On the Distribution of Zeros of the Derivatives of Dirichlet L-Functions

by

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Abstract

Zeros of the Riemann zeta function and its derivatives have been studied by many mathematicians. Among, zero-free regions, the number of zeros, and the distribution of the real part of non-real zeros of the derivatives of the Riemann zeta function have been investigated by R. Spira, B. C. Berndt, N. Levinson, H. L. Montgomery, and H. Akatsuka. Berndt, Levinson, and Montgomery investigated the general case, while Akatsuka gave sharper estimates for the first derivative of the Riemann zeta function under the truth of the Riemann hypothesis. Analogous results were also obtained by C. Y. Yıldırım for other Dirichlet L-functions associated with primitive Dirichlet characters. Yıldırım studied zero-free regions and the number of zeros of the derivatives of Dirichlet L-functions associated with primitive Dirichlet characters of modulo $q > 1$. In this dissertation, we briefly introduce these results and present the author's results on the zeros of higher order derivatives of the Riemann zeta function and of the first derivative of the Dirichlet L-functions associated with primitive Dirichlet characters.

We also present the author's collaborative result on an ergodic value distribution of a large class of zeta functions and L-functions. The value distributions of the Riemann zeta function, Dirichlet L-functions, and Hurwitz zeta functions were studied by M. Lifshitz, M. Weber, and T. Srichan by using the Cauchy random walk. Their results showed that the values of these functions are small on average, especially on the critical line. J. Steuding investigated an ergodic value distribution of the Riemann zeta function on vertical lines under the Boolean transformation. We are interested in extending this result of Steuding to a larger class of functions under more general transformations.

Contents

Preface

This thesis is about the distribution of zeros of Dirichlet L-functions and their derivatives associated with primitive Dirichlet characters.

Dirichlet L-functions are L-functions which are generalizations of the Riemann zeta function defined by B. Riemann as a complex meromorphic function. The Riemann zeta function $\zeta(s)$ was first known through Basel's problem solved by L. Euler in 1735. It is a function of s defined by the series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots$$

which converges when $s > 1$. Only the values of $\zeta(s)$ at positive integer points had been considered until Riemann [Rie59] defined it for complex variable s satisfying $\mathrm{Re}(s) > 1$ in 1859. Riemann used analytic methods to continue this function to the whole complex plane $\mathbb{C}$ except for a simple pole at $s = 1$. Riemann noticed that the distribution of some zeros of $\zeta(s)$ is closely related to the distribution of prime numbers and he proposed that all of these related zeros must lie on a straight line. This conjecture is well-known as the Riemann hypothesis (see Chapter 2 Section 2.2).

Dirichlet L-functions $L(s, \chi)$ are generalization of $\zeta(s)$ by using Dirichlet characters χ for some modulo q. They were first introduced by P. G. L. Dirichlet [Dir37] in 1837 for positive integer s in order to prove the infinitude of primes on arithmetic progressions which is later known as Dirichlet's theorem on primes in arithmetic progressions. For each character χ, $L(s, \chi)$ is analytically continued to $\mathbb{C}$ in a similar manner as $\zeta(s)$, except that it becomes an entire function on $\mathbb{C}$ when χ is non-principal (see Chapter 1 Section 1.3).

As in the case of $\zeta(s)$, for primitive characters χ, the distribution of some zeros of $L(s, \chi)$ is shown to be closely related to the distribution of prime numbers in arithmetic progressions. We note that there exists only one Dirichlet L-function modulo 1, the Riemann zeta function $\zeta(s)$. The Riemann hypothesis is expected to also hold for these L-functions, the conjecture, combined with the Riemann hypothesis itself, is commonly called the generalized Riemann hypothesis (see Chapter 2 Section 2.3).

It is known that the distribution of zeros of Dirichlet L-functions is related to the distribution of zeros of their derivatives. A. Speiser [Spe35] in 1935 showed that the Riemann hypothesis is equivalent to the assertion that the first derivative of $\zeta(s)$ has no non-real zeros in $\mathrm{Re}(s) < 1/2$, a striking result that invited analytic number theorists' attention to the study of the distribution of zeros of the derivatives of $\zeta(s)$. A stronger result was obtained by N. Levinson and H. L.

Montgomery in [LM74, Theorem 1]. The author and her collaborator H. Akatsuka [AS-p] showed this type of equivalence for $L(s,\chi)$ associated with primitive characters χ modulo $q > 1$ (Chapter 4 Section 4.2).

Zero-free regions of $\zeta^{(k)}(s)$, the k-th derivative of $\zeta(s)$ for any positive integer k, were first studied by R. Spira [Spi65, Spi70, Spi73]. B. C. Berndt [Ber70] in 1970 investigated the number of zeros, and in 1974, Levinson and Montgomery [LM74] studied the real part distribution of zeros of $\zeta^{(k)}(s)$. In 1996, C. Y. Yıldırım investigated the zeros of $L^{(k)}(s,\chi)$ associated with primitive characters χ modulo $q > 1$ in [Yıl96b] and the zeros of the $\zeta''(s)$ and $\zeta'''(s)$ in [Yıl96a, Yıl00].

In 2012, Akatsuka [Aka12], assuming the Riemann hypothesis, improved some of the above mentioned results for $\zeta'(s)$. The author showed that analogous results hold for any $\zeta^{(k)}(s)$ in [Sur15] (Chapter 3 Section 3.2) and for $L'(s,\chi)$ associated with primitive characters χ modulo $q > 1$ in [Sur-p2] (Chapter 4 Section 4.3). The author and Akatsuka [AS-p] improved the zero-free region obtained by Yıldırım [Yıl96b, Theorem 3] and showed unconditional results for the number of zeros and the distribution of the real part of zeros of $L'(s,\chi)$ (Chapter 4 Section 4.2).

The study of zeros of zeta functions and L-functions is not limited to the zeros themselves. It is also important to consider the value distribution of these functions, especially near the regions which are expected to have lots of zeros. The author is interested in studying the value distribution of zeta functions and L-functions along with their derivatives under some specific ergodic transformations. In 2009, M. Lifshitz and M. Weber [LW09] investigated the value distribution of $\zeta(s)$ by using the Cauchy random walk. Recently, T. Srichan [Sri15] investigated analogous results for $L(s,\chi)$ and Hurwitz zeta functions. They showed that these functions have small value in average on the critical line $\mathrm{Re}(s) = 1/2$.

J. Steuding [Ste12] in 2012 studied the ergodic value distribution of $\zeta(s)$ on vertical lines under the Boolean transformation. The author and her collaborator J. Lee in [LS-p] considered the value distribution of a larger class of meromorphic functions which includes but is not limited to the Selberg class (of zeta functions and L-functions) and their derivatives, on vertical lines under more general Boolean transformations (Chapter 5).

In Chapter 1 we first introduce some preliminary concepts on the study of zeta functions and L-functions, especially Dirichlet L-functions. We will mainly focus on their analytic properties. In Chapter 2 we introduce some results on their zeros. In Chapters 3 and 4, we introduce some results on the zeros of their derivatives, including the author's results, as mentioned in previous paragraphs. Finally in Chapter 5, we introduce the author's further research topic on an ergodic value distribution of zeta functions and L-functions.

Chapter 1

Preliminaries

In this chapter, we introduce some basic notions in the study of zeta functions and L-functions. Zeta functions and L-functions are often considered as complex meromorphic functions defined by some specific convergent series on some half-plane. These convergent series are called *Dirichlet series*. We first define and introduce a few basic properties of Dirichlet series. The rest of the chapter will be dedicated to introduce the two most basic functions defined by using Dirichlet series, the *Riemann zeta function* and *Dirichlet L-functions*.

1.1 Dirichlet series

In this dissertation, we define a Dirichlet series as a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.1}$$

where the coefficients a_n are any given numbers and s is a complex variable. We usually consider a Dirichlet series as a function of s in the region where the series is convergent. In other words, suppose that the series (1.1) converges (but not necessarily absolutely) for s satisfying $\mathrm{Re}(s) > \sigma_c$ and diverges if $\mathrm{Re}(s) < \sigma_c$ for some $\sigma_c \in \mathbb{R}$, then we consider (1.1) as a function of s in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma_c\}$. Referring to [Tit39, Sections 9.11 and 9.12], the series (1.1) is an analytic function of s when $\mathrm{Re}(s) > \sigma_c$. The line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = \sigma_c\}$ is called the *abscissa of convergence*, and the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma_c\}$ called the *half-plane of convergence* of the Dirichlet series (1.1). In the rest of this thesis, we often use the notation for a line and a half-plane as $\mathrm{Re}(s) = c$ and $\mathrm{Re}(s) > c$ respectively, for some $c \in \mathbb{R}$.

Suppose that there exists a real number $\sigma_a \in \mathbb{R}$ such that the Dirichlet series (1.1) is absolutely convergent in the half-plane $\mathrm{Re}(s) > \sigma_a$. Then the function

(1.1) is bounded in that *half-plane of absolute convergence* $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma_a\}$ (cf. [Tit39, Section 9.3]). That is to say that we can find an absolute constant $M > 0$ such that, for any $\sigma_a' > \sigma_a$,

$$\left| \sum_{n=1}^{\infty} \frac{a_n}{n^s} \right| \leq M$$

holds for all s satisfying $\mathrm{Re}(s) \geq \sigma_a'$.

Since a Dirichlet series can be regarded as an analytic function of some complex variable s, we are interested in analytically continuing it to a larger plane, such as the complex plane $\mathbb{C}$. On the line $\mathrm{Re}(s) = \sigma_c$, the series may not be convergent and thus may have singularities there. Therefore in most cases, we analytically continue a Dirichlet series into a meromorphic function on $\mathbb{C}$. We shall see concrete examples of these functions in later chapters, the Riemann zeta function and Dirichlet L-functions. We note that in Chapter 5, we may encounter more functions of this kind, namely zeta functions and L-functions, but we omit their details in this thesis.

It is interesting in the study of Dirichlet series, to see that many meromorphic functions $f(s)$ defined by some certain Dirichlet series in some half-plane $\sigma > \sigma_c$, satisfy the inequality of the form

$$f(\sigma + it) \ll |t|^{\nu(\sigma)}$$

for some function $\nu(\sigma)$ in another half-plane $\sigma > c$ which may be outside of the half-plane of convergence of the defining Dirichlet series. Here the sign $\ll$ is a symbol equivalent to the Landau O-symbol, that is:

$$f(\sigma + it) = O\left(|t|^{\nu(\sigma)}\right).$$

In the rest of this thesis, we use both symbols accordingly. When studying the function $\nu(\sigma)$, the following lemma of L. E. Phragmén and E. L. Lindelöf is useful:

Lemma 1.1 (Phragmén-Lindelöf theorem; cf. [Tit39, Section 5.6])**.** *If $\phi(s)$ is regular and $O(e^{\epsilon|t|})$, for every positive ϵ, in the strip $\sigma_1 \leq \sigma \leq \sigma_2$, and*

$$\phi(\sigma_1 + it) = O(|t|^{k_1}), \quad \phi(\sigma_2 + it) = O(|t|^{k_2}),$$

then

$$\phi(\sigma + it) = O(|t|^{k(\sigma)})$$

uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$, $k(\sigma)$ being the linear function of σ which takes the values k_1, k_2 for $\sigma = \sigma_1, \sigma_2$, respectively.

We end our discussion on general Dirichlet series here. In the following sections, we introduce the Riemann zeta function and Dirichlet L-functions.

1.2 The Riemann zeta function

Definition 1 (Riemann zeta function)**.** The Riemann zeta function $\zeta(s)$ is a meromorphic function of s defined by the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.2}$$

for any $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > 1$.

We can easily check that the series (1.2) defining $\zeta(s)$ is absolutely and compactly uniformly convergent in that region and thus defines an analytic function in $\mathrm{Re}(s) > 1$.

Lemma 1.2 (Euler product; cf. [Tit86, Equation (1.1.2)])**.** *For $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > 1$, we have*

$$\zeta(s) = \prod_{p:primes} \frac{1}{1 - \frac{1}{p^s}}.$$

Proof of Lemma 1.2. This *Euler product* expansion of $\zeta(s)$ is easily shown by using the uniqueness of prime factorization of natural numbers. □

Applying Lemma 1.2 we easily obtain:

Corollary 1.3. *$\zeta(s)$ has no zeros in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.*

We first show that $\zeta(s)$ can be analytically continued to a larger half-plane by using the following expression:

Lemma 1.4. *For $\mathrm{Re}(s) > 1$,*

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s\int_1^{\infty} \frac{[x] - x + 1/2}{x^{s+1}} dx. \tag{1.3}$$

Here $[x]$ denotes the greatest integer t satisfying $t \leq x$.

For this purpose we invoke the following lemma:

Lemma 1.5 (Euler's summation formula; cf. [BD04, Lemma 3.12])**.** *Let $f(x)$ be a continuous function on $[a, b]$ with a piecewise continuous derivative and let c be a constant. Then*

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)dt - (t - [t] - c)f(t)\Big|_a^b + \int_a^b (t - [t] - c)f'(t)dt.$$

Proof of Lemma 1.5. See [BD04, Proof of Lemma 3.12 (p. 48)]. □

Proof of Lemma 1.4. We first recall the Dirichlet series expression (1.2) of $\zeta(s)$. We apply Lemma 1.5 with $a = \lim_{t\uparrow 1} t$, any $b > 1$, and $c = 1/2$. Letting $b \to \infty$, we immediately obtain (1.3). $\square$

We can check that the integral in (1.3) defines an analytic function on $\mathrm{Re}(s) > 0$. We note that the right hand side of (1.3) is analytic for $\mathrm{Re}(s) > 0$ except for one simple pole at $s = 1$. Hence Lemma 1.4 gives an analytic continuation of $\zeta(s)$ to $\{s \in \mathbb{C}\backslash\{1\} \mid \mathrm{Re}(s) > 0\}$.

Now since $\zeta(s)$ is analytic on $\{s \in \mathbb{C}\backslash\{1\} \mid \mathrm{Re}(s) > 0\}$, we can analytically continue $\zeta(s)$ to $\mathbb{C}\backslash\{1\}$ by using the following lemma:

Lemma 1.6 (Asymmetric functional equation; cf. [BD04, Theorem 8.1])**.** *$\zeta(s)$ is an analytic function on $\mathbb{C}\backslash\{1\}$ and it satisfies there the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s). \tag{1.4}$$

Here $\Gamma(s)$ is the Euler gamma function (cf. [Dav00, Chapter 10]).

Proof of Lemma 1.6. See [BD04, Proof of Theorem 8.1 (pp. 183–185)]. $\square$

From the asymmetric functional equation for $\zeta(s)$, we can easily deduce the symmetric form of the functional equation for $\zeta(s)$:

Corollary 1.7 (Symmetric functional equation; cf. [BD04, Theorem 8.2])**.**

$$\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1.5}$$

for any $s \in \mathbb{C}$.

Remarks. If we define the function $\xi(s)$ on $\mathbb{C}$ as

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

then $\xi(s)$ is an entire function and satisfies the functional equation $\xi(s) = \xi(1-s)$. Furthermore, $\xi(s)$ is real valued on the real axis and on the line $\mathrm{Re}(s) = 1/2$.

The function $\xi(s)$ is often called the completed Riemann zeta function. We also note that non-real zeros of $\xi(s)$ are completely determined by non-real zeros of $\zeta(s)$. In other words, when $\mathrm{Im}(s) \neq 0$ the following relation holds:

$$\xi(s) = 0 \quad \Longleftrightarrow \quad \zeta(s) = 0.$$

Now we have obtained an analytic continuation of $\zeta(s)$ to $\mathbb{C}\backslash\{1\}$ and we have also seen that the simple pole $s = 1$ is the only singularity of $\zeta(s)$. Thus from now on we speak of the Riemann zeta function $\zeta(s)$ as a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ and no other singularities, defined by the Dirichlet series (1.2) on $\mathrm{Re}(s) > 1$.

1.3 Dirichlet L-functions

Dirichlet L-functions are a family of functions defined in a manner similar to the Riemann zeta function by using *Dirichlet characters*. In this section, we briefly introduce some properties of Dirichlet L-functions as introduced in the previous section for the Riemann zeta function.

Before we define Dirichlet L-functions, we first define Dirichlet characters.

Definition 2 (Dirichlet character)**.** Let q be a positive integer. A Dirichlet character χ modulo q is a complex valued function defined on the set of all rational integers $\mathbb{Z}$ satisfying:

1. $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$,
2. $\chi(n + q) = \chi(n)$ for all $n \in \mathbb{Z}$,
3. $\chi(n) = 0$ for any $n \in \mathbb{Z}$ satisfying $(n, q) > 1$,
4. $\chi(n_0) \neq 0$ for some positive integer n_0.

Here (n, q) denotes the greatest common divisor of n and q. A character χ is said to be *non-principal* if there exists a positive integer n_1 s.t. $\chi(n_1) \neq 1$, otherwise we say that it is *principal*.

We can now define Dirichlet L-functions.

Definition 3 (Dirichlet L-functions)**.** The Dirichlet L-function $L(s, \chi)$ associated with a Dirichlet character χ is a meromorphic function of s defined by the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{1.6}$$

for any $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) > 1$. If χ is non-principal, then the defining series (1.6) converges in $\mathrm{Re}(s) > 0$.

Remarks. There exists only one Dirichlet character modulo 1 and the associated Dirichlet L-function is the Riemann zeta function. Thus we can say that Dirichlet L-functions are a family of functions generalized from the Riemann zeta function by using Dirichlet characters. Note that the Riemann zeta function is a *principal* Dirichlet L-function.

