§3 Infinite cluster

For $\omega \in \Omega$

- $B \subset \mathbb{B}$ is a **ω-cluster** $\iff$ B is a maximal connected set of ω-open bonds.

- 〃 is a **ω-infinite cluster** (ω-i.c.) $\iff$ $\begin{cases} B \text{ is an } \omega\text{-cluster} \\ \& \ \# B = \infty \end{cases}$

Thm 3.1 (uniqueness of the infinite cluster)

$$N_\infty(\omega) \overset{\text{def}}{=} \#\text{ of } \omega\text{-i.c.'s}$$

Then $$\theta(p) \begin{cases} = 0 & \Rightarrow P(N_\infty(\omega) = 0) = 1 \\ > 0 & \Rightarrow P(N_\infty(\omega) = 1) = 1 \end{cases}$$

Prop. 3-1

a) $\theta(p) = 0 \iff P(N_\infty(\omega) = 0) = 1$ $\leftarrow$ This proves Thm 3-1 for $\theta(p) = 0$

b) $\theta(p) > 0 \iff P(N_\infty(\omega) \geq 1) = 1$

Proof a) ($\Rightarrow$) Let $A = \{ N_\infty(\omega) \geq 1 \}$

$$C_x(\omega) \stackrel{\text{def}}{=} \bigcup_{\gamma \in \Gamma_x(\omega)} \gamma$$

Then,

$$A = \bigcup_{x \in \mathbb{Z}^d} \{ \# C_x(\omega) = \infty \}$$

$$\theta(p) \underset{\substack{\uparrow \\ \text{Lem 2.2}}}{=} P(\# C_x(\omega) = \infty)$$

Thus,

$$P(A) \leq \sum_{x \in \mathbb{Z}^d} P(\# C_x(\omega) = \infty) \leq \sum_{x \in \mathbb{Z}^d} \theta(p) = 0$$

$(\Leftarrow)$ $\quad \{\omega : \# C_0(\omega) = \infty\} \subset A$ ∴ Thus, $\theta(p) \leq P(A) = 0$

b). $\quad A \in \mathcal{T}$. Thus, by Prop 2.4, $P(A) \in \{0, 1\}$ —— (1)

$$\theta(p) > 0 \overset{a)}{\Longleftrightarrow} P(A^c) < 1 \Longleftrightarrow P(A) > 0 \overset{(1)}{\Longleftrightarrow} P(A) = 1$$

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Thm 3-1 is the consequence of $\underbrace{\text{Prop 3-1}}_{\text{for } \theta(p)=0}$ and the following two propositions.

<u>Prop. 3.2</u> [Newman - Schulman, 1981]

Suppose that $\theta(p) > 0$, Then

$$P(N_\infty(\omega) < \infty) > 0 \Rightarrow P(N_\infty(\omega) = 1) = 1$$

<u>Prop. 3.3</u> [Aizenman - Kesten - Newman, 1987] $[\underbrace{\text{Burton - Keane 1989}}_{\text{simpler proof}}]$

$$P(N_\infty(\omega) \leq 2) = 1, \quad {}^\forall p \in (0,1)$$

Proof of Prop 3.2

$$0 < P(N_\infty(\omega) < \infty) \overset{\text{Prop 3.1}}{=} P(1 \leq N_\infty(\omega) < \infty) = P\left(\bigcup_{k \geq 1} \{N_\infty(\omega) = k\}\right)$$

Thus, $\exists\, k \in \mathbb{N} \setminus \{0\}$ s.t. $P(N_\infty(\omega) = k) > 0$ —— (1)

Note that $\{\omega : N_\infty(\omega) = k\} \in \mathcal{T}$. Thus, by Prop 2.4 and (1),

$$P(N_\infty(\omega) = l) = \begin{cases} 1 & l = k \\ 0 & l \neq k \end{cases} \qquad \text{——— (2)}$$

