# Algebraic part of motivic cohomology with compact supports

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# Abstract

We define the algebraic part of motivic cohomology group with compact supports $H_c^{2r}(X, \mathbb{Z}(r))$ of a smooth scheme $X$ over an algebraically closed field. This generalizes the classical notion of the algebraic part of the Chow group in codimension $r$ of a smooth proper variety. We then define algebraic representatives for these algebraic parts as the universal regular homomorphisms with targets in the category of semi-abelian varieties. We give a criterion for the existence of a universal regular homomorphism and show the existence for $r = 1, 2$ and $\dim X$. (For the codimension one and two cases, we assume that the scheme $X$ in question has a smooth compactification $\bar{X}$ with a simple normal crossing boundary divisor $Z$.) We prove that the algebraic representative in codimension one agrees with the semi-abelian variety obtained as the reduction of the identity component of the group scheme that represents the functor of relative Picard groups, i.e. the functor that sends a scheme $T$ to $Pic(T \times \bar{X}, T \times Z)$. This implies, as in the classical case, that the algebraic representative in codimension one is an isomorphism.

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# Contents

# Convention

*Schemes* are assumed separated and of finite type over some field.

A *curve* means a connected (not necessarily smooth) scheme of pure dimension one.

The symbol $k$ stands for an algebraically closed field except in Chapter 2.

By *resolution of singularities,* we mean that the base field $k$ "admits resolution of singularities" in the sense of [FV, Definition 3.4]:

- For any scheme $X$ over $k$, there is a proper surjective morphism $Y \longrightarrow X$ such that $Y$ is a smooth scheme over $k$.
- For any smooth scheme $X$ over $k$ and abstract blow-up $q : X' \longrightarrow X$, there exists a sequence of blow-ups with smooth centers $p : X_n \longrightarrow \cdots \longrightarrow X_1 = X$ such that $p$ factors through $q$.

These conditions are satisfied over any field of characteristic zero (Ibid. Proposition 3.5) by Hironaka's resolution of singularities ([Hi]).

# Chapter 1

# Introduction

One classical theorem in the study of Chow groups and their algebraic part is the theorem of Abel and Jacobi on smooth projective curves. It states that there is an isomorphism (known as the Abel-Jacobi map) from the degree zero part of the Chow group of zero cycles to the group of rational points of a certain abelian variety known as the Jacobian of the curve.

One generalization of the Abel-Jacobi map to higher dimensions is the concept of algebraic representatives. Suppose $X$ is a smooth proper connected scheme over an algebraically closed field $k$. The subgroup $A^r(X)$ of the Chow group $CH^r(X)$ consisting of cycles algebraically equivalent to zero is called the algebraic part of $CH^r(X)$ (See the equation (3.1) in Section 3.1 for a precise definition). A group homomorphism from the algebraic part $A^r(X)$ of the Chow group $CH^r(X)$ to the group of rational points of an abelian variety $A$ that is "continuous" in a certain sense (see Definition 3.1.1) is called a regular homomorphism. A regular homomorphism $\phi : A^r(X) \longrightarrow A(k)$ is said universal if, given any regular homomorphism $\phi' : A^r(X) \longrightarrow A'(k)$, there is a unique homomorphism of abelian varieties $h : A \longrightarrow A'$ such that $h \circ \phi = \phi'$ holds. If such a universal regular homomorphism exists, we call it the algebraic representative of $A^r(X)$. It is a classical result that the algebraic representative exists for $r = 1$ and $\dim X$. The case $r = 1$ is the theory of Picard varieties and the case $r = \dim X$ coincides with the Albanese variety. The existence for $r = 2$ is also known by Murre ([Mur, Theorem A]), but the existence in other codimensions is unknown.

This thesis defines algebraic part and algebraic representatives for arbitrary smooth schemes over $k$ and proves analogues of some of the classical results in codimension one known for smooth proper schemes. We replace Chow groups with motivic cohomology with compact supports[1].

Below, $DM^{-,eff}_{Nis}(k)$ is Voevodsky's triangulated category of effective motives over the base field $k$, and $M(X)$ (resp., $M^c(X)$) is the motive (resp., motive with compact supports) of a

[1] Note that the latter agrees with motivic homology for smooth schemes under resolution of singularities.

scheme $X$ over $k$. $DM_{Nis}^{-,eff}(k)$ has a tensor structure, which is denoted by $\otimes$. The motive $M(\text{Spec } k)$ of the base field is denoted by $\mathbb{Z}$ and it is the unit of the tensor structure on $DM_{Nis}^{-,eff}(k)$. Motivic cohomology with compact supports of $X$ is, by definition,

$$H_c^m(X, \mathbb{Z}(n)) := Hom_{DM_{Nis}^{-,eff}(k)}(M^c(X), \mathbb{Z}(n)[m]).$$

Recall that, if $X$ is smooth and proper, there is a canonical isomorphism

$$H_c^{2r}(X, \mathbb{Z}(r)) \cong CH^r(X).$$

We shall also write

$$H_0(X, \mathbb{Z})^0 := \ker\{H_0(X, \mathbb{Z}) \xrightarrow{str_*} H_0(\text{Spec } k, \mathbb{Z})\},$$

where $H_0(X, \mathbb{Z}) := Hom_{DM_{Nis}^{-,eff}(k)}(\mathbb{Z}, M(X))$. For more details, see Section 2.3.

We define the algebraic part of motivic cohomology with compact supports as follows.

**Definition 1.0.1** (Definition 3.2.1)**.** *Let $X$ be a smooth scheme over $k$. The* **algebraic part** *of the motivic cohomology group with compact supports $H_c^{2r}(X, \mathbb{Z}(r))$ is defined as*

$$H_{c,alg}^{2r}(X, \mathbb{Z}(r)) := \bigcup_{\substack{T,\ smooth \\ connected}} \text{im}\{H_0(T, \mathbb{Z})^0 \times Hom_{DM_{Nis}^{-,eff}(k)}(M(T) \otimes M^c(X), \mathbb{Z}(r)[2r]) \longrightarrow H_c^{2r}(X, \mathbb{Z}(r))\},$$

*where the map sends a pair $(z, Y)$ with*

$$z \in H_0(T, \mathbb{Z})^0 \subset Hom_{DM_{Nis}^{-,eff}(k)}(\mathbb{Z}, M(T))$$

*and*

$$Y \in Hom_{DM_{Nis}^{-,eff}(k)}(M(T) \otimes M^c(X), \mathbb{Z}(r)[2r])$$

*to the composition*

$$M^c(X) \cong \mathbb{Z} \otimes M^c(X) \xrightarrow{z \otimes id_{M^c(X)}} M(T) \otimes M^c(X) \xrightarrow{Y} \mathbb{Z}(r)[2r]$$

*in $Hom_{DM_{Nis}^{-,eff}(k)}(M^c(X), \mathbb{Z}(r)[2r]) \overset{def}{=} H_c^{2r}(X, \mathbb{Z}(r))$.*

This definition agrees with the classical algebraic part of Chow groups of smooth proper schemes over $k$ as shown in Proposition 3.2.4. (See also Proposition 3.2.5.) The notion of regular homomorphisms and algebraic representatives naturally generalizes to this setting.

**Definition 1.0.2** (Definition 3.3.1)**.** *Let $X$ be a smooth scheme over $k$ and let $S$ be a semi-*

*abelian variety over $k$, i.e. an extension of an abelian variety over $k$ by a torus $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ over $k$. A group homomorphism $\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k)$ is called* **regular** *if for any smooth connected scheme $T$ over $k$, $t_0 \in T(k)$ and $Y \in Hom_{DM^{-,eff}_{Nis}(k)}(M(T) \otimes M^c(X), \mathbb{Z}(r)[2r])$, the composition*

$$T(k) \xrightarrow{w_Y} H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \xrightarrow{\phi} S(k)$$

*is induced by some scheme morphism $T \longrightarrow S$. Here, $w_Y$ sends $t \in T(k)$ to*

$$Y \circ (t \otimes id_{M^c(X)}) - Y \circ (t_0 \otimes id_{M^c(X)}),$$

*where $t$ and $t_0$ are regarded as morphisms from $\mathbb{Z}$ to $M(T)$ in $DM^{-,eff}_{Nis}(k)$.*

**Definition 1.0.3** (Definition 3.3.2)**.** *A regular homomorphism $\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k)$, is said* **universal** *if for any regular homomorphism $\phi' : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S'(k)$, there is a unique homomorphism of semi-abelian varieties $a : S \longrightarrow S'$ such that $a \circ \phi = \phi'$. The universal regular homomorphism, if it exists, is called the* **algebraic representative** *of $H^{2r}_{c,alg}(X, \mathbb{Z}(r))$ or the* **algebraic representative with compact supports of $X$ in codimension** *$r$, and it is written as*

$$\Phi^r_{c,X} : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow Alg^r_{c,X}(k).$$

Our main results are as follows.

**Theorem 1.0.4** (Theorems 3.4.6 and 3.4.7)**.** *Let $X$ be a smooth connected scheme of dimension $d_X$ over an algebraically closed field $k$.*

*(i) If $X$ has a good compactification (see Definition 3.4.5) and $r = 1$ or $2$, then there is an algebraic representative*

$$\Phi^r_{c,X} : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow Alg^r_{c,X}(k).$$

*(ii) If $r = d_X$, then the algebraic representative exists for an arbitrary connected smooth scheme $X$.*

Along the way, we obtain a motivic proof (Corollary 3.2.9) of the classical fact that algebraically equivalent cycles can be parametrized by abelian varieties.

For the case $r = 1$, we interpret the algebraic representative of $H^2_{c,alg}(X, \mathbb{Z}(1))$ in terms of the relative Picard group of a good compactification of $X$. More precisely, we obtain the following theorem.

**Theorem 1.0.5** (Proposition 4.3.5, Theorem 4.3.6)**.** *Assume resolution of singularities. If $X$ is a smooth connected scheme over $k$, then the algebraic representative in codimension one has* *the form*

$$\phi_0 : H^2_{c,alg}(X, \mathbb{Z}(1)) \longrightarrow Pic^0_{\bar{X},Z,red}(k)$$

*and it is an isomorphism. Here,* $Pic^0_{\bar{X},Z,red}$ *is the reduction of the identity component of the group scheme representing the functor that sends* $T \in Sch/k$ *to the relative Picard group* $Pic(T \times \bar{X}, T \times Z)$*, where* $(\bar{X}, Z)$ *is a good compactification of* $X$*.*

This theorem, in particular, implies that $Pic^0_{\bar{X},Z,red}$ only depends on $X$.

# Chapter 2

# Preliminaries

In this preliminary chapter, we review concepts and notations from Voevodsky's homological theory of motives and collect results relevant to us later.

## 2.1 The category of finite correspondences

Let $k$ be a perfect field and $Sm/k$ be the category of smooth schemes. For a smooth schemes $X$ and $Y$, a closed integral subscheme $V$ of $X \times Y$ such that the projection of $V$ to $X$ is finite and surjective over some connected component of $X$ is called an **elementary correspondence**. The group $Cor_k(X,Y)$ of **finite correspondences** from $X$ to $Y$ is defined as the free abelian group generated by **elementary correspondences**. For $X, Y, Z \in Sm/k$, there is a well-defined homomorphism of abelian groups

$$Cor_k(X,Y) \times Cor_k(Y,Z) \longrightarrow Cor_k(X,Z)$$

that sends the pair $(V,W) \in Cor_k(X,Y) \times Cor_k(Y,Z)$ of elementary correspondences to $W \circ V := p_*((V \times Z) \cdot (X \times W))$, where $p_*$ is the pushforward along the projection $p : X \times Y \times Z \longrightarrow X \times Z$, and the dot "$\cdot$" indicates the intersection product. For finite correspondences $V$ and $W$, the cycles $V \times Z$ and $X \times W$ intersect properly on $X \times Y \times Z$. Thus, the intersection product is well-defined at the cycle level (see [MVW, Lemma 1.7]). The morphism $p$ is not a proper morphism in general, but it is proper along the support of $(V \times Z) \cdot (X \times W)$. This makes the pushforward possible.

We define the category $Cor_k$ of finite correspondences as the category whose objects are smooth schemes over $k$ and morphisms are the groups of finite correspondences, i.e.

$$Hom_{Cor_k}(X,Y) := Cor_k(X,Y).$$

$Cor_k$ is an additive category. We regard $Sm/k$ as a subcategory of $Cor_k$ by sending a scheme morphism $f : X \longrightarrow Y$ to its graph $\Gamma_f \subset X \times Y$.

An additive presheaf on $Cor_k$ with values in the category $Ab$ of abelian groups is called a **presheaf with transfers**. For a Grothendieck topology $\tau$ on $Sm/k$, a presheaf with transfers is called a **$\tau$-sheaf with transfers** if its restriction to $Sm/k$ is a sheaf with respect to the topology $\tau$.

An important example of presheaf with transfers is $\mathbb{Z}_{tr}(X) := Cor_k(-, X)$. Even when $X$ is not smooth, we may talk about presheaf $\mathbb{Z}_{tr}(X)$. Indeed, in order to define the group $Cor_k(U, X)$ of finite correspondences and contravariant functoriality in $U$, the smoothness of the first entry $U$ is enough and $X$ may be an arbitrary scheme.

**Proposition 2.1.1.** *For an arbitrary scheme $X$ over $k$, the presheaf with transfers $Cor_k(-, X)$ is a sheaf with respect to the étale topology; hence, a fortiori, for coarser topologies such as Zariski and Nisnevich.*

*Proof.* See [MVW, Lemma 6.2]. □

## 2.2 Motivic complexes

### 2.2.1 Voevodsky's motivic complex

The open subscheme of the affine line $\mathbb{A}^1_k$ with the origin 0 removed is denoted by $\mathbb{G}_m$. We consider $\mathbb{G}_m$ as a scheme pointed at $1 : \mathrm{Spec}\ k \longrightarrow \mathbb{G}_m$.

For a pointed scheme $(X, \mathrm{Spec}\ k \xrightarrow{x} X)$, we define the presheaf with transfers $\mathbb{Z}_{tr}(X, x)$ by

$$\mathbb{Z}_{tr}(X, x) := \mathrm{coker}\{\mathbb{Z}_{tr}(\mathrm{Spec}\ k) \xrightarrow{x} \mathbb{Z}_{tr}(X)\}.$$

Since $X$ is a scheme over $k$, the structure morphism induces a splitting of the exact sequence

$$0 \longrightarrow \mathbb{Z}_{tr}(\mathrm{Spec}\ k) \xrightarrow{x} \mathbb{Z}_{tr}(X) \longrightarrow \mathbb{Z}_{tr}(X, x) \longrightarrow 0.$$

Therefore, $\mathbb{Z}_{tr}(X, x)$ is a direct summand of $\mathbb{Z}_{tr}(X)$. This implies that $\mathbb{Z}_{tr}(X, x)$ is a sheaf if $\mathbb{Z}_{tr}(X)$ is.

In particular, $\mathbb{Z}_{tr}(\mathbb{G}_m, 1)$ is a sheaf with transfers in the étale topology by Proposition 2.1.1. Similarly, the following generalization is known.

**Definition 2.2.1.** *Let $(X, x)$ be a pointed scheme and $n$ be a positive integer. The presheaf* *with transfers* $\mathbb{Z}_{tr}((X,x)^{\wedge n})$ *is defined as the cokernel*

$$\mathbb{Z}_{tr}((X,x)^{\wedge n}) := \mathrm{coker}\{\bigoplus_{i=1,\cdots,n} \mathbb{Z}_{tr}(X\times\cdots\times X\times \mathrm{Spec}\ k\times X\times\cdots\times X) \overset{id\times\cdots\times id\times x\times id\times\cdots\times id}{\longrightarrow} \mathbb{Z}_{tr}(X\times\cdots\times X)\}.$$

**Lemma 2.2.2.** *With the same notation as above, the canonical surjection*

$$\mathbb{Z}_{tr}(X\times\cdots\times X) \longrightarrow \mathbb{Z}_{tr}((X,x)^{\wedge n})$$

*is split surjective. In particular,* $\mathbb{Z}_{tr}((X,x)^{\wedge n})$ *is a sheaf if* $\mathbb{Z}_{tr}(X\times\cdots\times X)$ *is a sheaf.*

*Proof.* See [MVW, Lemma 2.13]. □

Let $\Delta^n$ be the $n$-th algebraic simplex. Given a presheaf $F : Sm/k \longrightarrow Ab$, we define the complex $C_*F$ of presheaves as the complex

$$\cdots \longrightarrow F(-\times\Delta^{n+1}) \longrightarrow F(-\times\Delta^n) \longrightarrow F(-\times\Delta^{n-1}) \longrightarrow \cdots \mathbb{F}(-\times\Delta^0) \longrightarrow 0,$$

where this is regarded as a cochain complex with $F(-\times\Delta^n)$ placed in degree $-n$ and the differentials are given by the alternating sums of the maps induced by the face maps of algebraic simplices. As we can easily see, if $F$ is a presheaf (resp., sheaf) with transfers, then $C_*F$ is a complex of presheaves (resp., sheaves) with transfers.

