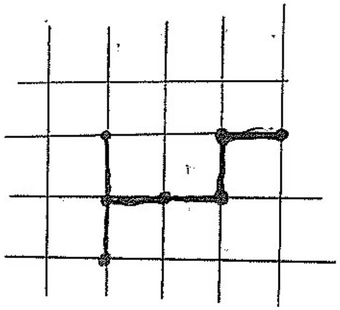
数理科学展望 III "Percolation"

(19, 26 May, 2, 9 June /2015)



§0 Introduction

Percolation [Broadbent & Hammerseley (1957)]

Suppose that each edge of $\mathbb{Z}^2$ are $\left\langle \begin{array}{lll} \text{open} & \text{with probability} & p \\ \text{closed} & \quad " \quad - & 1-p \end{array}\right.$

(segment between adjacent vertices)

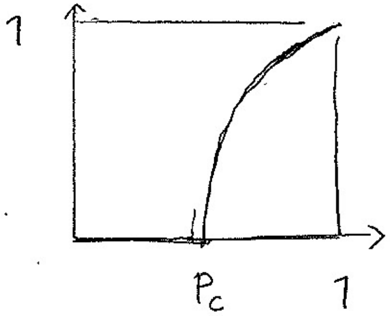
Question 1 Is the following probability positive?

$$\theta(p) \overset{\text{def}}{=} P\left(\begin{array}{l}\text{the origin is contained in an unbounded} \\ \underline{\text{connected}} \text{ set of open edges}\end{array}\right)$$

(obvious: $\theta(0) = 0$, $\theta(1) = 1$. Not obvious for $0 < p < 1$)

Answer 1 $\quad \exists\, p_c \in (0,1)$ called critical probability

s.t. $\theta(p) \begin{cases} = 0 & \text{if } p \leq p_c \\ > 0 & \text{if } p > p_c \end{cases}$



$\mapsto$ The situation changes abruptly at $p = p_c$ (an example of "critical phenomena")

↓

other example : The water $\begin{cases} \text{freezes at } 0°\text{C} \\ \text{boils} \quad // \quad 100°\text{C} \end{cases}$

Question 2 How is the following probability related to $\theta(p)$?

$$\psi(p) \overset{\text{def}}{=} P\left(\begin{array}{l} \exists \text{ unbounded connected set} \\ \quad \text{open edges} \end{array} \right)$$

If $\psi(p) > 0$, then, how many is the # of connected components ?

Answer 2

$$\psi(p) = \begin{cases} 0 & \Longleftrightarrow \theta(p) = 0 \\ 1 & \Longleftrightarrow \theta(p) > 0 \end{cases}$$

$$\theta(p) > 0 \Rightarrow P\left(\begin{array}{l} \exists 1 \text{ unbounded connected} \\ \quad \text{set of open edges} \end{array} \right) = 1$$

Rem We solve these question with the help of "ergodic theory"

Plan for the course

§1 The percolation prob. $\theta(p)$

§2 Ergodic theory in the context of percolation

§3 The uniqueness of the infinite cluster

§1 The percolation probability $\theta(p)$

Lattice

▷ $\mathbb{Z}^d = \{ x = (x_1, .. , x_d) \ ; \ x_j \in \mathbb{Z} \}$

For $x, y \in \mathbb{Z}^d$

▷ x and y are **adjacent** [ədʒéisnt] (denoted by $x \sim y$)

$$\overset{\text{def}}{\Longleftrightarrow} \quad \|x-y\| \overset{\text{def}}{=} \sum_{j=1}^{d} |x_j - y_j| = 1 \quad \left(, \ \exists 1 \ i , \underset{\text{s.t.}}{\wedge} \ |x_j - y_j| = \begin{cases} 1 & (j = i) \\ 0 & (j \neq i) \end{cases} \right)$$

▷ a segment $[x, y] \overset{\text{def}}{=} \{ tx + (1-t)y \ , \ 0 \leq t \leq 1 \}$ is called a **bond** or **edge**

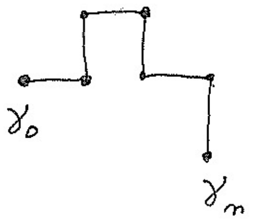
▷ $\mathbb{B}$ = the set of all bonds in $\mathbb{Z}^d$

▷ $\# A \overset{\text{def}}{=} \sum_{x \in A} 1$ for any countable set A $\left(\text{e.g. } A \subset \mathbb{Z}^d, \ A \subset \mathbb{B} \right)$

▷ $\gamma = \{b_1, \ldots, b_n\} \subset \mathbb{B}$ is a path with the length n $\left(|\gamma| \overset{\text{def}}{=} n \right)$

$\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \exists \gamma_0, \ldots, \exists \gamma_n \in \mathbb{Z}^d \text{ s.t.} \\ 1)\ b_j = [\gamma_{j-1}, \gamma_j] \quad (j = 1, \ldots, n) \\ 2)\ \gamma_j \neq \gamma_k \text{ if } 1 \leq k - j \leq n - 1 \quad (\gamma_0 = \gamma_n \text{ is allowed}) \end{cases}$

$n = 6$



▷ $\Gamma_{x,n}$ = the set of all path γ s.t. $\gamma_0 = x$, $|\gamma| = n$

▷ $\Gamma_x = \bigcup_{n \geq 1} \Gamma_{x,n}$ = the set of all path γ s.t. $\gamma_0 = x$

