}{x^2+(1-s)^2}$ for $x \in \mathbb{R} \setminus (-s/2, s/2)$. By simple computation, we can show

$$\sup_{|x|\geq s/2} |f_s(x)| \leq \frac{5}{2} \quad\text{and}\quad \sup_{|x|\geq s/2}|g_s(x)| \leq \frac{5}{4}.$$

Therefore, we obtain

$$\begin{aligned}\|(D_s+(1-s)\epsilon)^{-1}\| &\leq \|(D_s^2+(1-s)^2)^{-1}D_s\| + \|(1-s)(D_s^2+(1-s)^2)^{-1}\| \\ (*) \qquad &\leq \sup_{|x|\geq s/2}|f_s(x)| + \sup_{|x|\geq s/2}|g_s(x)| \leq 15/4\end{aligned}$$

for all $s \in [0,1]$. On the other hand, we have

$$\begin{aligned} u_s - u_{s'} =& \{(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}\} \begin{bmatrix} (1-s)(\phi-1) & i\phi' \\ 0 & (1-s)(\phi-1) \end{bmatrix} \\ &+ (D_{s'} + (1-s')\epsilon)^{-1} \begin{bmatrix} (s'-s)(\phi-1) & 0 \\ 0 & (s'-s)(\phi-1) \end{bmatrix} \end{aligned}$$

and then the second term converges to 0 with the operator norm as $s \to s'$, thus it suffices to show $\|(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}\| \to 0$ as $s \to s'$. But this is proved by $(*)$ as follows:

$$\begin{aligned} &\|(D_s + (1-s)\epsilon)^{-1} - (D_{s'} + (1-s')\epsilon)^{-1}\| \\ =&\|(D_s + (1-s)\epsilon)^{-1}((s-s')\epsilon + D_{s'} - D_s)(D_{s'} + (1-s')\epsilon)^{-1}\| \\ \leq&\frac{3}{2}|s-s'|\|(D_{s'} + (1-s')\epsilon)^{-1}\|\|(D_s + (1-s)\epsilon)^{-1}\| \\ \leq&32|s-s'| \to 0. \end{aligned}$$

□

Due to Proposition 5.7.1, we obtain

$$\operatorname{index}\left(\Pi(D+\epsilon)^{-1}\begin{bmatrix} \phi_k & 0 \\ 0 & 1 \end{bmatrix}(D+\epsilon)\Pi\right) = \operatorname{index}(\Pi u_{k,0}\Pi) = \operatorname{index}(\Pi u_{k,1}\Pi).$$

Set

$$\mathscr{T}_k = \Pi(\partial/\partial t + 1/2 - i\partial/\partial x)^{-1}\phi_k(\partial/\partial t + 1/2 - i\partial/\partial x)\Pi.$$

By $\Pi u_{k,1}\Pi = \begin{bmatrix} \Pi & 0 \\ 0 & \mathscr{T}_k \end{bmatrix}$, it suffices to calculate

$$\operatorname{index}\Big(\mathscr{T}_k : \Pi(L^2(\mathbb{R})) \otimes L^2(S^1)^l \to \Pi(L^2(\mathbb{R})) \otimes L^2(S^1)^l\Big).$$

Next, we treat the Fredholm index of $\mathscr{T}_k$. Since the Fourier transformation $\mathscr{F}$ induces an invertible operator from $\mathscr{H}_-$ to $\Pi(L^2(\mathbb{R}))$, so we obtain

$$\begin{aligned} &\operatorname{index}\Big(\mathscr{T}_k : \Pi(L^2(\mathbb{R})) \otimes L^2(S^1) \to \Pi(L^2(\mathbb{R})) \otimes L^2(S^1)\Big) \\ =&\operatorname{index}\Big(\mathscr{F}^{-1}\mathscr{T}_k\mathscr{F} : \mathscr{H}_- \otimes L^2(S^1) \to \mathscr{H}_- \otimes L^2(S^1)\Big). \end{aligned}$$

Set

$$\hat{\mathscr{T}}_k = \mathscr{F}^{-1}\mathscr{T}_k\mathscr{F} = \hat{P}(-it + 1/2 - i\partial/\partial x)^{-1}\phi_k(-it + 1/2 - i\partial/\partial x)\hat{P}^*.$$