A presheaf $F : Sm/k \longrightarrow Ab$ is called **homotopy invariant** if for any $X \in Sm/k$, the map $F(X) \longrightarrow F(X\times\mathbb{A}^1)$ induced by the projection is an isomorphism. Here is an elementary but important property.

**Proposition 2.2.3.** *The homology presheaves of* $C_*F$ *are homotopy invariant for any presheaf* $F$.

*Proof.* See [MVW, Corollary 2.19]. □

**Definition 2.2.4.** *The complex of presheaves with transfers* $\mathbb{Z}(n) := C_*\mathbb{Z}_{tr}((\mathbb{G}_m,1)^{\wedge n})[-n]$ *is called* **Voevodsky's motivic complex**.

**Proposition 2.2.5.** *Voevodsky's motivic complex* $\mathbb{Z}(n)$ *is a complex of sheaves with respect to the étale and, a fortiori, coarser topologies.*

*Proof.* Clear from Proposition 2.1.1 and Lemma 2.2.2. □

### 2.2.2 Suslin-Friedlander's motivic complex and Bloch's cycle complex

Let $X$ be any scheme over $k$. We define the presheaf

$$z_{equi}(X,r) : Sm/k \longrightarrow Ab$$

by sending $U \in Sm/k$ to the free abelian group generated by closed integral subschemes of $U \times X$ that is dominant and equidimensional of relative dimension $r$ over $U$. This is known to be a sheaf with respect to the étale topology and covariant in $X$ for proper morphisms and contravariant in $X$ for flat morphisms after appropriate shifting of dimension $r$ (see [MVW, Lecture 16, p.125]). Note that $z_{equi}(X, 0) = \mathbb{Z}_{tr}(X)$ if $X$ is a proper scheme.

**Definition 2.2.6.** *The complex of (pre)sheaves $\mathbb{Z}(n)^{SF} := C_* z_{equi}(\mathbb{A}^n, 0)[-2n]$ is called the* **Suslin-Friedlander's motivic complex.** *This is a complex of sheaves in the étale topology.*

Bloch's cycle complex, which can naturally be regarded as a complex of sheaves on a small Zariski site, is related to Suslin-Friedlander's motivic complex.

**Theorem 2.2.7** ([MVW, Theorem 19.8])**.** *Let $X$ be a smooth scheme over $k$. On the small Zariski site $X_{Zar}$, the map of complexes of Zariski sheaves*

$$\mathbb{Z}(n)^{SF}[2n] \longrightarrow z^n(- \times \mathbb{A}^n, \bullet)$$

*given by the inclusion is a quasi-isomorphism for all non-negative integers $n$.*

One notable feature of Bloch's cycle complex is the Zariski descent property:

**Theorem 2.2.8** ([MVW, Proposition 19.12])**.** *For any scheme $X$, Bloch's cycle complex $z^n(- \times T, \bullet)$ satisfies Zariski descent, i.e. the canonical map*

$$CH^n(X \times T, m) \overset{def}{=} H^{-m}(z^n(X \times T, \bullet)) \xrightarrow{\cong} H^{-m}_{Zar}(X, z^n(- \times T, \bullet))$$

*is an isomorphism. In particular, for $T = \mathbb{A}^n$, we have*

$$CH^n(X, m) \xrightarrow{\cong} CH^n(X \times \mathbb{A}^n, m) \xrightarrow{\cong} H^{-m}_{Zar}(X, z^n(- \times \mathbb{A}^n, \bullet)).$$

### 2.2.3 Voevodsky's and Susin-Friedlander's motivic complexes

**Theorem 2.2.9** ([MVW, Theorem 16.7])**.** *There is a quasi-isomorphism of complexes of sheaves on the big Zariski site $(Sm/k)_{Zar}$ :*

$$\mathbb{Z}(n) \simeq \mathbb{Z}(n)^{SF}.$$

Combining Theorems 2.2.7, 2.2.8 and 2.2.9, we obtain:

**Corollary 2.2.10** ([MVW, Theorem 19.1])**.** *For a smooth scheme $X$ over $k$, there is a natural isomorphism*

$$H^m_{Zar}(X, \mathbb{Z}(n)) \xrightarrow{\cong} CH^n(X, 2n - m).$$

The naturality of the isomorphism follows from [MVW, Proposition 19.16].

## 2.3 Voevodsky's triangulated category of motives

The construction of Voevodsky's triangulated category of motives starts with the category $Sm/k$ of smooth schemes over $k$. We imbed it into the category $Cor_k$ of finite correspondences and then into the category $Sh_{Nis}(Cor_k)$ of Nisnevich sheaves with transfers. Then, consider the bounded above cochain complexes of Nisnevich sheaves with transfers and take the derived category $D^-(Sh_{Nis}(Cor_k))$.

**Definition 2.3.1. Voevodsky's triangulated category of effective motives** $DM^{-,eff}_{Nis}(k)$ **over** *$k$ is the Verdier localization of the bounded above derived category $D^-(Sh_{Nis}(Cor_k))$ of Nisnevich sheaves with transfers with respect to the class of all morphisms of the form $\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \xrightarrow{proj.} \mathbb{Z}_{tr}(X)$ for $X \in Sm/k$.*

The image of $\mathbb{Z}_{tr}(X)$ (resp., $z_{equi}(X,0)$) in $DM^{-,eff}_{Nis}(k)$ is denoted by $M(X)$ (resp., $M^c(X)$).

By the definition of the Verdier localization, all morphisms in $D^-(Sh_{Nis}(Cor_k))$ whose cones belong to the smallest thick subcategory that contains all cones of $\mathbb{Z}_{tr}(X\times\mathbb{A}^1) \xrightarrow{\text{proj.}} \mathbb{Z}_{tr}(X)$ ($X \in Sm/k$) are invertible in $DM^{-,eff}_{Nis}(k)$. These morphisms are called **$\mathbb{A}^1$-weak equivalences**. It can be shown that for a complex $\mathcal{K}^\bullet$ of Nisnevich sheaves with transfers the canonical map $\mathcal{K}^\bullet \longrightarrow TotC_*\mathcal{K}^\bullet$ is an $\mathbb{A}^1$-weak equivalence ([MVW, Lemma 14.4]). In particular, $M(X)$ (resp., $M^c(X)$) is isomorphic to $C_*\mathbb{Z}_{tr}(X)$ (resp., $C_*z_{equi}(X,0)$) in $DM^{-,eff}_{Nis}(k)$.

The localization functor $D^-(Sh_{Nis}(Cor_k)) \longrightarrow DM^{-,eff}_{Nis}(k)$ is a tensor triangulated functor with appropriate tensor structures on both categories (see [MVW, Definition 14.2] and the subsequent discussion). These tensor structures respect the product of schemes, i.e. we have $M(X\times Y) \cong M(X)\otimes M(Y)$. Moreover, $M^c(X\times Y) \cong M^c(X)\otimes M^c(Y)$ holds ([MVW, Corollary 16.16]).

For an arbitrary scheme $X$ over $k$ and integers $m$ and $n$, four motivic homology theories are defined in [V00]:

- **motivic homology**: $H_m(X,\mathbb{Z}(n)) := Hom_{DM^{-,eff}_{Nis}(k)}(\mathbb{Z}(n)[m], M(X))$,
- **motivic cohomology**: $H^m(X,\mathbb{Z}(n)) := Hom_{DM^{-,eff}_{Nis}(k)}(M(X), \mathbb{Z}(n)[m])$,
- **motivic homology with compact supports**: $H^{BM}_m(X,\mathbb{Z}(n)) := Hom_{DM^{-,eff}_{Nis}}(\mathbb{Z}(n)[m], M^c(X))$,
- **motivic cohomology with compact supports**: $H^m_c(X,\mathbb{Z}(n)) := Hom_{DM^{-,eff}_{Nis}(k)}(M^c(X), \mathbb{Z}(n)[m])$.

The index $m$ is called a degree and $n$ a twist. For negative twist $n$, the above definition means, for example,

$$H^m(X,\mathbb{Z}(n)) := Hom_{DM^{-,eff}_{Nis}(k)}(M(X),\mathbb{Z}(n)[m]) := Hom_{DM^{-,eff}_{Nis}(k)}(M(X)\otimes\mathbb{Z}(-n),\mathbb{Z}[m])$$

(cf. [V10]).

An object $L \in D^-(Sh_{Nis}(Cor_k))$ is called $\mathbb{A}^1$**-local** if it does not see $\mathbb{A}^1$-weak equivalence in the sense that $Hom_{D^-(Sh_{Nis}(Cor_k))}(-,L)$ sends an $\mathbb{A}^1$- weak equivalence to an isomorphism. Consequently, we have

$$Hom_{D^-(Sh_{Nis}(Cor_k))}(K,L) \xrightarrow{\cong} Hom_{DM^{-,eff}_{Nis}(k)}(K,L)$$

for all $K \in D^-(Sh_{Nis}(Cor_k))$ if $L$ is $\mathbb{A}^1$-local. An $\mathbb{A}^1$-local object can be characterized as follows.

**Proposition 2.3.2.** *A bounded above complex $\mathcal{K}^\bullet$ of Nisnevich sheaves with transfers is $\mathbb{A}^1$-local if and only if the homology sheaves are strictly $\mathbb{A}^1$-homotopy invariant, i.e. for all $X \in Sm/k$ and $i$ the canonical maps*

$$H^m_{Nis}(X,H^i(\mathcal{K}^\bullet)_{Nis}) \longrightarrow H^m_{Nis}(X\times\mathbb{A}^1,H^i(\mathcal{K}^\bullet)_{Nis})$$

*is an isomorphism in every degree $m$.*

*Proof.* Since the Nisnevich cohomological dimension is finite, the hypercohomology spectral sequence converges. Thus, we may assume that the complex $\mathcal{K}^\bullet$ is just a single sheaf. This case is immediate from the definitions. □

One of the deepest results in the theory of presheaves with transfers is the following.

**Theorem 2.3.3** ([MVW, Theorem 24.1])**.** *Let $k$ be a perfect field as usual. If $F : Cor_k \longrightarrow Ab$ is a homotopy invariant presheaf with transfers, then $H^n_{Nis}(-,F_{Nis})$ is a homotopy invariant presheaf with transfers for all $n$.*

Note that the $n = 0$ case claims that the Nisnevich sheafification of a homotopy invariant presheaf with transfers is homotopy invariant.

**Corollary 2.3.4.** *Voevodsky's motivic complex $\mathbb{Z}(n)$ is $\mathbb{A}^1$-local. In particular, it computes motivic cohomology of a smooth scheme $X$ with non-negative twist as Nisnevich hypercohomology, i.e. for $n \geq 0$,*

$$H^m(X,\mathbb{Z}(n)) \overset{def}{=} Hom_{DM^{-,eff}_{Nis}}(M(X),\mathbb{Z}(n)[m]) \cong H^m_{Nis}(X,\mathbb{Z}(n)).$$

*Proof.* This follows from Definition 2.2.4, Propositions 2.2.3, 2.3.2, Theorem 2.3.3 and an easy lemma [MVW, Exercise 13.5], which claims that

$$Hom_{D^-(Sh_{Nis}(Cor_k))}(\mathbb{Z}_{tr}(X), \mathcal{K}^\bullet[m])) \cong H^m_{Nis}(X, \mathcal{K}^\bullet)$$

for any $X \in Sm/k$ and any complex $\mathcal{K}^\bullet$ of Nisnevich sheaves with transfers. $\square$

The Nisnevich hyercohomology group in the statement of Corollary 2.3.4 can actually be calculated in the Zariski topology.

**Lemma 2.3.5** ([MVW, Corollary 11.2]). *Let $F : Cor_k \longrightarrow Ab$ be a homotopy invariant presheaf with transfers. If $F(\text{Spec } K) = 0$ for all fields $K$, then $F_{Zar} = 0$.*

With this lemma, we can prove the following.

**Proposition 2.3.6.** *Let $F : Cor_k \longrightarrow Ab$ be a homotopy invariant Nisnevich sheaf with transfers. Then, for all smooth schemes $X$ over $k$ and integers $m$, the canonical map*

$$H^m_{Zar}(X, F) \longrightarrow H^m_{Nis}(X, F)$$

*is an isomorphism.*

*A similar statement holds for hypercohomology of bounded above complex of Nisnevich sheaves with transfers with homotopy invariant cohomology sheaves, for example Voevodsky's motivic complex $\mathbb{Z}(n)$.*

*Proof.* We prove the first half of the proposition. The second claim follows from the first by a formal argument. For details, see [MVW, Proposition 13.10].

Let $o$ be the forgetful functor sending Nisnevich sheaves to Zariski sheaves. It is enough to show that $R^i o_* F = 0$ for all $i > 0$. Now, $R^i o_* F$ is the Zariski sheafification of the presheaf with transfers

$$U \mapsto H^i_{Nis}(U, F).$$

Lemma 2.3.5 says that it suffices to that for any field $F$, we have $H^i_{Nis}(K, F) = 0$ for $i > 0$. But this is clear becaue any Nisnevich cover over a field Spec $F$ can be refined by the trivial cover $id : \text{Spec } F \longrightarrow \text{Spec } F$. $\square$

Combining Corollaries 2.2.10 and 2.3.4 and Proposition 2.3.6, we obtain

**Theorem 2.3.7.** *For any smooth scheme $X$ over $k$. There are natural isomorphisms*

$$H^m(X, \mathbb{Z}(n)) \cong H^m_{Nis}(X, \mathbb{Z}(n)) \cong H^m_{Zar}(X, \mathbb{Z}(n)) \cong CH^n(X, 2n - m).$$

# Chapter 3

# Algebraic representatives

The aim of this chapter is to define and study the existence of algebraic representatives for arbitrary smooth schemes. We replace Chow groups with motivic cohomology groups with compact supports and use Voevodsky's triangulated category of motives to generalize the notion of algebraic equivalence to this setting. To motivate our discussion, we review the classical case of smooth proper schemes.

## 3.1 The case of smooth proper schemes

For a smooth proper scheme $X$ over an algebraically closed field $k$, the free abelian group $Z^r(X)$ generated by the set of cycles of codimension $r$ is endowed with equivalence relations known as rational equivalence ($\sim_{\text{rat}}$) and algebraic equivalence ($\sim_{\text{alg}}$). The group of rational equivalence classes $Z^r(X)/_{\sim_{\text{rat}}}$ is denoted by $CH^r(X)$ and called the Chow group of $X$ in codimension $r$. The subset of cycles algebraically equivalent to zero forms a subgroup of $Z^r(X)$ and its image $A^r(X)$ in $CH^r(X)$ is called the algebraic part of $CH^r(X)$. In other words,

$$A^r(X) := \bigcup_{\substack{T,\ \text{smooth, proper} \\ \text{connected}}} \{CH_0(T)^0 \times CH^r(T \times X) \longrightarrow CH^r(X)\}, \tag{3.1}$$

where the map sends a pair $(\sum_i n_i t_i, Y) \in CH_0(T)^0 \times CH^r(T \times X)$ to $\sum n_i Y_{t_i} \in CH^r(X)$. $Y_{t_i}$ denotes the pullback of $Y$ along $t_i : \text{Spec } k \longrightarrow T$.

In order to study $A^r(X)$ by means of abelian varieties, the concept of regular homomorphisms was introduced ([Sam]). Roughly, a regular homomorphism is a group homomorphism from $A^r(X)$ to the group of rational points $A(k)$ of an abelian variety $A$ such that any family of codimension $r$ cycles on $X$ algebraically equivalent to zero that is parametrized by a smooth proper connected scheme $T$ gives rise to a scheme morphism from $T$ to $A$. We may say that a regular homomorphism $A^r(X) \longrightarrow A(k)$ records such a family of cycles on $X$ parametrized by $T$ as a $T$-point of the abelian variety $A$. Here is a precise definition.

**Definition 3.1.1** ([Sam, Section 2.5]). *For an abelian variety $A$ over $k$, a group homomorphism $\phi : A^r(X) \longrightarrow A(k)$ is called* **regular** *if, for any connected smooth proper scheme $T$ over $k$ pointed at a rational point $t_0$, and for any cycle $Y \in CH^r(T \times X)$, the composition*

$$T(k) \xrightarrow{w_Y} A^r(X) \xrightarrow{\phi} A(k)$$

*is induced by a scheme morphism $T \longrightarrow A$. Here, $w_Y$ maps $t \in T(k)$ to $Y_t - Y_{t_0}$. $Y_t$ stands for the image of the intersection of cycles $(t \times X) \cdot Y \in CH^{\dim T + r}(T \times X)$ under the proper pushforward along the projection $T \times X \longrightarrow X$.*

The definition in the above form can be found in [Ha, Section 4] and [Mur, Definition 1.6.1]. A regular homomorphism $\phi : A^r(X) \longrightarrow A(k)$ is said **universal** ([Sam, Section 2.5, Remarque (2)]) if for any regular homomorphism $\phi' : A^r(X) \longrightarrow A'(k)$, there is a unique homomorphism of abelian varieties $a : A \longrightarrow A'$ such that $a \circ \phi = \phi'$. The universal regular homomorphism, if it exists, is called the **algebraic representative** of $A^r(X)$ (or of $X$ in codimension $r$) and written as

$$\Phi^r_X : A^r(X) \longrightarrow Alg^r_X(k).$$

We also refer to the target abelian variety $Alg^r_X$ itself as the algebraic representative.