Exer 1.1 $\# \Gamma_{x,n} \leqq 2d\,(2d-1)^{n-1}$

▷ The following system $(\Omega, \mathcal{F}, P ; \{X_b\}_{b\in\mathbb{B}})$ is called the **percolation**

$$\begin{cases} (\Omega, \mathcal{F}, P) : \text{probability space} \ (\text{meas. sp. s.t. } P(\Omega) = 1) \\ X_b : \Omega \to \{0,1\} \ (b \in \mathbb{B}) \ \underbrace{\text{independent}}_{(*)} \ \underbrace{\text{random. variables.}}_{\overset{\|}{\text{measurable functions.}}} \\ \text{s.t. } P(X_b = 1) = p \ (\forall b \in \mathbb{B}) \end{cases}$$

(*) $\overset{\text{def}}{\iff}$ $\forall n \in \mathbb{N}$, $\forall b_j \in \mathbb{B}$, $\forall \varepsilon_j \in \{0,1\}$ $(j = 1, \cdots n)$

$$P\left(\bigcap_{j=1}^{n} \{X_{b_j} = \varepsilon_j\}\right) = \prod_{j=1}^{n} P(X_{b_j} = \varepsilon_j)$$

▷ For $\omega \in \Omega$, a set $B \subset \mathbb{B}$ is $(\omega\text{-})\underline{\underline{\text{open}}} \overset{def}{\iff} X_b(\omega) = 1 \quad \forall b \in B$

↑ often omitted

▷ $$C_0(\omega) = \bigcup_{\substack{\gamma \in \Gamma_0 \\ \gamma \text{ is } \omega\text{-open}}} \gamma \quad \subset \mathbb{B}$$

<u>Lem</u> 1.1 $\quad \# C_0(\omega) = \infty \iff \forall n \geq 1,\ \exists \gamma \in \Gamma_{0,n},\ \forall b \in \gamma;\quad X_b(\omega) = 1$

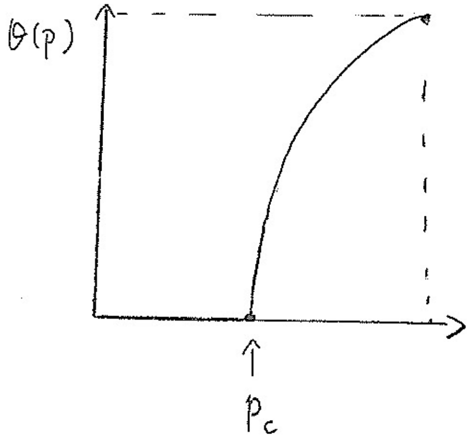
<u>Proof</u> obvious //

$\triangleright$ $\theta(p) = \mathbb{P}(\#C_0(\omega) = \infty)$ ← percolation prob

<u>FACT</u> $\theta : [0,1] \to [0,1]$ is ↗, $\theta(0) = 0$, $\theta(1) = 1$

Thm 1.2

a) $0 \leq p < \frac{1}{2d-1} \implies \theta(p) = 0$



In particular, $d = 1 \implies \theta(p) = 1$, $\forall\, p < 1$

b) $d \geq 2 \implies \theta(p) \nearrow 1$ as $p \nearrow 1$

Proof of Thm 1. 2 a) For $\forall n \geqslant 1$

$$\{\omega : \# C_0(\omega) = \infty\} \underset{\substack{\Uparrow \\ \text{Lem 1.1}}}{\subset} \bigcup_{\gamma \in \Gamma_{0,n}} \{\omega : \forall b \in \gamma , X_b(\omega) = 1\}$$

Thus,

$$\theta(p) \leqslant \sum_{\gamma' \in \Gamma_{0,n}} \underbrace{P(\forall b \in \gamma, X_b(\omega) = 1)}_{\substack{\shortparallel \\ p^n}} \overset{\substack{\text{Exer 1.1} \\ \downarrow}}{\leqslant} 2d(2d-1)^{n-1} p^n \underset{\substack{\uparrow \\ \left(p < \frac{1}{2d-1}\right)}}{\xrightarrow{n \to \infty}} 0$$

//

Reduction to Thm 1.2 (b) to d = 2

$$\begin{array}{ccc} \mathbb{Z}^2 & \cong H = \{ (\underbrace{x_1, x_2, 0, \cdots 0}_{d}) : (x_1, x_2) \in \mathbb{Z}^2 \} \subset & \mathbb{Z}^d \quad (d \geqslant 2) \\ \downarrow & & \downarrow \\ \theta_2(p) & & \theta_d(p) \end{array}$$

$$\theta_2(p) = P\begin{pmatrix} \text{0 is contained in un} \\ \text{ubbdd connected set} \\ \text{of open bonds inside } H \end{pmatrix}$$

$$\leqslant P\begin{pmatrix} \downarrow \\ // \\ \text{inside } \mathbb{Z}^d \end{pmatrix} = \theta_d(p)$$

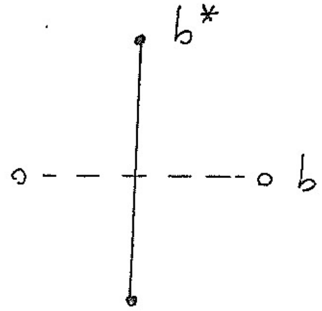
Thus. $\lim_{p \to 1} \theta_2(p) = 1 \Rightarrow \lim_{p \to 1} \theta_d(p) = 1$

Preparations for the proof of Thm 1.2 b) for $d=2$

▷ $(\mathbb{Z}^2)^* = (\frac{1}{2},\frac{1}{2}) + \mathbb{Z}^2 = \{(\frac{1}{2},\frac{1}{2}) + x \; ; \; x \in \mathbb{Z}^2\}$ (dual lattice)