In order to calculate the Fredholm index of $\hat{\mathscr{T}}_k$, we use Example A.5. Let $E_\lambda = \mathbb{C}\{e^{i\lambda x}\}$ be the λ-eigenspace of $-i\partial/\partial x$. On $\mathscr{H}_- \otimes E_\lambda$, $\hat{\mathscr{T}}_k$ acts as

$$\hat{P}(-it + 1/2 + \lambda + k)^{-1}(-it + 1/2 + \lambda)\hat{P}^* \otimes \phi_k.$$

Thus $\hat{\mathscr{T}}_k(\mathscr{H}_- \otimes E_\lambda)$ is contained in $\mathscr{H}_- \otimes E_{\lambda+k}$. Therefore, we obtain

$$\begin{aligned}&\operatorname{index}(\hat{\mathscr{T}}_k)\\ =&\sum_{\lambda=-\infty}^{\infty}\operatorname{index}\left(\hat{P}\frac{t+i(\lambda+1/2)}{t+i(\lambda+k+1/2)}\hat{P}^*\otimes\phi_k : \mathscr{H}_-\otimes E_\lambda\to\mathscr{H}_-\otimes E_{\lambda+k}\right)\\ =&-k\end{aligned}$$

by Example A.5.

Summarizing, we have

$$\langle \operatorname{Ind}(\phi_k, D), \zeta\rangle = \frac{k}{8\pi i}.$$

On the other hand, $\operatorname{index}(T_{\phi_k}) = -k$ calculates the right hand side of the main theorem.

APPENDIX A

The Hilbert transformation

In this appendix, we recall the Hilbert transformation and the index theorem for Wiener-Hopf operators. We can see its index theorem is a variant of that of the Toeplitz operators.

Let $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transformation:

$$\mathscr{F}[f](\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx.$$

Let $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Hilbert transformation [1]:

$$Hf(t) = -\frac{i}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{t-y}dy = -\frac{i}{\pi}\lim_{\epsilon\downarrow 0}\int_{|t-y|>\epsilon}\frac{f(y)}{t-y}dy,$$

where p.v. is the Cauchy principal value. Let sgn be the signature function, that is, $\text{sgn}(x)$ is equal to 1 if $x \geq 0$ or -1 if $x < 0$. Then H can be verified $H = -\mathscr{F}^{-1}\text{sgn}\mathscr{F}$. By this formula, we have $H^2 = \text{id}$ and $H^* = H$. Thus $L^2(\mathbb{R})$ is decomposed by $\mathscr{H}_+$, the 1-eigen space of H, and $\mathscr{H}_-$, the -1-eigen space of H. Denote by $\hat{P} : L^2(\mathbb{R}) \to \mathscr{H}_-$ the projection to $\mathscr{H}_-$, that is, we set $\hat{P} = \frac{1}{2}(\text{id} - H)$.

Proposition A.2. *We have the following.*

(*i*) *We assume $f \in L^2(\mathbb{R})$ can be extended to $\{z \in \mathbb{C}\,;\, \text{Im}(z) \geq 0\}$ as follows: f is holomorphic on $\{z \in \mathbb{C}\,;\, \text{Im}(z) \geq 0\}$ and there exists $C > 0$ such that we have $\int_{\mathbb{R}} |f(x+iy)|^2 dx < C$ for any $y \geq 0$. Then we have $Hf = -f$.*

(*ii*) *We assume $f \in L^2(\mathbb{R})$ can be extended to $\{z \in \mathbb{C}\,;\, \text{Im}(z) \leq 0\}$ as follows: f is holomorphic on $\{z \in \mathbb{C}\,;\, \text{Im}(z) \leq 0\}$ and there exists $C > 0$ such that we have $\int_{\mathbb{R}} |f(x+iy)|^2 dx < C$ for any $y \leq 0$. Then we have $Hf = f$.*

Proof. We show only (*i*). Set $C_r(t) = \{re^{i\omega} + t\,;\, 0 \leq \omega \leq \pi\}$ for $r > 0$ and $t \in \mathbb{R}$. We assume the orientation of $C_r(t)$ is counterclockwise. Take $a, b > |t|$ and sufficiently small $\epsilon > 0$, and let C be a integral cycle as in Figure A.1. Since f is a holomorphic function on the upper half plane, we have $\int_C \frac{f(z)}{z-t}dz = 0$.

[1] In literature, the coefficient of the Hilbert transformation in the right hand side is usually $1/\pi$. We need i times in order to get $H^2 = \text{id}$.