We can easily check that the series (1.6) defining $L(s, \chi)$ is absolutely and compactly uniformly convergent in $\mathrm{Re}(s) > 1$ and thus defines an analytic function there. If χ is non-principal, the series (1.6) defining $L(s, \chi)$ is not absolutely convergent in $\mathrm{Re}(s) > 0$, but is compactly uniformly convergent there, and thus defines an analytic function there.

Dirichlet L-functions also have Euler product expansions.

Lemma 1.8 (Cf. [MV06, Equation (4.21)]). *For $s \in \mathbb{C}$ satisfying* $\mathrm{Re}(s) > 1$*, we have*

$$L(s,\chi) = \prod_{p:primes} \frac{1}{1 - \frac{\chi(p)}{p^s}}.$$

From Lemma 1.8 we easily obtain:

Corollary 1.9. *$L(s,\chi)$ has no zeros in* $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.

We remark that all Dirichlet L-functions defined in Definition 3 can be analytically continued to $\mathbb{C}$, except possibly for a simple pole at $s = 1$. From now on, we speak of Dirichlet L-functions $L(s,\chi)$ as these meromorphic functions on $\mathbb{C}$. When χ is principal, $L(s,\chi)$ is a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ as its only singularity. When χ is non-principal, $L(s,\chi)$ is an entire function on $\mathbb{C}$.

Besides the Riemann zeta function (Dirichlet L-function of modulo 1), we are especially interested in Dirichlet L-functions associated with *primitive* Dirichlet characters. The reason can be seen from Lemma 1.11 below.

Definition 4 (primitive Dirichlet character). A non-principal Dirichlet character χ modulo q is said to be primitive if for any proper divisor d of q (that is, d is a positive integer satisfying $d|q$ and $d < q$), there exists some integer $n \equiv 1 \mod d$ such that $(n,q) = 1$ and $\chi(n) \neq 1$.

Each Dirichlet character which is not primitive can be expressed by a unique primitive character:

Lemma 1.10 (Cf. [MV06, Equation (9.1) and Theorem 9.2]). *Let χ be a Dirichlet character modulo q. Then there exists a unique primitive Dirichlet character χ^* modulo d for some $d|q$ such that*

$$\chi(n) = \begin{cases} \chi^*(n) & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, every Dirichlet L-function associated with an *imprimitive* Dirichlet character χ can be expressed by a unique Dirichlet L-function associated with a primitive character as in the following lemma.

Lemma 1.11 (Cf. [MV06, Equation (10.20)]). *Let χ be a Dirichlet character modulo q. Then there exists a primitive Dirichlet character χ^* modulo d for some $d|q$ such that*

$$L(s,\chi) = L(s,\chi^*) \prod_{\substack{p|q, \\ p:primes}} \left(1 - \frac{\chi^*(p)}{p^s}\right)$$

This lemma implies that it is sufficient for us to study Dirichlet L-functions associated with primitive characters.

As in Lemma 1.6, Dirichlet L-functions associated with primitive Dirichlet characters can also be analytically continued to $\mathbb{C}$ by using functional equations:

Lemma 1.12 (Cf. [MV06, Corollary 10.8])**.** *Let χ be a primitive Dirichlet character. $L(s,\chi)$ is an entire function on $\mathbb{C}$ and it satisfies there the functional equation*

$$L(s,\chi) = \epsilon(\chi)2^s\pi^{s-1}q^{1/2-s}\sin\left(\frac{\pi(s+\kappa)}{2}\right)\Gamma(1-s)L(1-s,\overline{\chi}), \tag{1.7}$$

where $\epsilon(\chi)$ is a factor that depends only on χ, satisfying $|\epsilon(\chi)| = 1$,

$$\kappa := \begin{cases} 0 & \text{when } \chi(-1) = 1; \\ 1 & \text{when } \chi(-1) = -1, \end{cases}$$

and $\Gamma(s)$ is the Euler gamma function as in Lemma 1.6.

As in the case of $\zeta(s)$, we can define the function $\xi(s,\chi)$ on $\mathbb{C}$ as

$$\xi(s,\chi) := \left(\frac{\pi}{q}\right)^{-(s+\kappa)/2}\Gamma\left(\frac{s+\kappa}{2}\right)L(s,\chi).$$

By using Lemma 1.12, we can easily show that $\xi(s,\chi)$ is an entire function satisfying the functional equation $\xi(s,\chi) = \epsilon(\chi)\xi(1-s,\overline{\chi})$. This gives the symmetric form of the functional equation for $L(s,\chi)$ associated with a primitive Dirichlet character χ.

Remark. In our definition of primitive characters (Definition 4), a primitive character is always non-principal. However, some texts treat the Dirichlet character modulo 1 as a primitive character. In later chapters, to avoid confusion, we mention "primitive Dirichlet character modulo $q > 1$" instead of only "primitive Dirichlet character".

Chapter 2

Zeros of the Riemann zeta function and Dirichlet L-functions

In this chapter we introduce some results on the distribution of zeros of the Riemann zeta function and Dirichlet L-functions. We will see in later chapters that many analogous results are obtained for their derivatives. Before we begin with the discussion on zeros, we introduce some useful tools in studying the distribution of zeros.

2.1 Some tools

In this section we introduce some tools we use for counting the number of zeros of meromorphic functions with proofs omitted.

The first lemma is due to J. Jensen. We state here the lemma in the form convenient for our purpose.

Lemma 2.1 (Jensen's theorem; cf. [Tit39, Section 3.61]). *Let $f(z)$ be analytic for $|z| < R$ and suppose that $f(0) \neq 0$. Let $n(x)$ denote the number of zeros of $f(z)$ in the disc $|z| \leq x$, then if $r < R$,*

$$\int_0^r \frac{n(x)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta - \log |f(0)|.$$

It is frequently convenient to count the number of zeros in a rectangle. The following lemma is due to J. E. Littlewood.

Lemma 2.2 (Littlewood's lemma; cf. [Tit39, Section 3.8]). *Let C denote the rectangle bounded by the lines $x = x_1$, $x = x_2$, $y = y_1$, and $y = y_2$, where $x_1 < x_2$, $y_1 < y_2$. Let $f(z)$ be analytic and not zero on C, and meromorphic inside it. We define the logarithm $\log f(z)$ by continuous variation along the line $y = y_0$ from*

$\log f(x_2 + iy_0)$ for $y_1 \le y_0 \le y_2$, provided that $[x + iy_0, x_2 + iy_0]$ does not contain any zero or pole of $f(z)$. Otherwise, we put $\log f(z) = \log f(z - i0)$.

Let $\nu(x')$ denote the number of zeros of $f(z)$ subtracted from the number of poles in the part of the rectangle with $x > x'$ (counted with multiplicity). Then

$$\int_C \log f(z)dz = -2\pi i \int_{x_1}^{x_2} \nu(x)dx = -2\pi i \sum_{\substack{\beta+i\gamma,\\ f(\beta+i\gamma)=0,\\ x_1<\beta<x_2,\, y_1<\gamma<y_2}} (\beta - x_1),$$

where the sum is counted with multiplicity.

2.2 Zeros of the Riemann zeta function and the Riemann hypothesis

From Lemma 1.6, we find that $\zeta(s) = 0$ for any negative even integer s ($s = -2, -4, -6, \cdots$) and we call these zeros the *trivial zeros* of $\zeta(s)$. Recall that Corollary 1.3 states that $\zeta(s) \neq 0$ when $\mathrm{Re}(s) > 1$. In view of the functional equation (1.4) (or (1.5)) for $\zeta(s)$, we find that $\zeta(s)$ is also nonzero when $\mathrm{Re}(s) < 0$ and $\mathrm{Im}(s) \neq 0$. Furthermore, referring to [BD04, Theorem 7.6], $\zeta(s) \neq 0$ when $\mathrm{Re}(s) = 1$ (note that $s = 1$ is a pole). Thus, again by the functional equation (1.4) (or (1.5)) for $\zeta(s)$, $\zeta(s) \neq 0$ for $\mathrm{Re}(s) = 0$. Therefore all other zeros, if exist (in fact, they exist (cf. [Hav03, Section 16.6])), they must all lie in the strip $0 < \mathrm{Re}(s) < 1$. It is further shown that

Theorem 2.3 (Cf. [BD04, Theorem 8.5])**.** *$\zeta(s)$ has infinitely many zeros in the strip $0 < \mathrm{Re}(s) < 1$.*

We call this strip the *critical strip* and we call the zeros in this strip the *non-trivial zeros* of $\zeta(s)$. We also note that all nontrivial zeros of $\zeta(s)$ are non-real, while all trivial zeros of $\zeta(s)$ are real as stated earlier. That is to say that "non-trivial zeros of $\zeta(s)$" and "non-real zeros of $\zeta(s)$" are equivalent terms. However, the exact location of these zeros remains an unsolved problem.

It is conjectured that all nontrivial zeros of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = 1/2$, called the *critical line*. This conjecture was proposed by B. Riemann in 1859 and is known as the *Riemann hypothesis* (cf. [BD04, p. 191], [Dav00, p. 60], or [MV06, p. 328]). This conjecture still remains unsolved and has been one of the strongest motivations in the study of the Riemann zeta function, especially for its close relation with the distribution of prime numbers. We shall not discuss this further, but we remark that the Riemann hypothesis gives the best possible estimate for the number of primes as shown by N. F. H. von Koch in 1901, more precisely:

Theorem 2.4 (Cf. [Koc01, pp. 181–182] or [MV06, Theorem 13.1 and the first line in Section 13.3])**.** *Let $\pi(x)$ denote the number of prime numbers at most x, and let*

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

Assume that the Riemann hypothesis is true. Then for $x \geq 2$,

$$\pi(x) = \mathrm{Li}(x) + O\left(x^{1/2} \log x\right).$$

It is known that the best possible error term in the above equation can be formulated as:

$$O\left(x^{1/2-\epsilon}\right)$$

for any $\epsilon > 0$ (cf. [MV06, Theorem 15.2 and Corollary 15.4]). Therefore, the above theorem also implies that the Riemann hypothesis gives the best possible version of the prime number theorem.

As we have seen earlier, $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \leq 0, s \neq -2, -4, -6, -8, \cdots\} \cup \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1\}$ is a trivial zero-free region for $\zeta(s)$. Below we introduce a more precise zero-free region for $\zeta(s)$.

Theorem 2.5 (Cf. [BD04, Theorem 8.8], [Dav00, Chapter 13], or [MV06, Theorem 6.6])**.** *There exists a constant $K > 0$ such that $\zeta(s) \neq 0$ in the region*

$$\left\{ s = \sigma + it \in \mathbb{C} \mid \sigma > 1 - \frac{K}{\log\left(|t| + 2\right)} \right\}.$$

We are interested in studying the distribution of the real part and the number of non-real zeros of the derivatives of the Riemann zeta function. Here we briefly introduce the corresponding results on the Riemann zeta function itself.

It is not difficult to see from the symmetric functional equation (1.5) that nontrivial zeros of $\zeta(s)$ are symmetric with respect to the critical line $\mathrm{Re}(s) = 1/2$. We also remark that they are symmetric with respect to the real line $\mathrm{Im}(s) = 0$, thus, recalling that they are non-real, we find that it is sufficient to study the nontrivial zeros of $\zeta(s)$ in the upper half-plane $\mathrm{Im}(s) > 0$. Since they all lie in the critical strip $0 < \mathrm{Re}(s) < 1$, we immediately obtain:

Theorem 2.6. *For $T > 0$, we have*

$$\sum_{\substack{\rho=\beta+i\gamma,\\ \zeta(\rho)=0,\, 0<\gamma\leq T}} \left(\beta - \frac{1}{2}\right) = 0$$

where the sum is counted with multiplicity.

We shall see in the next chapter that non-real zeros of the derivatives of $\zeta(s)$ are not so beautifully distributed as in Theorem 2.6 around the critical line.

Finally, we close this section by introducing two known results on the number of zeros of $\zeta(s)$. We let $N(T)$ denote the number of zeros of $\zeta(s)$ with $0 < \mathrm{Im}(s) \le T$, counted with multiplicity. The first result is due to H. C. F. von Mangoldt, proven in 1905.

Theorem 2.7 (Cf. [Dav00, pp. 59–60 and Chapter 15] or [MV06, Corollary 14.3])**.** *For $T \ge 2$, we have*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Assuming the truth of the Riemann hypothesis, we have a better estimate as shown by Littlewood in 1924:

Theorem 2.8 (Cf. [Lit24, Theorem 11] or [MV06, Corollary 14.4])**.** *Assume that the Riemann hypothesis is true. Then for $T \ge 2$,*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O\left(\frac{\log T}{\log\log T}\right).$$

2.3 Zeros of Dirichlet L-functions and the generalized Riemann hypothesis

In this section, we consider Dirichlet L-functions $L(s,\chi)$ associated with primitive Dirichlet characters χ modulo $q > 1$. Note that χ is non-principal under this condition (recall Definition 4 and the last remark in Chapter 1).

From Lemma 1.12, $L(s,\chi) = 0$ for $s = -\kappa, -\kappa-2, -\kappa-4, -\kappa-6, \cdots$, where κ is determined for each χ as in Lemma 1.12. These zeros are called the *trivial zeros* of $L(s,\chi)$. As in the case of $\zeta(s)$, we call all the other zeros the *nontrivial zeros* of $L(s,\chi)$. From Corollary 1.9 and the functional equation (1.7) for $L(s,\chi)$, we find that these nontrivial zeros of $L(s,\chi)$ must all lie in the strip $0 \le \mathrm{Re}(s) \le 1$. It is further known that $L(s,\chi) \ne 0$ on the lines $\mathrm{Re}(s) = 0, 1$ except at $s = 0$ itself (recall that this is a trivial zero when $\kappa = 0$). We can see this from a more precise zero-free region for $L(s,\chi)$ given in the following Theorem 2.9 and the functional equation (1.7). Hence all nontrivial zeros of $L(s,\chi)$ also lie in the critical strip $0 < \mathrm{Re}(s) < 1$.

Theorem 2.9 (Cf. [Dav00, Chapter 14] or [MV06, Theorem 11.3])**.** *There exists a constant $K > 0$ such that $L(s,\chi) \ne 0$ in the region*

$$\left\{ s = \sigma + it \in \mathbb{C} \mid \sigma > 1 - \frac{K}{\log\left(q(|t|+2)\right)} \right\}, \tag{2.1}$$

unless χ is a real non-principal character (also commonly called quadratic*, see [MV06, the first paragraph in Section 9.3]), in which case $L(s, \chi)$ has at most one real zero $\beta_0 < 1$ in the region* (2.1)*.*

An extension of the Riemann hypothesis, usually known as the *generalized Riemann hypothesis*, states that both $\zeta(s)$ and $L(s, \chi)$ satisfy the Riemann hypothesis, that is, all nontrivial zeros lie on the critical line $\mathrm{Re}(s) = 1/2$ (cf. [MV06, p. 333]). The truth of this hypothesis still remains unknown for both functions.

We are interested in studying the distribution of the real part and the number of non-real zeros of the derivatives of not only the Riemann zeta function, but also of Dirichlet L-functions associated with primitive characters. We close this section and this chapter by introducing the corresponding results on the Dirichlet L-functions themselves.

As in the case of $\zeta(s)$, the symmetric functional equation given by the function $\xi(s, \chi)$ results in the nontrivial zeros of $L(s, \chi)$ being symmetric with respect to the critical line $\mathrm{Re}(s) = 1/2$. However, we remark that they are ***not necessarily*** symmetric with respect to the real line $\mathrm{Im}(s) = 0$, thus we consider not only the nontrivial zeros of $L(s, \chi)$ in the upper half-plane, but also on the real line and in the lower half-plane. Since they all lie in the critical strip, we immediately obtain:

Theorem 2.10. *For $T > 0$, we have*

$$\sum_{\substack{\rho=\beta+i\gamma,\\ L(\rho,\chi)=0,\\ \beta>0,\, |\gamma|\le T}} \left(\beta - \frac{1}{2}\right) = 0$$

where the sum is counted with multiplicity.

Finally, analogous to the results we introduced for $\zeta(s)$, we introduce two known results on the number of zeros of $L(s, \chi)$. We let $N(T, \chi)$ denote the number of zeros of $L(s, \chi)$ with $0 < \mathrm{Re}(s) < 1$, $|\mathrm{Im}(s)| \le T$, counted with multiplicity.

Theorem 2.11 (Cf. [Dav00, Chapter 16] or [MV06, Corollary 14.7])**.** *For $T \ge 2$, we have*

$$N(T, \chi) = \frac{T}{\pi} \log \frac{qT}{2\pi} - \frac{T}{\pi} + O(\log (qT)).$$

Assuming the truth of the generalized Riemann hypothesis, we have a better estimate:

Theorem 2.12 (Cf. [MV06, Exercise 14.1.1])**.** *Assume that the generalized Riemann hypothesis is true. Then for $T \ge 2$,*

$$N(T, \chi) = \frac{T}{\pi} \log \frac{qT}{2\pi} - \frac{T}{\pi} + O\left(\frac{\log (qT)}{\log \log (qT)}\right).$$

Chapter 3

Zeros of the derivatives of the Riemann zeta function

In this chapter we introduce some results on the distribution of the k-th derivative of the Riemann zeta function, denoted by $\zeta^{(k)}(s)$ for positive integer k, especially on the distribution of the real part and the number of non-real zeros. We first introduce some known results in the first section. In Section 3.2, we introduce some conditional results and prove those which were shown by the author in [Sur15]. Throughout this chapter, only the results proven by the author are stated as theorems.

3.1 Unconditional results

We begin with our strongest motivation in studying zeros of the derivatives of the Riemann zeta function. A. Speiser [Spe35] in 1935 showed an equivalence between the distribution of nontrivial zeros of the Riemann zeta function $\zeta(s)$ and that of non-real zeros of its first derivative $\zeta'(s)$. More precisely, he proved that the Riemann hypothesis is equivalent to the statement that $\zeta'(s)$ has no non-real zeros in $\mathrm{Re}(s) < 1/2$.