▷ $\mathbb{B}^*$ = the set of all bonds in $(\mathbb{Z}^2)^*$

▷ $\mathbb{B} \to \mathbb{B}^*$ $(b \longmapsto b^*)$ bijection defined by



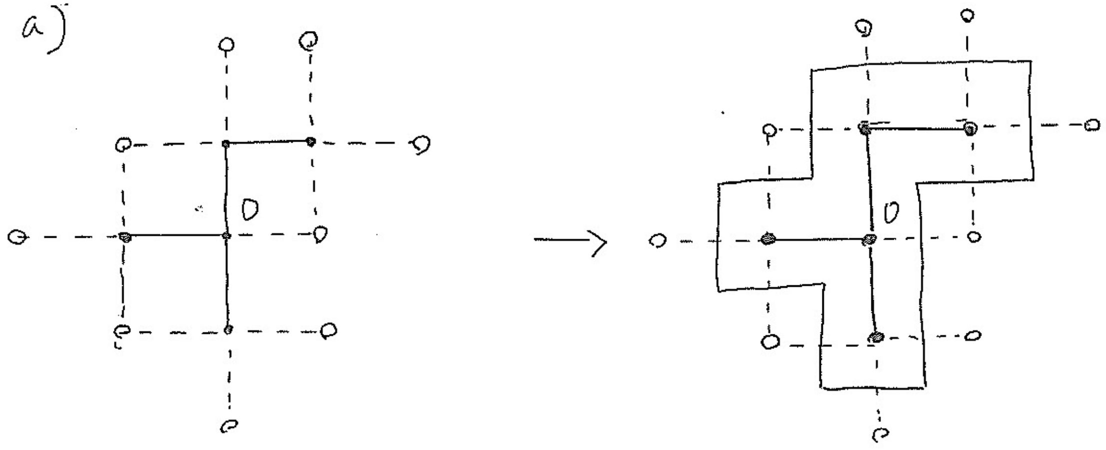
⇝ The notion of path in $(\mathbb{Z}^2)^*$ can be defined as before

⇝ $\Gamma^*_{x,n}$, Γ^*_x are defined accordingly.

Lem 1.3 a) $\# C_0(\omega) < \infty \iff \left\{ \begin{array}{l} \exists\, n \geqslant 4 \ \exists \text{ path } \gamma \text{ in } (\mathbb{Z}^2)^* \text{ s.t.} \\ 1)\ |\gamma|^* = n \\ 2)\ b^* \in \gamma \Rightarrow X_b(\omega) = 0 \\ 3)\ \gamma \text{ is a circuit } (\text{i.e. } \gamma_0 = \gamma_n) \text{ which} \\ \qquad \text{encloses } 0. \end{array} \right.$

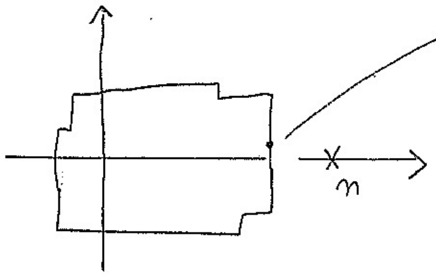
b) $\#\{\gamma : \gamma \text{ is a circuit in } (\mathbb{Z}^2)^*,\ |\gamma| = n,\ \gamma \text{ encloses } 0\} \leqslant 4n\,3^{n-1}$

a)



b) $\left.\begin{array}{l}\gamma \text{ encloses } 0 \\ |\gamma| = n\end{array}\right\} \Longrightarrow$ γ should contain some of

$$\left(x+\frac{1}{2}, \frac{1}{2}\right) \quad \underbrace{x = 0, 1, \dots, n-1}$$



For each x

$$\#\left\{ \gamma : \gamma \ni \left(x+\frac{1}{2}, \frac{1}{2}\right) \right\} \overset{\text{similar as Exer1.1}}{\leq} 4 \cdot 3^{n-1}$$

//

Proof of Thm 1.2 b)

$$1-\theta(p) = P(\#C_0(\omega) < \infty)$$

$$\underset{\substack{\uparrow \\ \text{Lem 1.3 a)}}}{=} P\left(\begin{array}{l} \exists n \geqslant 4, \exists \text{ circuit } \gamma, \text{ in } (\mathbb{Z}^2)^* \text{ s.t. } |\gamma| = n \\ \quad X_b(\omega) = 0 \text{ for } \forall b^* \in \gamma \end{array}\right)$$

$$\leqslant \sum_{n \geqslant 4} \sum_{\substack{\gamma : \text{circuit in } (\mathbb{Z}^2)^* \\ |\gamma| = n, \ \gamma \text{ encloses } 0}} \underbrace{P(X_b(\omega) = 0 \text{ for } \forall b^* \in \gamma)}_{\substack{\shortparallel \\ (1-p)^n}}.$$

$$\overset{\substack{\text{Lem 1.4 b)} \\ \downarrow}}{\leqslant} \sum_{n \geqslant 4} 4n3^{n-1}(1-p)^n \leqslant 4(1-p)^4 \sum_{n \geqslant 4} n3^{n-1}(1-p)^{n-4}$$

$$\underset{\substack{\uparrow \\ \text{if } p > 3/4}}{\leqslant} 4(1-p)^4 \sum_{n \geqslant 4} n3^{n-1}\left(\frac{1}{4}\right)^{n-4} = C(1-p)^4 \longrightarrow 0 \quad (p \nearrow 1)$$

§2 Ergodic theory in the context of percolation

▷ $\Omega = \{0,1\}^{\mathbb{B}} = \{ \omega = (\omega(b))_{b \in \mathbb{B}} \; ; \; \omega_b \in \{0,1\})$

↰ The canonical realization of $(\underline{\Omega}, \mathcal{F}, P \; ; \; \{X_b\}_{b \in B})$

▷ $C \subset \Omega$ is called a cylinder set

$$\overset{\text{def}}{\Longleftrightarrow} \left\{ \begin{array}{l} \exists \text{ finite set } B \subset \mathbb{B} , \; \exists \; \eta : B \to \{0,1\} \\ \text{s.t.} \quad C = \bigcap_{b \in B} \{ \omega \in \Omega : \; \omega(b) = \eta(b) \} \end{array} \right.$$

▷ $\mathcal{C}$ = the set of all cylinder sets.