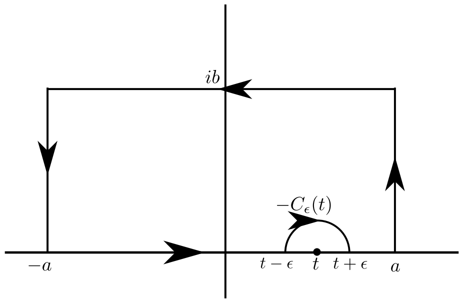

FIGURE A.1. Integral cycle C

Because of $f(\pm a+iy)\to 0$ as $a\to+\infty$ for any $y\geq 0$, we have

$$\left|\int_a^{a+ib}\frac{f(z)}{z-t}dz\right|\leq\int_0^b\frac{|f(a+iy)|}{|a+iy|-|t|}dy\to 0\quad\text{as } a\to+\infty \text{ and}$$

$$\left|\int_{-a+ib}^{-a}\frac{f(z)}{z-t}dz\right|\leq\int_0^b\frac{|f(-a+iy)|}{|-a+iy|-|t|}dy\to 0\quad\text{as } a\to+\infty.$$

On the other hand, $\int_{\mathbb{R}}|f(x+iy)|^2dx<C$ implies

$$\begin{aligned}\left|\int_{\infty+ib}^{-\infty+ib}\frac{f(z)}{z-t}dz\right| &\leq\int_{-\infty}^{\infty}\frac{|f(x+ib)|}{|x-t+ib|}dx\\ &\leq\left(\int_{-\infty}^{\infty}|f(x+ib)|^2dx\right)^{1/2}\left(\int_{-\infty}^{\infty}\frac{1}{(x-t)^2+b^2}dx\right)^{1/2}\\ &<\frac{\pi}{b}C^{1/2}\to 0\quad\text{as } b\to+\infty.\end{aligned}$$

Moreover, we have

$$\lim_{\epsilon\downarrow 0}\int_{C_\epsilon(t)}\frac{f(z)}{z-t}=\pi i f(t).$$

By the way, a formula

$$0=\int_C\frac{f(z)}{z-t}dz=\int_{-C_\epsilon(t)}+\int_{t+\epsilon}^{a}+\int_a^{a+ib}+\int_{a+ib}^{-a+ib}+\int_{-a+ib}^{-a}+\int_{-a}^{t-\epsilon}\frac{f(z)}{z-t}dz$$

and taking $\epsilon\to 0$ and $a\to\infty$ imply

$$0=-\pi i f(t)+\text{p.v.}\int_{\mathbb{R}}\frac{f(y)}{y-t}dy+\int_{\infty+ib}^{-\infty+ib}\frac{f(z)}{z-t}dz.$$

Finally, taking $b\to\infty$, we have

$$0=-\pi i f(t)+\text{p.v.}\int_{\mathbb{R}}\frac{f(y)}{y-t}dy.$$

Hence we have

$$\text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{t-y} dy = -\pi i f(t).$$

This implies $Hf = -f$.

□

For example, L^2-functions $a_n(t) = (t-i)^n/(t+i)^{n+1}$ satisfy conditions in Proposition A.2. These functions determine a basis of $L^2(\mathbb{R})$ which is defined by eigenvectors of H.

THEOREM A.3. [**41**, Theorem 1] *Set*

$$a_n(t) = \frac{(t-i)^n}{(t+i)^{n+1}}$$

for any $n \in \mathbb{Z}$ *and* $t \in \mathbb{R}$. *Then* $\{a_n/\sqrt{\pi}\}$ *is an orthonormal basis of* $L^2(\mathbb{R})$ *and we have* $\mathscr{H}_+ = \mathrm{Span}_{\mathbb{C}}\{a_n\,;\, n < 0\}$ *and* $\mathscr{H}_- = \mathrm{Span}_{\mathbb{C}}\{a_n\,;\, n \geq 0\}$.

PROOF. Since we have

$$\langle a_n, a_m \rangle_{L^2} = \int_{\mathbb{R}} \frac{(t-i)^{n-m-1}}{(t+i)^{n-m+1}} dt = \begin{cases} \pi & n = m \\ 0 & n \neq m \end{cases},$$

$\{a_n/\sqrt{\pi}\}$ is an orthonormal system of $L^2(\mathbb{R})$. Due to Proposition A.2, we have $Ha_n = a_n$ for any $n < 0$ and $Ha_n = -a_n$ for any $n \geq 0$.