In 1970, B. C. Berndt [Ber70, Theorem] proved that

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O(\log T) \tag{3.1}$$

where $N_k(T)$ denotes the number of zeros of $\zeta^{(k)}(s)$ with $0 < \mathrm{Im}(s) \le T$, counted with multiplicity. Further in 1974, N. Levinson and H. L. Montgomery [LM74, Theorem 10] showed that

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ \zeta^{(k)}(\rho^{(k)})=0,\,0<\gamma^{(k)}\le T}} \left(\beta^{(k)}-\frac{1}{2}\right) = \frac{kT}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)T - k\,\mathrm{Li}\left(\frac{T}{2\pi}\right) + O(\log T) \tag{3.2}$$

where the sum is counted with multiplicity and $\mathrm{Li}(x)$ is as defined in Theorem 2.4. In addition to the above result (3.2), Levinson and Montgomery [LM74] also studied the location of the zeros of $\zeta^{(k)}(s)$. There are many other papers on the zeros of $\zeta^{(k)}(s)$; for example, J. B. Conrey and A. Ghosh [CG90, Theorem 1] in 1989, studied the zeros of $\zeta^{(k)}(s)$ near the critical line.

3.2 Results obtained under the truth of the Riemann hypothesis

In 2012, H. Akatsuka [Aka12, Theorems 1 and 3] improved each of the error term of the results obtained by Berndt and by Levinson and Montgomery mentioned above (see (3.1) and (3.2)) for the case $k=1$ under the assumption of the truth of the Riemann hypothesis. More precisely, he showed that

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ \zeta'(\rho')=0,\,0<\gamma'\le T}} \left(\beta'-\frac{1}{2}\right) = \frac{T}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - \log\log 2\right)T - \mathrm{Li}\left(\frac{T}{2\pi}\right) + O((\log\log T)^2)$$

and

$$N_1(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O\left(\frac{\log T}{(\log\log T)^{1/2}}\right) \tag{3.3}$$

if the Riemann hypothesis is true. In this section[1], we generalize these two results of Akatsuka for any positive integer k.

Remark. Recently, F. Ge [Ge-p, Theorem 1] showed that we can improve the error term in (3.3) shown by Akatsuka [Aka12, Theorem 3] to

$$O\left(\frac{\log T}{\log\log T}\right).$$

[1] The content of this section is essentially the same as the manuscript [Sur15] published in *Functiones et Approximatio, Commentarii Mathematici* **53**, and is slightly modified in order to fit in the content and structure of this thesis.

This result is the current best estimate on the number of zeros of $\zeta'(s)$ under the Riemann hypothesis.

Throughout this section, the letter k is used as a fixed positive integer, unless otherwise specified. For simplicity, we denote by $\rho = \beta + i\gamma$ and $\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)}$ the nontrivial zeros of $\zeta(s)$ and the non-real zeros of $\zeta^{(k)}(s)$, respectively. In addition, $N(T)$ and $N_k(T)$ are as defined previously, that is they each count the number of zeros of $\zeta(s)$ and $\zeta^{(k)}(s)$, respectively, in $0 < \mathrm{Im}(s) \leq T$, with multiplicity.

The following results generalize Theorem 1, Corollary 2, and Theorem 3 of [Aka12], respectively. Note that each sum counts the non-real zeros of $\zeta^{(k)}(s)$ with multiplicity and that the implicit constant in $O_k(\cdot)$ depends only on k.

Theorem 3.1. *Assume that the Riemann hypothesis is true. Then for any $T > 2\pi$, we have*

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\leq T}} \left(\beta^{(k)} - \frac{1}{2}\right) = \frac{kT}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)T - k\,\mathrm{Li}\left(\frac{T}{2\pi}\right) + O_k((\log\log T)^2).$$

Corollary 3.2 (Cf. [LM74, Theorem 3])**.** *Assume that the Riemann hypothesis is true. Then for $0 < U < T$ (where T is restricted to satisfy $T > 2\pi$), we have*

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ T<\gamma^{(k)}\leq T+U}} \left(\beta^{(k)} - \frac{1}{2}\right) = \frac{kU}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)U + O\left(\frac{U^2}{T\log T}\right) + O_k\left((\log\log T)^2\right).$$

Here the implicit constant in the error term $O\left(U^2(T\log T)^{-1}\right)$ does not depend on any parameter.

Theorem 3.3. *Assume that the Riemann hypothesis is true. Then for $T \geq 2$, we have*

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O_k\left(\frac{\log T}{(\log\log T)^{1/2}}\right).$$

We write $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ (for any $s \in \mathbb{C}$) as σ and t, respectively. We abbreviate the Riemann hypothesis as RH, and finally, we define two functions $F(s)$ and $G_k(s)$ as follows:

Definition 5.

$$F(s) := 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s), \quad G_k(s) := (-1)^k \frac{2^s}{(\log 2)^k}\zeta^{(k)}(s).$$

By the above definition of $F(s)$, we can check easily that the functional equation for $\zeta(s)$ states

$$\zeta(s) = F(s)\zeta(1-s). \tag{3.4}$$

Remark. The function $F(s)$ appeared in [Aka12] and [LM74, Section 3] and the function $G_k(s)$ is the $\zeta^{(k)}$-version of the function $G(s)$ in [Aka12], which is denoted by $Z_k(s)$ in [LM74, Section 3]. Most of the symbols used in this section follow those used in [Aka12].

Since the steps of our proofs basically follow those given in [Aka12] with a few crucial modifications, instead of the outline of the proofs, below we present the main needed modifications related to the proofs.

First of all, condition 2 of Lemma 2.1 of [Aka12] is related to the functional equation for $\zeta'(s)$. In our case, we need to consider $\zeta^{(k)}(s)$ for any positive integer k. Thus, we obtain a function which consists of terms that are not logarithmic derivatives of some functions so we cannot easily follow the case of $\zeta'(s)$. In the present section, we take care of these terms in a way that does not involve any calculation on logarithmic derivatives.

Secondly, similar to condition 2, in condition 3 of Lemma 2.1 of [Aka12], the factor to be estimated was $(F'/F)(s)$ which is just the logarithmic derivative of $F(s)$, whereas in the present section, we need to take care of $(F^{(k)}/F)(s)$ which is not a logarithmic derivative of any function. Thus, as in condition 2, we estimate this term for any k in a way which does not require any calculation on logarithmic derivatives, and hence we need to take a suitable logarithmic branch of the function $\log(F^{(k)}/F)(s)$.

The next is condition 4 of Lemma 2.1 of [Aka12]. For $\zeta'(s)$, the term we need to estimate was $(\zeta'/\zeta)(s)$ which is just the logarithmic derivative of $\zeta(s)$. In [Aka12], the inequality $\mathrm{Re}((\zeta'/\zeta)(s)) < 0$ was obtained, however for $\zeta^{(k)}(s)$, the sign of $\mathrm{Re}\left((\zeta^{(k)}/\zeta)(s)\right)$ does not seem to stay unchanged in any region defined by $x \le \sigma < 1/2$, $t \ge y$ for some $x \le -1$ and large $y > 0$. Nevertheless, since it is sufficient to show that $(\zeta^{(k)}/\zeta)(s)$ is holomorphic and non-zero, and has bounded argument in some region of the above kind, we shall modify the condition in such a way.

Furthermore, with the modifications of these conditions of the first lemma, the choice of logarithmic branch of the function $\log\left(((F^{(k)}/F)(s))^{-1}(\zeta^{(k)}/\zeta)(s)\right)$ in the proof of Proposition 3.5 (which generalizes Proposition 2.2 in [Aka12]) must be taken more carefully so that these conditions can be used in our calculations. In order to evaluate the function $\log\left(((F^{(k)}/F)(s))^{-1}(\zeta^{(k)}/\zeta)(s)\right)$, we first define the functions $\log\big(((F^{(k)}/F)(s))^{-1}(\zeta^{(k)}/\zeta)(s)\big)$, $\log(F^{(k)}/F)(s)$, and $\log(\zeta^{(k)}/\zeta)(s)$ independently. Then using the continuities of $\arg\big(((F^{(k)}/F)(s))^{-1}(\zeta^{(k)}/\zeta)(s)\big)$, $\arg(F^{(k)}/F)(s)$, and $\arg(\zeta^{(k)}/\zeta)(s)$, we observe the difference

$$\arg\left(\frac{1}{\frac{F^{(k)}}{F}(s)}\frac{\zeta^{(k)}}{\zeta}(s)\right)-\left(-\arg\frac{F^{(k)}}{F}(s)+\arg\frac{\zeta^{(k)}}{\zeta}(s)\right)$$

in the region under evaluation (see the evaluation of I_{15} in Proposition 3.5).

Finally, the region $1/2<\sigma\le a$ considered in Lemma 2.3 of [Aka12] does not work well for $(\zeta^{(k)}/\zeta)(s)$. The reason is that the current best estimate of $(\zeta^{(k)}/\zeta)(s)$ depends on the usage of Cauchy's integral formula, hence we need to keep a certain distance between $1/2$ and the infimum of σ in the region. Therefore, we put here a small distance $\epsilon_0>0$ (see the statement of our Lemma 3.6).

3.2.1 Proof of Theorem 3.1 and Corollary 3.2

In this subsection we give the proofs of Theorem 1 and Corollary 2. For that purpose, we need a few lemmas and a proposition which are analogues of those in [Aka12]. The following lemma is a generalization of Lemma 2.1 of [Aka12] for the case of $\zeta^{(k)}(s)$.

Lemma 3.4. *Assume RH. Then there exist $a_k\ge 10$, $\sigma_k\le -1$, and $t_k\ge\max\{a_k^2,-\sigma_k\}$ such that the following conditions are satisfied:*

1. $|G_k(s)-1|\le\frac{1}{2}\left(\frac{2}{3}\right)^{\sigma/2}$, *for any $\sigma\ge a_k$;*

2. $\left|\sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)}\frac{\zeta^{(j)}}{\zeta}(1-s)\right|\le 2^{\sigma}$, *for $\sigma\le\sigma_k$ and $t\ge 2$;*

3. $\left|\frac{F^{(k)}}{F}(s)\right|\ge 1$ *holds in the region $\sigma_k\le\sigma\le 1/2$, $t\ge t_k-1$. Furthermore, we can take the logarithmic branch of $\log\frac{F^{(k)}}{F}(s)$ in that region such that it is holomorphic there and*

$$\frac{\alpha_k\pi}{6}<\arg\frac{F^{(k)}}{F}(s)<\frac{\beta_k\pi}{6}$$

holds, where

$$(\alpha_k,\beta_k)=\begin{cases}(5,7) & \text{if } k \text{ is odd,}\\ (-1,1) & \text{if } k \text{ is even;}\end{cases}$$

4. $\dfrac{\zeta^{(k)}}{\zeta}(s) \neq 0$ *holds in the region* $\sigma_k \leq \sigma < 1/2$, $t \geq t_k - 1$. *Furthermore, we can take the logarithmic branch of* $\log \dfrac{\zeta^{(k)}}{\zeta}(s)$ *in that region such that it is holomorphic there and*

$$\frac{k\pi}{2} < \arg \frac{\zeta^{(k)}}{\zeta}(s) < \frac{3k\pi}{2}$$

holds;

5. $\zeta(\sigma + it_k) \neq 0$, $\zeta^{(k)}(\sigma + it_k) \neq 0$, *for all* $\sigma \in \mathbb{R}$.

Proof. 1. See [LM74, (3.2) (p. 54)].

2. We start by estimating $(F^{(k)}/F^{(k-j)})(s)$ $(j = 1, 2, \cdots, k)$ in the region $\sigma < 1$, $t \geq 2$. We set

$$f(s) := \left(\frac{1}{2} - s\right)\left(\log(1-s) - \log(2\pi) + \frac{\pi i}{2}\right) + s + O(1),$$

where $f(s)$ is an analytic function and

$$f'(s) = -\log(1-s) + O(1), \qquad f^{(j)}(s) = O(1) \qquad (j \geq 2).$$

As in [LM74, pp. 54–55], we can write

$$F(s) = \exp(f(s)).$$

Using methods similar to [Gon84, Lemma 6 (p. 133)] and [LM74, pp. 54–55], we can show that

$$F^{(j)}(s) = F(s)(f'(s))^j \left(1 + O\left(\frac{1}{|\log s|^2}\right)\right) \tag{3.5}$$

holds for any positive integer j. In consequence, for $j = 1, 2, \cdots, k$, we have

$$\begin{aligned}\left|\frac{F^{(k)}}{F^{(k-j)}}(s)\right| &= \left|(f'(s))^j\left(1 + O\left(\frac{1}{|\log s|^2}\right)\right)\right| \\ &\geq (\log|1-s|)^j - \left|O\left((\log|1-s|)^{j-1}\right)\right|.\end{aligned}$$

Certainly, this also holds in the region $\sigma \leq -1$, $t \geq 2$, so for any positive integer k, we can take $\sigma_{k_1} \leq -1$ sufficiently small (i.e. sufficiently large in the negative direction) so that for any s with $\sigma \leq \sigma_{k_1}$ and $t \geq 2$, we have

$$\left|\frac{F^{(k)}}{F^{(k-j)}}(s)\right| \geq \frac{1}{2k}(\log|1-s|)^j \geq \frac{1}{2k}(\log(1-\sigma))^j. \tag{3.6}$$

Next we estimate $(\zeta^{(j)}/\zeta)(1-s)$ $(j=1,2,\cdots,k)$. In the region $\sigma \le -1$, $t \ge 2$, we have

$$\left|\zeta^{(j)}(1-s)\right| \le \left|\frac{(\log 2)^j}{2^{1-s}}\right| + \left|\sum_{n=3}^{\infty} \frac{(\log n)^j}{n^{1-s}}\right| \le \frac{1}{2}(\log 2)^j 2^\sigma + \int_2^\infty \frac{(\log x)^j}{x^{1-\sigma}}dx$$
$$= 2^\sigma \left(\frac{1}{2}(\log 2)^j + \sum_{l=0}^{j} \frac{(\log 2)^{j-l}\frac{j!}{(j-l)!}}{(-\sigma)^{l+1}}\right)$$

and

$$|\zeta(1-s)| \ge 1 - \left|\sum_{n=2}^{\infty} \frac{1}{n^{1-s}}\right| \ge 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} = 2 - \frac{\pi^2}{6}.$$

Thus,

$$\left|\frac{\zeta^{(j)}}{\zeta}(1-s)\right| \le \frac{2^\sigma}{2-\frac{\pi^2}{6}} \left(\frac{1}{2}(\log 2)^j + \sum_{l=0}^{j} \frac{(\log 2)^{j-l}\frac{j!}{(j-l)!}}{(-\sigma)^{l+1}}\right). \tag{3.7}$$

Now combining (3.6) and (3.7), for $\sigma \le \sigma_{k_1}$ and $t \ge 2$, we have

$$\left|\sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)} \frac{\zeta^{(j)}}{\zeta}(1-s)\right|$$
$$\le \sum_{j=1}^{k} \binom{k}{j} \frac{1}{\left|\frac{F^{(k)}}{F^{(k-j)}}(s)\right|} \left|\frac{\zeta^{(j)}}{\zeta}(1-s)\right|$$
$$\le 2^\sigma \frac{2k}{2-\frac{\pi^2}{6}} \sum_{j=1}^{k} \binom{k}{j} \frac{1}{(\log(1-\sigma))^j} \left(\frac{1}{2}(\log 2)^j + \sum_{l=0}^{j} \frac{(\log 2)^{j-l}\frac{j!}{(j-l)!}}{(-\sigma)^{l+1}}\right).$$

Since for any positive integer k,

$$\lim_{\sigma\to-\infty} \frac{2k}{2-\frac{\pi^2}{6}} \sum_{j=1}^{k} \binom{k}{j} \frac{1}{(\log(1-\sigma))^j} \left(\frac{1}{2}(\log 2)^j + \sum_{l=0}^{j} \frac{(\log 2)^{j-l}\frac{j!}{(j-l)!}}{(-\sigma)^{l+1}}\right) = 0,$$

we can take $\sigma_k \le \sigma_{k_1}$ (≤ -1) so that

$$\frac{2k}{2-\frac{\pi^2}{6}} \sum_{j=1}^{k} \binom{k}{j} \frac{1}{(\log(1-\sigma))^j} \left(\frac{1}{2}(\log 2)^j + \sum_{l=0}^{j} \frac{(\log 2)^{j-l}\frac{j!}{(j-l)!}}{(-\sigma)^{l+1}}\right) \le 1$$

holds for any $\sigma \le \sigma_k$. This implies that

$$\left|\sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)} \frac{\zeta^{(j)}}{\zeta}(1-s)\right| \le 2^\sigma$$

holds for $\sigma \leq \sigma_k$, $t \geq 2$.

Now with the above σ_k, we are going to find $t_k \geq \max\{a_k^2, -\sigma_k\}$ for which conditions 3 to 5 hold.

3. We start by examining condition 3. We first consider the region $\sigma_k \leq \sigma \leq 1/2$, $t \geq 99$. It follows from (3.5) that in this region,

$$F^{(k)}(s) = F(s)(-\log(1-s) + O(1))^k \left(1 + O\left(\frac{1}{|\log s|^2}\right)\right) \tag{3.8}$$

holds. This gives us,

$$\left|\frac{F^{(k)}}{F}(s)\right| \geq |(\log(1-s))^k| - \left|O_{\sigma_k}\left((\log t)^{k-1}\right)\right| \geq (\log t)^k - \left|O_{\sigma_k}\left((\log t)^{k-1}\right)\right|$$

for $\sigma_k \leq \sigma \leq 1/2$ and $t \geq 99$. Thus, for any integer $k \geq 1$, we can take $t_{k_1} \geq 100$ such that

$$\left|\frac{F^{(k)}}{F}(s)\right| \geq 1 \tag{3.9}$$

for $\sigma_k \leq \sigma \leq 1/2$ and $t \geq t_{k_1} - 1$.