▷ $\mathcal{F} = \sigma[\mathcal{C}]$ = the smallest σ-field on Ω that contains $\mathcal{C}$.

▷ $X_b(\omega) = \omega(b) \quad (b \in \mathbb{B})$

We henceforth assume that $(\Omega, \mathcal{F}, P : \{X_b\}_{b \in \mathbb{B}})$ is given as above.

> For $x \in \mathbb{Z}^d$,

▷ $\tau_x : \Omega \longrightarrow \Omega$ ("shift")

$(\tau_x \omega)(b) = \omega(b+x)$,

where $b+x = [x+y, x+z]$ for $b = [y,z]$

▷ $A \subset \Omega$ is shift-invariant $\overset{def}{\iff}$ $\tau_x A = A$, $\forall x \in \mathbb{Z}^d$

▷ $\mathcal{J} = \{ A \subset \Omega, \ A \text{ is shift-inv.} \}$

Exa $\{ \omega : \exists$ infinite $\underset{\omega\text{-}}{\wedge}$ connected set of open bonds $\} \in \mathcal{J}$

Prop 2.1 (P is shift-invariant)

$$\forall x \in \mathbb{Z}^d, \ \forall A \in \mathcal{F}, \ P(\tau_x A) = P(A).$$

Proof

Case 1 $\quad A = \bigcap_{j=1}^{n} \{ \omega(b_j) = \varepsilon_j \} \in \mathcal{C}$

$$\tau_x A = \bigcap_{j=1}^{n} \{ \omega(b_j + x) = \varepsilon_j \}$$

$$\leadsto P(\tau_x A) = p^{\varepsilon_1 + \cdots + \varepsilon_n} (1-p)^{n-(\varepsilon_1 + \cdots + \varepsilon_n)} = P(A)$$

Case 2 $A \in \mathcal{F}$

⑪ - 1 : $\mathcal{G} := \bigcap_{x \in \mathbb{Z}^d} \{ A \in \mathcal{F} \cdot P(\tau_x A) = P(A) \}$ is a σ-field

$\because$ easy

⑪ - 2 $\mathcal{C} \subset \mathcal{G}$

$\because$ by Case 1

⑪ - 3 $\mathcal{F} \subset \mathcal{G}$ (Thus, $\mathcal{F} = \mathcal{G}$)

$\because$ $\mathcal{F}$ is the smallest σ-field that contains $\mathcal{C}$

Lem 2.2

▷ $\mathcal{F}_0 \overset{\text{def}}{=} \{\phi\} \cup \{ \dot{\bigcup}_{j=1}^{n} C_j \; : \; n \geq 1, \; C_1, \cdots, C_n \in \mathcal{C} \}$

$$\text{Then, } \; {}^{\forall} A \in \mathcal{F}, \; \underbrace{{}^{\forall} \varepsilon > 0, \; {}^{\exists} \tilde{A} \in \mathcal{F}_0 \text{ s.t. } P(A \triangle \tilde{A}) < \varepsilon}_{(*)}$$

where $A \triangle \tilde{A} = (A \setminus \tilde{A}) \cup (\tilde{A} \setminus A)$

Proof (outline)

$\mathcal{G} = \{ A \in \mathcal{F} : (*) \text{ holds} \}$ is a σ-field that contains $\mathcal{C}$

Thus $\mathcal{F} = \mathcal{G}$

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Exer 2.1 $\quad A, B, A', B' \in \mathfrak{F}$

i) $|P(A) - P(B)| \leq P(A \Delta B)$

ii) $(A \cap A') \Delta (B \cap B') \subset (A \Delta B) \cup (A' \Delta B')$

iii) $|P(A \cap A') - P(B \cap B')| \leq P(A \Delta B) + P(A' \Delta B')$

Prop. 2.3

a) (P is mixing) $P(A \cap \tau_x B) \xrightarrow{|x| \to \infty} P(A)P(B)$ $\forall A, \forall B \in \mathcal{F}$

b) (P is ergodic) $A \in \mathcal{I} \Rightarrow P(A) \in \{0, 1\}$

<u>Proof</u> a) Fix $A, B \in \mathcal{F}$ and $\varepsilon > 0$. By Lem 2.3, $\exists \tilde{A}, \exists \tilde{B} \in \mathcal{F}_0$ s.t.

$$P(A \Delta \tilde{A}) + P(B \Delta \tilde{B}) < \varepsilon \qquad \text{——} \ (1)$$

