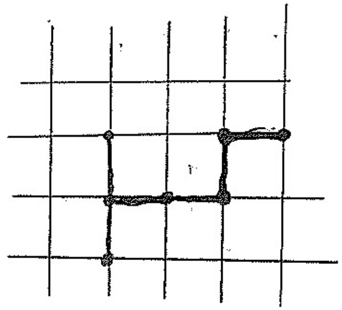
§0 Introduction



Percolation [Broadbent & Hammerseley (1957)]

Suppose that each edge of $\mathbb{Z}^2$ are $\begin{cases} \text{open with probability } p \\ \text{closed} \quad '' \quad - \quad 1-p \end{cases}$

(segment between adjacent vertices)

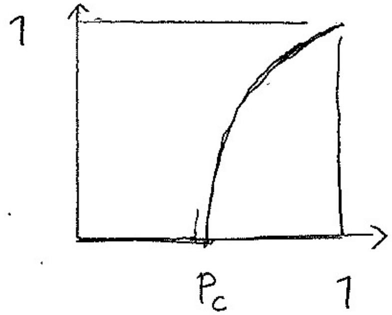
Question 1 Is the following probability positive?

$$\theta(p) \stackrel{\text{def}}{=} P\left(\begin{array}{l} \text{the origin is contained in an unbounded} \\ \underline{\text{connected}} \text{ set of open edges} \end{array} \right)$$

(obvious : $\theta(0) = 0$, $\theta(1) = 1$. Not obvious for $0 < p < 1$)

<u>Answer 1</u> $\exists\, p_c \in (0,1)$ called critical probability

s.t. $\theta(p) \begin{cases} = 0 & \text{if } p \leq p_c \\ > 0 & \text{if } p > p_c \end{cases}$



$\mapsto$ The situation changes abruptly at

$p = p_c$ (an example of "critical phenomena")

$\downarrow$

other example : The water $\begin{cases} \text{freezes at } 0^\circ C \\ \text{boils } 〃 \ 100^\circ C \end{cases}$

Question 2 How is the following probability related to $\theta(p)$?

$$\varphi(p) \stackrel{\text{def}}{=} P\left(\begin{array}{l} \exists \text{ unbounded connected set} \\ \quad \text{open edges} \end{array} \right)$$

If $\varphi(p) > 0$, then, how many is the # of connected components?

Answer 2

$$\varphi(p) = \begin{cases} 0 \iff \theta(p) = 0 \\ 1 \iff \theta(p) > 0 \end{cases}$$

$$\theta(p) > 0 \Rightarrow P\left(\begin{array}{l} \exists 1 \text{ unbounded connected} \\ \quad \text{set of open edges} \end{array} \right) = 1$$

Rem We solve these question with the help of "ergodic theory"

Plan for the course

§1 The percolation prob. $\theta(p)$

§2 Ergodic theory in the context of percolation

§3 The uniqueness of the infinite cluster