We prove a completeness of $\{a_n\}$. Basically, our proof is adopted from [**11**, p.99]. Let $c : \mathbb{R} \to S^1 \subset \mathbb{C}$ be the Cayley transformation, that is, we set $c(t) = \frac{t-i}{t+i}$. Define $\Phi : L^2(S^1) \to L^2(\mathbb{R})$ by

$$\Phi(g)(t) = \frac{1}{t+i} g(c(t)) \quad t \in \mathbb{R}.$$

Then Φ is linear and we have $\|\Phi(g)\|_{L^2(\mathbb{R})} = \|g\|_{L^2(S^1)}/\sqrt{2}$. On the other hand, define $\Psi : L^2(\mathbb{R}) \to L^2(S^1)$ by

$$\Psi(f)(z) = (c^{-1}(z) + i) f(c^{-1}(z)) \quad z \in S^1 \setminus \{1\}.$$

Then Ψ is the inverse of Φ. Thus we can see $L^2(S^1) \cong L^2(\mathbb{R})$ by Φ. By the way, since $\Phi(z^n) = a_n$ and $\{z^n\}$ is a basis of $L^2(S^1)$, $\{a_n\}$ is also a basis of $L^2(\mathbb{R})$.

□

By this proof, we can calculate the Fredholm index of the Wienor-Hopf operator.

REMARK A.4. *Let* $f \in C(\mathbb{R}; GL_l(\mathbb{C}))$ *such that*

$$\lim_{t \to \infty} f(t) = \lim_{t \to -\infty} f(t) \in GL_l(\mathbb{C}).$$

By $\Phi^{-1} f \Phi = f \circ c^{-1}$ *and* $\Phi^{-1}\hat{P}\Phi = P$ *as linear operators on* $L^2(S^1)^l$, *we have*

$$\mathrm{index}(\hat{P} f \hat{P}^* : \mathscr{H}_-^l \to \mathscr{H}_-^l) = \mathrm{index}(T_{f\circ c^{-1}}) = -\deg(\det(f \circ c^{-1})).$$

Here the last equality is obtained by the index theorem of the Toeplitz operator on S^1.

In particular, we can calculate directly the following Fredholm indices.

EXAMPLE A.5. *For any* $\alpha, \beta \neq 0$, $\hat{P}\dfrac{t+i\beta}{t+i\alpha}\hat{P}^* \in \mathcal{L}(\mathscr{H}_-)$ *is a Fredholm operator and one has*

$$\mathrm{index}\left(\hat{P}\frac{t+i\beta}{t+i\alpha}\hat{P}^*\right) = \begin{cases} 0 & \textit{if } \alpha\beta > 0 \\ -1 & \textit{if } \alpha > 0, \beta < 0 \\ 1 & \textit{if } \alpha < 0, \beta > 0 \end{cases}.$$

PROOF. We can calculate

$$\left|\frac{t+i\beta}{t+i\alpha}\right|^2 = \frac{t^2+\beta^2}{t^2+\alpha^2} > 0$$

and $\lim_{t\to\pm\infty} \frac{t+i\beta}{t+i\alpha} = 1$. Therefore, $\hat{P}\frac{t+i\beta}{t+i\alpha}\hat{P}^*$ is a Fredholm operator.

We calculate $\mathrm{index}(\hat{P}\frac{t+i\beta}{t+i\alpha}\hat{P}^*)$. Set $\mathrm{sgn}(\alpha) = \begin{cases} 1 & \text{if } \alpha \geq 0 \\ -1 & \text{if } \alpha < 0 \end{cases}$. Then we define a homotopy of Fredholm operators from $\hat{P}\frac{t+i\beta}{t+i\alpha}\hat{P}^*$ to $\hat{P}\frac{t+i\mathrm{sgn}(\beta)}{t+i\mathrm{sgn}(\alpha)}\hat{P}^*$ by

$$\hat{P}\frac{t+i(s\beta+(1-s)\mathrm{sgn}(\beta))}{t+i(s\alpha+(1-s)\mathrm{sgn}(\alpha))}\hat{P}^*$$

for $s \in [0,1]$. Therefore, we obtain

$$\mathrm{index}\left(\hat{P}\frac{t+i\beta}{t+i\alpha}\hat{P}^*\right) = \mathrm{index}\left(\hat{P}\frac{t+i\mathrm{sgn}(\beta)}{t+i\mathrm{sgn}(\alpha)}\hat{P}^*\right) = \begin{cases} 0 & \text{if } \alpha\beta > 0 \\ -1 & \text{if } \alpha > 0, \beta < 0 \\ 1 & \text{if } \alpha < 0, \beta > 0 \end{cases}$$

by $\mathscr{H}_- = \mathrm{Span}_{\mathbb{C}}\{a_n \,;\, n \geq 0\}$.

∎

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