Write:

$$|P(A \cap \tau_x B) - P(A)P(B)| \leq p_1 + p_2 + p_3$$

$$p_1 = |P(A \cap \tau_x B) - P(\tilde{A} \cap \tau_x \tilde{B})| , \quad p_2 = |P(\tilde{A} \cap \tau_x \tilde{B}) - P(\tilde{A})P(\tilde{B})|$$

$$p_3 = |P(\tilde{A})P(\tilde{B}) - P(A)P(B)|$$

$$p_1 \overset{\substack{\text{Exer2.1}\\ \downarrow}}{\leq} P(A \Delta \tilde{A}) + \underbrace{P(\tau_x B \Delta \tau_x \tilde{B})}_{\substack{\Vert \leftarrow \text{Lem2.2} \\ P(B \Delta \tilde{B})}} \overset{(1)}{<} \varepsilon \qquad \text{——} \ (2)$$

$$\left.\begin{aligned} P_3 &\le P(\tilde{A})\,|P(\tilde{B}) - P(B)| + P(B)\,|P(\hat{A}) - P(A)| \\ &\overset{\text{Exer 2.1}}{\le} P(B \Delta \tilde{B}) + P(A \Delta \tilde{A}) \overset{(1)}{<} \varepsilon \end{aligned}\right\} \text{—— } (3)$$

$$\left.\begin{aligned} &\tilde{A}, \tilde{B} \in \exists_0 \rightsquigarrow {}^{\exists}\, \Gamma \subset\subset \mathbb{B} \text{ s.t. } \tilde{A}, \tilde{B} \in \sigma[\omega(b) : b \in \Gamma] \\ &\top \qquad \rightsquigarrow \tau_x \tilde{B} \in \sigma[\omega(b) : b \in x + \Gamma] \end{aligned}\right\} \text{—— } (4)$$

Take $|x|$ large enouch s.t. $(x+\Gamma) \cap \Gamma = \phi$.

Then, $\tilde{A}$ and $\tau_x \tilde{B}$ are indep. by (4). Thus

$$P(\tilde{A} \cap \tau_x \tilde{B}) = P(\tilde{A}) P(\tau_x \tilde{B}) \underset{\substack{\uparrow \\ \text{Prop.2.1}}}{=} P(\tilde{A}) P(\tilde{B}) \quad \text{—— } (5)$$

Therefore $P_2 = 0$

By (2), (3), (5), we have that

$$\overline{\lim_{|x|\to\infty}} \left| P(A \cap \tau_x B) - P(A)P(B) \right| \leq 2\varepsilon$$

b) Let $A \in \mathcal{J}$. Then, by applying a) with $A = B$,

$$P(A) = P(A \cap \tau_x A) \xrightarrow{|x|\to\infty} P(A)^2 .$$

Thus $P(A) = P(A)^2$ //

Prop 2.4 Let $V_n = \mathbb{Z}^d \cap [-n, n]^d$. Then, for ${}^{\forall} A \in \mathcal{I}$,

$$M_n \overset{\text{def}}{=} \frac{1}{(2n+1)^d} \sum_{x \in V_n} \mathbb{1}_A \circ \tau_x \xrightarrow{n \to \infty} P(A) \text{ in } L^2(P)$$

Rem This is a special case of "von-Neumann's ergodic theorem".

Proof $m := \ddot{P}(A)$ for simple. Then

$$|M_n - m|^2 = \left| \frac{1}{(2n+1)^d} \sum_{x \in V_n} (\mathbb{1}_A \circ \tau_x - m) \right|^2$$

$$= \frac{1}{(2n+1)^{2d}} \sum_{x, y \in V_n} (\mathbb{1}_A \circ \tau_x - m)(\mathbb{1}_A \circ \tau_y - m)$$

$$= \frac{1}{(2n+1)^{2d}} \sum_{x, y \in V_n} \left\{ (\mathbb{1}_A \circ \tau_x)(\mathbb{1}_A \circ \tau_y) - m\, \mathbb{1}_A \circ \tau_x - m\, \mathbb{1}_A \circ \tau_y + m^2 \right)$$

$$E[\cdots] = \int \cdots \, dP$$

Note that
$$\left\{ \begin{aligned} & E[\mathbb{1}_A \circ \tau_x] = P(\tau_{-x} A) = P(A) = m. \\ & E[(\mathbb{1}_A \circ \tau_x)(\mathbb{1}_A \circ \tau_y)] = P(\tau_{-x} A \cap \tau_{-y} A) \\ & \qquad = P(A \cap \tau_{x-y} A) \end{aligned} \right.$$

Thus,

$$E\left[\,|M_n - m|^2\right] = \frac{1}{(2n+1)^{2d}} \sum_{x,y\in V_n} \overbrace{\left(P(A\cap \tau_{x-y} A) - m^2 \right)}^{(1)}$$

By Prop. 2-3, $P(A\cap \tau_z A) \longrightarrow m^2 \quad (|z|\to\infty)$

Therefore $\forall \varepsilon>0,\ \exists \ell\in \mathbb{N}$, s.t. $|x-y|>\ell \Rightarrow |(1)| \leq \varepsilon$ —— (2)

For $\forall x\in V_n$ (fixed)

$$\sum_{y\in V_n} |(1)| = \sum_{\substack{y\in V_n \\ |x-y|\leq \ell}} |(1)| + \sum_{\substack{y\in V_n \\ |x-y|>\ell}} |(1)| \leq (2\ell+1)^d + \varepsilon\,(2n+1)^d$$

Hence

$$\sum_{x,y\in V_n} |(1)| \leq (2\ell+1)^d (2n+1)^d + \varepsilon\,(2n+1)^{2d}.$$

and thus.

$$E\left[|M_n - m|^2 \right] \leqslant \frac{(2\ell+1)^d}{(2n+1)^d} + \varepsilon$$