**Remark 3.1.2** ([Sam, Section 2.5, Remarque (2)][Mur, Section 1.8 and Theorem A]). *The algebraic representatives exist if $r = 1, 2$ or $d_X$ for any smooth proper connected scheme $X$ of dimension $d_X$.*

*For $r = 1$, it is given by the isomorphism*

$$w_{\mathcal{P}}^{-1} : A^1(X) \xrightarrow{\cong} Pic^0_{X,red}(k),$$

*where $\mathcal{P} \in CH^1(Pic^0_{X,red} \times X)$ is the divisor corresponding to the Poincaré bundle on $Pic^0_{X,red} \times X$.*

*The case $r = d_X$ coincides with the Albanese map*

$$alb_X : A^{d_X}(X) \longrightarrow Alb_X(k),$$

*i.e. the map[1] sending $\sum_i n_i \cdot x_i \in A^{d_X}(X)$ to $\sum_i n_i a_p(x_i) \in Alb_X(k)$, where $a_p : X \longrightarrow Alb_X$ is the canonical map that sends $p \in X(k)$ to the unit $0 \in Alb_X(k)$.*

[1] As $a_p = a_q + a_p(q)$ for all rational points $p$ and $q$ of $X$ (the universality of Albanese varieties), the Albanese map $alb_X$ is independent of the choice of $p$.

The algebraic representatives encode properties of the algebraic parts of Chow groups in the following sense.

**Theorem 3.1.3.** *Let $X$ be as above. The algebraic representative $\Phi_X^r : A^r(X) \longrightarrow Alg_X^r(k)$ is an isomorphism if $r = 1$ and induces an isomorphism on torsion parts if $r = d_X$. If, in addition, $k = \mathbb{C}$, it is an isomorphism on torsion for $r = 2$ as well.*

*Proof.* The case $r = 1$ is the theory of Picard varieties (see [Kl, Proposition 9.5.10]). The case $r = d_X$ is known as Rojtman's theorem [R, Bl, Mi82], and the codimension 2 case is [Mur, Theorem C]. □

## 3.2 The algebraic part

In this section, we define the algebraic part of motivic cohomology groups with compact supports $H_c^{2r}(X, \mathbb{Z}(r))$. We simply write $DM$ for Voevodsky's (tensor) triangulated category $DM_{Nis}^{-,eff}(k)$ of effective motives; see Section 2.3.

Let $X$ be a smooth scheme over $k$. We consider the map

$$H_0(T, \mathbb{Z}) \times Hom_{DM}(M(T) \otimes M^c(X), \mathbb{Z}(r)[2r]) \longrightarrow H_c^{2r}(X, \mathbb{Z}(r))$$

that sends a pair $(z, Y)$ with $z \in H_0(T, \mathbb{Z}) \stackrel{\text{def}}{=} Hom_{DM}(\mathbb{Z}, M(T))$ and $Y \in Hom_{DM}(M(T) \otimes M^c(X), \mathbb{Z}(r)[2r])$ to $Y \circ (z \otimes id_{M^c(X)}) \in Hom_{DM}(M^c(X), \mathbb{Z}(r)[2r]) \stackrel{\text{def}}{=} H_c^{2r}(X, \mathbb{Z}(r))$.

The structure morphism of $X$ induces the degree map on the zeroth motivic homology group $deg : H_0(X, \mathbb{Z}) \longrightarrow H_0(k, \mathbb{Z}) \cong \mathbb{Z}$. We set $H_0(X, \mathbb{Z})^0 := \ker(deg)$.

**Definition 3.2.1.** *Let $X$ be a smooth scheme over $k$ and $\mathfrak{T}$ be a class of connected $k$-schemes. The* **algebraic part by $\mathfrak{T}$-parametrization** *of the motivic cohomology group with compact supports $H_c^{2r}(X, \mathbb{Z}(r))$ is defined as*

$$H_{c,\mathfrak{T}}^{2r}(X, \mathbb{Z}(r)) := \bigcup_{T \in \mathfrak{T}} \operatorname{im}\{H_0(T, \mathbb{Z})^0 \times Hom_{DM}(M(T) \otimes M^c(X), \mathbb{Z}(r)[2r]) \longrightarrow H_c^{2r}(X, \mathbb{Z}(r))\}.$$

*If $\mathfrak{T}$ is the class of connected smooth schemes, $H_{c,\mathfrak{T}}^{2r}(X, \mathbb{Z}(r))$ is written as $H_{c,alg}^{2r}(X, \mathbb{Z}(r))$ and simply called the* **algebraic part** *of $H_c^{2r}(X, \mathbb{Z}(r))$.*

**Proposition 3.2.2.** *Let $X$ be a smooth scheme over $k$. Then, $H_{c,alg}^{2r}(X, \mathbb{Z}(r))$ is a subgroup of $H_c^{2r}(X, \mathbb{Z}(r))$.*

*Proof.* We need to show that $H_{c,alg}^{2r}(X, \mathbb{Z}(r))$ is closed under addition and taking inverses. For taking inverses, let $x \in H_{c,alg}^{2r}(X, \mathbb{Z}(r))$. Then, there is a smooth connected scheme $T$,

$Z \in H_0(T,\mathbb{Z})^0$ and $Y \in Hom_{DM}(M(T)\otimes M^c(X),\mathbb{Z}(r)[2r])$ such that $x = Y\circ(Z\otimes id_{M^c(X)})$. Now, by the additivity of the category $DM$, we have $-x = Y\circ(-Z\otimes id_{M^c(X)}) \in H^{2r}_{c,alg}(X,\mathbb{Z}(r))$.

For the closedness under addition, take another $x' \in H^{2r}_{c,alg}(X,\mathbb{Z}(r))$, and choose a smooth connected scheme $T'$, an element $Z' \in H_0(T,\mathbb{Z})^0$ and $Y' \in Hom_{DM}(M(T')\otimes M^c(X),\mathbb{Z}(r)[2r])$ such that $x' = Y'\circ(Z'\otimes id_{M^c(X)})$. It is clear that $x+x'$ belongs to $H^{2r}_{c,alg}(X,\mathbb{Z}(r))$ if $Y = Y'$. We shall reduce the general case to this.

Let us write $Z = \sum_i n_i t_i$ and $Z' = \sum_i n'_i t'_i$ with $n_i, n'_i \in \mathbb{Z}$, $t_i \in T(k)$ and $t'_i \in T'(k)$, and choose $s \in T(k)$ and $s' \in T'(k)$. Define

$$Y'' := Y\circ(p\otimes id_{M^c(X)}) + Y'\circ(p'\otimes id_{M^c(X)}),$$

where $p : M(T\times T') \longrightarrow M(T)$ and $p' : M(T\times T') \longrightarrow M(T')$ are the morphisms induced by the projections. Consider the diagram

$$\begin{tikzcd}
 & & M(T)\otimes M^c(X) \arrow[dr, "Y"] & \\
M^c(X)\cong\mathbb{Z}\otimes M^c(X) \arrow[r, "\sum n_i(t_i\times s')\otimes id"] \arrow[urr, bend left, "\sum n_i t_i\otimes id"] \arrow[drr, bend right, "\sum n_i s'\otimes id=0"'] & M(T\times T')\otimes M^c(X) \arrow[ur, "p\otimes id"] \arrow[dr, "p'\otimes id"'] \arrow[rr, "Y''"] & & \mathbb{Z}(r)[2r] \\
 & & M(T')\otimes M^c(X) \arrow[ur, "Y'"'] &
\end{tikzcd}$$

with $\sum n_i s'\otimes id = 0$ because $\sum_i n_i = 0$. We have

$$\begin{aligned}
x &= Y\circ\big(\sum_i n_i t_i\otimes id_{M^c(X)}\big)\\
&= Y\circ\big(\sum_i n_i t_i\otimes id_{M^c(X)}\big) + Y'\circ\big(\sum_i n_i s'\otimes id_{M^c(X)}\big)\\
&= Y\circ(p\otimes id_{M^c(X)})\circ\big(\sum_i n_i(t_i\times s')\otimes id_{M^c(X)}\big) + Y'\circ(p'\otimes id_{M^c(X)})\circ\big(\sum_i n_i(t_i\times s')\otimes id_{M^c(X)}\big)\\
&= \big(Y\circ(p\otimes id_{M^c(X)}) + Y'\circ(p'\otimes id_{M^c(X)})\big)\circ\big(\sum_i n_i(t_i\times s')\otimes id_{M^c(X)}\big)\\
&= Y''\circ\big(\sum_i n_i(t_i\times s')\otimes id_{M^c(X)}\big).
\end{aligned} \tag{3.2}$$

Similarly, we have

$$x' = Y''\circ\big(\sum_i n'_i(t'_i\times s')\otimes id_{M^c(X)}\big).$$

Therefore, we conclude that

$$x+x' = Y''\circ\big(\big(\sum_i n_i(t_i\times s') + \sum_i n'_i(t'_i\times s')\big)\otimes id_{M^c(X)}\big) \in H^{2r}_{c,alg}(X,\mathbb{Z}(r)).$$

$\square$

For smooth proper schemes, our definition of algebraic part agrees with the classical notion.

**Lemma 3.2.3.** *Let $\mathfrak{T}_{(1)}$ be the subclass of $\mathfrak{T}$ consisting of schemes of dimension one. If $\mathfrak{T}$ is either the class of connected smooth schemes or that of connected smooth proper schemes, we have*

$$H^{2r}_{c,\mathfrak{T}}(X,\mathbb{Z}(r)) = H^{2r}_{c,\mathfrak{T}_{(1)}}(X,\mathbb{Z}(r)).$$

*Proof.* Any two rational points of $T \in \mathfrak{T}$ belong to the image of some smooth connected curve $C$ ([Mum, Chapter II, Section 6, Lemma]). (We may choose $C$ to be additionally proper if $T$ is proper.) Therefore, the lemma follows from the commutativity of the following diagram:

$$\begin{tikzcd}
H_0(C,\mathbb{Z})^0 \arrow[d] & \times & Hom_{DM}(M(C)\otimes M^c(X),\mathbb{Z}(r)[2r]) \arrow[r] & H^{2r}_c(X,\mathbb{Z}(r)) \arrow[d, equal] \\
H_0(T,\mathbb{Z})^0 & \times & Hom_{DM}(M(T)\otimes M^c(X),\mathbb{Z}(r)[2r]) \arrow[u] \arrow[r] & H^{2r}_c(X,\mathbb{Z}(r)).
\end{tikzcd}$$

$\square$

**Proposition 3.2.4.** *Suppose $X$ is a smooth proper scheme over $k$. Then, there is a natural isomorphism*

$$H^{2r}_{c,alg}(X,\mathbb{Z}(r)) \xrightarrow{\cong} A^r(X).$$

*Proof.* The isomorphism is given as a restrictions of the natural isomorphism

$$H^{2r}_c(X,\mathbb{Z}(r)) = H^{2r}(X,\mathbb{Z}(r)) \xrightarrow{F} CH^r(X)$$

in Theorem 2.3.7.

Recall that, by definition,

$$A^r(X) := \bigcup_{\substack{T,\text{ sm, conn}\\ \text{proper}}} \operatorname{im}\{CH_0(T)^0 \times CH^r(T\times X) \longrightarrow CH^r(X)\},$$

where the map sends the pair of $\sum_i n_i t_i \in CH_0(X)^0$ and $Y \in CH^r(T\times X)$ to $\sum_i n_i Y_{t_i}$. Since $Y_{t_i}$ is the pullback of $Y$ along $t_i \times id_X$, the (contravariant) naturality of the comparison map $F$ implies the commutativity of the diagram

$$\begin{tikzcd}
H_0(T,\mathbb{Z})^0 \arrow[d, equal] & \times & Hom_{DM}(M(T)\otimes M^c(X),\mathbb{Z}(r)[2r]) \arrow[d, "F"', "\cong"] \arrow[r] & H^{2r}_c(X,\mathbb{Z}(r)) \arrow[d, "F"', "\cong"] \\
CH_0(T)^0 & \times & CH^r(T\times X) \arrow[r] & CH^r(X)
\end{tikzcd}$$

for all smooth and proper $X$ and $T$. Therefore, $F$ induces an isomorphism

$$H^{2r}_{c,\mathfrak{P}}(X) \longrightarrow A^r(X),$$

where $\mathfrak{P}$ is the class of smooth proper connected schemes.

We claim that $H^{2r}_{c,\mathfrak{P}}(X) = H^{2r}_{c,alg}(X)$. By definition, we have the inclusion $H^{2r}_{c,\mathfrak{P}}(X) \subset H^{2r}_{c,alg}(X)$. For the other inclusion $H^{2r}_{c,\mathfrak{P}}(X) \supset H^{2r}_{c,alg}(X)$, by Lemma 3.2.3, it is enough to prove

$$H^{2r}_{c,\{\text{smooth proper curves}\}}(X) \supset H^{2r}_{c,\{\text{smooth curves}\}}(X).$$

It is enough to observe the surjectivity of $i^*$ in the following commutative diagram for a smooth curve $C$ and its smooth compactification $i : C \hookrightarrow \bar{C}$ :

$$\begin{tikzcd}
H_0(C,\mathbb{Z})^0 \arrow[d, "i_*"'] & \times & Hom_{DM}(M(C)\otimes M^c(X),\mathbb{Z}(r)[2r]) \arrow[r] & H^{2r}_c(X,\mathbb{Z}(r)) \arrow[d, equal] \\
H_0(\bar{C},\mathbb{Z})^0 & \times & Hom_{DM}(M(\bar{C})\otimes M^c(X),\mathbb{Z}(r)[2r]) \arrow[u, "i^*"] \arrow[r] & H^{2r}_c(X,\mathbb{Z}(r)).
\end{tikzcd}$$

But the map $i^*$ is surjective because there is a commutative diagram

$$\begin{tikzcd}
Hom_{DM}(M(\bar{C})\otimes M^c(X),\mathbb{Z}(r)[2r]) \arrow[r, "i^*"] \arrow[d, "F"', "\cong"] & Hom_{DM}(M(C)\otimes M^c(X),\mathbb{Z}(r)[2r]) \arrow[d, "F"', "\cong"] \\
CH^r(C\times X) \arrow[r, two heads, "i^*"] & CH^r(\bar{C}\times X)
\end{tikzcd}$$

□

For a smooth proper connected scheme $X$ over $k$, we have $A^{d_X}(X) = CH_0(X)^0$. This extends to our non-proper situation.

**Proposition 3.2.5.** *Let $X$ be a smooth connected scheme of dimension $d_X$ over $k$. Under the assumption of resolution of singularities, there is a canonical morphism (induced by the duality isomorphism)*

$$H^{2d_X}_{c,alg}(X,\mathbb{Z}(d_X)) \cong H_0(X,\mathbb{Z})^0.$$

*Proof.* Under resolution of singularities, the duality isomorphism ([V00, Theorem 4.3.7 (3)])

$$M^c(X)^* \cong M(X)(-d_X)[-2d_X]$$

in the category $DM^-(k)$ gives the equality

$$H^{2d_X}_{c,alg}(X,\mathbb{Z}(d_X)) = \bigcup_{\substack{T,\text{ smooth} \\ \text{connected}}} \mathrm{im}\{H_0(T,\mathbb{Z})^0 \times Hom_{DM}(M(T),M(X)) \xrightarrow{\text{composition}} H_0(X,\mathbb{Z})\}.$$

Here, we used the fact that $DM$ is a full tensor triangulated subcategory of the closed tensor triangulated category $DM^-(k)$. The fully faithfulness is Voevodsky's cancellation theorem ([V10]), and the compatibility of the two tensor triangulated structures follows from the construction of $DM^-(k)$ together with [MVW, Exercise 8A.8, Corollaries 8A.11 and 15.8].

Let $\{X_i\}$ be the set of connected components of $X$. Since $X_i$ is smooth and connected,

$$\begin{aligned} H^{2d_X}_{c,alg}(X,\mathbb{Z}(d_X)) &\supset \mathrm{im}\{H_0(X_i,\mathbb{Z})^0 \times Hom_{DM}(M(X_i),M(X)) \xrightarrow{\text{composition}} H_0(X,\mathbb{Z})\} \\ &\supset \mathrm{im}\{H_0(X_i,\mathbb{Z})^0 \times \{M(inc) : M(X_i) \to M(X)\} \xrightarrow{\text{inclusion}} H_0(X,\mathbb{Z})\} \\ &= H_0(X_i,\mathbb{Z})^0 \end{aligned}$$

for all $i$. Hence, $H^{2d_X}_{c,alg}(X,\mathbb{Z}(d_X)) \supset H_0(X,\mathbb{Z})^0$.