We note from (3.8) that $(F^{(k)}/F)(s) = (-1)^k(\log t)^k + O\left((\log t)^{k-1}\right)$ when $\sigma_k \leq \sigma \leq 1/2$ and $t \geq 99$. Consequently, for odd integer $k \geq 1$, we can find sufficiently large $t'_{k_2} \geq 100$ such that

$$\frac{5\pi}{6} < \arg \frac{F^{(k)}}{F}(s) < \frac{7\pi}{6}$$

holds for $\sigma_k \leq \sigma \leq 1/2$ and $t \geq t'_{k_2} - 1$. Similarly, when k is even, we can also find sufficiently large $t''_{k_2} \geq 100$ such that

$$-\frac{\pi}{6} < \arg \frac{F^{(k)}}{F}(s) < \frac{\pi}{6}$$

holds for $\sigma_k \leq \sigma \leq 1/2$ and $t \geq t''_{k_2} - 1$. Since all zeros and poles of $F(s)$ lie on $\mathbb{R}$, $(F^{(k)}/F)(s)$ has no poles for $t > 0$. This along with (3.9) implies that $\log(F^{(k)}/F)(s)$ is holomorphic in the region with this branch. Thus setting

$$(\alpha_k, \beta_k) := \begin{cases} (5,7) & \text{if } k \text{ is odd,} \\ (-1,1) & \text{if } k \text{ is even;} \end{cases}$$

and

$$t_{k_2} := \begin{cases} t'_{k_2} & \text{if } k \text{ is odd,} \\ t''_{k_2} & \text{if } k \text{ is even;} \end{cases}$$

we find that $\log \frac{F^{(k)}}{F}(s)$ is holomorphic and that

$$\frac{\alpha_k \pi}{6} < \arg \frac{F^{(k)}}{F}(s) < \frac{\beta_k \pi}{6}$$

holds in the region $\sigma_k \le \sigma \le 1/2$, $t \ge t_{k_2} - 1$.

By the above calculations, we see that $\max\{t_{k_1}, t_{k_2}, a_k^2, -\sigma_k\}$ is a candidate for t_k. Thus we have proven that $t_k \ge \max\{a_k^2, -\sigma_k\}$ for which condition 3 holds exists. Since we want t_k to also satisfy conditions 4 and 5, we need to examine those conditions to completely prove the existence of t_k.

4. Referring to [LM74, Corollary of Theorem 7 (p. 51)], we know that RH implies that for any positive integer j, $\zeta^{(j)}(s)$ has at most a finite number of non-real zeros in $\sigma < 1/2$. Hence we can number all the non-real zeros of $\zeta^{(j)}(s)$ in $\sigma < 1/2$ as $\rho_1^{(j)}, \rho_2^{(j)}, \rho_3^{(j)}, \cdots, \rho_{m_j}^{(j)}$ $(\rho_l^{(j)} = \beta_l^{(j)} + i\gamma_l^{(j)})$ for some integer $m_j \ge 2$ (note that if $\zeta^{(j)}\left(\rho^{(j)}\right) = 0$, then $\zeta^{(j)}\left(\overline{\rho^{(j)}}\right) = 0$, so $m_j \ge 2$) in the order such that $\gamma_l^{(j)} \le \gamma_{l+1}^{(j)}$ for all $1 \le l \le m_j - 1$. Therefore, $\zeta^{(j)}(s) \ne 0$ when $\sigma < 1/2$ and $t \ge \gamma_{m_j}^{(j)} + 1$. We set $t_{k_3} := \max\limits_{1 \le j \le k} (\gamma_{m_j}^{(j)} + 2)$, then for all $j = 1, 2, \cdots, k$, we have
$$\zeta^{(j)}(s) \ne 0 \tag{3.10}$$
in the region $\sigma < 1/2$, $t \ge t_{k_3} - 1$.

   Next we show that we can take the logarithmic branch of $\log(\zeta^{(k)}/\zeta)(s)$ in the region $\sigma_k \le \sigma < 1/2$, $t \ge t_{k_4} - 1$ for some $t_{k_4} \ge 100$, so that it is holomorphic there and

$$\frac{k\pi}{2} < \arg \frac{\zeta^{(k)}}{\zeta}(s) < \frac{3k\pi}{2}$$

   holds there by first claiming that we can find some $t_{k_4} \ge t_{k_3}$ for which

$$\mathrm{Re}\left(\frac{\zeta^{(j)}}{\zeta^{(j-1)}}(s)\right) < 0 \qquad (\sigma_k \le \sigma < 1/2,\ t \ge t_{k_4} - 1) \tag{3.11}$$

   holds for all $j = 1, 2, \cdots, k$. We first note that for any $j = 1, 2, \cdots, k$, $(\zeta^{(j)}/\zeta^{(j-1)})(s)$ is holomorphic and has no zeros in the region defined by $\sigma < 1/2$ and $t \ge t_{k_3} - 1$.

   To show this, we refer to [LM74, pp. 64–65] and we can show that for any $j = 1, 2, \cdots, k$,

$$\mathrm{Re}\left(\frac{\zeta^{(j)}}{\zeta^{(j-1)}}(s)\right) \le -\frac{2}{9}\log|s| + O_{\sigma_k}(1)$$

holds when $\sigma_k \leq \sigma < 1/2$, and $t \geq t_{k_3} - 1$. Thus, we can take $t_{k_4} \geq t_{k_3}$ such that (3.11) holds for all $j = 1, 2, \cdots, k$.

The above immediately implies that for each $j = 1, 2, \cdots, k$, there exists an integer l_j such that

$$\frac{\pi}{2} + 2l_j\pi < \arg \frac{\zeta^{(j)}}{\zeta^{(j-1)}}(s) < \frac{3\pi}{2} + 2l_j\pi \tag{3.12}$$

holds for $\sigma_k \leq \sigma < 1/2$, $t \geq t_{k_4} - 1$. We then choose the logarithmic branch of each $\log(\zeta^{(j)}/\zeta^{(j-1)})(s)$ such that each l_j in (3.12) is zero and take the logarithmic branch of $\log(\zeta^{(k)}/\zeta)(s)$ so that

$$\arg \frac{\zeta^{(k)}}{\zeta}(s) = \sum_{j=1}^{k} \arg \frac{\zeta^{(j)}}{\zeta^{(j-1)}}(s)$$

holds in the region $\sigma_k \leq \sigma < 1/2$, $t \geq t_{k_4} - 1$. Note that from (3.10) and the analyticity of $\zeta^{(k)}(s)$ in $\sigma < 1/2$ (also note that we are assuming RH thus $\zeta(s) \neq 0$ when $\sigma < 1/2$ and $t \geq t_{k_4} - 1$), $\log(\zeta^{(k)}/\zeta)(s)$ is holomorphic in this region with this branch. We then obtain a holomorphic function $\log(\zeta^{(k)}/\zeta)(s)$ with inequalities

$$\frac{k\pi}{2} < \arg \frac{\zeta^{(k)}}{\zeta}(s) < \frac{3k\pi}{2}$$

in the region $\sigma_k \leq \sigma < 1/2$, $t \geq t_{k_4} - 1$.

Combining the proof of condition 3 and the above calculations, we find that $\max\{t_{k_1}, t_{k_2}, t_{k_4}, a_k^2, -\sigma_k\}$ is a candidate for t_k. Therefore we have proven that $t_k \geq \max\{a_k^2, -\sigma_k\}$ for which conditions 3 and 4 hold exists.

5. Now we set $t_{k_5} := \max\{t_{k_1}, t_{k_2}, t_{k_4}, a_k^2, -\sigma_k\}$.

   - Since we are assuming RH, $\zeta(\sigma + it) \neq 0$ for any $t > 0$ if $\sigma \neq 1/2$.
   - According to [Spi65, Table 1 (p. 678)], $\zeta'(\sigma + it) \neq 0$ for any $t \in \mathbb{R}$ if $\sigma \geq 3$ and $\zeta''(\sigma + it) \neq 0$ for any $t \in \mathbb{R}$ if $\sigma \geq 5$. According to [Spi65, Theorem 1], for $k \geq 3$, $\zeta^{(k)}(\sigma + it) \neq 0$ for any $t \in \mathbb{R}$ if $\sigma \geq 7k/4 + 2$. Indeed, we can check that for $k = 1$, $7/4 + 2 > 3$ and for $k = 2$, $7/2 + 2 > 5$, thus for any positive integer k,

   $$\zeta^{(k)}(\sigma + it) \neq 0 \qquad (\sigma \geq \frac{7}{4}k + 2,\ t \in \mathbb{R}).$$

   - Since $t_{k_5} \geq t_{k_3}$, from (3.10), we have $\zeta^{(k)}(\sigma + it) \neq 0$ for $\sigma < 1/2$ and $t \geq t_{k_5}$.

Hence, for any positive integer k, we only need to find $t_k \in [t_{k_5}+1, t_{k_5}+2]$ for which

$$\zeta\left(\frac{1}{2}+it_k\right) \neq 0 \qquad \text{and} \qquad \zeta^{(k)}(\sigma+it_k) \neq 0 \quad \text{for } \frac{1}{2} \leq \sigma \leq \frac{7}{4}k+2$$

hold. Note that this is possible by the identity theorem for complex analytic functions. Thus, we have shown that t_k defined above satisfies $t_k \geq \max\{a_k^2, -\sigma_k\}$ and also conditions 3 to 5.

□

Remark. For $k=1$ and $k=2$, more precise results are known. Refer to [Aka12] and [Sur-p1], respectively. These results are obtained based on the works of Speiser [Spe35], R. Spira [Spi73], and C. Y. Yıldırım [Yıl96a] (also [Yıl00]) on the zeros of $\zeta'(s)$ and $\zeta''(s)$.

Proposition 3.5. *Assume RH. Take a_k and t_k which satisfy all conditions of Lemma 3.4. Then for $T \geq t_k$ which satisfies $\zeta^{(k)}(\sigma+iT) \neq 0$ and $\zeta(\sigma+iT) \neq 0$ for any $\sigma \in \mathbb{R}$, we have*

$$\begin{aligned}\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\leq T}} \left(\beta^{(k)}-\frac{1}{2}\right) = {} & \frac{kT}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)T \\ & - k\,\mathrm{Li}\left(\frac{T}{2\pi}\right) \\ & + \frac{1}{2\pi}\int_{1/2}^{a_k} \left(-\arg\zeta(\sigma+iT) + \arg G_k(\sigma+iT)\right) d\sigma \\ & + O_k(1),\end{aligned}$$

where the logarithmic branches are taken so that $\log\zeta(s)$ and $\log G_k(s)$ tend to 0 as $\sigma \to \infty$ and are holomorphic in $\mathbb{C}\backslash\{z+\lambda \mid \zeta(z)=0 \text{ or } \infty,\ \lambda \leq 0\}$ and $\mathbb{C}\backslash\{z+\lambda \mid \zeta^{(k)}(z)=0 \text{ or } \infty,\ \lambda \leq 0\}$, respectively.

Proof. The steps of the proof generally follow the proof of Proposition 2.2 of [Aka12]. We first take a_k, σ_k, and t_k as in Lemma 3.4 and fix them. Then, we take $T \geq t_k$ such that $\zeta^{(k)}(\sigma+iT) \neq 0$ and $\zeta(\sigma+iT) \neq 0$ ($^\forall\sigma \in \mathbb{R}$). We also let $\delta \in (0, 1/2]$ and put $b := 1/2-\delta$. We consider the rectangle with vertices $b+it_k$, a_k+it_k, a_k+iT, and $b+iT$, and then we apply Littlewood's lemma (see Lemma 2.2 or [Tit39, Section 3.8]) to $G_k(s)$ there. By taking the imaginary part, we obtain

$$2\pi \sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ t_k<\gamma^{(k)}\le T}} (\beta^{(k)}-b) = \int_{t_k}^{T} \log|G_k(b+it)|dt - \int_{t_k}^{T} \log|G_k(a_k+it)|dt$$
$$- \int_{b}^{a_k} \arg G_k(\sigma+it_k)d\sigma + \int_{b}^{a_k} \arg G_k(\sigma+iT)d\sigma$$
$$=: I_1+I_2+I_3+\int_{b}^{a_k} \arg G_k(\sigma+iT)d\sigma \tag{3.13}$$

where the sum is counted with multiplicity. By the same reasoning as in [Aka12, p. 2246], we have

$$I_2 = O_{a_k}(1), \quad I_3 = O_{a_k,t_k}(1).$$

Now we only need to estimate I_1. From the functional equation (3.4) for $\zeta(s)$, we can deduce that

$$\zeta^{(k)}(s) = F^{(k)}(s)\zeta(1-s)\left(1-\sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)}\frac{\zeta^{(j)}}{\zeta}(1-s)\right)$$
$$= F(s)\frac{F^{(k)}}{F}(s)\zeta(1-s)\left(1-\sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)}\frac{\zeta^{(j)}}{\zeta}(1-s)\right).$$

Hence,

$$I_1 = \int_{t_k}^{T}\log|G_k(b+it)|dt = \int_{t_k}^{T}\log\frac{2^b}{(\log 2)^k}|\zeta^{(k)}(b+it)|dt$$
$$= \int_{t_k}^{T}\log\frac{2^b}{(\log 2)^k}dt + \int_{t_k}^{T}\log|\zeta^{(k)}(b+it)|dt$$
$$= (b\log 2 - k\log\log 2)(T-t_k) + \int_{t_k}^{T}\log|F(b+it)|dt + \int_{t_k}^{T}\log\left|\frac{F^{(k)}}{F}(b+it)\right|dt$$
$$+ \int_{t_k}^{T}\log|\zeta(1-b-it)|dt$$
$$+ \int_{t_k}^{T}\log\left|1-\sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(b+it)}\frac{\zeta^{(j)}}{\zeta}(1-b-it)\right|dt$$
$$=: ((b\log 2 - k\log\log 2)T + O_{t_k}(1)) + I_{12}+I_{13}+I_{14}+I_{15}. \tag{3.14}$$

As shown in [Aka12, pp. 2247–2249],

$$I_{12} = \left(\frac{1}{2}-b\right)\left(T\log\frac{T}{2\pi}-T\right)+O_{t_k}(1),$$

$$I_{14} = -\int_{1-b}^{a_k} \arg\zeta(\sigma+iT)d\sigma + O_{a_k,t_k}(1).$$

Below we estimate I_{13} and I_{15}.

We begin with the estimation of I_{13}. We consider for $0<\sigma<1/2$ and $t\geq 100$. We first show that

$$F^{(k)}(s) = F(s)(f'(s))^k \left(1 + O\left(\frac{e^{-t}}{|\log s|^2}\right) + O\left(\frac{1}{|s||\log s|^2}\right)\right) \tag{3.15}$$

holds in the region $\sigma<1$, $t\geq 100$. It is obvious that the above error estimate is more precise than that in (3.5). The proof is similar to the proof of condition 2 of Lemma 3.4. We begin by taking the logarithmic branch of $\log(\sin(\pi s/2))$ as

$$\log\left(\sin\frac{\pi s}{2}\right) = -\frac{\pi i s}{2} - \log 2 + \frac{\pi i}{2} - \sum_{n=1}^{\infty}\frac{e^{\pi ins}}{n} \tag{3.16}$$

in the region $0<\sigma<1$, $t\geq 2$ and analytically continue it to the region $\sigma<1$, $t\geq 2$. Next, we apply Stirling's formula to $\Gamma(1-s)$ in the region $-\pi/2<\arg(1-s)<\pi/2$. Substituting these into $F(s)$, we obtain

$$F(s) = \exp\left(\frac{\pi i}{4} - 1 + \left(\frac{1}{2}-s\right)\log\frac{(1-s)i}{2\pi} + s + O(e^{-t}) + O\left(\frac{1}{|s|}\right)\right)$$

for $\sigma<1$ and $t\geq 100$, where the term $O(e^{-t})$ comes from the term $\sum_{n=1}^{\infty} e^{\pi ins}n^{-1}$ in (3.16) and the term $O(1/|s|)$ originates from the Stirling's formula.

