§1 The percolation probability $\theta(p)$

Lattice

▷ $\mathbb{Z}^d = \{ x = (x_1, \ldots, x_d) \; ; \; x_j \in \mathbb{Z} \}$

For $x, y \in \mathbb{Z}^d$

▷ x and y are <u>adjacent</u> [ədʒéisnt] (denoted by $x \sim y$)

$\overset{def}{\Longleftrightarrow}$ $\quad \|x-y\| \overset{def}{=} \sum_{j=1}^{d} |x_j - y_j| = 1 \quad \left(\exists 1\, i, \underset{s.t.}{\wedge} \; |x_j - y_j| = \begin{cases} 1 & (j = i) \\ 0 & (j \neq i) \end{cases} \right)$

▷ a segment $[x, y] \overset{def}{=} \{ tx + (1-t)y \, , \; 0 \le t \le 1 \}$ is called a <u>bond</u> or <u>edge</u>

▷ $\mathbb{B}$ = the set of all bonds in $\mathbb{Z}^d$

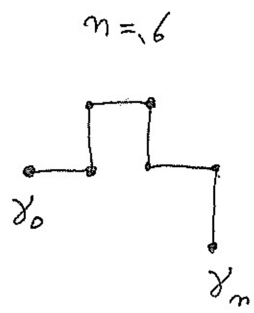
▷ $\# A \overset{def}{=} \sum_{x \in A} 1$ for any countable set A (e.g. $A \subset \mathbb{Z}^d$, $A \subset \mathbb{B}$)

▷ $\gamma = \{b_1, \ldots, b_n\} \subset \mathbb{B}$ is a path with the length n ($|\gamma| \overset{\text{def}}{=} n$)

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists \gamma_0, \ldots, \exists \gamma_n \in \mathbb{Z}^d$ s.t.

1) $b_j = [\gamma_{j-1}, \gamma_j]$ $(j = 1, \ldots, n)$

2) $\gamma_j \neq \gamma_k$ if $1 \leq k - j \leq n-1$ ($\gamma_0 = \gamma_n$ is allowed)



▷ $\Gamma_{x,n}$ = the set of all path γ s.t. $\gamma_0 = x$, $|\gamma| = n$

▷ $\Gamma_x = \bigcup_{n \geq 1} \Gamma_{x,n}$ = the set of all path γ s.t. $\gamma_0 = x$

Exer 1.1 $\# \Gamma_{x,n} \leq 2d\,(2d-1)^{n-1}$

▷ The following system $(\Omega, \mathcal{F}, P ; \{X_b\}_{b \in \mathbb{B}})$ is called the **percolation**

$$\begin{cases} (\Omega, \mathcal{F}, P) : \text{probability space} \quad (\text{meas. sp. s.t. } P(\Omega) = 1) \\[1em] X_b : \Omega \to \{0,1\} \quad (b \in \mathbb{B}) \quad \underbrace{\text{independent}}_{(*)} \quad \underbrace{\text{random variables.}}_{\substack{\| \\ \text{measurable functions.}}} \\[1em] \text{s.t.} \quad P(X_b = 1) = p \quad (\forall b \in \mathbb{B}) \end{cases}$$

$$\left((*) \overset{\text{def}}{\iff} \quad \forall n \in \mathbb{N}, \ \forall b_j \in \mathbb{B}, \ \forall \varepsilon_j \in \{0,1\} \quad (j = 1, \cdots n) \right.$$

$$\left. \qquad P\left(\bigcap_{j=1}^{n} \{X_{b_j} = \varepsilon_j\} \right) = \prod_{j=1}^{n} P(X_{b_j} = \varepsilon_j) \right.$$

▷ For $\omega \in \Omega$, a set $B \subset \mathbb{B}$ is $(\omega\text{-})$**open** $\overset{\text{def}}{\iff} X_b(\omega) = 1 \quad \forall b \in B$

often omitted

▷ $$C_0(\omega) = \bigcup_{\substack{\gamma \in \Gamma_0 \\ \gamma \text{ is } \omega\text{-open}}} \gamma \quad \subset \mathbb{B}$$

<u>Lem 1.1</u> $\quad \# C_0(\omega) = \infty \iff \forall n \geq 1, \ \exists \gamma \in \Gamma_{0,n}, \ \forall b \in \gamma; \quad X_b(\omega) = 1$