For the other inclusion, we need to show that for any smooth connected scheme $T$, the composition of any morphism $a \in Hom_{DM}(\mathbb{Z}, M(T))$ that satisfies $str \circ a = 0$ ($str_T : M(T) \to \mathbb{Z}$ is the morphism induced by the structure morphism of $T$) and any $b \in Hom_{DM}(M(T), M(X))$ belongs to $H_0(X,\mathbb{Z})^0$, i.e. the large triangle of the diagram in $DM$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "a"] \arrow[dr, "0"'] & M(T) \arrow[r, "b"] \arrow[d, "str_T"] & M(X) \arrow[dl, "str_X"] \\ & \mathbb{Z} & \end{tikzcd}$$

is commutative if the left triangle is commutative.

Since $T$ is smooth and connected, the group $Hom_{DM}(M(T),\mathbb{Z}) \cong \mathbb{Z}$ is generated by $str_T$. Thus, there is an integer $n$ such that $str_X \circ b = n \cdot str_T$. Hence, $str_X \circ b \circ a = n \cdot str_T \circ a = 0$. $\square$

**Proposition 3.2.6.** *Let $X$ be a smooth scheme over $k$. Then, the group $H^{2r}_{c,alg}(X,\mathbb{Z}(r))$ is divisible.*

*Proof.* The algebraic part is generated by the images of $H_0(C,\mathbb{Z})^0$ with $C$ being smooth curves by Lemma 3.2.3. Thus, it suffices to show the divisibility of $H_0(C,\mathbb{Z})^0$. If $C$ is proper, it is a consequence of the Abel-Jacobi theorem. If $C$ is not proper, take the smooth compactification $C \hookrightarrow \bar{C}$ with $Z := \bar{C} \backslash C$ endowed with the induced reduced structure. The localization sequence for motivic cohomology with compact supports yields

$$\cdots \longrightarrow \bigoplus k^* \longrightarrow H_0(C,\mathbb{Z}) \xrightarrow{f} H_0(\bar{C},\mathbb{Z}) \longrightarrow 0.$$

This gives the short exact sequence

$$0 \longrightarrow \ker f \longrightarrow H_0(C,\mathbb{Z})^0 \longrightarrow H_0(\bar{C},\mathbb{Z})^0 \longrightarrow 0$$

The kernel of $f$ is divisible because it is an image of $\bigoplus k^*$, and $H_0(\bar{C},\mathbb{Z})^0$ is also divisible by the smooth proper case. Hence, the middle group is divisible as well. □

In the rest of this section, we show that certain classes of algebraic groups are enough to define the algebraic part and algebraic part by proper parametrization.

**Lemma 3.2.7.** *Suppose* $0 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} H \longrightarrow 0$ *be an exact sequence of smooth commutative algebraic groups over* $k$*. Then,* $G$ *is an* $(N \times_k H)$*-torsor over* $H$ *in the fppf topology.*

*Proof.* Since $p$ is a morphism between connected smooth schemes and all fibers have the same dimension, it is flat by [Ma, Corollary to Theorem 23.1]. For $p$ is also surjective, it is an fppf cover. We claim that the map $G \times_k N \longrightarrow G \times_H G$ that sends $(g,n)$ to $(g, g\cdot n)$ is an isomorphism.

Let $\mathcal{N},\mathcal{G}$ and $\mathcal{H}$ be the fppf sheaves on $Sch/k$ represented respectively by $N, G$ and $H$. It is enough to prove that the corresponding map

$$\mathcal{G} \times \mathcal{N} \longrightarrow \mathcal{G} \times_{\mathcal{H}} \mathcal{G} \cong \mathcal{G} \times_{\widetilde{\mathcal{G}/\mathcal{N}}} \mathcal{G}$$

of fppf sheaves is an isomorphism. ($\widetilde{\mathcal{G}/\mathcal{N}}$ denotes the fppf sheafification of the quotient $\mathcal{G}/\mathcal{N}$ as presheaves.) Now, this map is nothing but the sheafification of the map of presheaves

$$\mathcal{G} \times \mathcal{N} \longrightarrow \mathcal{G} \times_{\mathcal{G}/\mathcal{N}} \mathcal{G}$$

that sends, for each scheme $U$, a pair of sections $(g_U, n_U) \in \mathcal{G}(U) \times \mathcal{N}(U)$ to $(g_U, g_U \cdot n_U) \in \mathcal{G}(U) \times_{\mathcal{G}(U)/\mathcal{N}(U)} \mathcal{G}(U)$, but this is clearly an isomorphism. □

**Proposition 3.2.8.** *Let* $X$ *be a smooth scheme over* $k$ *and* $\mathfrak{S}$ *be the class of semi-abelian varieties over* $k$*. Then,*

$$H^{2r}_{c,alg}(X,\mathbb{Z}(r)) = H^{2r}_{c,\mathfrak{S}}(X,\mathbb{Z}(r)).$$

*Proof.* The inclusion "⊃" is obvious. By the same argument as in the proof of Proposition 3.2.2, $H^{2r}_{c,\mathfrak{S}}(X,\mathbb{Z}(r))$ is a subgroup of $H^{2r}_c(X,\mathbb{Z}(r))$. Thus, for the other inclusion, Lemma 3.2.3 implies that it is enough to show

$$H^{2r}_{c,\{\text{smooth curves}\}}(X,\mathbb{Z}(r)) \subset H^{2r}_{c,\mathfrak{S}}(X,\mathbb{Z}(r)).$$

Let $x \in H^{2r}_{c,\{\text{smooth curves}\}}(X, \mathbb{Z}(r))$. Then there are a smooth affine curve $C$, $Z \in H_0(C, \mathbb{Z})^0$ and $Y \in Hom_{DM}(M(C) \otimes M^c(X), \mathbb{Z}(r)[2r])$ such that $x = Y \circ (Z \otimes id_{M^c(X)})$. (If we find a proper curve $C$, just remove one point not supporting the divisor $Z$.)

Let $C^l$ be the $l$-th power and $C^{(l)}$ the $l$-th symmetric power of the curve $C$. Write the quotient morphism as $f : C^l \longrightarrow C^{(l)}$ and the diagonal as $\Delta : C \longrightarrow C^l$. Consider the following commutative diagram.

$$\begin{array}{ccccccc}
H_0(C^{(l)}, \mathbb{Z})^0 & \times & Hom_{DM}(M(C^{(l)}) \otimes M^c(X), \mathbb{Z}(r)[2r]) & \xrightarrow{\alpha} & H^{2r}_c(X, \mathbb{Z}(r)) \\
{\scriptstyle f\circ -}\uparrow & & {\scriptstyle -\circ f}\downarrow \uparrow{\scriptstyle -\circ^t\Gamma_f} & & \| \\
H_0(C^l, \mathbb{Z})^0 & \times & Hom_{DM}(M(C^l) \otimes M^c(X), \mathbb{Z}(r)[2r]) & \xrightarrow{\beta} & H^{2r}_c(X, \mathbb{Z}(r)) \\
{\scriptstyle \Delta\circ -}\uparrow & & {\scriptstyle -\circ \Delta}\downarrow \uparrow{\scriptstyle -\circ P} & & \| \\
H_0(C, \mathbb{Z})^0 & \times & Hom_{DM}(M(C) \otimes M^c(X), \mathbb{Z}(r)[2r]) & \xrightarrow{\gamma} & H^{2r}_c(X, \mathbb{Z}(r)).
\end{array}$$

Here, $P := \sum_{i=1,\cdots,l} p_i$, where $p_i : C^l \longrightarrow C$ is the $i$-th projection and the summation is taken in $Hom_{DM}(M(C^l), M(C))$. ${}^t\Gamma_f$ is a finite correspondence because $f : C^l \longrightarrow C^{(l)}$ is a finite surjective morphism ([Mi86b, Propositions 3.1 and 3.2]).

By inspection, we may see that ${}^t\Gamma_f \circ f \circ \Delta =^t \Gamma_f \circ (f \circ \Delta) = m \cdot \Delta$ in $Cor_k(C, C^l)$ ($m$ is some intersection multiplicity). Since $P \circ \Delta = l \cdot id_C$, we have $P \circ^t \Gamma_f \circ f \circ \Delta = P \circ (m \cdot \Delta) = m \cdot l \cdot id_C$.

Therefore, since $\gamma(Z, Y) = x$, the commutativity of the diagram gives

$$\begin{aligned}
& \alpha(f \circ \Delta \circ Z, Y \circ (P \otimes id_{M^c(X)}) \circ ({}^t\Gamma_f \otimes id_{M^c(X)})) \\
= \ & \gamma(Z, Y \circ (P \otimes id_{M^c(X)}) \circ ({}^t\Gamma_f \otimes id_{M^c(X)}) \circ (f \otimes id_{M^c(X)}) \circ (\Delta \otimes id_{M^c(X)})) \\
= \ & \gamma(Z, Y \circ ((P \circ^t \Gamma_f \circ f \circ \Delta) \otimes id_{M^c(X)})) \\
= \ & \gamma(Z, l \cdot m \cdot Y) \\
= \ & l \cdot m \cdot x.
\end{aligned}$$

Thus, we have a commutative diagram

$$\begin{tikzcd}
H_0(C^{(l)}, \mathbb{Z})^0 \arrow[r, "\alpha_{Y\circ P\circ^t\Gamma_f}"] & H^{2r}_c(X, \mathbb{Z}(r)) \\
H_0(C, \mathbb{Z})^0 \arrow[u, "f\circ\Delta\circ -"] \arrow[ur, "\gamma_{l\cdot m\cdot Y}"']
\end{tikzcd}$$

where $\alpha_{Y\circ P\circ^t\Gamma_f} := \alpha(-, Y \circ (P \otimes id_{M^c(X)}) \circ ({}^t\Gamma_f \otimes id_{M^c(X)})$ and $\gamma_{l\cdot m\cdot Y} := \gamma(-, l \cdot m \cdot Y)$. Therefore,

$$\mathrm{im}(\alpha_{Y\circ P\circ^t\Gamma_f}) \supset \mathrm{im}(\gamma_{l\cdot m\cdot Y}) = l \cdot m \cdot \mathrm{im}(\gamma_Y) = \mathrm{im}(\gamma_Y).$$

The last equality holds because $H_0(C,\mathbb{Z})^0$ is divisible. Since this is true for all $Y \in Hom_{DM}(M(C)\otimes M^c(X), \mathbb{Z}(r)[2r])$, we conclude that $\mathrm{im}(\alpha) \supset \mathrm{im}(\gamma)$.

Now, since $C$ is an affine curve, $C^{(l)}$ is an affine bundle over the smooth connected commutative algebraic group $Pic^0(C^+)$ if $l$ is sufficiently large by [Wi, Appendix] ($C^+ := \bar{C}\coprod_{\bar{C}\setminus C} \mathrm{Spec}\ k$ for a smooth compactification $\bar{C}$ of $C$). By Chevalley's theorem (see [BLR, Chapter 9, Section 2, Theorem 1]), there is a smooth connected affine commutative algebraic subgroup $L$ of $Pic^0(C^+)$. By [Bo, Theorem 10.6 (i) and (ii)], $L$ has a connected unipotent algebraic subgroup $U$ such that the quotient $L/U$ is a torus. Hence, the quotient $Pic^0(C^+)/U$ is a semi-abelian variety. Moreover, by [Bo, Corollary 15.5 (ii)], there is a composition series consisting of connected algebraic subgroups

$$U = U_0 \supset U_1 \supset \cdots \supset U_n = \{e\}$$

such that each quotient algebraic group $U_i/U_{i+1}$ is isomorphic to $\mathbb{G}_a$. We are given with the exact sequence

$$0 \longrightarrow \mathbb{G}_a(\cong U_{n-1}) \longrightarrow Pic^0(C^+) \longrightarrow Pic^0(C^+)/U_{n-1} \longrightarrow 0$$

of algebraic groups. Now, $Pic^0(C^+)$ is a $\mathbb{G}_a$-torsor (in the fppf topology) over $Pic^0(C^+)/U_{n-1}$ by Lemma 3.2.7. We claim that it is locally trivial in the Zariski topology as well.

By [Mi80, Chapter III, Proposition 3.7], the canonical map

$$H^1_{Zar}(Pic^0(C^+)/U_{n-1}, \mathbb{G}_a) \longrightarrow H^1_{fppf}(Pic^0(C^+)/U_{n-1}, \mathbb{G}_a)$$

is an isomorphism because $\mathbb{G}_a$ is coherent. Now, let $PHS^{\mathbb{G}_a}(Pic^0(C^+)/U_{n-1})$ (resp. $PHS^{\mathbb{G}_a}_{Zar}(Pic^0(C^+)/U_{n-1})$) denote the isomorphism classes of $\mathbb{G}_a$-torsors over $Pic^0(C^+)/U_{n-1}$ locally trivial in the fppf (resp. Zariski) topology. Consider the following commutative diagram

$$\begin{tikzcd}
PHS^{\mathbb{G}_a}(Pic^0(C^+)/U_{n-1}) \arrow[r, "\cong", "\text{Yoneda}"'] & \{\text{sheaf } \mathbb{G}_a\text{-torsors on } \big(Pic^0(C^+)/U_{n-1}\big)_{fppf}\} \arrow[r, "\cong", "a"'] & H^1_{fppf}(Pic^0(C^+)/U_{n-1}, \mathbb{G}_a) \\
PHS^{\mathbb{G}_a}_{Zar}(Pic^0(C^+)/U_{n-1}) \arrow[u, hook, "\iota,\ \text{inclusion}"] \arrow[r, "\cong", "\text{Yoneda}"'] & \{\text{sheaf } \mathbb{G}_a\text{-torsors on } \big(Pic^0(C^+)/U_{n-1}\big)_{Zar}\} \arrow[r, "\cong", "b"'] & H^1_{Zar}(Pic^0(C^+)/U_{n-1}, \mathbb{G}_a) \arrow[u, "\cong"']
\end{tikzcd}$$

The Yoneda imbeddings are isomorphisms By [Mi80, Chapter III, Theorem 4.3(a)] and its variant in the Zariski topology, and the maps $a$ and $b$ are isomorphisms by [ibid., Proposition 4.6] and its Zariski variant. Therefore, the inclusion $\iota$ is an isomorphism. This means that the $\mathbb{G}_a$-torsor $Pic^0(C^+)$ over $Pic^0(C^+)/U_{n-1}$ is locally trivial already in the Zariski topology.

By Mayer-Vietoris exact triangle ([MVW, (14.5.1)]) and $\mathbb{A}^1$-homotopy invariance in $DM$, we may see that the canonical map

$$M(q) : M(Pic^0(C^+)) \longrightarrow M(Pic^0(C^+)/U_{n-1})$$

is an isomorphism. By repeating this process, we obtain the canonical isomorphism

$$M(Pic^0(C^+)) \xrightarrow{\cong} M(Pic^0(C^+)/U).$$

Now, the isomorphisms

$$M(C^{(l)}) \xrightarrow{\cong} M(Pic^0(C^+)) \xrightarrow{\cong} M(Pic^0(C^+)/U)$$

mean

$$\mathrm{im}(\alpha) = \mathrm{im}\{H_0(Pic^0(C^+)/U, \mathbb{Z})^0 \times Hom_{DM}(M(Pic^0(C^+)/U) \otimes M^c(X), \mathbb{Z}(r)[2r]) \longrightarrow H_c^{2r}(X, \mathbb{Z}(r))\}.$$

This equality holds for all smooth affine curves $C$. Since $Pic^0(C^+)/U$ is a semi-abeian variety, we conclude

$$H^{2r}_{c,\mathfrak{S}}(X, \mathbb{Z}(r)) \supset H^{2r}_{c,\{\text{smooth affine curves}\}}(X, \mathbb{Z}(r)) = H^{2r}_{c,\{\text{smooth curves}\}}(X, \mathbb{Z}(r)).$$

□

We recover the following classical result (see, for example, [La, p.60, Theorem 1]) on the algebraic part $A^r(X)$ of the Chow group of cycles of codimension $r$.