We now write

$$f(s) := \left(\frac{1}{2}-s\right)\log\frac{(1-s)i}{2\pi} + s + O(e^{-t}) + O\left(\frac{1}{|s|}\right)$$

and differentiate it with respect to s to obtain

$$f'(s) = -\log\frac{(1-s)i}{2\pi} + \frac{1}{2(1-s)} + O(e^{-t}) + O\left(\frac{1}{|s|^2}\right)$$

and

$$f^{(j)}(s) = O(e^{-t}) + O\left(\frac{1}{|s|^{j-1}}\right)$$

for $j\geq 2$. (3.15) immediately follows. As a consequence to (3.15),

$$\frac{F^{(k)}}{F}(b+it) = \left(-\log\frac{t+(1-b)i}{2\pi} + \frac{1}{2(1-b-it)} + O\left(\frac{1}{t^2}\right)\right)^k \times\left(1 + O\left(\frac{1}{t(\log t)^2}\right)\right)$$

$$= \left(-\log \frac{t}{2\pi} + \frac{t^2 - 2(1-b)((1-b)^2 + t^2)}{2((1-b)^2 + t^2)t} i + O\left(\frac{1}{t^2}\right) \right)^k$$
$$\times \left(1 + O\left(\frac{1}{t(\log t)^2} \right) \right).$$

This gives us

$$\log \frac{F^{(k)}}{F}(b+it) = k \log\log \frac{t}{2\pi} + k \log(-1)$$
$$+ k \log \left(1 - \frac{t^2 - 2(1-b)((1-b)^2 + t^2)}{2((1-b)^2 + t^2)t \log \frac{t}{2\pi}} i + O\left(\frac{1}{t^2 \log t} \right) \right)$$
$$+ O\left(\frac{1}{t(\log t)^2} \right)$$
$$= k \log\log \frac{t}{2\pi} + k \log(-1) - k \frac{t^2 - 2(1-b)((1-b)^2 + t^2)}{2((1-b)^2 + t^2)t \log \frac{t}{2\pi}} i$$
$$+ O\left(\frac{1}{t(\log t)^2} \right).$$

Consequently we have

$$\operatorname{Re}\left(\log \frac{F^{(k)}}{F}(b+it) \right) = k \log\log \frac{t}{2\pi} + O\left(\frac{1}{t(\log t)^2} \right).$$

Hence,

$$I_{13} = \int_{t_k}^{T} \log \left| \frac{F^{(k)}}{F}(b+it) \right| dt = \int_{t_k}^{T} \operatorname{Re}\left(\log \frac{F^{(k)}}{F}(b+it) \right) dt$$
$$= k \int_{t_k}^{T} \log\log \frac{t}{2\pi} dt + O\left(\int_{t_k}^{T} \frac{dt}{t(\log t)^2} \right)$$
$$= kT \log\log \frac{T}{2\pi} - 2\pi k \operatorname{Li}\left(\frac{T}{2\pi} \right) + O_{t_k}(1).$$

Finally, we estimate I_{15}. Again from the functional equation (3.4) for $\zeta(s)$, we have

$$\zeta^{(k)}(s) = F^{(k)}(s)\zeta(1-s) \left(1 - \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)} \frac{\zeta^{(j)}}{\zeta}(1-s) \right)$$
$$= \frac{F^{(k)}}{F}(s)\zeta(s) \left(1 - \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)} \frac{\zeta^{(j)}}{\zeta}(1-s) \right)$$

which gives us

$$\frac{1}{\frac{F^{(k)}}{F}(s)}\frac{\zeta^{(k)}}{\zeta}(s) = 1 - \sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)}\frac{\zeta^{(j)}}{\zeta}(1-s). \tag{3.17}$$

It follows from condition 2 of Lemma 3.4 that the right hand side of (3.17) is holomorphic and has no zeros in the region defined by $\sigma \le \sigma_k$ and $t \ge 2$. Moreover from conditions 3 and 4 of Lemma 3.4, the left hand side of (3.17) is holomorphic and has no zeros in the region defined by $\sigma_k \le \sigma < 1/2$ and $t \ge t_k - 1$. Thus, we can determine

$$\log\left(1 - \sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)}\frac{\zeta^{(j)}}{\zeta}(1-s)\right)$$

so that it tends to 0 as $\sigma \to -\infty$ which follows from condition 2 of Lemma 3.4, and is holomorphic in the region $\sigma < 1/2$, $t > t_k - 1$.

Now we consider the trapezoid C with vertices $b + it_k$, $b + iT$, $-T + iT$, and $-t_k + it_k$ (as in [Aka12, p. 2247]). Then by Cauchy's integral theorem,

$$\int_C \log\left(1 - \sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)}\frac{\zeta^{(j)}}{\zeta}(1-s)\right)ds = 0. \tag{3.18}$$

By using condition 2 of Lemma 3.4, we can also show that (cf. [Aka12, p. 2248])

$$\left(\int_{\sigma_k+iT}^{-T+iT} + \int_{-T+iT}^{-t_k+it_k} + \int_{-t_k+it_k}^{\sigma_k+it_k}\right)$$
$$\log\left(1 - \sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)}\frac{\zeta^{(j)}}{\zeta}(1-s)\right)ds = O(1).$$

Next we estimate the integral from $\sigma_k + it_k$ to $b + it_k$ trivially and we obtain

$$\int_{\sigma_k+it_k}^{b+it_k} \log\left(1 - \sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)}\frac{\zeta^{(j)}}{\zeta}(1-s)\right)ds = O_{t_k}(1).$$

Substituting the above two equations into (3.18) and taking the imaginary part, we obtain

$$\begin{aligned}I_{15} &= \int_{t_k}^{T} \log\left|1 - \sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(b+it)}\frac{\zeta^{(j)}}{\zeta}(1-b-it)\right|dt \\ &= \int_{\sigma_k}^{b} \arg\left(1 - \sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(\sigma+iT)}\frac{\zeta^{(j)}}{\zeta}(1-\sigma-iT)\right)d\sigma + O_{t_k}(1)\end{aligned}$$

$$\overset{(3.17)}{=} \int_{\sigma_k}^{b} \arg\left(\frac{1}{\frac{F^{(k)}}{F}(\sigma+iT)}\frac{\zeta^{(k)}}{\zeta}(\sigma+iT)\right)d\sigma + O_{t_k}(1).$$

Now we determine the logarithmic branch of $\log(F^{(k)}/F)(s)$ and $\log(\zeta^{(k)}/\zeta)(s)$ in the region $\sigma_k \le \sigma < 1/2$, $t \ge t_k - 1$ as in conditions 3 and 4, respectively, of Lemma 3.4. Note that

$$\log\left|\frac{1}{\frac{F^{(k)}}{F}(s)}\frac{\zeta^{(k)}}{\zeta}(s)\right| = -\log\left|\frac{F^{(k)}}{F}(s)\right| + \log\left|\frac{\zeta^{(k)}}{\zeta}(s)\right|$$

holds in the region $\sigma_k \le \sigma < 1/2$, $t \ge t_k - 1$. Furthermore, since

$$\log\left(\frac{1}{\frac{F^{(k)}}{F}(s)}\frac{\zeta^{(k)}}{\zeta}(s)\right) = \log\left(1 - \sum_{j=1}^{k}\binom{k}{j}(-1)^{j-1}\frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)}\frac{\zeta^{(j)}}{\zeta}(1-s)\right),$$

$\log(F^{(k)}/F)(s)$, and $\log(\zeta^{(k)}/\zeta)(s)$ are holomorphic in this region, we know that $\arg\big(((F^{(k)}/F)(s))^{-1}(\zeta^{(k)}/\zeta)(s)\big)$, $\arg(F^{(k)}/F)(s)$, and $\arg(\zeta^{(k)}/\zeta)(s)$ are continuous there. Since the region $\sigma_k \le \sigma < 1/2$, $t \ge t_k - 1$ is connected, there exists a constant $n \in \mathbb{Z}$ such that

$$\arg\left(\frac{1}{\frac{F^{(k)}}{F}(s)}\frac{\zeta^{(k)}}{\zeta}(s)\right) = -\arg\frac{F^{(k)}}{F}(s) + \arg\frac{\zeta^{(k)}}{\zeta}(s) + 2n\pi$$

holds in $\sigma_k \le \sigma < 1/2$, $t \ge t_k - 1$.

From this choice of logarithmic branch, we have

$$\frac{(3k-\beta_k)}{6}\pi + 2n\pi < \arg\left(\frac{1}{\frac{F^{(k)}}{F}(\sigma+iT)}\frac{\zeta^{(k)}}{\zeta}(\sigma+iT)\right) < \frac{(9k-\alpha_k)}{6}\pi + 2n\pi \quad (3.19)$$

for $\sigma_k \le \sigma < 1/2$. Here, α_k and β_k are the constants given in Lemma 3.4, that is,

$$(\alpha_k, \beta_k) = \begin{cases} (5,7) & \text{if } k \text{ is odd,} \\ (-1,1) & \text{if } k \text{ is even.} \end{cases}$$

Since n does not depend on s, $n = O_k(1)$. Therefore

$$\arg\left(\frac{1}{\frac{F^{(k)}}{F}(\sigma+iT)}\frac{\zeta^{(k)}}{\zeta}(\sigma+iT)\right) = O_k(1).$$

From this, we can easily show that

$$I_{15} = O_k(1),$$

for σ_k and t_k are fixed constants that depend only on k.

Inserting the estimates of I_{12}, I_{13}, I_{14}, and I_{15} into (3.14), we obtain

$$I_1 = (b\log 2 - k\log\log 2)T + \left(\frac{1}{2} - b\right)\left(T\log\frac{T}{2\pi} - T\right) + kT\log\log\frac{T}{2\pi} \\ - 2k\pi\,\mathrm{Li}\left(\frac{T}{2\pi}\right) - \int_{1-b}^{a_k} \arg\zeta(\sigma + iT)d\sigma + O_k(1),$$

since a_k and t_k are fixed constants that depend only on k.

To finalize the proof of Proposition 3.5, we insert the estimates of I_1, I_2, and I_3 into (3.13) to obtain

$$2\pi \sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\le T}} (\beta^{(k)} - b) = kT\log\log\frac{T}{2\pi} + (b\log 2 - k\log\log 2)T - 2k\pi\,\mathrm{Li}\left(\frac{T}{2\pi}\right) \\ + \left(\frac{1}{2} - b\right)\left(T\log\frac{T}{2\pi} - T\right) - \int_{1-b}^{a_k} \arg\zeta(\sigma + iT)d\sigma \\ + \int_b^{a_k} \arg G_k(\sigma + iT)d\sigma + O_k(1).$$

Taking the limit $\delta \to 0$, we have $b \to 1/2$, thus

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\le T}} \left(\beta^{(k)} - \frac{1}{2}\right) = \frac{kT}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)T \\ - k\,\mathrm{Li}\left(\frac{T}{2\pi}\right) \\ + \frac{1}{2\pi}\int_{1/2}^{a_k} (-\arg\zeta(\sigma + iT) + \arg G_k(\sigma + iT))d\sigma \\ + O_k(1).$$

□

Remark. The proof of Proposition 3.5 (and thus of Proposition 2.2 of [Aka12]) actually, more or less, follows the proof of Theorem 10 given in [LM74, Section 3]. One obvious difference is that we did not estimate the fourth integral in (3.13) while Levinson and Montgomery estimated the corresponding integral (the fourth integral in (3.1) of [LM74, Section 3]) as $O(\log T)$. As in Akatsuka's [Aka12] did, it turns out that this term contributes to the integral appearing in Proposition 3.5 which will be estimated in the following few lemmas. This integral will contribute to the error term in Theorem 3.1 and in the proofs of the following lemmas, we shall use the assumption of RH to reduce the upper bound of this integral.

Remark. In contrast to the proof of Theorem 10 of [LM74], in this section (and in [Aka12] as well), we describe some important estimates, such as those on $G_k(s)$, $(F^{(k)}/F)(s)$, and $(\zeta^{(k)}/\zeta)(s)$, which are related to the existence of fixed constants a_k, σ_k, and t_k in Lemma 3.4 for the sake of clarity. Furthermore, we also explicitly state Proposition 3.5 since it clearly points out the main terms of Theorem 3.1 and thus this gives the readers clear information of the term that in the current research contributes to the error term which is to be possibly improved in future research.

To complete the proof of Theorem 3.1, we need to estimate

$$\int_{1/2}^{a_k} (-\arg\zeta(\sigma+iT)+\arg G_k(\sigma+iT))d\sigma$$

in Proposition 3.5. For that purpose, similar to the method used in [Aka12], below we give two bounds for $-\arg\zeta(\sigma+iT)+\arg G_k(\sigma+iT)$. We write

$$-\arg\zeta(\sigma+iT)+\arg G_k(\sigma+iT)=\arg\frac{G_k}{\zeta}(\sigma+iT)$$

where the argument on the right hand side is taken so that $\log(G_k/\zeta)(s)$ tends to 0 as $\sigma\to\infty$ and is holomorphic in $\mathbb{C}\backslash\{z+\lambda \mid (\zeta^{(k)}/\zeta)(z)=0 \text{ or } \infty,\ \lambda\le 0\}$.

Lemma 3.6. *Assume RH and let $T\ge t_k$. Then for any $\epsilon_0>0$ satisfying $\epsilon_0<\frac{1}{2\log T}$ (since $T\ge t_k\ge 100$, $\epsilon_0<1/8$), we have for $1/2+\epsilon_0<\sigma\le a_k$,*

$$\arg\frac{G_k}{\zeta}(\sigma+iT)=O_{a_k,t_k}\left(\frac{\log\frac{\log T}{\epsilon_0}}{\sigma-1/2-\epsilon_0}\right).$$

Proof. To begin with, we note that $(G_k/\zeta)(s)$ is uniformly convergent to 1 as $\sigma\to\infty$ for $t\in\mathbb{R}$, so we can take a number $c_k\in\mathbb{R}$ satisfying $a_k+1\le c_k\le t_k/2$ and $1/2\le \mathrm{Re}((G_k/\zeta)(s))\le 3/2$ when $\sigma\ge c_k$. In fact, we can check that taking $c_k=10+k^2$ is sufficient.

The proof also proceeds similarly to the proof of Lemma 2.3 of [Aka12]. We let $\sigma\in(1/2+\epsilon_0,a_k]$ and let $q_{G_k/\zeta}=q_{G_k/\zeta}(\sigma,T)$ denote the number of times $\mathrm{Re}((G_k/\zeta)(u+iT))$ vanishes in $u\in[\sigma,c_k]$. Then,

$$\left|\arg\frac{G_k}{\zeta}(\sigma+iT)\right|\le\left(q_{G_k/\zeta}+1\right)\pi.$$

Now we estimate $q_{G_k/\zeta}$. For that purpose, we set

$$H_k(z)=H_{k_T}(z):=\frac{1}{2}\left(\frac{G_k}{\zeta}(z+iT)+\frac{G_k}{\zeta}(z-iT)\right)\quad(z\in\mathbb{C})$$

and $n_{H_k}(r) := \sharp\{z \in \mathbb{C} \mid H_k(z) = 0, |z - c_k| \le r\}$. Then, we have $q_{G_k/\zeta} \le n_{H_k}(c_k - \sigma)$ for $1/2 + \epsilon_0 < \sigma \le a_k$. For each $\sigma \in (1/2 + \epsilon_0, a_k]$, we take $\epsilon = \epsilon_{\sigma,T}$ satisfying $0 < \epsilon < \sigma - 1/2 - \epsilon_0$, then $H_k(z)$ is holomorphic in the region $\{z \in \mathbb{C} \mid |z - c_k| \le c_k - \sigma + \epsilon\}$. As in [Aka12, p. 2250], by using Jensen's theorem (see Lemma 2.1 or [Tit39, Section 3.61]), we can show that

$$\begin{aligned} n_{H_k}(c_k - \sigma) &\le \frac{1}{C_1\epsilon}\int_0^{c_k-\sigma+\epsilon} \frac{n_{H_k}(r)}{r} dr \\ &= \frac{1}{C_1\epsilon}\frac{1}{2\pi}\int_0^{2\pi} \log|H_k(c_k + (c_k - \sigma + \epsilon)e^{i\theta})| d\theta - \frac{1}{C_1\epsilon}\log|H_k(c_k)| \end{aligned}$$

for some constant $C_1 > 0$, which by our choice of c_k gives us

$$n_{H_k}(c_k - \sigma) \le \frac{1}{C_1\epsilon}\frac{1}{2\pi}\int_0^{2\pi} \log|H_k(c_k + (c_k - \sigma + \epsilon)e^{i\theta})| d\theta + \frac{1}{\epsilon}O_{a_k,t_k}(1). \quad (3.20)$$

Finally we estimate

$$\frac{1}{2\pi}\int_0^{2\pi} \log|H_k(c_k + (c_k - \sigma + \epsilon)e^{i\theta})| d\theta.$$

From [Tit86, Theorems 9.2 and 9.6(A)] (similar to what stated in [Aka12, p. 2250]),

$$\frac{\zeta'}{\zeta}(\sigma \pm it) = O\left(\frac{\log T}{\sigma - \frac{1}{2}}\right)$$

holds for $1/2 < \sigma \le 2c_k$ and $T/2 \le t \le 2T$. Thus, for $1/2 + \epsilon_0 < \sigma \le 2c_k$ and $T/2 \le t \le 2T$, we have

$$\frac{\zeta'}{\zeta}(\sigma \pm it) = O\left(\frac{\log T}{\epsilon_0}\right). \quad (3.21)$$

With this estimate, we show that

$$\frac{\zeta^{(k)}}{\zeta}(s) = O\left(\frac{(\log T)^k}{\epsilon_0^k}\right)$$

holds for $1/2 + \epsilon_0 < \sigma < 2c_k$ and $T/2 \le |t| \le 2T$. We use induction on k in the equation. For $k = 1$, $(\zeta'/\zeta)(\sigma \pm it) = O\left(\epsilon_0^{-1}\log T\right)$ follows from (3.21). Suppose that $(\zeta^{(n)}/\zeta)(s) = O\left(\epsilon_0^{-n}(\log T)^n\right)$ holds in the region $1/2 + \epsilon_0 < \sigma < 2c_k$, $T/2 \le |t| \le 2T$ for a positive integer n, then

$$\left(\frac{\zeta^{(n)}}{\zeta}(s)\right)' = \frac{1}{2\pi i}\int_{|z-s|=\epsilon_0} \frac{\frac{\zeta^{(n)}}{\zeta}(z)}{(z-s)^2} dz = O\left(\frac{(\log T)^n}{\epsilon_0^{n+1}}\right). \quad (3.22)$$

Meanwhile,

$$\left(\frac{\zeta^{(n)}}{\zeta}(s)\right)' = \frac{\zeta^{(n+1)}}{\zeta}(s) - \frac{\zeta^{(n)}}{\zeta}(s)\frac{\zeta'}{\zeta}(s)$$

holds in the region.