This implies that

$$\overline{\lim_n}\, E\left[|M_n - m|^2 \right] \leqslant \varepsilon$$

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§3 Infinite cluster

For $\omega \in \Omega$

- ▷ $B \subset \mathbb{B}$ is a **ω-cluster** $\iff$ B is a maximal connected set of ω-open bonds.
- ▷ 〃 is a **ω-infinite cluster** **(ω-i.c.)** $\iff$ $\begin{cases} B \text{ is an } \omega\text{-cluster} \\ \& \ \#B = \infty \end{cases}$

Thm 3.1 (uniqueness of the infinite cluster)

$$N_\infty(\omega) \overset{\text{def}}{=} \# \text{ of } \omega\text{-i.c.'s}$$

Then $$\theta(p)\begin{cases} = 0 & \Longrightarrow \ P(N_\infty(\omega) = 0) = 1 \\ > 0 & \Longrightarrow \ P(N_\infty(\omega) = 1) = 1 \end{cases}$$

Prop. 3 - 1

a) $\theta(p) = 0 \iff P(N_\infty(\omega) = 0) = 1$

b) $\theta(p) > 0 \iff P(N_\infty(\omega) \geqslant 1) = 1$

$\leftarrow \Big\{$ This proves Thm 3 - 1 for $\theta(p) = 0$

Proof a) $(\Rightarrow)$ Let $A = \{ N_\infty(\omega) \geqslant 1 \}$

$$C_x(\omega) \overset{\text{def}}{=} \bigcup_{\gamma \in \Gamma_x(\omega)} \gamma$$

Then,

$$\left\{ \begin{aligned} & A = \bigcup_{x \in \mathbb{Z}^d} \{ \# C_x(\omega) = \infty \} \\ & \theta(p) \underset{\substack{\uparrow \\ \text{Lem 2.2}}}{=} P(\# C_x(\omega) = \infty) \end{aligned} \right.$$

Thus,

$$P(A) \leqslant \sum_{x \in \mathbb{Z}^d} P(\# C_x(\omega) = \infty) \leqslant \sum_{x \in \mathbb{Z}^d} \theta(p) = 0$$

($\Leftarrow$) $\{\omega : \#C_0(\omega) = \infty\} \subset A$. Thus, $\theta(p) \leq P(A) = 0$

b). $A \in \mathcal{T}$. Thus, by Prop 2.4, $P(A) \in \{0, 1\}$ —— (1)

$$\theta(p) > 0 \overset{a)}{\iff} P(A^c) < 1 \iff P(A) > 0 \overset{(1)}{\iff} P(A) = 1$$

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Thm 3-1 is the consequence of $\underbrace{\text{Prop 3-1}}_{\text{for } \theta(p)=0}$ and the following two propositions.

Prop. 3.2 [Newman - Schulman, 1981]

Suppose that $\theta(p) > 0$, Then

$$P(N_\infty(\omega) < \infty) > 0 \Rightarrow P(N_\infty(\omega) = 1) = 1$$

Prop. 3.3 [Aizenman - Kesten - Newman, 1987] [$\underbrace{\text{Burton - Keane 1989}}_{\text{simpler proof}}$]

$$P(N_\infty(\omega) \leq 2) = 1, \quad {}^\forall p \in (0, 1)$$

Proof of Prop 3.2

$$0 < P(N_\infty(\omega) < \infty) \overset{\text{Prop 3.1}}{=} P(1 \leq N_\infty(\omega) < \infty) = P\Big(\bigcup_{k \geq 1} \{N_\infty(\omega) = k\}\Big)$$

Thus, $\exists\, k \in \mathbb{N} \setminus \{0\}$ s.t. $P(N_\infty(\omega) = k) > 0$ —— (1)

Note that $\{\omega : N_\infty(\omega) = k\} \in \mathcal{T}$. Thus, by Prop 2.4 and (1),

$$P(N_\infty(\omega) = \ell) = \begin{cases} 1 & \ell = k \\ 0 & \ell \neq k \end{cases} \qquad \text{———} \quad (2)$$

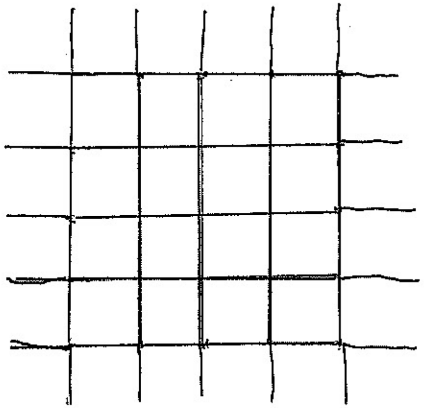
We will prove that $k = 1$. Let $\underbrace{I_1, \dots, I_k \text{ be } \omega\text{-i.c.}}$ and

These exist with prob one, by (2)

$$B_n = \{ b \in \mathbb{B},\ b \subset [-n, n]^d \}$$

$$\partial B_n = \{ b \in \mathbb{B} \setminus B_n,\ b \cap [-n, n]^d \neq \phi \}$$

Since $I_1, \dots I_k$ are i.c.'s, $\begin{cases} B_n \cap I_j \neq \phi \quad \forall j = 1, \dots, k \quad \text{—} \quad (3) \\ \text{for all sufficiently large } n\text{'s.} \end{cases}$



Thus,

$$\begin{cases} 1 = P\Big(\bigcup_{n \geq 1} \underbrace{\{ B_n \cap I_j \neq \phi \ , j = 1, \dots, k \}}_{\substack{\| \\ E_n}} \Big) \qquad \left(\Rightarrow E_n \subset E_{n+1} \leftarrow (B_n \subset B_{n+1}) \right) \\ = \lim_{n \to \infty} P(E_n) \end{cases}$$