<u>Proof</u> obvious //

▷ $\theta(p) = \mathbb{P}(\# C_0(\omega) = \infty) \leftarrow$ percolation prob

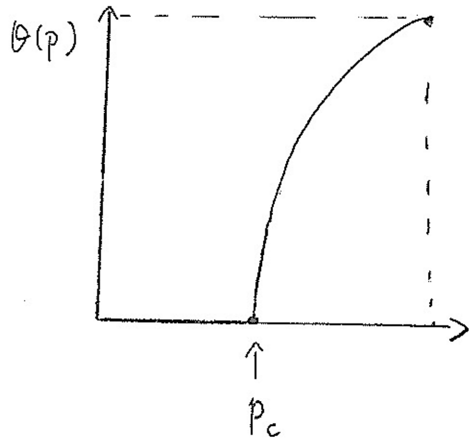
<u>FACT</u> $\theta : [0,1] \to [0,1]$ is $\nearrow$, $\theta(0) = 0$, $\theta(1) = 1$

<u>Thm 1.2</u>

a) $0 \leq p < \dfrac{1}{2d-1} \implies \theta(p) = 0$

In particular, $d = 1 \implies \theta(p) = 1, \ \forall p < 1$

b) $d \geqslant 2 \implies \theta(p) \nearrow 1$ as $p \nearrow 1$



Proof of Thm 1. 2 a)   For $\forall n \geq 1$

$$\{\omega : \# C_0(\omega) = \infty\} \underset{\text{Lem 1.1}}{\subset} \bigcup_{\gamma \in \Gamma_{0,n}} \{\omega : \forall b \in \gamma ,\ X_b(\omega) = 1\}$$

Thus,

$$\theta(p) \leq \sum_{\gamma' \in \Gamma_{0,n}} \underbrace{P(\forall b \in \gamma,\ X_b(\omega) = 1)}_{= p^n} \overset{\text{Exer 1.1}}{\leq} 2d\,(2d-1)^{n-1} p^n \xrightarrow{n \to \infty} 0$$

$$\left(p < \frac{1}{2d-1}\right)$$

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Reduction to Thm 1.2 (b) to $d = 2$

$$\mathbb{Z}^2 \cong H = \{ \underbrace{(x_1, x_2, 0, \cdots 0)}_{d} : (x_1, x_2) \in \mathbb{Z}^2 \} \subset \mathbb{Z}^d \quad (d \geq 2)$$

$$\mathbb{Z}^2 \downarrow \theta_2(p) \qquad \mathbb{Z}^d \downarrow \theta_d(p)$$

$$\theta_2(p) = P\left(\begin{array}{l}0 \text{ is contained in an} \\ \text{unbdd connected set} \\ \text{of open bonds inside } H\end{array}\right)$$

$$\leq P\left(\begin{array}{l} \qquad // \qquad \text{inside } \mathbb{Z}^d \end{array}\right) = \theta_d(p)$$

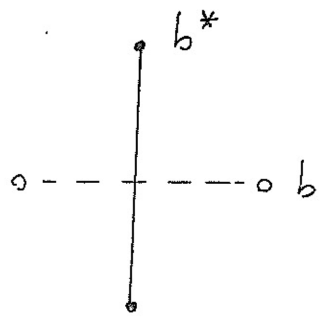
Thus. $\lim_{p \to 1} \theta_2(p) = 1 \Rightarrow \lim_{p \to 1} \theta_d(p) = 1$

Preparations for the proof of Thm 1.2 b) for $d=2$

▷ $(\mathbb{Z}^2)^* = \left(\frac{1}{2},\frac{1}{2}\right) + \mathbb{Z}^2 = \left\{ \left(\frac{1}{2},\frac{1}{2}\right) + x \; ; \; x \in \mathbb{Z}^2 \right\}$ (dual lattice)