Therefore, by (3.22) and by the induction hypothesis,

$$\begin{aligned}\frac{\zeta^{(n+1)}}{\zeta}(s) &= \left(\frac{\zeta^{(n)}}{\zeta}(s)\right)' + \frac{\zeta^{(n)}}{\zeta}(s)\frac{\zeta'}{\zeta}(s) = O\left(\frac{(\log T)^n}{\epsilon_0^{n+1}}\right) + O\left(\frac{(\log T)^{n+1}}{\epsilon_0^{n+1}}\right) \\ &= O\left(\frac{(\log T)^{n+1}}{\epsilon_0^{n+1}}\right)\end{aligned}$$

holds for $1/2 < \sigma \leq 2c_k$ and $T/2 \leq |t| \leq 2T$. Hence, by induction, we find that

$$\frac{\zeta^{(k)}}{\zeta}(s) = O\left(\frac{(\log T)^k}{\epsilon_0^k}\right)$$

holds in the region defined by $1/2 + \epsilon_0 < \sigma < 2c_k$ and $T/2 \leq |t| \leq 2T$. This immediately gives us

$$|H_k(c_k + (c_k - \sigma + \epsilon)e^{i\theta})| \ll_{a_k,t_k} \frac{(\log T)^k}{\epsilon_0^k},$$

and so

$$|H_k(c_k + (c_k - \sigma + \epsilon)e^{i\theta})| \leq C_2(a_k, t_k)\frac{(\log T)^k}{\epsilon_0^k}$$

for some constant $C_2 > 0$ which depends only on a_k and t_k. Thus,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log|H_k(c_k + (c_k - \sigma + \epsilon)e^{i\theta})|d\theta &\leq \log C_2(a_k, t_k) + k\log\frac{\log T}{\epsilon_0} \\ &\ll_{a_k,t_k} \log\frac{\log T}{\epsilon_0}.\end{aligned}$$

Applying this to (3.20), we obtain

$$n_{H_k}(c_k - \sigma) = \frac{1}{\epsilon}O_{a_k,t_k}\left(\log\frac{\log T}{\epsilon_0}\right)$$

which implies

$$\arg\frac{G_k}{\zeta}(\sigma + iT) = \frac{1}{\epsilon}O_{a_k,t_k}\left(\log\frac{\log T}{\epsilon_0}\right).$$

Taking $\epsilon = (\sigma - 1/2 - \epsilon_0)/2$ $(< \sigma - 1/2 - \epsilon_0)$, we obtain

$$\arg\frac{G_k}{\zeta}(\sigma + iT) = O_{a_k,t_k}\left(\frac{\log\frac{\log T}{\epsilon_0}}{\sigma - 1/2 - \epsilon_0}\right).$$

□

Lemma 3.7. *Assume RH and let $A \geq 2$ be fixed. Then there exists a constant $C_0 > 0$ such that*

$$\left|\zeta^{(k)}(\sigma+it)\right| \leq \exp\left(C_0\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T} + (\log T)^{1/10}\right)\right)$$

holds for $T \geq t_k$, $T/2 \leq t \leq 2T$, $1/2 - (\log\log T)^{-1} \leq \sigma \leq A$.

Proof. Referring to [Tit86, (14.14.2), (14.14.5), and the first equation on p. 384] (cf. [Aka12, pp. 2251–2252]), we know that

$$|\zeta(\sigma+it)| \leq \exp\left(C_3\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right) + (\log T)^{1/10}\right) \tag{3.23}$$

holds for $1/2 - 2(\log\log T)^{-1} \leq \sigma \leq A+1$, $T/3 \leq t \leq 3T$ for some constant $C_3 > 0$.

Applying Cauchy's integral formula, we see that

$$\zeta^{(k)}(s) = \frac{k!}{2\pi i}\int_{|z-s|=\epsilon} \frac{\zeta(z)}{(z-s)^{k+1}}dz \qquad \text{for} \quad 0 < \epsilon < \frac{1}{2}$$

holds in the region defined by $1/2 - (\log\log T)^{-1} \leq \sigma \leq A$ and $T/2 \leq t \leq 2T$. Applying (3.23) and by taking $\epsilon = (2(\log\log T)^{1/k})^{-1}$ ($< 1/2$), we obtain Lemma 3.7.

□

Lemma 3.8. *Assume RH and let $T \geq t_k$. Then for any $1/2 \leq \sigma \leq 3/4$, we have*

$$\arg G_k(\sigma+iT) = O_{a_k}\left(\frac{(\log T)^{2(1-\sigma)}}{(\log\log T)^{1/2}}\right).$$

Proof. The proof proceeds in the same way as the proof of Lemma 2.4 of [Aka12]. Refer to [Aka12, pp. 2252–2253] for the detailed proof and use Lemma 3.7 above in place of Lemma 2.6 of [Aka12].

□

Remark. The restrictions of the lower bound of T we gave in Lemmas 3.6, 3.7, and 3.8 are not essential, but they are sufficient for our purpose. We may let T be any positive number in Lemmas 3.6, 3.7, and 3.8, however in that case, we need to modify some calculations in the proofs. Thus we used these restrictions for our convenience.

Proof of Theorem 3.1. First of all, we consider for $T \geq t_k$ which satisfies $\zeta^{(k)}(\sigma+iT) \neq 0$ and $\zeta(\sigma+iT)\neq 0$ for any $\sigma \in \mathbb{R}$. By Lemma 3.6, we have

$$\int_{1/2+2\epsilon_0}^{a_k} \arg \frac{G_k}{\zeta}(\sigma+iT)d\sigma \ll_{a_k,t_k} \int_{1/2+2\epsilon_0}^{a_k} \frac{\log \frac{\log T}{\epsilon_0}}{\sigma-1/2-\epsilon_0}d\sigma \ll_{a_k} \log\frac{\log T}{\epsilon_0}\log\frac{1}{\epsilon_0}.$$

Next, by Lemma 3.8,

$$\arg G_k(\sigma+iT) = O_{a_k}\left(\frac{(\log T)^{2(1-\sigma)}}{(\log\log T)^{1/2}}\right) \qquad \text{for} \quad \frac{1}{2}\leq\sigma\leq\frac{3}{4}$$

and from (2.23) of [Aka12, p. 2251] (cf. [Tit86, (14.14.3) and (14.14.5)]), RH implies that

$$\arg\zeta(\sigma+iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right)$$

holds uniformly for $1/2 \leq \sigma \leq 3/4$. Thus,

$$\int_{1/2}^{1/2+2\epsilon_0} \arg\frac{G_k}{\zeta}(\sigma+iT)d\sigma \ll_{a_k} \frac{\log T}{(\log\log T)^{1/2}}\epsilon_0.$$

Now we take $\epsilon_0 = (4\log T)^{-1}$ $(< (2\log T)^{-1})$, then we have

$$\int_{1/2}^{a_k} \arg\frac{G_k}{\zeta}(\sigma+iT)d\sigma \ll_{a_k,t_k} (\log\log T)^2.$$

Applying this to Proposition 3.5 and noting that a_k and t_k are fixed constants that depend only on k, we have

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\leq T}} \left(\beta^{(k)}-\frac{1}{2}\right) = \frac{kT}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)T \\ - k\,\mathrm{Li}\left(\frac{T}{2\pi}\right) + O_k((\log\log T)^2). \tag{3.24}$$

Secondly, for $2\pi < T < t_k$, we are adding some finite number of terms which depend on t_k, and thus depend only on k so this can be included in the error term.

Thirdly, for $T \geq t_k$ such that $\zeta^{(k)}(\sigma+iT)=0$ or $\zeta(\sigma+iT)=0$ for some $\sigma\in\mathbb{R}$, we start by taking a small $0<\epsilon<(\log\log T)^{-1}$ such that $\zeta^{(k)}(\sigma+i(T\pm\epsilon))\neq 0$ and $\zeta(\sigma+i(T\pm\epsilon))\neq 0$ for any $\sigma\in\mathbb{R}$. We first note that the inequalities

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ t_k-1<\gamma^{(k)}\leq T-\epsilon}} \left(\beta^{(k)}-\frac{1}{2}\right) \leq \sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ t_k-1<\gamma^{(k)}\leq T}} \left(\beta^{(k)}-\frac{1}{2}\right) \leq \sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ t_k-1<\gamma^{(k)}\leq T+\epsilon}} \left(\beta^{(k)}-\frac{1}{2}\right)$$

and that

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ t_k-1<\gamma^{(k)}\le T+\epsilon}}\left(\beta^{(k)}-\frac{1}{2}\right)=\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ t_k-1<\gamma^{(k)}\le T-\epsilon}}\left(\beta^{(k)}-\frac{1}{2}\right)+\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ T-\epsilon<\gamma^{(k)}\le T+\epsilon}}\left(\beta^{(k)}-\frac{1}{2}\right)$$

hold. According to (3.24),

$$\begin{aligned}\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\le T\pm\epsilon}}\left(\beta^{(k)}-\frac{1}{2}\right)=&\frac{k(T\pm\epsilon)}{2\pi}\log\log\frac{T\pm\epsilon}{2\pi}\\ &+\frac{1}{2\pi}\left(\frac{1}{2}\log 2-k\log\log 2\right)(T\pm\epsilon)\\ &-k\operatorname{Li}\left(\frac{T\pm\epsilon}{2\pi}\right)+O_k((\log\log T)^2).\end{aligned}$$

Since

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\le T}}\left(\beta^{(k)}-\frac{1}{2}\right)=\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ t_k-1<\gamma^{(k)}\le T}}\left(\beta^{(k)}-\frac{1}{2}\right)+O_k(1),$$

we find that

$$\begin{aligned}\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\le T}}\left(\beta^{(k)}-\frac{1}{2}\right)=&\frac{kT}{2\pi}\log\log\frac{T}{2\pi}+\frac{1}{2\pi}\left(\frac{1}{2}\log 2-k\log\log 2\right)T\\ &-k\operatorname{Li}\left(\frac{T}{2\pi}\right)+O_k((\log\log T)^2)\end{aligned}$$

also holds for this case.

□

Proof of Corollary 3.2. This is an immediate consequence of Theorem 3.1. For the proof, refer to [LM74, p. 58 (the ending part of Section 3)].

□

3.2.2 Proof of Theorem 3.3

In this subsection we give the proof of Theorem 3.3. We first show the following proposition.

Proposition 3.9. *Assume RH. Then for $T\ge 2$ which satisfies $\zeta(\sigma+iT)\ne 0$ and $\zeta^{(k)}(\sigma+iT)\ne 0$ for all $\sigma\in\mathbb{R}$, we have*

$$N_k(T)=\frac{T}{2\pi}\log\frac{T}{4\pi}-\frac{T}{2\pi}+\frac{1}{2\pi}\arg G_k\left(\frac{1}{2}+iT\right)+\frac{1}{2\pi}\arg\zeta\left(\frac{1}{2}+iT\right)+O_k(1)$$

where the arguments are taken as in Proposition 3.5.

Proof. The steps of the proof also follow those of the proof of Proposition 3.1 of [Aka12]. We take a_k, σ_k, t_k, T, δ, and b as in the beginning of the proof of Proposition 3.5. We let $b' := 1/2 - \delta/2$. Replacing b by b' in (3.13), we have

$$2\pi \sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\leq T}} (\beta^{(k)} - b') = \int_{t_k}^{T} \log|G_k(b'+it)|dt - \int_{t_k}^{T} \log|G_k(a_k+it)|dt$$
$$- \int_{b'}^{a_k} \arg G_k(\sigma+it_k)d\sigma + \int_{b'}^{a_k} \arg G_k(\sigma+iT)d\sigma.$$

Subtracting this from (3.13), we have

$$\begin{aligned}\pi\delta(N_k(T) - N_k(t_k)) &= \int_{t_k}^{T} \log|G_k(b+it)|dt - \int_{t_k}^{T} \log|G_k(b'+it)|dt \\ &\quad - \int_{b}^{b'} \arg G_k(\sigma+it_k)d\sigma + \int_{b}^{b'} \arg G_k(\sigma+iT)d\sigma \\ &=: J_1 + J_2 + J_3 + \int_{b}^{b'} \arg G_k(\sigma+iT)d\sigma.\end{aligned} \tag{3.25}$$

Referring to the estimate of I_3 in the proof of Proposition 3.5 (cf. [Aka12, p. 2246]), we can easily show that

$$J_3 = O_{t_k}(\delta).$$

Now we estimate $J_1 + J_2$. From (3.14), we have

$$\begin{aligned} J_1 + J_2 &= \int_{t_k}^{T} \log|G_k(b+it)|dt - \int_{t_k}^{T} \log|G_k(b'+it)|dt \\ &= ((b-b')\log 2)(T-t_k) + \int_{t_k}^{T} (\log|F(b+it)| - \log|F(b'+it)|)dt \\ &\quad + \int_{t_k}^{T} \left(\log\left|\frac{F^{(k)}}{F}(b+it)\right| - \log\left|\frac{F^{(k)}}{F}(b'+it)\right| \right) dt \\ &\quad + \int_{t_k}^{T} (\log|\zeta(1-b-it)| - \log|\zeta(1-b'-it)|)dt \\ &\quad + \int_{t_k}^{T} \left(\log\left|1 - \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(b+it)} \frac{\zeta^{(j)}}{\zeta}(1-b-it)\right| \right. \\ &\qquad \left. - \log\left|1 - \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(b'+it)} \frac{\zeta^{(j)}}{\zeta}(1-b'-it)\right| \right) dt \end{aligned}$$

$$=: \left(\left(-\frac{\delta}{2} \log 2 \right) T + O_{t_k}(\delta) \right) + J_{12} + J_{13} + J_{14} + J_{15}.$$

Referring to [Aka12, pp. 2255–2256], we have

$$J_{12} = \frac{\delta}{2} \left(T \log \frac{T}{2\pi} - T \right) + O_{t_k}(\delta),$$

$$J_{14} = \int_{1-b'}^{1-b} \arg \zeta(\sigma + iT) d\sigma + O_{t_k}(\delta).$$

We only need to estimate J_{13} and J_{15}. We begin with the estimation of J_{13}. We determine the logarithmic branch of $\log (F^{(k)}/F)(s)$ for $0 < \sigma < 1/2$ and $t > t_k - 1$ as in condition 3 of Lemma 3.4. We then have $\arg (F^{(k)}/F)(s) \in (\alpha_k \pi/6, \beta_k \pi/6)$, where the pair (α_k, β_k) is defined as in Lemma 3.4.

As in [Aka12, p. 2255], we apply Cauchy's integral theorem to $\log (F^{(k)}/F)(s)$ on the rectangle with vertices $b + it_k$, $b' + it_k$, $b' + iT$, and $b + iT$ and take the imaginary part, then we obtain

$$J_{13} = \int_b^{b'} \arg \frac{F^{(k)}}{F}(\sigma + it_k) d\sigma - \int_b^{b'} \arg \frac{F^{(k)}}{F}(\sigma + iT) d\sigma = O_k(\delta).$$

Finally, we estimate J_{15}. We determine the logarithmic branch of

$$\log \left(1 - \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)} \frac{\zeta^{(j)}}{\zeta}(1-s) \right)$$

in the same manner as that in the estimation of I_{15} in the proof of Proposition 3.5, then it is holomorphic in the region $0 < \sigma < 1/2$, $t > t_k - 1$. Applying Cauchy's integral theorem to it on the path taken for estimating J_{13}, we have

$$J_{15} = \int_b^{b'} \arg \left(1 - \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(\sigma + it_k)} \frac{\zeta^{(j)}}{\zeta}(1 - \sigma - it_k) \right)$$
$$- \int_b^{b'} \arg \left(1 - \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(\sigma + iT)} \frac{\zeta^{(j)}}{\zeta}(1 - \sigma - iT) \right) d\sigma.$$

Again using (3.17),

$$J_{15} = \int_b^{b'} \arg \left(\frac{1}{\frac{F^{(k)}}{F}(\sigma + it_k)} \frac{\zeta^{(k)}}{\zeta}(\sigma + it_k) \right) d\sigma$$
$$- \int_b^{b'} \arg \left(\frac{1}{\frac{F^{(k)}}{F}(\sigma + iT)} \frac{\zeta^{(k)}}{\zeta}(\sigma + iT) \right) d\sigma.$$

Applying (3.19), we obtain

$$J_{15} = O_k(\delta).$$

Hence, since t_k is a fixed constant that depends only on k,

$$J_1 + J_2 = \frac{\delta}{2}\left(T\log\frac{T}{4\pi} - T\right) + \int_{1-b'}^{1-b} \arg\zeta(\sigma+iT)d\sigma + O_k(\delta).$$

Inserting the estimates of $J_1 + J_2$ and J_3 into (3.25), we have

$$\begin{aligned} N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + \frac{1}{\pi\delta}\left(\int_{1-b'}^{1-b}\arg\zeta(\sigma+iT)d\sigma + \int_{b}^{b'}\arg G_k(\sigma+iT)d\sigma\right) \\ + O_k(1). \end{aligned} \tag{3.26}$$

Taking the limit $\delta \to 0$ and applying the mean value theorem,

$$\lim_{\delta\to 0}\frac{1}{\pi\delta}\int_{1-b'}^{1-b}\arg\zeta(\sigma+iT)d\sigma = \frac{1}{2\pi}\arg\zeta\left(\frac{1}{2}+iT\right)$$

by noting that $b = 1/2 - \delta$ and $b' = 1/2 - \delta/2$. Similarly,

$$\lim_{\delta\to 0}\frac{1}{\pi\delta}\int_{b}^{b'}\arg G_k(\sigma+iT)d\sigma = \frac{1}{2\pi}\arg G_k\left(\frac{1}{2}+iT\right).$$

Substituting these into (3.26), we have

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + \frac{1}{2\pi}\arg G_k\left(\frac{1}{2}+iT\right) + \frac{1}{2\pi}\arg\zeta\left(\frac{1}{2}+iT\right) + O_k(1).$$

If $2 \le T < t_k$, then $N_k(T) \le N_k(t_k) = O_{t_k}(1) = O_k(1)$. Hence the above equation holds for any $T \ge 2$ which satisfies the conditions of Proposition 3.9. □

Proof of Theorem 3.3. Firstly we consider for $T \ge 2$ which satisfies $\zeta^{(k)}(\sigma + iT) \neq 0$ and $\zeta(\sigma+iT) \neq 0$ for any $\sigma \in \mathbb{R}$. By Lemma 3.8,

$$\arg G_k\left(\frac{1}{2}+iT\right) = O_{a_k}\left(\frac{\log T}{(\log\log T)^{1/2}}\right)$$

and again from equation (2.23) of [Aka12, p. 2251], we have

$$\arg\zeta\left(\frac{1}{2}+iT\right) = O\left(\frac{\log T}{\log\log T}\right).$$

Substituting these into Proposition 3.9, we obtain

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O_k\left(\frac{\log T}{(\log\log T)^{1/2}}\right). \tag{3.27}$$

Next, if $\zeta(\sigma+iT)=0$ or $\zeta^{(k)}(\sigma+iT)=0$ for some $\sigma\in\mathbb{R}$ when $T\geq 2$, then we again take a small $0<\epsilon<(\log T)^{-1}$ such that $\zeta^{(k)}(\sigma+i(T\pm\epsilon))\neq 0$ and $\zeta(\sigma+i(T\pm\epsilon))\neq 0$ for any $\sigma\in\mathbb{R}$ as in the proof of Theorem 3.1. Then noting that

$$N_k(T-\epsilon)\leq N_k(T)\leq N_k(T-\epsilon)+\left(N_k(T+\epsilon)-N_k(T-\epsilon)\right),$$

similarly we can show that (3.27) also holds in this case.