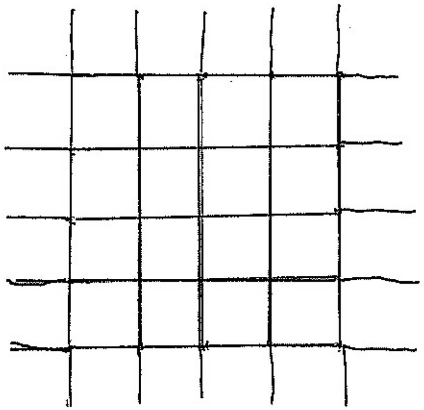
We will prove that $k = 1$. Let $I_1, \dots, I_k$ be ω-i.c. and

These exist with prob one, by (2)

$$B_n = \{ b \in \mathbb{B},\ b \subset [-n, n]^d \}$$

$$\partial B_n = \{ b \in \mathbb{B} \setminus B_n,\ b \cap [-n, n]^d \neq \phi \}$$

Since $I_1, \ldots I_k$ are i.c.'s, $\begin{cases} B_n \cap I_j \neq \phi \quad {}^{\forall} j = 1, \ldots, k \quad \text{—— (3)} \\ \text{for all sufficiently large } n\text{'s.} \end{cases}$



Thus,

$$\begin{cases} 1 = P\left(\bigcup_{n \geq 1} \underbrace{\{ B_n \cap I_j \neq \phi, \ j = 1, \ldots, k \}}_{\shortparallel \atop E_n} \right) \quad \left(\Rightarrow E_n \subset E_{n+1} \ \leftarrow (B_n \subset B_{n+1}) \right) \\ = \lim_{n \to \infty} P(E_n) \end{cases}$$

Therefore, $\exists n \in \mathbb{N}$ s.t. $P(E_n) \geq 1/2$ ——— (4)

$\begin{cases} \text{Again, since } I_1, \ldots I_k \text{ are i.c.'s, (3) implies that} \\ \exists B_n \cap I_j \neq \phi. \ {}^{\forall} j = 1, \ldots, k \end{cases}$

Therefore

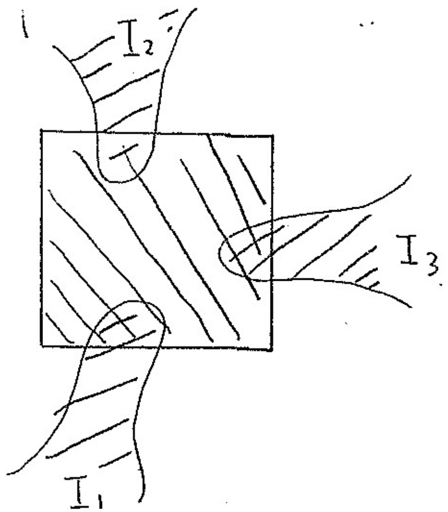
$$\frac{1}{2} \le P(E_n) \le P\left(\partial B_n \cap I_j \ne \emptyset \quad \forall j = 1, \dots, k \right)$$

$$\le P\left(\begin{array}{l} \exists\ \omega\text{-i.c. in } \mathbb{B} \setminus B_n \ \& \\ \underbrace{\partial B_n \cap I \ne \emptyset \ \text{ for all } \omega\text{-i.c. } I \text{ in } \mathbb{B} \setminus B_n}_{= F_n} \end{array} \right)$$

On the other hand $G_n \overset{\text{def}}{=} \{ \omega \equiv 1 \text{ on } B_n \}$ is indep of F_n

and $P(G_n) = p^{|B_n|} > 0$. Therefore (explained by picture (next page))

$$0 < P(G_n)\, P(F_n) = P(G_n \cap F_n) \le P(N_\infty(\omega) = 1) \quad \text{— } (5)$$



$(2), (5) \Longrightarrow k = 1$

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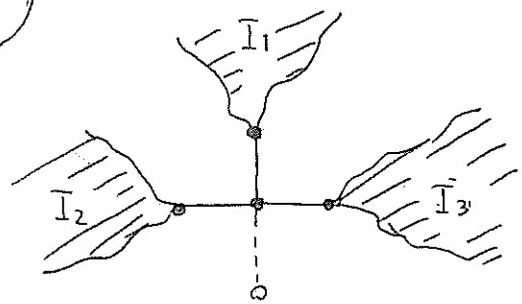
Preparation for the proof of Prop. 3.3 [Burton-Keane 1989]