**Corollary 3.2.9.** *Let $X$ be a smooth proper scheme over $k$ and $\mathfrak{A}$ be the class of abelian varieties over $k$. Then,*

$$A^r(X) = \bigcup_{A \in \mathfrak{A}} \mathrm{im}\{CH_0(A)^0 \times CH^r(A \times X) \longrightarrow CH^r(X)\}.$$

*Proof.* The right hand side is equal to $H^{2r}_{c,\mathfrak{A}}(X, \mathbb{Z}(r))$. By Propositions 3.2.4 and 3.2.8, the left hand side is equal to $H^{2r}_{c,\mathfrak{S}}(X, \mathbb{Z}(r))$. Thus, it is enough to show that

$$\begin{aligned} &\bigcup_{S \in \mathfrak{S}} \mathrm{im}\{H_0(S, \mathbb{Z})^0 \times Hom_{DM}(M(S) \otimes M(X), \mathbb{Z}(r)[2r]) \longrightarrow H^{2r}(X, \mathbb{Z}(r))\} \\ = &\bigcup_{A \in \mathfrak{A}} \mathrm{im}\{H_0(A, \mathbb{Z})^0 \times Hom_{DM}(M(A) \otimes M(X), \mathbb{Z}(r)[2r]) \longrightarrow H^{2r}(X, \mathbb{Z}(r))\}. \end{aligned}$$

We claim that for a semi-abelian variety $S$ with the Chevalley decomposition $0 \to \mathbb{G}_m^s \to$

$S \to A \to 0$, there is an inclusion

$$\begin{aligned} & \mathrm{im}\{H_0(S,\mathbb{Z})^0 \times Hom_{DM}(M(S)\otimes M(X), \mathbb{Z}(r)[2r]) \longrightarrow H^{2r}(X,\mathbb{Z}(r))\} \\ \subset \quad & \mathrm{im}\{H_0(A,\mathbb{Z})^0 \times Hom_{DM}(M(A)\otimes M(X), \mathbb{Z}(r)[2r]) \longrightarrow H^{2r}(X,\mathbb{Z}(r))\}. \end{aligned}$$

We prove this by induction on the torus rank $s$. If $s = 0$, the claim is trivially true. Suppose that the claim is true for semi-abelian varieties of torus rank $s-1$, and let $S$ be a semi-abelian variety with torus rank $s$. There is a short exact sequence of algebraic groups

$$0 \longrightarrow \mathbb{G}_m \longrightarrow S \longrightarrow S' \longrightarrow 0.$$

By a similar argument in the proof of Proposition 3.2.8 this time with Hilbert's Satz 90 ([Mi80, Chapter III, Proposition 4.9]) instead of [Ibid., Chapter III, Proposition 3.7], we can see that $S$ is a $\mathbb{G}_m$-torsor over $S'$ in the Zariski topology. Hence, there is an associated line bundle $p : E \longrightarrow S'$ with a zero section $s : S' \longrightarrow E$ such that $E \setminus s(S') \cong S$. By $\mathbb{A}^1$-homotopy invariance, $p$ induces an isomorphism of motives $M(E) \xrightarrow{\cong} M(S')$.

Hence, there is a commutative diagram

$$\begin{array}{ccccccc} H_0(S,\mathbb{Z})^0 & \times & Hom_{DM}(M(S)\otimes M(X), \mathbb{Z}(r)[2r]) & \xrightarrow{\alpha} & H^{2r}(X,\mathbb{Z}(r)) \\ {\scriptstyle inc_*}\downarrow & & \uparrow{\scriptstyle -\circ(inc\otimes id_{M(X)})} & & \| \\ H_0(E,\mathbb{Z})^0 & \times & Hom_{DM}(M(E)\otimes M(X), \mathbb{Z}(r)[2r]) & \longrightarrow & H^{2r}(X,\mathbb{Z}(r)) \\ {\scriptstyle p_*}\downarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle -\circ(p\otimes id_{M(X)})}\,{\scriptstyle \cong} & & \| \\ H_0(S',\mathbb{Z})^0 & \times & Hom_{DM}(M(S')\otimes M(X), \mathbb{Z}(r)[2r]) & \xrightarrow{\beta} & H^{2r}(X,\mathbb{Z}(r)). \end{array}$$

The upper middle map is surjective because it can be identified with the pullback $CH^r(E \times X) \longrightarrow CH^r(S \times X)$ along the open immersion $S \times X \hookrightarrow E \times X$. The commutativity of the diagram implies that

$$\mathrm{im}(\alpha) \subset \mathrm{im}(\beta).$$

Since the torus rank of $S'$ is $s-1$, the induction hypothesis gives

$$\mathrm{im}(\beta) \subset \mathrm{im}\{H_0(A,\mathbb{Z})^0 \times Hom_{DM}(M(A)\otimes M(X), \mathbb{Z}(r)[2r]) \longrightarrow H^{2r}(X,\mathbb{Z}(r))\}.$$

Hence, the claim is proved. □

## 3.3 Regular homomorphisms

The classical definition (Definition 3.1.1) naturally generalizes to our setting.

**Definition 3.3.1.** *Let $X$ be a smooth scheme over $k$ and $S$ be a semi-abelian variety over $k$. A group homomorphism $\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k)$ is called* **regular** *if for any smooth connected scheme $T$ pointed at $t_0 \in T(k)$ and $Y \in Hom_{DM}(M(T) \otimes M^c(X), \mathbb{Z}(r)[2r])$, the composition*

$$T(k) \xrightarrow{w_Y} H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \xrightarrow{\phi} S(k)$$

*is induced by some scheme morphism $T \longrightarrow S$. Here, $w_Y$ sends $t \in T(k) = Hom_{Sch/k}(\mathrm{Spec}\ k, T)$ to*

$$Y \circ (t \otimes id_{M^c(X)}) - Y \circ (t_0 \otimes id_{M^c(X)}),$$

*i.e.*

$$M^c(X) \cong \mathbb{Z} \otimes M^c(X) \stackrel{t \otimes id_{M^c(X)} - t_0 \otimes id_{M^c(X)}}{\longrightarrow} M(T) \otimes M^c(X) \xrightarrow{Y} \mathbb{Z}(r)[2r]$$

*where $t$ and $t_0$ are regarded as morphisms from $\mathbb{Z}$ to $M(T)$ in $DM$.*

A regular homomorphism $\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k)$ is said **universal** if for any regular homomorphism $\phi' : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S'(k)$, there is a unique homomorphism of semi-abelian varieties $a : S \longrightarrow S'$ such that $a \circ \phi = \phi'$.

**Definition 3.3.2.** *The universal regular homomorphism, if it exists, is called the* **algebraic representative** *of $H^{2r}_{c,alg}(X, \mathbb{Z}(r))$ or the* **algebraic representative with compact supports of $X$ in codimension** *$r$, and it is written as*

$$\Phi^r_{c,X} : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow Alg^r_{c,X}(k).$$

*The target semi-abelian variety $Alg^r_{c,X}$ itself is often referred to as the algebraic representative with compact supports.*

**Proposition 3.3.3.** *Let $X$ be a smooth scheme over $k$. Then, given a regular homomorphism $\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k)$, there is a semi-abelian variety $S_0$ (pointed at the unit) and $Y_0 \in Hom_{DM}(M(S_0) \otimes M^c(X), \mathbb{Z}(r)[2r])$ such that* $\mathrm{im}(\phi \circ w_{Y_0}) = \mathrm{im}(\phi)$.

*Proof.* We follows the method of [Mur, Proof of Lemma 1.6.2 (i)]. Consider the diagram

$$S'(k) \xrightarrow{w_{Y'}} H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \xrightarrow{\phi} S(k)$$

where $S'$ is a semi-abelian variety pointed at the unit and $Y' \in Hom_{DM}(M(S') \otimes M^c(X), \mathbb{Z}(r)[2r])$. Since the composition is induced by the homomorphism of semi-abelian varieties, the image of

$\phi \circ w_{Y'}$ has a structure of a semi-abelian variety.

Choose $S_0$ and $Y_0$ so that the dimension of $\mathrm{im}(\phi \circ w_{Y_0})$ becomes maximal among such diagrams. We claim that they have the desired property $\mathrm{im}(\phi \circ w_{Y_0}) = \mathrm{im}(\phi)$.

Suppose that $\mathrm{im}(\phi \circ w_{Y_0}) \neq \mathrm{im}(\phi)$. Then, there is an element $x \in H^{2r}_{c,alg}(X, \mathbb{Z}(r))$ such that $\phi(x) \notin \mathrm{im}(\phi \circ w_{Y_0})$. Using Proposition 3.2.8, we find a semi-abelian variety $S_1$ and $Y_1 \in Hom_{DM}(M(S_1) \otimes M^c(X), \mathbb{Z}(r)[2r])$ for which $\phi(x) \in \mathrm{im}(\phi \circ w_{Y_1})$.

Now, let

$$S_2 := S_0 \times S_1$$

and

$$Y_2 := Y_0 \circ (p_0 \otimes id_{M^c(X)}) + Y_1 \circ (p_1 \otimes id_{M^c(X)}),$$

where $p_i : M(S_0 \times S_1) \longrightarrow M(S_i)$ is the morphism induced by the projection. Then, $\mathrm{im}(\phi \circ w_{Y_2})$ contains $\phi(x)$, but

$$\mathrm{im}(\phi \circ w_{Y_2}) \supset \mathrm{im}(\phi \circ w_{Y_0}) \not\ni \phi(x).$$

This contradicts the maximality of the dimension of $\mathrm{im}(\phi \circ w_{Y_0})$. $\square$

**Corollary 3.3.4.** *If $\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k)$ is a regular homomorphism, then the image of $\phi$ has a structure of a semi-abelian subvariety of $S$.*

*Proof.* By Proposition 3.3.3, there are semi-abelian varieties $S_0$ and $Y_0 \in Hom_{DM}(M(S_0) \otimes M^c(X), \mathbb{Z}(r)[2r])$ such that $\mathrm{im}(\phi) = \mathrm{im}(\phi \circ w_{Y_0})$. Since $\phi \circ w_{Y_0}$ is a homomorphism between semi-abelian varieties, its image $\mathrm{im}(\phi)$ is a semi-abelian variety. $\square$

**Proposition 3.3.5.** *Suppose $\Phi^r_{c,X} : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow Alg^r_{c,X}(k)$ is an algebraic representative. Then, it is surjective, and it also induces a surjective homomorphism on the torsion parts.*

*Proof.* The surjectivity of $\Phi^r_{c,X}$ is immediate from Corollary 3.3.4. As for the claim on the torsion parts, by Proposition 3.3.3, there is a semi-abelian variety $S_0$ and a surjective homomorphism, say, $f : S_0(k) \longrightarrow Alg^r_{c,X}(k)$ that factors through $H^{2r}_{c,alg}(X, \mathbb{Z}(r))$. Since the kernel of $f$ is an extension of a finite group by a divisible group, $\ker(f) \otimes \mathbb{Q}/\mathbb{Z} = 0$. This implies that $f$ induces a surjection on the torsion parts. $\square$

## 3.4 Existence of algebraic representatives

We prove the existence of algebraic representatives $Alg^r_{c,X}$ of $H^{2r}_{c,alg}(X, \mathbb{Z}(r))$ for $r = 1$ and 2 (for smooth $X$ with a good compactification) and $r = d_X$ (for all smooth $X$). We use the method of Serre [Se], Saito [Sai] and Murre [Mur].

We define an analogue of a maximal morphism (cf. [Se, Définition 2]), which we shall call a maximal homomorphism (Definition 3.4.1). We characterize the algebraic representatives as the maximal homomorphisms whose target semi-abelian variety has the maximal dimension (Proposition 3.4.4). This is a generalization of [Sai, Theorem 2.2] as presented in [Mur, Proposition 2.1] to our non-proper setting. It then remains to bound the dimension of the targets of maximal homomorphisms to obtain the existence of algebraic representatives. To achieve this, we use the Beilinson-Lichtenbaum conjecture, which is now a theorem by the work of Rost and Voevodsky and others (cf. [GL, Corollary 2.1]).

Throughout this section, $X$ is a smooth connected scheme over $k$.

**Definition 3.4.1.** *A regular homomorphism* $\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k)$ *is called* **maximal** *if it is surjective and for any factorization*

$$\begin{tikzcd} & S'(k) \arrow[d, "\forall \pi,\ isogeny"] \\ H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \arrow[ur, two heads, "\forall\ regular"] \arrow[r, two heads, "\phi"'] & S(k), \end{tikzcd}$$

$\pi$ *is an isomorphism.*

**Lemma 3.4.2.** *Let* $\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k)$ *be a regular homomorphism. Then, there is a factorization*

$$\begin{tikzcd} H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \arrow[rr, "\phi"] \arrow[dr, "g"'] & & S(k) \\ & S'(k) \arrow[ur, "h"'] & \end{tikzcd}$$

*where* $g$ *is a maximal homomorphism and* $h$ *is a finite morphism.*

*Proof.* We follow the proof of [Se, Théorème 1]. By Corollary 3.3.4, we may assume that $\phi$ is surjective. If $\phi$ is maximal, there is nothing to prove.

If $\phi$ is not maximal, there is a factorization

$$\begin{tikzcd} & S_1(k) \arrow[d, "\pi_1"] \\ H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \arrow[ur, two heads, "\phi_1,\ \text{regular}"] \arrow[r, two heads, "\phi"'] & S(k), \end{tikzcd}$$

where $\pi_1$ is an isogeny that is not an isomorphism. Here, $\phi_1$ is surjective because $\pi_1$ is an isogeny. If $\phi_1$ is maximal, there is nothing more to do.

Repeat this process. Suppose that we obtain an infinite tower

$$
\begin{tikzcd}
 & \vdots \arrow[d, "{\pi_3,\ \text{isog., not an isom.}}"] \\
 & S_2(k) \arrow[d, "{\pi_2,\ \text{isog., not an isom.}}"] \\
 & S_1(k) \arrow[d, "{\pi_1,\ \text{isog, not an isom.}}"] \\
H^{2r}_{c,alg}(X,\mathbb{Z}(r)) \arrow[r, two heads, "\phi"'] \arrow[ur, two heads, "{\phi_1,\ \text{reg.}}"] \arrow[uur, two heads, "{\phi_2,\ \text{reg.}}"] & S(k).
\end{tikzcd}
$$

By Proposition 3.3.3, choose a semi-abelian variety $S_0$ and $Y_0 \in Hom_{DM}(M(S_0)\otimes M^c(X), \mathbb{Z}(r)[2r])$ such that $\phi \circ w_{Y_0}$ is surjective. Then, since $\pi_i$'s are isogeny, $\phi_i \circ w_{Y_0}$ is surjective for all $i$.

Then, we obtain the diagram of function fields

$$
\begin{tikzcd}
 & \vdots \\
 & K(S_2) \arrow[u, hook, "\text{not an isom.}"] \arrow[ddl, hook] \\
 & K(S_1) \arrow[u, hook, "\text{not an isom.}"] \arrow[dl, hook] \\
K(S_0) & K(S) \arrow[u, hook, "\text{not an isom.}"] \arrow[l, hook']
\end{tikzcd}
$$

Thus, we have

$$K(S_0) \supset \bigcup_{i\geq 1} K(S_i) \supset K(S),$$

where the extension $\bigcup_{i\geq 1} K(S_i)/K(S)$ is not finitely generated. However, the extension $K(S_0)/K(S)$ is finitely generated. Since a subextension of a finitely generated field extension is finitely generated ([Se, Lemme 1]), this is a contradiction. $\square$

We need one more lemma before giving a criterion for the existence of an algebraic representative.

**Lemma 3.4.3.** *Let* $\phi : H^{2r}_{c,alg}(X,\mathbb{Z}(r)) \longrightarrow S(k)$ *be a surjective regular homomorphism and* $\phi' : H^{2r}_{c,alg}(X,\mathbb{Z}(r)) \longrightarrow S'(k)$ *be any regular homomorphism. Then, there is at most one* *scheme morphism $f : S \longrightarrow S'$ that makes the following diagram commutative:*

$$\begin{tikzcd} H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \arrow[r, two heads, "\phi"] \arrow[dr, "\phi'"'] & S(k) \arrow[d, "f"] \\ & S'(k). \end{tikzcd}$$

*Proof.* Choose a semi-abelian variety $S_0$ and $Y_0 \in Hom_{DM}(M(S_0) \otimes M(X), \mathbb{Z}(r)[2r])$ as in Remark 3.3.3. Then, since $\phi \circ w_{Y_0} : S_0 \longrightarrow S$ is a morphism between connected smooth schemes and all fibers have the same dimension, it is flat by [Ma, Corollary to Theorem 23.1]. It is also surjective, so it is a strict epimorphism. In particular,

$$Hom_{Sch}(S, S') \overset{-\circ\phi\circ w_{Y_0}}{\longrightarrow} Hom_{Sch}(S_0, S')$$

is injective. Hence, the lemma follows. □

**Proposition 3.4.4.** *There is an algebraic representative of $H^{2r}_{c,alg}(X, \mathbb{Z}(r))$ if and only if there is a constant $c$ such that $\dim S \leq c$ for any maximal homomorphism*

$$\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k).$$

*In fact, the maximal homomorphism with the maximal dimensional target is the algebraic representative.*

*Proof.* " $\Rightarrow$ " is clear. We prove the converse by combining the arguments for [Se, Théorème 2] and [Mur, Proposition 2.1].

Let $\phi_0 : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S_0(k)$ be a maximal homomorphism with the maximal dimensional target $S_0$. Suppose that $\phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow S(k)$ is a regular homomorphism.

Now, $\phi_0 \times \phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \longrightarrow (S_0 \times S)(k)$ is a regular homomorphism. By Lemma 3.4.2, there is a factorization

$$\phi_0 \times \phi : H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \overset{g,\ \text{max.}}{\longrightarrow} S_1(k) \overset{i,\ \text{fin.}}{\longrightarrow} (S_0 \times S)(k)$$

with some maximal homomorphism $g$.