Therefore

$$N_k(T)=\frac{T}{2\pi}\log\frac{T}{4\pi}-\frac{T}{2\pi}+O_k\left(\frac{\log T}{(\log\log T)^{1/2}}\right)$$

holds for any $T\geq 2$.

□

Remark. It is well-known that in the case of the Riemann zeta function $\zeta(s)$, the number $N(T)$ of zeros of $\zeta(s)$ is estimated as

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+S(T)+O\left(\frac{1}{T}\right),$$

where

$$S(T)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right)$$

with a standard branch (cf. [Tit86, Section 9.3]). Thus, the function $S(T)$ determines the error term in the estimate of $N(T)$. Under RH, we have

$$S(T)=O\left(\frac{\log T}{\log\log T}\right) \tag{3.28}$$

(cf. [Tit86, (14.13.1) of Theorem 14.13]). In comparison to the above estimate, the term that determines the error term of $N_k(T)$ is

$$\frac{1}{2\pi}\arg G_k\left(\frac{1}{2}+iT\right)+\frac{1}{2\pi}\arg\zeta\left(\frac{1}{2}+iT\right)$$

by Proposition 3.9 and under RH, they are currently estimated as follows:

$$\arg G_k\left(\frac{1}{2}+iT\right)=O_k\left(\frac{\log T}{(\log\log T)^{1/2}}\right),\quad \arg\zeta\left(\frac{1}{2}+iT\right)=O\left(\frac{\log T}{\log\log T}\right). \tag{3.29}$$

This estimate of $\arg G_k(1/2+iT)$ determines the error term of $N_k(T)$ and it results in $N_k(T)$ having error term slightly greater in magnitude than that of $N(T)$. However, this is the best known estimate on $N_k(T)$ under RH at present.

Furthermore, the size of the implied O-constant in (3.28) has been studied in many papers, such as [CCM13] and [FGH07]. In contrast to this, we currently have no information about the implied O-constant in the first equation of (3.29).

Chapter 4

Zeros of the derivatives of Dirichlet L-functions

In this chapter we introduce results analogous to those introduced in Chapter 3, extended to the k-th derivative of Dirichlet L-functions associated with primitive Dirichlet characters χ modulo $q > 1$, denoted by $L^{(k)}(s, \chi)$ for positive integer k. We recall that there exists only one Dirichlet character modulo 1 and the associated Dirichlet L-function is the Riemann zeta function, whose results are given in the previous chapter.

Throughout this chapter, we denote by $L^{(k)}(s, \chi)$ the k-th derivative of Dirichlet L-function associated with a primitive Dirichlet character χ modulo $q > 1$. We also denote by m the smallest integer $n \geq 2$ such that $\chi(n) \neq 0$. We easily see that $m = \min\{n \in \mathbb{Z}_{\geq 2} \mid (n, q) = 1\}$ holds. This together with the prime number theorem yields $m \ll \log q$. We recall from Lemma 1.12 that $\kappa \in \{0, 1\}$ is a factor associated to χ determined as $\chi(-1) = (-1)^{\kappa}$, that is, $\kappa = 0$ is equal to saying that χ is an even character and $\kappa = 1$ describes χ being an odd character. Finally, it is to be noted that only the results proven by the author and her collaborator H. Akatsuka are stated as theorems in this chapter.

We first introduce a few results shown by C. Y. Yıldırım [Yıl96b] in Section 4.1. In Section 4.2, we prove some improved unconditional results shown by Akatsuka and the author in [AS-p]. We also prove results, analogous to Speiser's theorem [Spe35], for Dirichlet L-functions; that is we show an equivalence between the generalized Riemann hypothesis and the distribution of zeros of $L'(s, \chi)$. In Section 4.3, we prove some conditional results as shown by the author in [Sur-p2].

4.1 Unconditional results

We have seen in the previous chapter that many results on the zeros of the derivatives of the Riemann zeta function $\zeta^{(k)}(s)$ are known. Unfortunately, not many results are known for $L^{(k)}(s,\chi)$. Yıldırım [Yıl96b] in 1996, studied many properties of zeros of $L^{(k)}(s,\chi)$. Among them, he [Yıl96b, Theorem 2] showed that $L^{(k)}(s,\chi) \neq 0$ for

$$\mathrm{Re}(s) > 1 + \frac{m}{2}\left(1+\sqrt{1+\frac{4k^2}{m\log m}}\right).$$

Furthermore he [Yıl96b, Theorem 3] proved that for any given $\epsilon > 0$, there exists a constant $K = K_{k,\epsilon,\kappa}$ that depends only on k, ϵ, and κ such that $L^{(k)}(s,\chi) \neq 0$ in the region $|s| > q^K$, $\mathrm{Re}(s) < -\epsilon$, $|\mathrm{Im}(s)| > \epsilon$.

Remark. Since $\kappa \in \{0,1\}$, it is easy to see that we can take the constant K described in [Yıl96b, Theorem 3] such that it is independent of κ.

With the above zero-free regions, Yıldırım in [Yıl96b] classified the zeros of $L^{(k)}(s,\chi)$ as:

1. trivial zeros in $\mathrm{Re}(s) \leq -q^K$, $|\mathrm{Im}(s)| \leq \epsilon$;

2. vagrant zeros in $|s| \leq q^K$, $\mathrm{Re}(s) < -\epsilon$;

3. nontrivial zeros in $\mathrm{Re}(s) \geq -\epsilon$

(see [Yıl96b, the first paragraph in Section 7]).

Based on this classification, in the same paper he [Yıl96b, Theorem 4] proved an estimate of the number of vagrant and nontrivial zeros of $L^{(k)}(s,\chi)$ in $|\mathrm{Im}(s)| \leq T$, denoted by $N_k(T,\chi)$ as follows:

$$N_k(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} + O(q^K \log T). \tag{4.1}$$

We shall see in the next section that we can improve these results when $k = 1$. We can also further improve the error term in (4.1) under the truth of the generalized Riemann hypothesis (see Section 4.3).

In [Yıl96b], Yıldırım proved two other results by assuming the generalized Riemann hypothesis. He [Yıl96b, Theorem 1] proved that if it is supposed that the generalized Riemann hypothesis is true,

1. if $\kappa = 0$ and $q \geq 216$, then $L'(s,\chi)$ has exactly one zero in $0 \leq \mathrm{Re}(s) < 1/2$ at
$$\frac{1}{\log q} + O\left(\frac{\log\log q}{\log^2 q}\right);$$

2. if $\kappa = 1$ and $q \geq 23$, then $L'(s,\chi)$ has no zeros in $0 \leq \mathrm{Re}(s) < 1/2$.

Finally in [Yıl96b, Theorem 5], it is proven that there exist only at most finitely many zeros of $L^{(k)}(s,\chi)$ in the strip $-\epsilon \leq \mathrm{Re}(s) < 1/2$ under the truth of the generalized Riemann hypothesis.

4.2 Improved unconditional results for $k = 1$ and equivalence results

The first aim of this section is to remove the possibility of vagrant zeros of $L'(s, \chi)$. In order to state our result precisely, we put

$$\Theta(\chi) := \sup\{\mathrm{Re}(\rho) \mid \rho \in \mathbb{C}, L(\rho, \chi) = 0\}.$$

It is easy to check that the following properties hold:

- $1/2 \le \Theta(\chi) \le 1$.
- $\Theta(\overline{\chi}) = \Theta(\chi)$.
- For each primitive Dirichlet character χ, the Riemann hypothesis for $L(s, \chi)$ is equivalent to $\Theta(\chi) = 1/2$.

One of our main results can be stated as follows:

Theorem 4.1. *$L'(s, \chi)$ has no zeros on $s \in \mathcal{D}_1(\chi) \cup \mathcal{D}_2(\chi)$, where*

$$\mathcal{D}_1(\chi) = \left\{\sigma + it \mid \sigma \le 1 - \Theta(\chi),\ |t| \ge \frac{6}{\log q}\right\} \setminus \{\rho \in \mathbb{C} \mid L(\rho, \chi) = 0\},$$
$$\mathcal{D}_2(\chi) = \left\{\sigma + it \mid \sigma \le -q^2,\ |t| \ge \frac{12}{\log |\sigma|}\right\}.$$

Remark. Apparently the constants 6 and 12 in $\mathcal{D}_1(\chi)$ and $\mathcal{D}_2(\chi)$ can be replaced by smaller constants.

Theorem 4.2. *For each positive integer j the following assertions hold:*

- *$L'(s, \chi)$ has a unique zero in the strip $s \in \{\sigma + it \mid -2j - \kappa - 1 < \sigma < -2j - \kappa + 1,\ t \in \mathbb{R}\}$.*
- *$L'(s, \chi)$ has no zeros on $\mathrm{Re}(s) = -2j - \kappa + 1$.*

Let χ be a non-principal primitive Dirichlet character. For $j \in \mathbb{Z}_{\ge 1}$ we denote the zero of $L'(s, \chi)$ in $\{\sigma + it \mid -2j - \kappa - 1 < \sigma < -2j - \kappa + 1\}$ by $\alpha_j(\chi)$. Then we have

Theorem 4.3. *Retain the notation. Then we have*

$$\alpha_j(\chi) = -2j - \kappa + O\left(\frac{1}{\log(jq)}\right),$$

where the implied constant is absolute.

Theorem 4.4. *1. If $\kappa = 0$ and $q \geq 7$, then $L'(s,\chi)$ has no zeros on $\{\sigma + it \mid -1 \leq \sigma \leq 0,\ t \in \mathbb{R}\}$.*

2. If $\kappa = 1$ and $q \geq 23$, then $L'(s,\chi)$ has a unique zero on $\{\sigma + it \mid -2 \leq \sigma \leq 0,\ t \in \mathbb{R}\}$.

We remark that from Theorems 4.1–4.4, each zero of $L'(s,\chi)$ in $\mathrm{Re}(s) \leq 0$ corresponds to a trivial zero of $L(s,\chi)$, except for only finitely many zeros. Thus it is natural to consider these zeros as *trivial zeros* of $L'(s,\chi)$. This implies that we have excluded, for $k = 1$, the possibility of vagrant zeros stated by Yıldırım in [Yıl96b]. In [Yıl96b] the possibility of these vagrant zeros prevents us from investigating *nontrivial zeros* of $L'(s,\chi)$. Hence Theorems 4.1–4.4 allow us to study nontrivial zeros of $L'(s,\chi)$, i.e. zeros of $L'(s,\chi)$ in $\mathrm{Re}(s) > 0$. This is the second aim of this section. We explain our main results on nontrivial zeros of $L'(s,\chi)$ below. For $T > 0$ we denote by $N_1(T,\chi)$ the number of zeros of $L'(s,\chi)$ on $\{\sigma + it \mid \sigma > 0, -T \leq t \leq T\}$, counted with multiplicity. With this notation we have

Theorem 4.5. *Retain the notation above. Then for $T \geq 2$ we have*

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2\pi m} - \frac{T}{\pi} + O(m^{1/2}\log(qT)),$$

where the implied constant is absolute.

We remark that this notation $N_1(T,\chi)$ differs from that defined by Yıldırım in [Yıl96b] for $k = 1$ since he counted not only nontrivial zeros, but also vagrant zeros of $L'(s,\chi)$.

Next we show an asymptotic formula on the sum of the horizontal distance between nontrivial zeros of $L'(s,\chi)$ and the critical line $\mathrm{Re}(s) = 1/2$.

Theorem 4.6. *Retain the notation. Then for $T \geq 2$ it holds that*

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ \beta'>0,-T\leq\gamma'\leq T}} \left(\beta' - \frac{1}{2}\right) = \frac{T}{\pi}\log\log\frac{qT}{2\pi} + \frac{T}{\pi}\left(\frac{1}{2}\log m - \log\log m\right)$$
$$- \frac{2}{q}\,\mathrm{Li}\left(\frac{qT}{2\pi}\right) + O(m^{1/2}\log(qT)),$$

where $\rho' = \beta' + i\gamma'$ runs over all zeros of $L'(s,\chi)$ satisfying $\beta' > 0$ and $-T \leq \gamma' \leq T$, counted with multiplicity and

$$\mathrm{Li}(x) = \int_2^x \frac{du}{\log u}$$

as defined in Theorem 2.4. Here the implied constant is absolute.

As mentioned in the previous chapter, A. Speiser [Spe35] in 1935 proved that the Riemann hypothesis is equivalent to the assertion that $\zeta'(s)$ has no non-real zeros in $\mathrm{Re}(s) < 1/2$. The final aim of this section is to extend this result to $L(s,\chi)$. To show our theorems, we first extend a quantitative version of Speiser's theorem shown by N. Levinson and H. L. Montgomery in [LM74, Theorem 1]. To state this result, we denote by $N^-(T,\chi)$ (*resp.* $N_1^-(T,\chi)$) the number of zeros of $L(s,\chi)$ (*resp.* $L'(s,\chi)$) on $\{\sigma + it \mid 0 < \sigma < 1/2, |t| \le T\}$, counted with multiplicity. Then we have

Theorem 4.7. *For $T \ge 2$ we have*

$$N^-(T,\chi) = N_1^-(T,\chi) + O(m^{1/2}\log(qT)), \tag{4.2}$$

where the implied constant is absolute.

Remark. In [GS15, Theorem 1.2] R. Garunkštis and R. Šimėnas have obtained (4.2) for *fixed* χ. The new element of this section is to give *uniform* estimates with respect to χ.

Theorem 4.8. *Let $\kappa = 0$ and $q \ge 216$. Then the following conditions (i) and (ii) are equivalent:*

(i) $L(s,\chi) \ne 0$ *in* $0 < \mathrm{Re}(s) < 1/2$.

(ii) $L'(s,\chi)$ *has a unique zero in* $0 < \mathrm{Re}(s) < 1/2$.

Theorem 4.9. *Let $\kappa = 1$ and $q \ge 23$. Then the following conditions (i) and (ii) are equivalent:*

(i) $L(s,\chi) \ne 0$ *in* $0 < \mathrm{Re}(s) < 1/2$.

(ii) $L'(s,\chi)$ *has no zeros in* $0 < \mathrm{Re}(s) < 1/2$.

Remark. The implications (i)$\Longrightarrow$(ii) in Theorems 4.8 and 4.9 have been obtained by Yıldırım [Yıl96b, Theorem 1] (recall also from Section 4.1). Our contribution in this section is to establish the implications (ii)$\Longrightarrow$(i).

Theorems 8 and 9 only give us results analogous to Speiser's theorem [Spe35] for q large to some extent. However, it is highly possible to formulate similar assertions for $q < 216$ (if χ is even, or $q < 23$ if χ is odd) by investigating the change in $\arg{(L'/L)(s,\chi)}$ on $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1/2$ through numerical calculations.

Throughout this section we denote nontrivial zeros of $L(s,\chi)$ (i.e., zeros in $\{\sigma + it \mid 0 < \sigma < 1\}$) by $\rho = \beta + i\gamma$ and zeros of $L'(s,\chi)$ by $\rho' = \beta' + i\gamma'$. We put $\Theta(\chi) := \sup_\rho \mathrm{Re}(\rho)$ and c_E is the Euler–Mascheroni constant.

4.2.1 Proof of Theorem 4.1

In this subsection we prove Theorem 4.1. It suffices to show the following:

Proposition 4.10. $\mathrm{Re}((L'/L)(s,\chi)) < 0$ *holds on* $s \in \mathcal{D}_1(\chi) \cup \mathcal{D}_2(\chi)$.

In order to show Proposition 4.10, we recall the Hadamard product expression of $L(s,\chi)$. The logarithmic derivative of the Hadamard product expression for $L(s,\chi)$ is given by (see [MV06, Corollary 10.18])

$$\frac{L'}{L}(s,\chi) = B(\chi) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa}{2}\right) - \frac{1}{2}\log\frac{q}{\pi} + \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right), \tag{4.3}$$

where $B(\chi)$ is a constant depending only on χ, which satisfies

$$\mathrm{Re}(B(\chi)) = -\sum_{\rho}\mathrm{Re}\left(\frac{1}{\rho}\right).$$

We use the Hadamard product (4.3) in the following form.

