§2 Ergodic theory in the context of percolation

- $\Omega = \{0,1\}^{\mathbb{B}} = \{ \omega = (\omega(b))_{b \in \mathbb{B}} \; ; \; \omega_b \in \{0,1\} \}$

The canonical realization of $(\underline{\Omega}, \mathcal{F}, P \; ; \; \{X_b\}_{b \in B})$

- $C \subset \Omega$ is called a cylinder set

$$\overset{\text{def}}{\iff} \begin{cases} \exists \text{ finite set } B \subset \mathbb{B}, \; \exists \, \eta : B \to \{0,1\} \\ \text{s.t.} \quad C = \bigcap_{b \in B} \{\omega \in \Omega : \omega(b) = \eta(b)\} \end{cases}$$

- $\mathcal{C}$ = the set of all cylinder sets.

- $\mathcal{F} = \sigma[\mathcal{C}]$ = the smallest σ-field on Ω that contains $\mathcal{C}$.

- $X_b(\omega) = \omega(b) \quad (b \in \mathbb{B})$

We henceforth assume that $(\Omega, \mathcal{F}, P : \{X_b\}_{b \in \mathbb{B}})$ is given as above.

- For $x \in \mathbb{Z}^d$,
- $\tau_x : \Omega \longrightarrow \Omega$ ("shift")

  $(\tau_x \omega)(b) = \omega(b + x)$,

  where $b + x = [x+y, x+z]$ for $b = [y, z]$

- $A \subset \Omega$ is <u>shift-invariant</u> $\overset{\text{def}}{\iff}$ $\tau_x A = A$, $\forall x \in \mathbb{Z}^d$

- $\mathcal{I} = \{ A \subset \Omega,\ A \text{ is shift-inv.} \}$

<u>Exa</u> $\{ \omega : \exists$ infinite ω-connected set of open bonds $\} \in \mathcal{I}$

Prop 2.1 (P is shift-invariant)

$$\forall x \in \mathbb{Z}^d,\ \forall A \in \mathcal{F},\ P(\tau_x A) = P(A).$$

Proof

Case 1 $\quad A = \bigcap_{j=1}^{n} \{\omega(b_j) = \varepsilon_j\} \in \mathcal{C}$

$$\tau_x A = \bigcap_{j=1}^{n} \{\omega(b_j + x) = \varepsilon_j\}$$

$$\leadsto P(\tau_x A) = p^{\varepsilon_1 + \cdots + \varepsilon_n} (1-p)^{n-(\varepsilon_1 + \cdots + \varepsilon_n)} = P(A)$$

<u>Case 2</u> $A \in \mathcal{F}$

⑪ - 1 : $\mathcal{G} := \bigcap_{x \in \mathbb{Z}^d} \{ A \in \mathcal{F} \cdot P(\tau_x A) = P(A) \}$ is a σ-field

∵ easy

⑪ - 2 $\mathcal{C} \subset \mathcal{G}$

∵ by Case 1

⑪ - 3 $\mathcal{F} \subset \mathcal{G}$ (Thus, $\mathcal{F} = \mathcal{G}$)

∵ $\mathcal{F}$ is the smallest σ-field that contains $\mathcal{C}$

Lem 2.2

▷ $\mathcal{F}_0 \stackrel{\text{def}}{=} \{\phi\} \cup \left\{ \bigcup_{j=1}^{n} C_j \;:\; n \geq 1,\ C_1, \cdots, C_n \in \mathcal{C} \right\}$

Then, $\forall A \in \mathcal{F},\ \underbrace{\forall \varepsilon > 0,\ \exists \tilde{A} \in \mathcal{F}_0 \text{ s.t. } P(A \Delta \tilde{A}) < \varepsilon}_{(*)}$

where $A \Delta \tilde{A} = (A \setminus \tilde{A}) \cup (\tilde{A} \setminus A)$

Proof (outline)

$\mathcal{G} = \{ A \in \mathcal{F} : (*) \text{ holds} \}$ is a σ-field that contains $\mathcal{C}$

Thus $\mathcal{F} = \mathcal{G}$

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Exer 2.1 $\quad A, B, A', B' \in \mathcal{F}$

i) $|P(A) - P(B)| \leq P(A \triangle B)$

ii) $(A \cap A') \triangle (B \cap B') \subset (A \triangle B) \cup (A' \triangle B')$

iii) $|P(A \cap A') - P(B \cap B')| \leq P(A \triangle B) + P(A' \triangle B')$

Prop. 2.3

a) (P is mixing) $P(A \cap \tau_x B) \xrightarrow{|x| \to \infty} P(A)\,P(B) \quad \forall A, \forall B \in \mathcal{F}$

b) (P is ergodic) $A \in \mathcal{I} \Rightarrow P(A) \in \{0, 1\}$

<u>Proof</u> a) Fix $A, B \in \mathcal{F}$ and $\varepsilon > 0$. By Lem 2.3, $\exists \tilde{A}, \exists \tilde{B} \in \mathcal{F}_0$ s.t.

$$P(A \triangle \tilde{A}) + P(B \triangle \tilde{B}) < \varepsilon \qquad \text{——} \ (1)$$