Consider the commutative diagram

$$\begin{tikzcd}
 & & & S_0(k) \\
H^{2r}_{c,alg}(X,\mathbb{Z}(r)) \arrow[rrru, two heads, bend left, "\phi_0,\ \text{max.}"] \arrow[r, two heads, "g,\ \text{max.}"] \arrow[rrrd, bend right, "\phi"'] & S_1(k) \arrow[rru, "\text{isom.}"] \arrow[r, "i"] \arrow[rrd] & & (S_0 \times S)(k) \arrow[u, "p_0"'] \arrow[d, "p"] \\
 & & & S(k)
\end{tikzcd}$$

Since $p_0 \circ i$ is surjective and $S_0$ has the maximal dimension, we must have $\dim S_0 = \dim S_1$. Hence, $p_0 \circ i$ is an isogeny. Since $\phi_0$ is a maximal homomorphism, $p_0 \circ i$ is an isomorphism. Let us put

$$r := (p_0 \circ i)^{-1} : S_0 \longrightarrow S_1,$$

and define $h := p \circ i \circ r : S_0 \longrightarrow S$. Then,

$$\begin{aligned} h \circ \phi_0 &= p \circ i \circ r \circ \phi_0 \\ &= p \circ i \circ g \\ &= \phi. \end{aligned}$$

By Lemma 3.4.3, $h$ is the only scheme morphism for which $\phi = h \circ \phi_0$ holds. Therefore, $\phi_0$ is the algebraic representative of $H^{2r}_{c,alg}(X,\mathbb{Z}(r))$. $\square$

**Definition 3.4.5.** *A smooth connected scheme $X$ over $k$ is said to have a* **good compactification** *if there is a smooth proper scheme $\bar{X}$ with an open immersion $X \hookrightarrow \bar{X}$ such that $Z := \bar{X} \setminus X$ is a simple normal crossing divisor on $\bar{X}$.*

**Theorem 3.4.6.** *Let $X$ be a smooth connected scheme over $k$ with a good compactification. Then, there is an algebraic representative of $H^{2r}_{c,alg}(X,\mathbb{Z}(r))$ if $r = 1$ or $2$.*

*Proof.* By Proposition 3.4.4, it suffices to show that the dimensions of the target semi-abelian varieties of surjective regular homomorphisms are bounded.

Let $\phi : H^{2r}_{c,alg}(X,\mathbb{Z}(r)) \longrightarrow S(k)$ be a surjective regular homomorphism. By Proposition 3.3.3, we may choose a semi-abelian variety $S'$ and $Y \in Hom_{DM}(M(S') \otimes M^c(X), \mathbb{Z}(r)[2r])$ such that the composition $f : S'(k) \xrightarrow{w_Y} H^{2r}_{c,alg}(X,\mathbb{Z}(r)) \xrightarrow{\phi} S(k)$ becomes a surjective homomorphism. We shall write the corresponding homomorphism of semi-abelian varieties by the same symbol $f$.

Let $l$ be a prime relatively prime to the characteristic of the base field $k$ and the index $(\ker f : \ker f^0)$, where $\ker f^0$ is the identity component (it is a semi-abelian variety) of the group scheme $\ker f$.

Let us look at the $l$-torsion parts:

$$\begin{tikzcd} {}_lS'(k) \arrow[r, "{}_lw_Y"] \arrow[rr, bend right, "{}_lf"'] & {}_lH^{2r}_{c,alg}(X,\mathbb{Z}(r)) \arrow[r, "{}_l\phi"] & {}_lS(k). \end{tikzcd}$$

We claim that ${}_l\phi$ is surjective. In fact, ${}_lf$ is surjective. For this, by the snake lemma, it is enough to show that $\ker f/l\cdot\ker f = 0$. Since $\ker f^0$ is a semi-abelian variety and $(\ker f : \ker f^0)$ is prime to $l$, another application of the snake lemma to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker f^0 \arrow[r] \arrow[d, "l"] & \ker f \arrow[r] \arrow[d, "l"] & \ker f/\ker f^0 \arrow[r] \arrow[d, "l"] & 0 \\ 0 \arrow[r] & \ker f^0 \arrow[r] & \ker f \arrow[r] & \ker f/\ker f^0 \arrow[r] & 0 \end{tikzcd}$$

yields the desired result.

Now, we have

$$H^{2r-1}_c(X,\mathbb{Z}/l(r)) \twoheadrightarrow {}_lH^{2r}_c(X,\mathbb{Z}(r)) \supset {}_lH^{2r}_{c,alg}(X,\mathbb{Z}(r)) \xrightarrow{{}_l\phi} {}_lS(k). \tag{3.3}$$

Since the dimension of $S$ less than or equal to the $l$-rank of $S$, it is now enough to prove that $H^{2r-1}_c(X,\mathbb{Z}/l(r))$ is a finite group.

Since $l \neq \operatorname{char} k$, there is a localization sequence ([Ke, Proposition 5.5.5])

$$\cdots \longrightarrow H^{2r-2}(Z,\mathbb{Z}/l(r)) \longrightarrow H^{2r-1}_c(X,\mathbb{Z}/l(r)) \longrightarrow H^{2r-1}(\bar{X},\mathbb{Z}/l(r)) \longrightarrow \cdots. \tag{3.4}$$

Therefore, it suffices to show the finiteness of $H^{2r-2}(Z,\mathbb{Z}/l(r))$ and $H^{2r-1}(\bar{X},\mathbb{Z}/l(r))$.

Since $\bar{X}$ is smooth and $2r \leq r+2$, the finiteness of the latter group follows from the injectivity of the Geisser-Levine cycle map ([GL, Corollary 2.1])

$$H^{2r-1}(\bar{X},\mathbb{Z}/l(r)) \hookrightarrow H^{2r-1}_{\acute{e}t}(\bar{X},\mathbb{Z}/l(r)) \tag{3.5}$$

and the finiteness of the étale cohomology group $H^{2r-1}_{\acute{e}t}(\bar{X},\mathbb{Z}/l(r))$ by [Mi80, Chapter VI, Corollary 2.8].

For the finiteness of $H^{2r-2}(Z,\mathbb{Z}/l(r))$ ($Z$ is a simple normal crossing divisor) consider the abstract blow-up

$$\begin{tikzcd} Z_1 \times (\bigcup_{i\neq 1} Z_i) \arrow[r] \arrow[d] & \bigcup_{i\neq 1} Z_i \arrow[d, "p"] \\ Z_1 \arrow[r, "\text{inc.}"'] & Z = \bigcup_{i=1,\cdots,r} Z_i, \end{tikzcd}$$

where $Z_i$'s are the irreducible components of $Z$. Now, $Z_1 \times (\bigcup_{i \neq 1} Z_i)$ is a simple normal crossing divisor (on $Z_1$) of dimension less than that of $Z$. Hence, by using the abstract blow-up sequence ([Ke, Proposition 5.5.4]) associated with this square, the induction on the dimension and on the number of irreducible components of $Z$ reduces the finiteness of $H^{2r-2}(Z, \mathbb{Z}/l(r))$ to the smooth proper case. $\square$

Let us deal with the zero cycle case.

**Theorem 3.4.7.** *If $X$ is a connected smooth scheme of dimension $d$, $H^{2d}_{c,alg}(X, \mathbb{Z}(d))$ has an algebraic representative.*

*Proof.* By proceeding as in the proof of Theorem 3.4.6, we need to show the finiteness of $H^{2d-1}_c(X, \mathbb{Z}/l(d))$. Since $l$ is prime to the characteristic of $k$, this group is isomorphic to the motivic homology group $H_1(X, \mathbb{Z}/l)$ by [Ke, Theorem 5.5.14] ([V00, Theorem 4.3.7(3)] under resolution of singularities). But $H_1(X, \mathbb{Z}/l)$ is finite because its dual is isomorphic to the finite group $H^1_{\acute{e}t}(X, \mathbb{Z}/l)$ ([MVW, Theorem 10.9]; cf. [SV96, Corollary 7.8] under resolution of singularities). $\square$

# Chapter 4

# Study in codimension one

The theory of algebraic representatives of smooth proper schemes in codimension one is part of the theory of Picard schemes (see Remark 3.1.2). We give a similar interpretation of $Alg^1_{c,X}$ for an arbitrary smooth scheme.

## 4.1 Motivic cohomology with compact supports as cdh hypercohomology

For our purpose, we need to interpret motivic cohomology with compact supports as sheaf hypercohomology. Let us begin with the definition of cdh cohomology with compact supports. $Sch/k$ is the category of schemes over $k$, and $D^-_{cdh}$ stands for the derived category $D^-(Sh_{cdh}(Sch/k))$ of the bounded above complex of cdh sheaves on $Sch/k$.

**Definition 4.1.1** ([FV, Section 3]). *For a bounded above complex $\mathcal{F}$ of cdh sheaves on $Sch/k$, the* **cdh cohomology with compact supports of** $X \in Sch/k$ **with coefficients in** $\mathcal{F}$ *is defined as*

$$H^m_c(X_{cdh}, \mathcal{F}) := Hom_{D^-_{cdh}}(\mathbb{Z}^c(X)_{cdh}, \mathcal{F}[m]),$$

*where $\mathbb{Z}^c(X)_{cdh}$ is the cdh sheafification of the presheaf that sends an irreducible scheme $U$ to the free abelian group $\mathbb{Z}^c(X)(U)$ generated by closed subschemes $Z$ of $U \times X$ such that the projection $Z \longrightarrow U$ is an open immersion.*

**Proposition 4.1.2.** *Under resolution of singularities, for any $X \in Sch/k$ and non-negative integers $m$ and $n$, there is an isomorphism*

$$H^m_c(X, \mathbb{Z}(n)) \xrightarrow{\cong} H^m_c(X_{cdh}, \mathbb{Z}(n)_{cdh}).$$

*Proof.* If $X$ is proper, this is [SV, Theorem 5.14].

For a non-proper $X$, choose a compactification $X \hookrightarrow \bar{X}$ with $Z := \bar{X} \setminus X$. Then the proposition follows from the exact sequence of cdh sheaves ([FV, Corollary 3.9]):

$$0 \longrightarrow \mathbb{Z}(Z)_{cdh} \longrightarrow \mathbb{Z}(\bar{X})_{cdh} \longrightarrow \mathbb{Z}^c(X)_{cdh} \longrightarrow 0, \tag{4.1}$$

where $\mathbb{Z}(S)$ is the presheaf of abelian groups freely generated by the presheaf of sets represented by $S$ in $Sch/k$. □

**Remark 4.1.3.** *Let $\mathcal{F}$ be a bounded above complex of cdh sheaves on $Sch/k$ and let $\mathcal{I}^\bullet$ be the total complex of a Cartan-Eilenberg resolution of $\mathcal{F}$ in $Sh_{cdh}(Sch/k)$. In view of the short exact sequence (4.1) in the proof of Proposition 4.1.2, we can express the cdh cohomology with compact supports more explicitly as:*

$$H^m_c(X_{cdh}, \mathcal{F}) \cong H^n(cone(\mathcal{I}^\bullet(\bar{X}) \longrightarrow \mathcal{I}^\bullet(Z))[-1]).$$

In this chapter, we are interested in the cohomology group $H^2_c(X, \mathbb{Z}(1))$ for a smooth scheme $X$. In order to study this group with Proposition 4.1.2, we would like to explicitly know what $\mathbb{Z}(1)_{cdh}$ is.

**Lemma 4.1.4.** *Suppose that $X \in Sch/k$ is a simple normal crossing divisor on some smooth scheme. Then, under resolution of singularities, the restriction $\mathbb{Z}(1)_{cdh,X}$ of $\mathbb{Z}(1)_{cdh}$ to the small Zariski site on $X$ is quasi-isomorphic to the Zariski sheaf $\mathbb{G}_{m,X}[-1]$ of units on $X$.*

*Proof.* There is a quasi-isomorphism $\mathbb{Z}(1) \xrightarrow{qis} \mathbb{G}_m[-1]$ of complexes of presheaves on $Sm/k$ ([MVW, Theorem 4.1]). Therefore, we need to show that the restriction of the cdh sheafification of $\mathbb{G}_m$ to the small Zariski site on $X$ agrees with $\mathbb{G}_{m,X}$, i.e. the canonical map $a : \mathbb{G}_{m,X} \longrightarrow \mathbb{G}_{m,cdh,X}$ on $X_{Zar}$ is an isomorphism. Note that, if $X$ is smooth, this follows from [MVW, Proposition 13.27] since $\mathbb{G}_m$ has the structure of a homotopy invariant Nisnevich sheaf with transfers.

For the injectivity, it is enough to show that for any affine open subscheme $U = \text{Spec } A$ of $X$, the map $a$ induces an injection $a_U : \mathbb{G}_{m,X}(U) \hookrightarrow \mathbb{G}_{m,cdh,X}(U)$. Let $U_j$'s be the irreducible components of $U$ corresponding to the minimal ideals $\mathfrak{p}_j$'s of $A$. Note that $\{U_j \longrightarrow U\}_j$ is a cdh cover by smooth schemes $U_j$ as $X$ is a strict normal crossing divisor. Now, consider the composition

$$A^* = \mathbb{G}_{m,X}(U) \xrightarrow{a_U} \mathbb{G}_{m,cdh,X}(U) \overset{res}{\hookrightarrow} \prod_j \mathbb{G}_{m,cdh,X}(U_j) = \prod (A/\mathfrak{p}_j)^*.$$

The last equality follows from the smooth case. Suppose that $s \in \mathbb{G}_{m,X}(U) = A^*$ is mapped to the unit under $a_U$. Then, the image of $s$ under the above composition of maps is also, of course, the unit 1. This means that $s - 1 \in \bigcap_j \mathfrak{p}_j = \sqrt{(0)}$. Since $U$ is reduced, we conclude that $s = 1$.

For the surjectivity, first note that $X$ is equidimensional, and the lemma is true if $d = 0$ or $r = 1$ (i.e. the case where $X$ is smooth). We prove the lemma by induction on the number $r$ of irreducible components of $X$ and the dimension $d$ of $X$.

Suppose now that the lemma holds for $r \leq r_0$ and for dimensions less than that of $X$. We prove the surjectivity of $a$ for a strict normal crossing divisor $X$ with $r_0 + 1$ irreducible components $X_0, X_1, \cdots, X_{r_0}$. Let us put $Y := X_1 \cup \cdots \cup X_{r_0}$ and consider the abstract blow-up with all arrows closed immersions

$$\begin{tikzcd} X_0 \cap Y \arrow[r, "i'"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ X_0 \arrow[r, "i"'] & X \end{tikzcd}$$

Put $f := p \circ i'$. There is a commutative diagram of Zariski sheaves on $X$ :

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{G}_{m,cdh,X} \arrow[r, "(p^\sharp, i^\sharp)"] & p_*\mathbb{G}_{m,cdh,Y} \oplus i_*\mathbb{G}_{m,cdh,X_0} \arrow[r, "\frac{p'^\sharp}{i'^\sharp}"] & f_*\mathbb{G}_{m,cdh,X_0\cap Y} \\ & \mathbb{G}_{m,X} \arrow[u, hook, "a"] \arrow[r, "(p^\sharp, i^\sharp)"] & p_*\mathbb{G}_{m,Y} \oplus i_*\mathbb{G}_{m,X_0} \arrow[u, equal, "\text{ind. hypo.}"'] \arrow[r, "\frac{p'^\sharp}{i'^\sharp}"] & f_*\mathbb{G}_{m,X_0\cap Y} \arrow[u, equal, "\text{smaller dim. case}"'] \end{tikzcd} \tag{4.2}$$

where the upper row is exact because the blow-up square is a cdh cover.

For the surjectivity of $a$, it suffices to show the exactness of the lower row in the diagram (4.2). We may do this at the stalks. Let $x \in X$ be a closed point of $X$ and $R := \mathcal{O}_{X,x}$ be the stalk of the structure sheaf at $x$. Let us only deal with the case where $x$ lies in $X_0 \cap Y$ because the other cases are simpler[1].

In this case, since $X$ is a simple normal crossing divisor, $\mathcal{O}_{X_0.x} = R/(f)$, where $f$ is the defining equation of $X_0$, and $\mathcal{O}_{Y_x} = R/(g)$ for $g := \prod_{i=1,\cdots s} g_i$ where $g_i$ is the defining equations of the irreducible components of $Y$ passing through $x$. We need to show that

$$R^* \longrightarrow (R/(f))^* \oplus (R/(g))^* \longrightarrow (R/(f,g))^*$$

is exact. Suppose that $(\bar{t}, \bar{t}') \in (R/(f))^* \oplus (R/(g))^*$ is mapped to the unit in $(R/(f,g))^*$. Since $R$ is a local ring, $\bar{t}$ and $\bar{t}'$ are respectively represented by units $t$ and $t'$ in $R$. Therefore, there exist elements $a$ and $b$ in $R$ such that $t/t' - 1 = af + bg$. Hence, we have $t - t'af = t' + t'bg =: t_0$. The element $t_0$ is invertible in the local ring $R$ because $t$ is invertible and $t'af$ belongs to the

[1] If $x \notin X_0 \cap Y$, then the map $(p^\sharp, i^\sharp) : \mathbb{G}_{m,X} \longrightarrow p_*\mathbb{G}_{m,Y} \oplus i_*\mathbb{G}_{m,X_0}$ becomes an isomorphism.

maximal ideal. Since $t_0 \in R^*$ is mapped to $(\bar{t}, \bar{t}')$ under the first arrow, the exactness follows. $\square$

## 4.2 Relative Picard groups

We need the following results on relative Picard groups.