$x \in \mathbb{Z}^d$, $\omega \in \Omega$,

▷ x is a ω-<u>trifurcation</u>

$\Leftrightarrow$ $\exists$ ω-i.c.'s I_j $(j = 1, 2, 3)$ in $\{b \in \mathbb{B},\ b \not\ni x\}$

s.t. $\{b \in C_x(\omega) \,;\ b \not\ni x\} = \bigcup_{j=1}^{3} I_j$



$x \in \mathbb{Z}^d$

▷ $T_x = \{\omega \in \Omega : x \text{ is a } \omega\text{-trifurcation}\}$

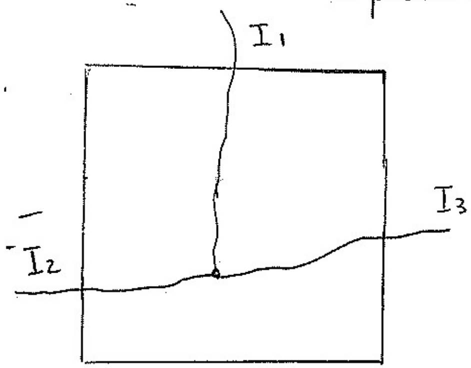
<u>Lem 3.4</u> $\forall x \in \mathbb{Z}^d, \quad P(T_x) = P(T_0) = 0$

Proof Since $T_x = \tau_{-x} T_0$, we have $P(T_x) \overset{\text{Prop 3.1}}{=} P(T_0)$ $V_n = \mathbb{Z}^d \cap [-n,n]^d$

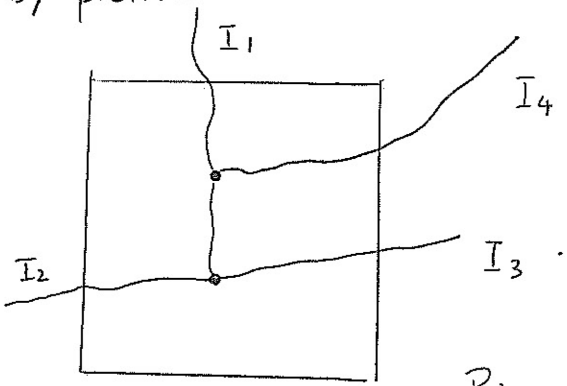
⑪-1

$$\underbrace{\sum_{x\in V_n} \mathbb{1}_{T_x}(\omega)}_{\substack{\parallel \\ \#\text{ of } \omega\text{-trifurcations in } V_n}} = k \implies \#\text{ of } \omega\text{-i.c's in } \mathbb{B}\setminus B_n \geq k+2$$

Explanation by pictures



$k = 1$



$k = 2$

... and so on

Rigorous proof → [Grimmett, p200–]

// ⑪-1

⑪-2 $\displaystyle\sum_{x\in V_n} \mathbb{1}_{T_x}(\omega) + 2 \leq \# \partial B_n$

(∵) Since distinct ω-i.c.'s in $\mathbb{B}\setminus B_n$ are disjoint,

$\underbrace{\text{each } b\in\partial B_n \text{ can be contained in at most one } \omega\text{-i.c. in } \mathbb{B}\setminus B_n}_{(1)}$

Thus

$$\#\partial B_n \overset{(1)}{\geq} \#\text{ of } \omega\text{-i.c.'s in } \mathbb{B}\setminus B_n \overset{⑪-1}{\geq} \sum_{x\in V_n} \mathbb{1}_{T_x} + 2$$

// ⑪-2

⑪-3 $P(T_0) = 0$

(∵) By Prop 2-4, $\displaystyle\frac{1}{(2n+1)^d}\sum_{x\in V_n} \mathbb{1}_{T_x}(\omega) \xrightarrow{n\to\infty} P(T_0)$ in $L^2(P)$

On the other hand, $\displaystyle\sum_{x\in V_n} \mathbb{1}_{T_x}(\omega) \overset{⑪-2}{\leq} \#\partial B_n = O(n^{d-1})$. Thus, $P(T_0) = 0$ // ⑪-3

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Proof of Prop 3-3

We assume that $P(N_\infty(\omega) \geq 3) > 0$ and conclude that $P(T_0) > 0$, which contradicts with Lem 3.4. Note that $\{N_\infty(\omega) \geq 3\} \in \mathcal{T}$.