Write:

$$|P(A \cap \tau_x B) - P(A)P(B)| \le p_1 + p_2 + p_3$$

$$p_1 = |P(A \cap \tau_x B) - P(\tilde{A} \cap \tau_x \tilde{B})|, \quad p_2 = |P(\tilde{A} \cap \tau_x \tilde{B}) - P(\tilde{A})P(\tilde{B})|$$

$$p_3 = |P(\tilde{A})P(\tilde{B}) - P(A)P(B)|$$

$$p_1 \overset{\text{Exer 2.1}}{\le} P(A \triangle \tilde{A}) + \underbrace{P(\tau_x B \triangle \tau_x \tilde{B})}_{\overset{\text{Lem 2.2}}{=} P(B \triangle \tilde{B})} \overset{(1)}{<} \varepsilon \qquad \text{——} \ (2)$$

$$P_3 \le P(\tilde{A})\,|P(\tilde{B}) - P(B)| + P(B)\,|P(\tilde{A}) - P(A)|$$
$$\overset{\text{Exer 2.1}}{\le} P(B \Delta \tilde{B}) + P(A \Delta \tilde{A}) \overset{(1)}{<} \varepsilon$$
— (3)

$$\tilde{A}, \tilde{B} \in \mathcal{F}_0 \rightsquigarrow \exists\, \Gamma \subset\subset \mathbb{B} \text{ s.t. } \tilde{A}, \tilde{B} \in \sigma[\omega(b) : b \in \Gamma]$$
$$\top \qquad \rightsquigarrow \tau_x \tilde{B} \in \sigma[\omega(b) : b \in x + \Gamma]$$
— (4)

Take $|x|$ large enough s.t. $(x + \Gamma) \cap \Gamma = \phi$.

Then, $\tilde{A}$ and $\tau_x \tilde{B}$ are indep. by (4). Thus

$$P(\tilde{A} \cap \tau_x \tilde{B}) = P(\tilde{A})P(\tau_x \tilde{B}) \underset{\text{Prop.2.1}}{=} P(\tilde{A})P(\tilde{B})$$
— (5)

Therefore $P_2 = 0$

By (2), (3), (5), we have that

$$\overline{\lim_{|x|\to\infty}} \left| P(A \cap \tau_x B) - P(A)P(B) \right| \leq 2\varepsilon$$

b) Let $A \in \mathcal{J}$. Then, by applying a) with $A = B$,

$$P(A) = P(A \cap \tau_x A) \xrightarrow{|x|\to\infty} P(A)^2 .$$

Thus $P(A) = P(A)^2$ //

Prop 2.4 Let $V_n = \mathbb{Z}^d \cap [-n, n]^d$. Then, for ${}^\forall A \in \mathcal{F}$,

$$M_n \overset{\text{def}}{=} \frac{1}{(2n+1)^d} \sum_{x \in V_n} \mathbb{1}_A \circ \tau_x \xrightarrow{n \to \infty} P(A) \quad \text{in } L^2(P)$$

Rem This is a special case of "von-Neumann's ergodic theorem".

<u>Proof</u>. $m := \ddot{P}(A)$ for simple. Then

$$|M_n - m|^2 = \left| \frac{1}{(2n+1)^d} \sum_{x \in V_n} (\mathbb{1}_A \circ \tau_x - m) \right|^2$$

$$= \frac{1}{(2n+1)^{2d}} \sum_{x, y \in V_n} (\mathbb{1}_A \circ \tau_x - m)(\mathbb{1}_A \circ \tau_y - m)$$

$$= \frac{1}{(2n+1)^{2d}} \sum_{x, y \in V_n} \left\{ (\mathbb{1}_A \circ \tau_x)(\mathbb{1}_A \circ \tau_y) - m \mathbb{1}_A \circ \tau_x - m \mathbb{1}_A \circ \tau_y + m^2 \right\}$$

$E[\cdots] = \int \cdots dP$

Note that

$$\begin{cases} E[\mathbb{1}_A \circ \tau_x] = P(\tau_{-x} A) = P(A) = m. \\ E[(\mathbb{1}_A \circ \tau_x)(\mathbb{1}_A \circ \tau_y)] = P(\tau_{-x} A \cap \tau_{-y} A) \\ \qquad = P(A \cap \tau_{x-y} A) \end{cases}$$

Thus,

$$E\left[\,|M_n - m|^2\,\right] = \frac{1}{(2n+1)^{2d}} \sum_{x,y\in V_n} \Big(\overbrace{P(A\cap \tau_{x-y} A) - m^2}^{(1)} \Big)$$

By Prop. 2-3, $\quad P(A\cap \tau_z A) \longrightarrow m^2 \quad (|z|\to\infty)$

Therefore $\forall \varepsilon > 0,\ \exists\, l \in \mathbb{N}$, s.t. $|x-y| > l \Rightarrow |(1)| \leq \varepsilon$ —— (2)

For $\forall x \in V_n$ (fixed)

$$\sum_{y\in V_n} |(1)| = \sum_{\substack{y\in V_n \\ |x-y|\leq l}} |(1)| + \sum_{\substack{y\in V_n \\ |x-y|> l}} |(1)| \leq (2l+1)^d + \varepsilon\,(2n+1)^d$$

Hence

$$\sum_{x,y\in V_n} |(1)| \leq (2l+1)^d (2n+1)^d + \varepsilon\,(2n+1)^{2d}.$$

and thus,

$$E\left[\,|M_n - m|^2\,\right] \leq \frac{(2\ell+1)^d}{(2n+1)^d} + \varepsilon$$

This implies that

$$\overline{\lim_{n}}\, E\left[\,|M_n - m|^2\,\right] \leq \varepsilon$$

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