**Definition 4.2.1** ([SV96, Section 2]). *For $X \in Sch/k$ and a closed subscheme $Z \overset{i}{\hookrightarrow} X$, the* **relative Picard group** *$Pic(X, Z)$ is the group consisting of isomorphism classes of pairs $(\mathcal{L}, u)$, where $\mathcal{L}$ is a line bundle on $X$ and $u$ is a trivialization $u : \mathcal{L}|_Z \xrightarrow{\cong} \mathcal{O}_Z$. The group structure is given by the tensor product. The pair $(\mathcal{L}, u)$ is called a line bundle on $(X, Z)$.*

**Lemma 4.2.2.** *Suppose that $i : Z \hookrightarrow X$ is a closed subscheme of a scheme $X$ over $k$. Then, there is a canonical isomorphism*

$$Pic(X, Z) \cong H^1_{Nis}(X, cone(\mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,Z})[-1]).$$

*Proof.* By [SV96, Lemma 2.1] we have canonical isomorphisms

$$Pic(X, Z) \cong H^1_{Zar}(X, cone(\mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,Z})[-1]) \xrightarrow{\cong} H^1_{\acute{e}t}(X, cone(\mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,Z})[-1]).$$

The above arrow, which is induced by the change of sites, factors as

$$\begin{tikzcd}
H^1_{Zar}(X, cone(\mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,Z})[-1]) \arrow[r, "\cong"] \arrow[d, "\text{change of sites}"'] & H^1_{\acute{e}t}(X, cone(\mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,Z})[-1]) \\
H^1_{Nis}(X, cone(\mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,Z})[-1]) \arrow[ur, "\text{change of sites}"'] &
\end{tikzcd}$$

Since the change of sites maps of a sheaf cohomology in degree one are injective, the lemma follows. $\square$

**Definition 4.2.3.** *Let $X$ and $Z$ as above. The* **relative Picard functor of the pair** $(X, Z)$ *is the functor*

$$Pic_{X,Z} : Sch/k \longrightarrow Ab$$

*that sends $T \in Sch/k$ to the relative Picard group $Pic(T \times X, T \times Z)$.*

On the representability of the relative Picard functor, the following result is known.

**Proposition 4.2.4** ([B-VS, Lemma 2.1 and Appendix]). *Let $X$ be a connected smooth proper scheme over $k$ and $Z \neq \emptyset$ be a simple normal crossing divisor on $X$. Then, the relative Picard functor $Pic_{X,Z}$ is representable by a group scheme locally of finite type over $k$.*

The group scheme representing the relative Picard functor $Pic_{X,Z}$ is denoted by the same symbol $Pic_{X,Z}$. The identity component $Pic^0_{X,Z}$ has the following structure.

**Proposition 4.2.5** ([B-VS, Proposition 2.2])**.** *Let $X$ be a connected smooth proper scheme over $k$ and $Z$ be a simple normal crossing divisor with irreducible components $Z_i$ on $X$. Then, there is an exact sequence of semi-abelian varieties over $k$*

$$0 \longrightarrow T_{X,Z} \longrightarrow Pic^0_{X,Z,red} \longrightarrow A_{X,Z} \longrightarrow 0,$$

*where $T_{X,Z}$ is the torus over $k$ representing the functor*

$$Sch/k \ni T \mapsto \mathrm{coker}\{\mathbb{G}_m(T \times X) \longrightarrow \mathbb{G}_m(T \times Z)\} \in Ab,$$

*and $A_{X,Z}$ is the abelian variety* $(\ker\{Pic^0_X \longrightarrow \bigoplus_i Pic^0_{Z_i}\})^0_{red}$.

## 4.3 Algebraic representatives in codimension one

By relative Nisnevich cohomology, we mean the following:

**Definition 4.3.1.** *Let $\mathcal{F}$ be a bounded above complex of Nisnevich sheaves on $Sch/k$. For a closed immersion $Z \hookrightarrow X$, the* **relative Nisnevich cohomology of the pair** $(X, Z)$ **with coefficients in** $\mathcal{F}$ *is defined as*

$$H^m_{Nis}(X, Z; \mathcal{F}) := H^m(cone(\mathcal{I}^\bullet(X) \longrightarrow \mathcal{I}^\bullet(Z))[-1]),$$

*where $\mathcal{I}^\bullet$ is the total complex of a Cartan-Eilenberg resolution of $\mathcal{F}$ in $Sh_{Nis}(Sch/k)$.*

Unlike the cdh case, this does not only depend on $X \setminus Z$. It is clear from the definition that there is a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^m_{Nis}(X, \mathcal{F}) \longrightarrow H^m_{Nis}(Z, \mathcal{F}) \longrightarrow H^{m+1}_{Nis}(X, Z; \mathcal{F}) \longrightarrow H^{m+1}_{Nis}(X, \mathcal{F}) \longrightarrow \cdots .$$

We shall interpret relative Picard groups in terms of relative Nisnevich cohomology.

**Proposition 4.3.2.** *Let $i : Z \hookrightarrow X$ be a closed subscheme of $X$ over $k$. Then, there is a canonical isomorphism*

$$Pic(X, Z) \cong H^1_{Nis}(X, Z; \mathbb{G}_m).$$

*Proof.* Let $\mathbb{G}_m \longrightarrow \mathcal{I}^\bullet$ be an injective resolution in $Sh_{Nis}(Sch/k)$. Then, its restriction $\mathbb{G}_{m,X} \longrightarrow \mathcal{I}^\bullet_X$ to the small Nisnevich site on $X$ is also an injective resolution. Similarly, so is $\mathbb{G}_{m,Z} \longrightarrow \mathcal{I}^\bullet_Z$.

Since $i : Z \hookrightarrow X$ is a closed immersion, $i_* : Sh(Z_{Nis}) \longrightarrow Sh(X_{Nis})$ is exact and preserves injectives (as its left adjoint $i^*$ is exact). Hence, $i_*\mathbb{G}_{m,Z} \longrightarrow i_*\mathcal{I}^\bullet_Z$ is an injective resolution on $X_{Nis}$. Therefore,

$$\begin{aligned} Pic(X,Z) &\cong H^1_{Nis}(X, cone(\mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,Z})[-1]) \ \ \text{(by Lemma 4.2.2)} \\ &\cong H^1_{Nis}(X, cone(\mathcal{I}^\bullet_X \longrightarrow i_*\mathcal{I}^\bullet_Z)[-1]) \\ &= H^1(cone(\mathcal{I}^\bullet(X) \longrightarrow \mathcal{I}^\bullet(Z))[-1]) \\ &= H^1_{Nis}(X, Z; \mathbb{G}_m). \end{aligned}$$

□

Suppose that $\bar{X}$ is a smooth proper scheme and $Z$ is a simple normal crossing divisor on $\bar{X}$. We shall give a motivic interpretation (on $Sm/k$) of the relative Picard functor of the pair $(\bar{X}, Z)$. We start with a lemma.

**Lemma 4.3.3.** *Under resolution of singularities, for any schemes $X$ and $T$, there is a canonical isomorphism natural in $T$*

$$Hom_{DM}(M(T)\otimes M^c(X), \mathbb{Z}(n)[m]) \xrightarrow{\cong} Hom_{D^-(Sh_{cdh}(Sch/k))}(\mathbb{Z}(T)_{cdh}\otimes\mathbb{Z}^c(X)_{cdh}, \mathbb{Z}(n)_{cdh}[m]).$$

*Proof.* Let us write as $(Sm/k)_t$ the restriction of the cdh topology to $Sm/k$, and recall that $Sh_t(Sm/k)$ is equivalent to $Sh_{cdh}(Sch/k)$ ([FV, Proof of Lemma 3.6]). The reason is that any scheme has a smooth cdh cover by resolution of singularities.

For any scheme $S \in Sch/k$, there is a composition $f^{m,n}_{T,S}$ of canonical maps

$$\begin{aligned} Hom_{DM}(M(T) \otimes M^c(S), \mathbb{Z}(n)[m]) &\overset{\text{def}}{=} Hom_{DM}(\mathbb{Z}_{tr}(T) \otimes \mathbb{Z}^c_{tr}(S), \mathbb{Z}(n)[m]) \\ &\overset{(a)}{=} Hom_{D^-(Sh^{tr}_{Nis}(Sm/k))}(\mathbb{Z}_{tr}(T) \otimes z_{equi}(S,0), \mathbb{Z}(n)[m]) \\ &\overset{(b)}{\to} Hom_{D^-(Sh_{Nis}(Sm/k))}(\mathbb{Z}(T) \otimes \mathbb{Z}^c(S), \mathbb{Z}(n)[m]) \\ &\overset{(c)}{\to} Hom_{D^-(Sh_t(Sm/k))}(\mathbb{Z}(T)_t \otimes \mathbb{Z}^c(S)_t, \mathbb{Z}(n)_t[m]) \\ &\overset{(d)}{\cong} Hom_{D^-(Sh_{cdh}(Sch/k))}(\mathbb{Z}(T)_{cdh} \otimes \mathbb{Z}^c(S)_{cdh}, \mathbb{Z}(n)_{cdh}[m]). \end{aligned}$$

Here, (a) is an equality because $\mathbb{Z}(n)$ is an $\mathbb{A}^1$-local object by Corollary 2.3.4. (b) is induced by the inclusions $\mathbb{Z}(T) \hookrightarrow \mathbb{Z}_{tr}(T)$ and $\mathbb{Z}^c(S) \hookrightarrow z_{equi}(S,0)$. (c) is induced by the $t$-sheafification, and (d) is due to the equivalence of categories between $Sh_t(Sm/k)$ and $Sh_{cdh}(Sch/k)$. By the construction, $f^{m,n}_{T,S}$ is functorial in $T$ with respect to pushforwards along all morphisms and functorial in $S$ with respect to pushforwards along proper morphisms and pullbacks along flat morphisms. Therefore, the localization triangles in $DM$ ([MVW, Theorem 16.15]) and in

$D^-(Sh_{cdh}(Sch/k))$ ([FV, Corollary 3.9]) associated with a good compactification $X \hookrightarrow \bar{X}$ of $X$ with the boundary divisor $Z$ give rise to the commutative diagram (Note that $DM$ is a tensor triangulated category ([MVW, p.110]), and $-\otimes \mathbb{Z}(T)_{cdh}$ is exact in $Sh_{cdh}(Sch/k)$ because $\mathbb{Z}(T)$ is a presheaf of free abelian groups and sheafification is exact.)

$$\begin{tikzcd}
\arrow[r] & H^{m-1}(T\times \bar{X}, \mathbb{Z}(n)) \arrow[r] \arrow[d, "f^{m-1,n}_{T,\bar{X}}"', "\cong"] & H^{m-1}(T\times Z, \mathbb{Z}(n)) \arrow[r] \arrow[d, "f^{m-1,n}_{T,Z}"', "\cong"] & Hom_{DM}(M(T)\otimes M^c(X), \mathbb{Z}(n)[m]) \arrow[d, "f^{m,n}_{T,X}"'] \\
\arrow[r] & H^{m-1}_{cdh}(T\times \bar{X}, \widetilde{\mathbb{Z}(n)}) \arrow[r] & H^{m-1}_{cdh}(T\times Z, \widetilde{\mathbb{Z}(n)}) \arrow[r] & Hom_{D^-_{cdh}}(\widetilde{\mathbb{Z}(T)}\otimes \widetilde{\mathbb{Z}^c(X)}, \widetilde{\mathbb{Z}(n)}[m])
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & H^{m}(T\times \bar{X}, \mathbb{Z}(n)) \arrow[r] \arrow[d, "f^{m,n}_{T,\bar{X}}"', "\cong"] & H^{m}(T\times Z, \mathbb{Z}(n)) \arrow[r] \arrow[d, "f^{m,n}_{T,Z}"', "\cong"] & {} \\
\arrow[r] & H^{m}_{cdh}(T\times \bar{X}, \widetilde{\mathbb{Z}(n)}) \arrow[r] & H^{m}_{cdh}(T\times Z, \widetilde{\mathbb{Z}(n)}) \arrow[r] & {}
\end{tikzcd}$$

where " $\sim$ " stands for the cdh sheafification. The four arrows between the cohomology groups are isomorphisms by [MVW, Theorem 14.20], so the middle map is also an isomorphism. □

**Proposition 4.3.4.** *Let $X$ and $T$ be smooth schemes over $k$ and let $\bar{X}$ be a good compactification of $X$ with the boundary divisor $Z$. Under resolution of singularities, there is a natural isomorphism in $T \in Sm/k$*

$$F : Pic(T\times \bar{X}, T\times Z) \xrightarrow{\cong} Hom_{DM}(M(T)\otimes M^c(X), \mathbb{Z}(1)[2])$$

*such that $F$ is compatible with the change of sites maps in the sense that the following diagram is commutative:*

$$\begin{tikzcd}
H^0_{Nis}(T\times Z, \mathbb{G}_m) \arrow[r] \arrow[d, "\cong"', "change\ of\ sites"] & Pic(T\times \bar{X}, T\times Z) \arrow[r] \arrow[d, "F"'] & H^1_{Nis}(T\times \bar{X}, \mathbb{G}_m) \arrow[d, "\cong"', "change\ of\ sites"] \\
H^1_{cdh}(T\times Z, \mathbb{Z}(1)_{cdh}) \arrow[r] & Hom_{DM}(M(T)\otimes M^c(X), \mathbb{Z}(1)[2]) \arrow[r] & H^2_{cdh}(T\times \bar{X}, \mathbb{Z}(1)_{cdh}).
\end{tikzcd}$$

*Proof.* Let $\mathbb{G}_{m,cdh} \longrightarrow \mathcal{I}^\bullet$ be an injective resolution in $Sh_{cdh}(Sch/k)$ and $\mathbb{G}_m \longrightarrow \mathcal{J}^\bullet$ in $Sh_{Nis}(Sch/k)$. Since $\mathcal{I}^\bullet$ is still a complex of injective sheaves when restricted to the Nisnevich site, there is an augmentation-preserving chain map (unique up to chain homotopy) $f : \mathcal{J}^\bullet \longrightarrow \mathcal{I}^\bullet$ of complexe of Nisnevich sheaves on $Sch/k$.

The short exact sequence

$$0 \longrightarrow \mathbb{Z}(T)_{cdh}\otimes \mathbb{Z}(Z)_{cdh} \longrightarrow \mathbb{Z}(T)_{cdh}\otimes \mathbb{Z}(\bar{X})_{cdh} \longrightarrow \mathbb{Z}(T)_{cdh}\otimes \mathbb{Z}^c(X)_{cdh} \longrightarrow 0$$

in $Sh_{cdh}(Sch/k)$ induces the top horizontal sequence which is exact in each degree in the following commutative diagram of complexes of presheaves in $T \in Sm/k$ with values in $Ab$ :

$$\begin{tikzcd}
0 \arrow[r] & Hom_{Sh_{cdh}}(\widetilde{\mathbb{Z}(T)} \otimes \widetilde{\mathbb{Z}^c(X)}, \mathcal{I}^\bullet) \arrow[r] & Hom_{Sh_{cdh}}(\widetilde{\mathbb{Z}(T \times \bar{X})}, \mathcal{I}^\bullet) \arrow[r] & Hom_{Sh_{cdh}}(\widetilde{\mathbb{Z}(T \times Z)}, \mathcal{I}^\bullet) \arrow[r] & 0 \\
& & Hom_{Sh_{Nis}}(\mathbb{Z}(T \times \bar{X}), \mathcal{I}^\bullet) \arrow[r] \arrow[u, "\text{adjunction}"', "\cong"] & Hom_{Sh_{Nis}}(\mathbb{Z}(T \times Z), \mathcal{I}^\bullet) \arrow[u, "\text{adjunction}"', "\cong"] & \\
& cone(h)[-1] \arrow[r] \arrow[uu, dotted, "\exists\ k \text{ in } D^-(PSh(Sm/k))"] & Hom_{Sh_{Nis}}(\mathbb{Z}(T \times \bar{X}), \mathcal{J}^\bullet) \arrow[r, "h"'] \arrow[u, "f\circ -"] & Hom_{Sh_{Nis}}(\mathbb{Z}(T \times Z), \mathcal{J}^\bullet) \arrow[u, "f\circ -"] &
\end{tikzcd}$$

Here, " $\sim$ " stands for the cdh sheafification. The dotted arrow $k$ exists in the derived category $D^-(PSh(Sm/k))$ of presheaves on $Sm/k$. Now, taking cohomology groups, we obtain the following commutative diagram natural[2] in $T \in Sm/k$ :

$$\begin{tikzcd}
H^0_{Nis}(T \times \bar{X}, \mathbb{G}_m) \arrow[r] \arrow[d, "\cong"', "a"] & H^0_{Nis}(T \times Z, \mathbb{G}_m) \arrow[r] \arrow[d, "\cong"', "b"] & H^1_{Nis}(T \times \bar{X}, T \times Z; \mathbb{G}_m) \arrow[d, "k'\text{, induced by } k"] \\
H^0_{cdh}(T \times \bar{X}, \widetilde{\mathbb{G}_m}) \arrow[r] & H^0_{cdh}(T \times Z, \widetilde{\mathbb{G}_m}) \arrow[r] & Hom_{D^-_{cdh}}(\widetilde{\mathbb{Z}(T)} \otimes \widetilde{\mathbb{Z}^c(X)}, \widetilde{\mathbb{G}_m}[1])
\end{tikzcd} \tag{4.3}$$

$$\begin{tikzcd}
{} \arrow[r] & H^1_{Nis}(T \times \bar{X}, \mathbb{G}_m) \arrow[r] \arrow[d, "\cong"', "c"] & H^1_{Nis}(T \times Z, \mathbb{G}_m) \arrow[r] \arrow[d, "d"] & {} \\
{} \arrow[r] & H^1_{cdh}(T \times \bar{X}, \widetilde{\mathbb{G}_m}) \arrow[r] & H^1_{cdh}(T \times Z, \widetilde{\mathbb{G}_m}) \arrow[r] & {}
\end{tikzcd}$$

where all solid vertical arrows induced by the change of sites. The maps $a$ and $c$ are isomorphisms by [MVW, Theorem 14.20] and $b$ is an isomorphism by Lemma 4.1.4.