Therefore, $\exists n \in \mathbb{N}$ s.t. $P(E_n) \geq 1/2$ ——— (4)

$$\begin{cases} \text{Again, since } I_1, \dots I_k \text{ are i.c.'s, (3) implies that} \\ \partial B_n \cap I_j \neq \phi \quad \forall j = 1, \dots, k \end{cases}$$

Therefore

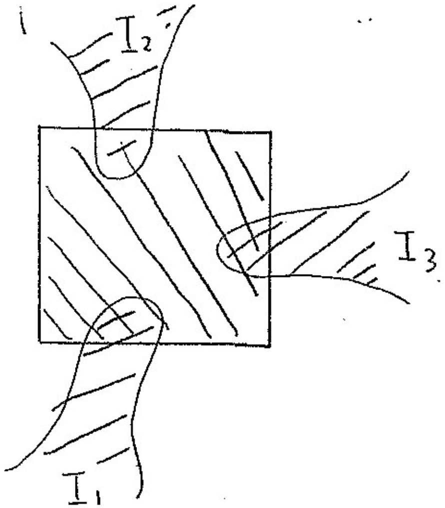
$$\frac{1}{2} \le P(E_m) \le P\left(\partial B_m \cap I_j \neq \phi \quad \forall j = 1, .., k\right)$$

$$\le P\left(\begin{array}{l} \exists\ \omega\text{-i.c. in } \mathbb{B}\setminus B_m \ \& \\ \underbrace{\partial B'_m \cap I \neq \phi \text{ for all } \omega\text{-ic. } I \text{ in } \mathbb{B}\setminus B_m}_{= F_n} \end{array}\right)$$

On the other hand $G_m \overset{\text{def}}{=} \{ \omega \equiv 1 \text{ on } B_m \}$ is indep of F_n

and $P(G_m) = p^{|B_m|} > 0$. Therefore explained by picture (next page)

$$0 < P(G_m)\, P(F_n) = P(G_m \cap F_n) \le P(N_\infty(\omega) = 1) \quad \text{— (5)}$$



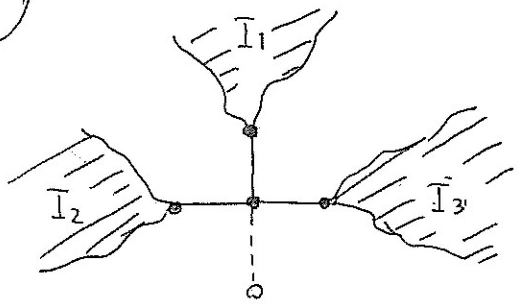
$(2), (5) \Rightarrow k = 1$

//

Preparation for the proof of Prop. 3.3 [Burton-Keane 1989]

$\left\{\begin{array}{l} x \in \mathbb{Z}^d, \ \omega \in \Omega, \\ \triangleright\ x \text{ is a } \omega\text{-}\underline{\underline{\text{trifurcation}}} \\ \quad \Longleftrightarrow \left\{\begin{array}{l} \exists\ \omega\text{-i.c.'s } I_j \ (j = 1, 2, 3) \\ \text{s.t. } \{ b \in C_x(\omega) \,;\ b \not\ni x \} = \bigcup_{j=1}^{3} I_j \end{array}\right. \end{array}\right.$

in $\{ b \in \mathbb{B},\ b \not\ni x \}$



$\left\{\begin{array}{l} x \in \mathbb{Z}^d \\ \triangleright\ T_x = \{ \omega \in \Omega : x \text{ is a } \omega\text{-trifurcation} \} \end{array}\right.$

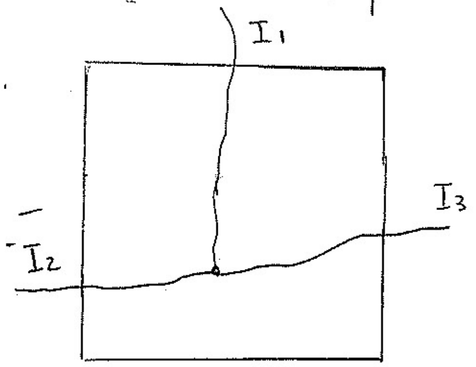
Lem 3.4 $\forall x \in \mathbb{Z}^d, \quad P(T_x) = P(T_0) = 0$

Proof Since $T_x = \tau_{-x} T_0$, we have $P(T_x) \overset{\text{Prop3:1}}{=} P(T_0)$ $\quad V_n = \mathbb{Z}^d \cap [-n,n]^d$

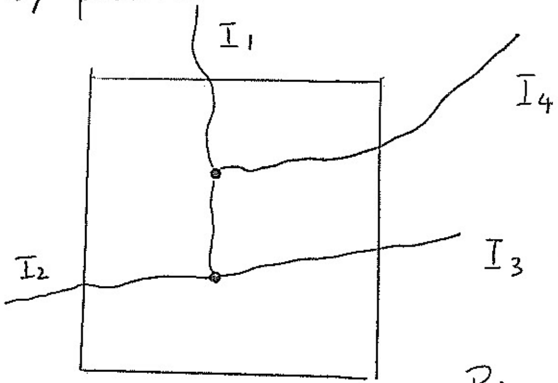
⑪-1

$$\underbrace{\sum_{x \in V_n} \mathbb{1}_{T_x}(\omega)}_{\substack{\parallel \\ \#\text{ of } \omega\text{-trifurcations in } V_n}} = k \Longrightarrow \#\text{ of } \omega\text{-i.c's in } \mathbb{B} \setminus B_n \geq k+2$$