Thus, $P(N_\infty(\omega) \geq 3) = 1$ by Prop. 2.3

⑪ - 1
$$\begin{cases} IC_n(\omega) = \{ I : I \text{ is an } \omega\text{-l.c. in } \mathbb{B} \setminus B_n \text{ s.t. } I \cap \partial B_n \neq \phi \} \\ \Longrightarrow \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ P(\# IC_n(\omega) \geq 3) \geq \frac{1}{2} \end{cases}$$

∵ same as in the proof of Prop 3.2 // ⑪ - 1

Let $\omega_m = (\omega(b))_{b \in B_m} \in \{0,1\}^{B_m}$, $\omega_m' = (\omega(b))_{b \in \mathbb{B} \setminus B_m} \in \{0,1\}^{\mathbb{B} \setminus B_m}$

Then $\omega = (\omega_m, \omega_m')$ and $IC_m(\omega) = IC_m(\omega_m')$

⑪-2 $\quad \# IC_m(\omega) \geq 3 \Rightarrow \exists \eta = \eta(\omega_m') \in \{0,1\}^{B_m}$ s.t. $(\eta, \omega_m') \in T_0$

(∵) Suppose that $I_j \in IC_m(\omega)$ $(j=1,2,3)$

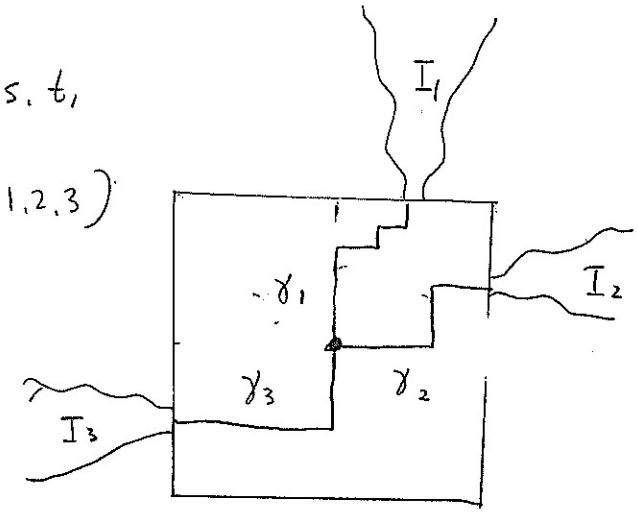
Then $\exists \eta \in \{0,1\}^{B_m}$ s.t.

$\exists$ η-open path $\gamma_j \subset B_m$ $(j=1,2,3)$ s.t.

i) 0 is connected to $I_j \cap \partial B_m$ by γ_j $(j=1,2,3)$

ii) $j \neq k \Rightarrow \gamma_j \cap \gamma_k = \{0\}$

iii) $b \in B_m \setminus \bigcup_{j=1}^{3} \gamma_j \Rightarrow \eta(b) = 0$



Thus; $(\eta, \omega_m') \in T_0$ // ⑪-2

⑪-3 $P(T_0) > 0$

∵ $P(T_0) \overset{⑪\text{-}2}{\geqq} P(\#IC_m(\omega) \geqq 3,\ \omega_m = \eta(\omega_m'))$

$\geqq \min_{\eta \in \{0,1\}^{B_m}} P(\#IC_m(\omega) \geqq 3,\ \omega_m \equiv \eta)$ indep

$= P(\#IC_m \geqq 3) \cdot \min_{\eta \in \{0,1\}^{B_m}} P(\omega_m \equiv \eta)$

$\overset{⑪\text{-}2}{\geqq} \frac{1}{2} \min_{\eta \in \{0,1\}^{B_m}} P(\omega_m \equiv \eta) > 0$ // ⑪-3

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