In view of Proposition 4.3.2 and Lemma 4.3.3, it remains to show that the map $k'$ is an isomorphism. For this, we claim that $d$ is injective.

It is enough to prove that the composition

$$f : H^1_{Zar}(T \times Z, \mathbb{G}_m) \overset{\text{change of sites}}{\hookrightarrow} H^1_{Nis}(T \times Z, \mathbb{G}_m) \xrightarrow{d} H^1_{cdh}(T \times Z, \widetilde{\mathbb{G}_m})$$

is injective because the first injective map is an isomorphism by Hilbert's Satz 90 ([Mi80, Chapter III, Proposition 4.9]). By the construction of $d$, $f$ factors through $H^1_{Zar}(T \times Z, \widetilde{\mathbb{G}_m})$ as

$$\begin{tikzcd}
H^1_{Zar}(T \times Z, \mathbb{G}_m) \arrow[r, "f"] \arrow[d, "i"', "\cong"] & H^1_{cdh}(T \times X, \widetilde{\mathbb{G}_m}) \\
H^1_{Zar}(T \times Z, \widetilde{\mathbb{G}_m}) \arrow[ur, hook, "g\text{, change of sites}"'] &
\end{tikzcd}$$

where $i$ is an isomorphism by Lemma 4.1.4 and $g$ is injective because it is a change of sites in

[2]The naturality in $T$ follows because the map $k$ was constructed in the derived category of presheaves.

degree one. Therefore, $d$ is injective. $\square$

We are now ready to compare the algebraic representative with compact supports of a smooth scheme with relative Picard variety.

**Proposition 4.3.5.** *Assume resolution of singularities. For any connected smooth scheme $X$ over $k$ with a good compactification $\bar{X}$ with the boundary divisor $Z$, the canonical homomorphism*

$$H^2_{c,alg}(X,\mathbb{Z}(1)) \overset{i:=inc}{\hookrightarrow} H^2_c(X,\mathbb{Z}(1)) \xrightarrow{g} Pic(\bar{X},Z) \overset{\psi}{\cong} Pic_{\bar{X},Z}(k)$$

*($g$ is the inverse of $F$ in Proposition 4.3.4 evaluated at $T = \mathrm{Spec}\ k$, and $\psi$ is as in Proposition 4.2.4) factors through $Pic^0_{\bar{X},Z,red}(k)$ as*

$$\begin{tikzcd} H^2_{c,alg}(X,\mathbb{Z}(1)) \arrow[rr, "\psi\circ g\circ i"] \arrow[dr, "\phi_0"'] & & Pic_{\bar{X},Z}(k) \\ & Pic^0_{\bar{X},Z,red}(k) \arrow[ur] & \end{tikzcd}$$

*and the homomorphism $\phi_0$ is regular.*

*Proof.* It is enough to show that for any smooth connected scheme $T$ over $k$ pointed at $t_0 \in T(k)$ and $Y \in Hom_{DM}(M(T)\otimes M^c(X),\mathbb{Z}(1)[2])$, the composition

$$T(k) \xrightarrow{w_Y} H^2_{c,alg}(X,\mathbb{Z}(1)) \xrightarrow{\psi\circ g\circ i} Pic_{\bar{X},Z}(k)$$

is induced by a scheme morphism, where $w_Y$ as in Definition 3.3.1. Indeed, it is because the image of $T$ is connected and contains the identity and $Pic^0_{\bar{X},Z,red}$ is a semi-abelian variety by Proposition 4.2.5.

Observe that there is a commutative diagram

$$\begin{tikzcd} T(k) \arrow[r, "w_Y"] \arrow[dd, equal] & H^2_{c,alg}(X,\mathbb{Z}(1)) \arrow[d, hook, "i"] & \\ & H^2_c(X,\mathbb{Z}(1)) \arrow[d, "g"] & \\ T(k) \arrow[r, "B_{F^{-1}(Y)}"] & Pic(\bar{X},Z) \arrow[r, "\psi"] & Pic_{\bar{X},Z}(k). \end{tikzcd}$$

Here, $F$ is the map defined in Proposition 4.3.4. Let $(\mathcal{L}, u : \mathcal{L}|_{T\times Z} \cong \mathcal{O}_{T\times Z})$ be the line bundle on the pair $(T\times\bar{X}, T\times Z)$ that represents $F^{-1}(Y)$. $B_{F^{-1}(Y)}$ is defined as the map that sends $t \in T(k)$ to $(\mathcal{L}_t \otimes \check{\mathcal{L}}_{t_0}, u_t \otimes \check{u}_{t_0})$, where $\mathcal{L}_t$ means the pullback of the line bundle along $t : \mathrm{Spec}\ k \longrightarrow T$, $u_t$ is the restriction of $u$ to $\{t\}\times X$, and $\check{-}$ signifies dual invertible sheaves.

The commutativity of the diagram follows from the naturality of $F$.

Now, let $(\mathcal{P}, p)$ be the Poincaré bundle, which is by definition the line bundle $(\mathcal{P}, p)$ representing the class in $Pic(Pic_{\bar{X},Z}\times\bar{X}, Pic_{\bar{X},Z}\times Z)$ corresponding to the identity in $Hom_{Sch/k}(Pic_{\bar{X},Z}, Pic_{\bar{X},Z})$. The representability of the relative Picard functor means that $F^{-1}(Y)$ is the pullback of $(\mathcal{P}, p)$ along some scheme morphism $h : T \longrightarrow Pic_{\bar{X},Z}$. Hence, we are given with the commutative diagram

$$\begin{tikzcd}
T(k) \arrow[r, "B_{F^{-1}(Y)}"] \arrow[dr, "h"'] & Pic(\bar{X}, Z) \arrow[r, "\psi"] & Pic_{\bar{X},Z}(k) \\
& Pic_{\bar{X},Z}(k) \arrow[u, "W_{(\mathcal{P},p)}"'] \arrow[ur, "id-h(t_0)"'] &
\end{tikzcd} \tag{4.4}$$

where $Pic_{\bar{X},Z}$ is considered pointed at $h(t_0)$ in defining $B_{(\mathcal{P},p)}$.

Since $\psi \circ B_{F^{-1}(Y)} = (id - h(t_0)) \circ h$, we conclude that $\psi \circ B_{F^{-1}(Y)}$ is induced by a scheme morphism. $\square$

**Theorem 4.3.6.** *Assume resolution of singularities. If $X$ is a smooth connected scheme over $k$ with a good compactification $\bar{X}$ with the boundary divisor $Z$, then the regular homomorphism in Proposition 4.3.5*

$$\phi_0 : H^2_{c,alg}(X, \mathbb{Z}(1)) \longrightarrow Pic^0_{\bar{X},Z,red}(k)$$

*is an isomorphism. In particular, it is the algebraic representative with compact supports in codimension one.*

*Proof.* The injectivity of $\phi_0$ follows because $\psi \circ g : H^2_c(X, \mathbb{Z}(1)) \longrightarrow Pic_{\bar{X},Z}(k)$ is an isomorphism. For the surjectivity, observe that

$$\mathrm{im}(g \circ i) = \bigcup_{\substack{T,\ \text{smooth} \\ \text{connected}}} \mathrm{im}\{H_0(T, \mathbb{Z})^0 \times Pic(T \times \bar{X}, T \times Z) \stackrel{\text{pullback}}{\longrightarrow} Pic(\bar{X}, Z)\} =: Pic_{alg}(\bar{X}, Z).$$

Thus, it remains to show that the elements of $Pic_{alg}(\bar{X}, Z)$ correspond to the rational points on $Pic^0_{\bar{X},Z,red}$. If $Z = \emptyset$, it is the classical smooth proper case. See [Kl, Proposition 9.5.10].

The case $Z \neq \emptyset$ is simpler because the relative Picard functor is already an fppf sheaf without taking the sheafification. We include the proof for the convenience of the reader. Let us consider $(\mathcal{L}, u) \in Pic(\bar{X}, Z)$ and the corresponding rational point $\alpha \in Pic_{\bar{X},Z}(k)$.

Suppose that $(\mathcal{L}, u)$ belongs to $Pic_{alg}(\bar{X}, Z)$. Then, there are a smooth connected scheme $T$, rational points $t_0, t_1 \in T(k)$ and $(\mathcal{M}, v) \in Pic(T\times\bar{X}, T\times Z)$ (cf. the proof of Proposition 3.2.2) such that

$$(\mathcal{L}, u) = (\mathcal{M}_{t_1}, v_{t_1}) - (\mathcal{M}_{t_0}, v_{t_0}).$$

Now, $(\mathcal{M}, v)$ defines a morphism $\tau : T \longrightarrow Pic_{\bar{X},Z}$, and we have $\alpha = \tau(t_1) - \tau(t_0)$. Since $\tau$ is continuous, $T$ is connected, and the image of the map $\sigma := \tau(-) - \tau(t_0) : T(k) \longrightarrow Pic_{\bar{X},Z}(k)$ contains the identity, the image of $\sigma$ is contained in the identity component of $Pic_{\bar{X}.Z}$, i.e. $\sigma(T(k)) \subset Pic^0_{\bar{X},Z}(k)$. In particular,

$$\alpha = \tau(t_1) - \tau(t_0) \overset{\text{def}}{=} \sigma(t_1) \in Pic^0_{\bar{X},Z}(k).$$

Conversely, suppose that $\alpha \in Pic^0_{\bar{X},Z}(k)$. The inclusion $Pic^0_{\bar{X},Z} \hookrightarrow Pic_{\bar{X},Z}$ corresponds to an element $(\mathcal{N}, w) \in Pic(Pic^0_{\bar{X},Z} \times \bar{X}, Pic^0_{\bar{X},Z} \times Z)$. Then, $(\mathcal{N}_\alpha, w_\alpha) = (\mathcal{L}, u)$ and $(\mathcal{N}_0, w_0) = (\mathcal{O}_{\bar{X}}, id_{\mathcal{O}_Z})$ (The subscript 0 signifies the identity of $Pic^0_{\bar{X},Z}$) because for any rational point $p \in Pic^0_{\bar{X},Z}(k)$, the diagram

$$\begin{tikzcd}
\mathcal{N} \in Pic(Pic^0_{\bar{X},Z} \times \bar{X}, Pic^0_{\bar{X},Z} \times Z) \arrow[r, phantom, "\cong"] \arrow[d, "(-)_t"'] & Hom_{Sch/k}(Pic^0_{\bar{X},Z}, Pic_{\bar{X},Z}) \ni inc \arrow[d, "-\circ p"] \\
\mathcal{N}_t \in Pic(\text{Spec } k \times \bar{X}, \text{Spec } k \times Z) \arrow[r, phantom, "\cong"] & Hom_{Sch/k}(\text{Spec } k, Pic_{\bar{X},Z}) \ni t
\end{tikzcd}$$

commutes. Therefore, we have

$$(\mathcal{L}, u) = (\mathcal{L}, u) - (\mathcal{O}_{\bar{X}}, id_{\mathcal{O}_Z}) = (\mathcal{N}_\alpha, w_\alpha) - (\mathcal{N}_0, w_0) \in Pic_{alg}(\bar{X}, Z).$$

The last assertion of the theorem follows from the first part because there is a commutative diagram:

$$\begin{tikzcd}
H^2_{c,alg}(X, \mathbb{Z}(1)) \arrow[r, two heads, "\Phi^1_{c,X}"] \arrow[dr, "\phi_0", "\text{isom.}"'] & Alg^1_{c,X}(k) \arrow[d, dotted, "\exists! \text{ by universality}"] \\
& Pic^0_{\bar{X},Z,red}(k)
\end{tikzcd}$$

$\square$

# Bibliography


[B-VS] Barbieri-Viale, L., Srinivas, V., Albanese and Picard 1-motives., Mémoires de la SMF 87 (2001), vi-104.

[Bl] Bloch, S., Torsion algebraic cycles and a theorem of Roitman. Compositio Math. 39 (1979), no. 1, 107-127.

[Bo] Borel, A., *Linear algebraic groups*, 2nd enl. ed., Graduate Texts in Mathematics, 126, Springer, 1991.

[BLR] Bosch, S., Lütkebohmert, W., Raynaud, M., *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, Springer-Verlag, Berlin, 1990.

[FV] Friedlander, E., Voevodsky, V., Bivariant cycle cohomology, in: *Cycles, transfers, and motivic homology theories,* Ann. of Math. Stud. 143, Princeton Univ. Press, Princeton, NJ, 2000, 138-187.

[GL] Geisser, T., Levine, M., The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky. J. Reine Angew. Math. 530 (2001), 55-103.

[Ha] Hartshorne, R., Equivalence relations on algebraic cycles and subvarieties of small codimension. Proceedings of Symposia in Pure Mathematics 29 (1975), 129-164.

[Hi] Hironaka, H., Resolution of singularities of an abstract algebraic variety over a field of characteristic zero: I, II. Ann. of Math. 79 (1964), 109-203; ibid. 79 (1964), 205-326.

[Ke] Kelly, S., Triangulated categories of motives in positive characteristic, Ph.D. thesis, Université Paris 13, 2012, Available at: `http://arxiv.org/abs/1305.5349.`

[Kl] Kleiman, S. L., The Picard scheme, in: *Fundamental algebraic geometry. Grothendieck's FGA explained,* Mathematical Surveys and Monographs 123, Amer. Math. Soc. 2005, 235-321.

[La] Lang, S., *Abelian varieties*, Interscience Tracts in Pure and Applied Mathematics. 7, Interscience Pubilshers, INC., New York, 1959.

[Ma] Matsumura, H., *Commutative ring theory*, Cambridge Studies in Advanced Mathematics. 8, Cambridge University Press, Cambridge, 1986.

[MVW] Mazza, C., Voevodsky, V., Weibel, C., *Lectures on motivic cohomology*, Clay Mathematics Monographs. 2, American Mathematical Society, Providence, RI, 2006.

[Mi80] Milne, J. S., *Étale cohomology*, Princeton Mathematical Series. 33, Princeton University Press, Princeton, New Jersey, 1980.

[Mi82] Milne, J. S., Zero cycles on algebraic varieties in nonzezro characteristic: Rojtman's theorem. Compositio Math. 47 (1982), no. 3, 271-287.

[Mi86b] Milne, J. S., Jacobian varieties, in: *Arithmetic geometry (Storrs, Conn.,1984),* Springer, New York, 1986, 167-212.

[Mum] Mumford, D., *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics. 5, Oxford University Press, London, 1974.

[Mur] Murre, J. P., Application of algebraic $K$-theory to the theory of algebraic cycles, in: *Proc. Algebraic Geometry Sitges (Barcelona) 1983,* Lecture Notes in Math. 1124, Springer-Verlag, Berlin 1985, 216-261.

[R] Rojtman, A. A., The torsion of the group of 0-cycles modulo rational equivalence. Ann. of Math. (2) 111 (1980), no. 3, 553-569.

[Sam] Samuel, P., Relations d'equivalence en géométrie algébrique. Proc. Internat. Congress Math. 1958, 470-487.

[Sai] Saito, H., Abelian varieties attached to cycles of intermediate dimension. Nagoya Math. J. 75 (1979), 95-119.

[Se] Serre, J.-P., Morphismes universels et variété d'Albanese. Séminaire Claude Schevalley, tome 4 (1958-1959), exp. no. 10, 1-22.

[SV96] Suslin, A., Voevodsky, V., Singular homology of abstract algebraic varieties. Invent. Math. 123 (1996), 61-94.

[SV] Suslin, A., Voevodsky, V., Bloch-Kato conjecture and motivic cohomology with finite coefficients. Available at: `http://www.math.uiuc.edu/K-theory/0341/`.

[V00] Voevodsky, V., Triangulated categories of motives over a field, in: *Cycles, transfers, and motivic homology theories,* Ann. of Math. Stud. 143, Princeton Univ. Press, Princeton, NJ, 2000, 188-238.

[V10] Voevodsky, V., Cancellation theorem. Documenta Math. Extra Volume: Andrei A. Suslin's Sixtieth birthday, (2010), 671-685.

[Wi] Wickelgren, K., 2-nilpotent real section conjecture. Math. Ann. 358 (2014), no. 1, 361-387.