▷ $\mathbb{B}^* =$ the set of all bonds in $(\mathbb{Z}^2)^*$

▷ $\mathbb{B} \to \mathbb{B}^*$ $(b \mapsto b^*)$ bijection defined by



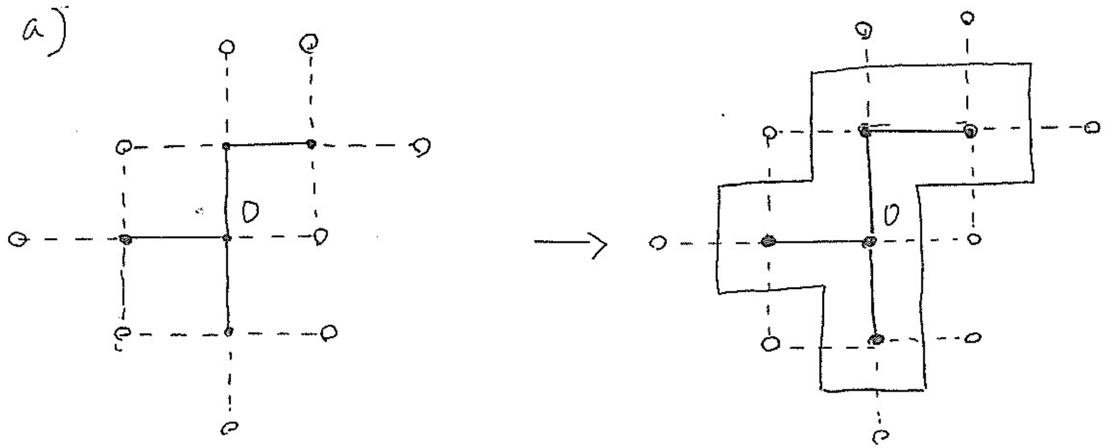
⇝ The notion of path in $(\mathbb{Z}^2)^*$ can be defined as before

⇝ $\Gamma^*_{x,n}$, Γ^*_x are defined accordingly.

Lem 1.3 a) $\# C_0(\omega) < \infty \iff \begin{cases} \exists n \geq 4 \ \exists \text{ path } \gamma \text{ in } (\mathbb{Z}^2)^* \text{ s.t.} \\ 1)\ |\gamma| = n \\ 2)\ b^* \in \gamma \Rightarrow X_b(\omega) = 0 \\ 3)\ \gamma \text{ is a circuit (i.e. } \gamma_0 = \gamma_n) \text{ which} \\ \quad \text{encloses } 0. \end{cases}$

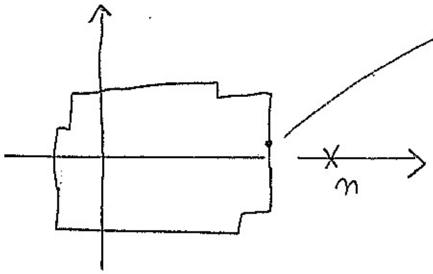
b) $\#\{\gamma : \gamma \text{ is a circuit in } (\mathbb{Z}^2)^*,\ |\gamma| = n,\ \gamma \text{ encloses } 0\} \leq 4n\,3^{n-1}$

a)



b) $\left.\begin{array}{l}\gamma \text{ encloses } 0 \\ |\gamma| = n\end{array}\right\} \Longrightarrow$ γ should contain some of

$$\underbrace{\left(x+\frac{1}{2}, \frac{1}{2}\right) \quad x = 0, 1, \dots, n-1}$$



For each x

$$\#\left\{\gamma : \gamma \ni \left(x+\frac{1}{2}, \frac{1}{2}\right)\right\} \overset{\text{similar as Exer 1.1}}{\leq} 4 \cdot 3^{n-1}$$

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Proof of Thm 1.2 b)

$$1-\theta(p) = P\left(\#C_0(\omega) < \infty\right)$$

$$\underset{\text{Lem 1.3 a)}}{=} P\left(\begin{array}{l}\exists\, n \geq 4;\ \exists \text{ circuit } \gamma \text{ in } (\mathbb{Z}^2)^* \text{ s.t. } |\gamma| = n, \\ \quad X_b(\omega) = 0 \text{ for } \forall\, b^* \in \gamma\end{array}\right)$$

$$\leq \sum_{n \geq 4} \sum_{\substack{\gamma:\ \text{circuit in } (\mathbb{Z}^2)^* \\ |\gamma| = n,\ \gamma \text{ encloses } 0}} \underbrace{P\left(X_b(\omega) = 0 \text{ for } \forall\, b^* \in \gamma\right)}_{= (1-p)^n}$$

$$\overset{\text{Lem 1.4 b)}}{\leq} \sum_{n \geq 4} 4n3^{n-1}(1-p)^n \leq 4(1-p)^4 \sum_{n \geq 4} n3^{n-1}(1-p)^{n-4}$$

$$\underset{\text{if } p > 3/4}{\leq} 4(1-p)^4 \sum_{n \geq 4} n3^{n-1}\left(\frac{1}{4}\right)^{n-4} = C(1-p)^4 \longrightarrow 0 \quad (p \nearrow 1)$$