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主　論　文　の　要　旨

論文題目　On the Distribution of Zeros of the Derivatives of Dirichlet L-Functions

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論　文　内　容　の　要　旨

This thesis is about the distribution of zeros of Dirichlet L-functions and their derivatives associated with primitive Dirichlet characters.

Dirichlet L-functions are L-functions which are generalizations of the Riemann zeta function defined by B. Riemann as a complex meromorphic function. The Riemann zeta function $\zeta(s)$ was first known through Basel's problem solved by L. Euler in 1735. It is a function of s defined by the series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots$$

which converges when $s > 1$. Only the values of $\zeta(s)$ at positive integer points had been considered until Riemann [Rie59] defined it for complex variable s satisfying $\mathrm{Re}(s) > 1$ in 1859. Riemann used analytic methods to continue this function to the whole complex plane $\mathbb{C}$ except for a simple pole at $s = 1$. Riemann noticed that the distribution of some zeros of $\zeta(s)$ is closely related to the distribution of prime numbers and he proposed that all of these related zeros must lie on a straight line. This conjecture is well-known as the Riemann hypothesis (see Chapter 2 Section 2.2).

Dirichlet L-functions $L(s, \chi)$ are generalization of $\zeta(s)$ by using Dirichlet characters χ for some modulo q. They were first introduced by P. G. L. Dirichlet [Dir37] in 1837 for positive integer s in order to prove the infinitude of primes on arithmetic progressions which is later known as Dirichlet's theorem on primes in arithmetic progressions. For each character χ, $L(s, \chi)$ is analytically continued to $\mathbb{C}$ in a similar manner as $\zeta(s)$, except that it becomes an entire function on $\mathbb{C}$ when χ is non-principal (see Chapter 1 Section 1.3).

As in the case of $\zeta(s)$, for primitive characters χ, the distribution of some zeros of $L(s, \chi)$ is shown to be closely related to the distribution of prime numbers in arithmetic progressions. We note that there exists only one Dirichlet L-function modulo 1, the Riemann zeta function $\zeta(s)$. The Riemann hypothesis is expected to also hold for these L-functions, the conjecture, combined with the Riemann hypothesis itself, is commonly called the generalized Riemann hypothesis (see Chapter 2 Section 2.3).

It is known that the distribution of zeros of Dirichlet L-functions is related to the distribution of zeros of their derivatives. A. Speiser [Spe35] in 1935 showed that the

Riemann hypothesis is equivalent to the assertion that the first derivative of $\zeta(s)$ has no non-real zeros in $\mathrm{Re}(s) < 1/2$, a striking result that invited analytic number theorists' attention to the study of the distribution of zeros of the derivatives of $\zeta(s)$. A stronger result was obtained by N. Levinson and H. L. Montgomery in [LM74, Theorem 1]. The author and her collaborator H. Akatsuka [AS-p] showed this type of equivalence for $L(s,\chi)$ associated with primitive characters χ modulo $q > 1$ (Chapter 4 Section 4.2).

Zero-free regions of $\zeta^{(k)}(s)$, the k-th derivative of $\zeta(s)$ for any positive integer k, were first studied by R. Spira [Spi65, Spi70, Spi73]. B. C. Berndt [Ber70] in 1970 investigated the number of zeros, and in 1974, Levinson and Montgomery [LM74] studied the real part distribution of zeros of $\zeta^{(k)}(s)$. In 1996, C. Y. Yıldırım investigated the zeros of $L^{(k)}(s,\chi)$ associated with primitive characters χ modulo $q > 1$ in [Yil96b] and the zeros of the $\zeta''(s)$ and $\zeta'''(s)$ in [Yil96a, Yil00].

In 2012, Akatsuka [Aka12], assuming the Riemann hypothesis, improved some of the above mentioned results for $\zeta'(s)$. The author showed that analogous results hold for any $\zeta^{(k)}(s)$ in [Sur15] (Chapter 3 Section 3.2) and for $L'(s,\chi)$ associated with primitive characters χ modulo $q > 1$ in [Sur-p2] (Chapter 4 Section 4.3). The author and Akatsuka [AS-p] improved the zero-free region obtained by Yıldırım [Yil96b, Theorem 3] and showed unconditional results for the number of zeros and the distribution of the real part of zeros of $L'(s,\chi)$ (Chapter 4 Section 4.2).

The study of zeros of zeta functions and L-functions is not limited to the zeros themselves. It is also important to consider the value distribution of these functions, especially near the regions which are expected to have lots of zeros. The author is interested in studying the value distribution of zeta functions and L-functions along with their derivatives under some specific ergodic transformations. In 2009, M. Lifshitz and M. Weber [LW09] investigated the value distribution of $\zeta(s)$ by using the Cauchy random walk. Recently, T. Srichan [Sri15] investigated analogous results for $L(s,\chi)$ and Hurwitz zeta functions. They showed that these functions have small value in average on the critical line $\mathrm{Re}(s) = 1/2$.

J. Steuding [Ste12] in 2012 studied the ergodic value distribution of $\zeta(s)$ on vertical lines under the Boolean transformation. The author and her collaborator J. Lee in [LS-p] considered the value distribution of a larger class of meromorphic functions which includes but is not limited to the Selberg class (of zeta functions and L-functions) and their derivatives, on vertical lines under more general Boolean transformations (Chapter 5).

In Chapter 1 we first introduce some preliminary concepts on the study of zeta functions and L-functions, especially Dirichlet L-functions. We will mainly focus on their analytic properties. In Chapter 2 we introduce some results on their zeros. In Chapters 3 and 4, we introduce some results on the zeros of their derivatives, including the author's results, as mentioned in previous paragraphs. Finally in Chapter 5, we introduce the author's further research topic on an ergodic value distribution of zeta functions and L-functions.