☺ Explanation by pictures



$k = 1$



$k = 2$

... and so on

Rigorous proof → [Grimmett, P200-]

// ⑪-1

⑪-2 $$\sum_{x\in V_n} \mathbb{1}_{T_x}(\omega) + 2 \leqslant \# \partial B_n$$

(∵) Since distinct ω-i.c.'s in $\mathbb{B}\setminus B_n$ are disjoint,

$$\underbrace{\text{each } b\in\partial B_n \text{ can be contained in at most one } \omega\text{-i.c. in } \mathbb{B}\setminus B_n}_{(1)}$$

Thus

$$\#\partial B_n \overset{(1)}{\geqslant} \#\text{ of } \omega\text{-i.c.'s in } \mathbb{B}\setminus B_n \overset{\text{⑪-1}}{\overset{\downarrow}{\geqslant}} \sum_{x\in V_n} \mathbb{1}_{T_x} + 2$$

// ⑪-2

⑪-3 $P(T_0) = 0$

(∵) By Prop 2-4, $\frac{1}{(2n+1)^d}\sum_{x\in V_n}\mathbb{1}_{T_x}(\omega) \xrightarrow{n\to\infty} P(T_0)$ in $L^2(P)$

On the other hand, $\sum_{x\in V_n}\mathbb{1}_{T_x}(\omega) \overset{\text{⑪-2}}{\leqslant} \#\partial B_n = O(n^{d-1})$. Thus, $P(T_0) = 0$ // ⑪-3

//

Proof of Prop 3-3

We assume that $P(N_\infty(\omega) \geqslant 3) > 0$ and conclude that $P(T_0) > 0$, which contradicts with Lem 3.4. Note that $\{N_\infty(\omega) \geqslant 3\} \in \mathcal{T}$.

Thus, $P(N_\infty(\omega) \geqslant 3) = 1$ by Prop. 2 3

⑪ - 1

$$\begin{cases} IC_n(\omega) = \{ I : I \text{ is an } \omega\text{-l.c. in } \mathbb{B} \setminus B_n \text{ s.t. } I \cap \partial B_n \neq \phi \} \\ \Longrightarrow \exists n_0 \in \mathbb{N},\ \forall n \geqslant n_0,\ P(\# IC_n(\omega) \geqslant 3) \geqslant \frac{1}{2} \end{cases}$$

∵ same as in the proof of Prop 3.2 // ⑪ - 1

Let $\omega_n = (\omega(b))_{b \in B_n} \in \{0,1\}^{B_n}$, $\omega_n' = (\omega(b))_{b \in \mathbb{B} \setminus B_n} \in \{0,1\}^{\mathbb{B} \setminus B_n}$

Then $\omega = (\omega_n, \omega_n')$ and $IC_n(\omega) = IC_n(\omega_n')$

⑪-2 $\# IC_n(\omega) \geqslant 3 \Longrightarrow \exists \eta = \eta(\omega_n') \in \{0,1\}^{B_n}$ s.t. $(\eta, \omega_n') \in T_0$

∵ Suppose that $I_j \in IC_n(\omega)$ $(j=1,2,3)$

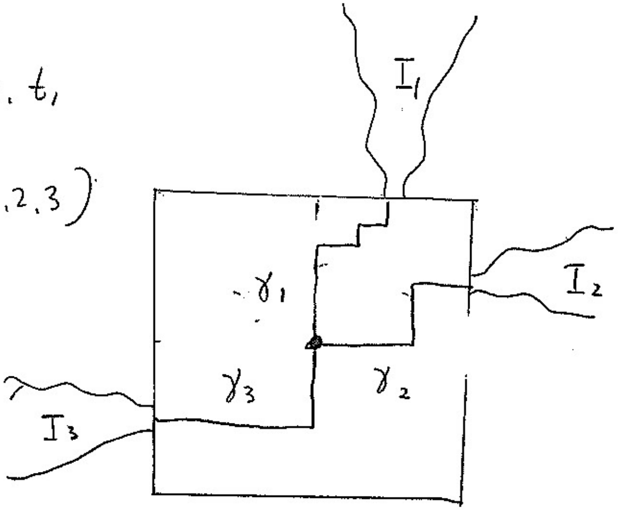
Then $\exists \eta \in \{0,1\}^{B_n}$ s.t.

$\exists$ η-open path $\gamma_j \subset B_n$ $(j=1,2,3)$ s.t.

i) 0 is connected to $I_j \cap \partial B_n$ by γ_j $(j=1,2,3)$

ii) $j \neq k \Rightarrow \gamma_j \cap \gamma_k = \{0\}$

iii) $b \in B_n \setminus \bigcup_{j=1}^{3} \gamma_j \Longrightarrow \eta(b) = 0$



Thus; $(\eta, \omega_m') \in T_0$ // ⑪-2

⑪-3 $P(T_0) > 0$ ⑪-2

∵ $P(T_0) \geq P(\#IC_m(\omega) \geq 3, \omega_m = \eta(\omega_m'))$

indep

$$\geq \min_{\eta \in \{0,1\}^{B_m}} P(\#IC_m(\omega) \geq 3, \omega_m \equiv \eta)$$

$$= P(\#IC_m \geq 3) \cdot \min_{\eta \in \{0,1\}^{B_m}} P(\omega_m \equiv \eta)$$

⑪-2

$$\geq \frac{1}{2} \min_{\eta \in \{0,1\}^{B_m}} P(\omega_m \equiv \eta) > 0$$

// ⑪-3

//