Lemma 4.11. *Suppose that* $s = \sigma + it$ *satisfies* $\sigma \le 1 - \Theta(\chi)$ *and* $L(s,\chi) \neq 0$. *Then we have*

$$\mathrm{Re}\left(\frac{L'}{L}(s,\chi)\right) \le -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa}{2}\right)\right). \tag{4.4}$$

Proof. Taking the real part on (4.3),

$$\mathrm{Re}\left(\frac{L'}{L}(s,\chi)\right) = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa}{2}\right)\right) + \sum_{\rho=\beta+i\gamma}\frac{\sigma-\beta}{|s-\rho|^2}. \tag{4.5}$$

By the definition of $\Theta(\chi)$ and the functional equation, $\beta \ge 1 - \Theta(\chi)$ holds for any ρ. Thus we find $\sigma - \beta \le 0$ if $\sigma \le 1 - \Theta(\chi)$. This says that the sum over nontrivial zeros is nonpositive, which is nothing but the result. □

To estimate the digamma function $(\Gamma'/\Gamma)(z)$ on (4.4), we use the following inequality:

Lemma 4.12. *For* $z = x + iy$ *with* $x \in \mathbb{R}$ *and* $y \in \mathbb{R} \setminus \{0\}$ *we have*

$$\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}(z)\right) \ge \log|z| - \frac{\pi}{2|y|}. \tag{4.6}$$

Proof. We start with the logarithmic derivative of the Hadamard product expression for $\Gamma(z)$ (see [MV06, Equation (C.10) in p. 522]):

$$\frac{\Gamma'}{\Gamma}(z) = -c_E - \sum_{n=0}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n+1}\right). \tag{4.7}$$

Suppose $z = x + iy \in \mathbb{C} \setminus (-\infty, 0]$. Applying the Euler–Maclaurin summation formula (that is, [MV06, Theorem B.5 when $K = 1$]), we have

$$\lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{1}{n+z} - \operatorname{Log}(N+z)\right) = -\operatorname{Log} z + \frac{1}{2z} - \int_0^\infty \frac{u-[u]-\frac{1}{2}}{(u+z)^2}du, \tag{4.8}$$

where $\operatorname{Log} z$ is the principal logarithmic branch of z. We note that (4.8) with $z = 1$ implies

$$c_E = \frac{1}{2} - \int_0^\infty \frac{u-[u]-\frac{1}{2}}{(u+1)^2}du. \tag{4.9}$$

Inserting (4.8), (4.8) with $z = 1$, and (4.9) into (4.7), we have

$$\frac{\Gamma'}{\Gamma}(z) = \operatorname{Log} z - \frac{1}{2z} + \int_0^\infty \frac{u-[u]-\frac{1}{2}}{(u+z)^2}du. \tag{4.10}$$

Taking the real part, we obtain

$$\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}(z)\right) = \log|z| - \frac{1}{2}\operatorname{Re}\left(\frac{1}{z}\right) + \operatorname{Re}\left(\int_0^\infty \frac{u-[u]-\frac{1}{2}}{(u+z)^2}du\right). \tag{4.11}$$

We consider the case $x \geq 0$. Then we have

$$\left|\operatorname{Re}\left(\frac{1}{z}\right)\right| \leq \frac{1}{|z|} \leq \frac{1}{|y|},$$

$$\left|\operatorname{Re}\left(\int_0^\infty \frac{u-[u]-\frac{1}{2}}{(u+z)^2}du\right)\right| \leq \frac{1}{2}\int_0^\infty \frac{du}{|u+z|^2} \leq \frac{1}{2}\int_0^\infty \frac{du}{u^2+y^2} = \frac{\pi}{4|y|}.$$

Inserting these into (4.11), we obtain

$$\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}(z)\right) \geq \log|z| - \left(\frac{1}{2} + \frac{\pi}{4}\right)\frac{1}{|y|}.$$

This yields the result in the case $x \geq 0$.

We consider the case $x < 0$. In this case $\operatorname{Re}(1/z)$ is negative. We also note that a standard estimate gives

$$\left|\operatorname{Re}\left(\int_0^\infty \frac{u-[u]-\frac{1}{2}}{(u+z)^2}du\right)\right| \leq \frac{1}{2}\int_{-\infty}^\infty \frac{du}{u^2+y^2} = \frac{\pi}{2|y|}.$$

Applying these to (4.11), we obtain the result in the case $x < 0$. □

Proof of Proposition 4.10. Suppose that $s = \sigma + it$ satisfies $\sigma \leq 1 - \Theta(\chi)$ and $L(s, \chi) \neq 0$. Inserting (4.6) into (4.4),

$$\operatorname{Re}\left(\frac{L'}{L}(s,\chi)\right) \leq -\frac{1}{2}\log\left|\frac{q(s+\kappa)}{2\pi}\right| + \frac{\pi}{2|t|}.$$

Thus we obtain the following inequalities:

$$\operatorname{Re}\left(\frac{L'}{L}(s,\chi)\right) \leq -\frac{1}{2}\log\frac{q|t|}{2\pi} + \frac{\pi}{2|t|}, \tag{4.12}$$

$$\operatorname{Re}\left(\frac{L'}{L}(s,\chi)\right) \leq -\frac{1}{2}\log\frac{q|\sigma+\kappa|}{2\pi} + \frac{\pi}{2|t|}. \tag{4.13}$$

If $|t| \geq 6/\log q$, then (4.12) is bounded above by

$$\leq -\frac{1}{2}\left(1-\frac{\pi}{6}\right)\log q + \frac{1}{2}\log\log q + \frac{1}{2}\log\frac{\pi}{3}.$$

Since $x^\alpha \geq \alpha e \log x$ for $x \geq 1$ and $\alpha > 0$, this is

$$\leq -\frac{1}{2}\log\left(1-\frac{\pi}{6}\right) - \frac{1}{2} + \frac{1}{2}\log\frac{\pi}{3} < -0.106.$$

This confirms that $\operatorname{Re}((L'/L)(s,\chi))$ is negative on $s \in \mathcal{D}_1(\chi)$. It is easy to check from (4.13) that $\operatorname{Re}((L'/L)(s,\chi))$ is negative on $s \in \mathcal{D}_2(\chi)$, whose detail is omitted. □

Proof of Theorem 4.1. Theorem 4.1 is an immediate consequence of Proposition 4.10. □

4.2.2 Proof of Theorem 4.2

In this subsection we prove Theorem 4.2. First of all we show

Proposition 4.13. *Keep the notation in Theorem 4.2. Then for each $j \in \mathbb{Z}_{\geq 1}$, $\operatorname{Re}((L'/L)(s,\chi)) < 0$ holds on $\operatorname{Re}(s) = -2j - \kappa + 1$.*

Proof. We start with the logarithmic derivative of the functional equation for $L(s,\chi)$, which can be written as (see [MV06, p. 352])

$$\frac{L'}{L}(s,\chi) = -\frac{L'}{L}(1-s,\overline{\chi}) - \log\frac{q}{2\pi} - \frac{\Gamma'}{\Gamma}(1-s) + \frac{\pi}{2}\cot\left(\frac{\pi(s+\kappa)}{2}\right). \tag{4.14}$$

We take $j \in \mathbb{Z}_{\geq 1}$, $t \in \mathbb{R}$ and put $s = -2j - \kappa + 1 + it$ on (4.14). Then we take the real part. Since the last term on (4.14) is purely imaginary on $\operatorname{Re}(s) = -2j - \kappa + 1$, we have

$$\begin{aligned}\operatorname{Re}&\left(\frac{L'}{L}(-2j-\kappa+1+it,\chi)\right)\\&=-\operatorname{Re}\left(\frac{L'}{L}(2j+\kappa-it,\overline{\chi})\right)-\log\frac{q}{2\pi}-\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}(2j+\kappa-it)\right).\end{aligned}\tag{4.15}$$

Firstly we treat the first term on the right. Taking the logarithmic derivative of the Euler product for $L(s,\overline{\chi})$, in $\operatorname{Re}(s)>1$ we have

$$\frac{L'}{L}(s,\overline{\chi})=-\sum_{p:\text{primes}}\frac{\overline{\chi}(p)\log p}{p^s-\overline{\chi}(p)}.\tag{4.16}$$

Thus we put $s=2j+\kappa-it$ and estimate it trivially, so that

$$\left|\operatorname{Re}\left(\frac{L'}{L}(2j+\kappa-it,\overline{\chi})\right)\right|\le\sum_{p\nmid q}\frac{\log p}{p^{2j+\kappa}-1}.\tag{4.17}$$

Next we deal with the last term on (4.15). It follows from (4.7) that for $z=x+iy$ with $x>0$, $y\in\mathbb{R}$

$$\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}(z)\right)-\frac{\Gamma'}{\Gamma}(x)=y^2\sum_{n=0}^{\infty}\frac{1}{(n+x)\{(n+x)^2+y^2\}}\ge 0.\tag{4.18}$$

Thus, putting $x=2j+\kappa$ and $y=-t$, we see that

$$\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}(2j+\kappa-it)\right)\ge\frac{\Gamma'}{\Gamma}(2j+\kappa)=-c_E+\sum_{a=1}^{2j+\kappa-1}\frac{1}{a}.\tag{4.19}$$

Applying (4.17) and (4.19) to (4.15), we obtain

$$\operatorname{Re}\left(\frac{L'}{L}(-2j-\kappa+1+it)\right)\le A(q,\kappa;j)+B(q,\kappa;j),\tag{4.20}$$

where

$$\begin{aligned}A(q,\kappa;j)&:=c_E-\sum_{a=1}^{2j+\kappa-1}\frac{1}{a}-\log\frac{q}{2\pi},\\B(q,\kappa;j)&:=\sum_{p\nmid q}\frac{\log p}{p^{2j+\kappa}-1}.\end{aligned}$$

We consider the case $\kappa=1$. Since $q\ge 3$ and $j\ge 1$, we have

$$A(q,1;j)\le c_E-\frac{3}{2}-\log\frac{3}{2\pi}<-0.183,$$

$$B(q,1;j) \leq \sum_{p} \frac{\log p}{p^3-1} < 0.165.$$

This implies the desired result when $\kappa = 1$.

We treat the case $\kappa = 0$. We note that $\kappa = 0$ implies $q \geq 5$ and that there are no primitive Dirichlet characters modulo 6. When $q \geq 8$ and $j \geq 1$, we have

$$A(q,0;j) \leq c_E - 1 - \log \frac{8}{2\pi} < -0.664,$$
$$B(q,0;j) \leq \sum_{p} \frac{\log p}{p^2-1} < 0.570.$$

When $q = 7$ and $j \geq 1$, we have

$$A(7,0;j) \leq c_E - 1 - \log \frac{7}{2\pi} < -0.530,$$
$$B(7,0;j) \leq \sum_{p \neq 7} \frac{\log p}{p^2-1} < 0.530.$$

Thus when $q \geq 7$ and $\kappa = 0$, we obtain the desired result.

It remains to show the assertion in the case $q = 5$ and $\kappa = 0$. Then χ is determined uniquely and given in terms of the Kronecker symbol by $\chi(n) = \chi_5(n) := \left(\frac{5}{n}\right)$. For $j \geq 2$ we have

$$A(5,0;j) \leq c_E - \frac{11}{6} - \log \frac{5}{2\pi} < -1.027,$$
$$B(5,0;j) \leq \sum_{p \neq 5} \frac{\log p}{p^4-1} < 0.062.$$

Thus we obtain the desired result in the case $j \geq 2$. We consider the case $j = 1$. Since χ_5 is real, $\mathrm{Re}((L'/L)(-1+it,\chi_5)) = \mathrm{Re}((L'/L)(-1-it,\chi_5))$ holds for $t \in \mathbb{R}$. Thus it suffices to show that $\mathrm{Re}((L'/L)(-1+it,\chi_5))$ is negative for $t \geq 0$. For this purpose we use (4.15) with $\chi = \chi_5$ and $j = 1$:

$$\mathrm{Re}\left(\frac{L'}{L}(-1+it,\chi_5)\right) = -\,\mathrm{Re}\left(\frac{L'}{L}(2-it,\chi_5)\right) - \log\frac{5}{2\pi} - \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}(2-it)\right). \tag{4.21}$$

First of all we treat the case $t \geq 3/2$. By the same manner as (4.17) we have

$$\left|\mathrm{Re}\left(\frac{L'}{L}(2-it,\chi_5)\right)\right| \leq \sum_{p \neq 5} \frac{\log p}{p^2-1} < 0.5029. \tag{4.22}$$

It is easy to see that the right-hand side of (4.18) is monotonically decreasing on $y \leq -3/2$, so that

$$\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}(2-it)\right) \geq 1 - c_E + \frac{9}{4}\sum_{n=0}^{\infty}\frac{1}{(n+2)\{(n+2)^2+9/4\}} > 0.7523 \qquad (4.23)$$

holds on $t \geq 3/2$. Here in the first inequality we used $(\Gamma'/\Gamma)(2) = 1 - c_E$. Inserting (4.22), (4.23), and $-\log(5/2\pi) < 0.2285$ into (4.21), we see that $\mathrm{Re}((L'/L)(-1+it, \chi_5))$ is negative for $t \geq 3/2$.

Finally we treat the case $0 \leq t < 3/2$. We deal with the first term on the right-hand side of (4.21). Using (4.16), we compute $\mathrm{Re}((L'/L)(s,\chi))$ numerically at some points on $\mathrm{Re}(s) = 2$ as follows:

$$\begin{aligned}
&\frac{L'}{L}(2,\chi_5) > 0.2869, &&\mathrm{Re}\left(\frac{L'}{L}\left(2-\frac{i}{2},\chi_5\right)\right) > 0.2527,\\
&\mathrm{Re}\left(\frac{L'}{L}(2-i,\chi_5)\right) > 0.1686, &&\mathrm{Re}\left(\frac{L'}{L}\left(2-\frac{5}{4}i,\chi_5\right)\right) > 0.1188, \qquad (4.24)\\
&\mathrm{Re}\left(\frac{L'}{L}\left(2-\frac{11}{8}i,\chi_5\right)\right) > 0.0936, &&\mathrm{Re}\left(\frac{L'}{L}\left(2-\frac{3}{2}i,\chi_5\right)\right) > 0.0688.
\end{aligned}$$

We note that for $t \in \mathbb{R}$ and $t_0 \in \{0, 1/2, 1, 5/4, 11/8, 3/2\}$

$$\mathrm{Re}\left(\frac{L'}{L}(2-it,\chi_5)\right) = \mathrm{Re}\left(\frac{L'}{L}(2-it_0,\chi_5)\right) + \mathrm{Im}\int_{t_0}^{t}\left(\frac{L'}{L}\right)'(2-iv,\chi_5)dv. \quad (4.25)$$

Numerical computation gives that for $v \in \mathbb{R}$

$$\left|\left(\frac{L'}{L}\right)'(2-iv,\chi_5)\right| \leq \sum_{p\neq 5}\frac{p^{-2}(\log p)^2}{(1-p^{-2})^2} < 0.7721. \qquad (4.26)$$

We see from (4.24), (4.25) and (4.26) that $\mathrm{Re}((L'/L)(2-it,\chi_5)) > 0$ for $0 \leq t < 3/2$. This together with (4.18) and (4.21) yields

$$\mathrm{Re}\left(\frac{L'}{L}(-1+it,\chi_5)\right) < -\log\frac{5}{2\pi} - 1 + c_E < -0.1943 < 0$$

for $0 \leq t < 3/2$. This completes the proof. □

Proof of Theorem 4.2. Let $j \in \mathbb{Z}_{\geq 1}$. Proposition 4.13 implies that $L'(s,\chi)$ does not vanish on $\mathrm{Re}(s) = -2j - \kappa - 1$. We show that $L'(s,\chi)$ has a unique zero in the strip $-2j-\kappa-1 < \mathrm{Re}(s) < -2j-\kappa+1$. We take the path determined by the rectangle with vertices at $-2j-\kappa \pm 1 \pm 1000i$. Then by Propositions 4.10 and 4.13 we find that $\mathrm{Re}((L'/L)(s,\chi))$ is negative on the path. Thus the argument principle gives that the number of zeros of $L'(s,\chi)$ inside the path equals that of $L(s,\chi)$. Since $L(s,\chi)$ has a unique zero $s = -2j-\kappa$ inside the path, $L'(s,\chi)$ has also a unique zero inside the path. This together with Theorem 4.1 gives the first claim of Theorem 4.2, which completes the proof. □

4.2.3 Proof of Theorem 4.3

In this subsection we prove Theorem 4.3.

Proof of Theorem 4.3. We take $\varepsilon \in (0, 1/2)$. Let $\mathcal{C} = \mathcal{C}_{j,\varepsilon}$ be the path determined by the circle centered at $-2j - \kappa$ with radius ε. Then it is easy to see from (4.14) and Stirling's formula that

$$\mathrm{Re}\left(\frac{L'}{L}(s, \chi)\right) = -\log(jq) + \mathrm{Re}\left(\frac{1}{\eta}\right) + O(1)$$

holds on $s = -2j - \kappa + \eta \in \mathcal{C}$, where the implied constant is absolute. Suppose that jq is sufficiently large and we choose $\varepsilon = 2/\log(jq)$. Then we find that $\mathrm{Re}((L'/L)(s, \chi))$ is negative on $s \in \mathcal{C}$. Thus the argument principle says that there is a unique zero of $L'(s, \chi)$ inside $\mathcal{C}$ thanks to the trivial zero $s = -2j - \kappa$ of $L(s, \chi)$. Since the zero of $L'(s, \chi)$ inside $\mathcal{C}$ coincides with $\alpha_j(\chi)$, we obtain $|\alpha_j(\chi) + 2j + \kappa| < 2/\log(jq)$. This completes the proof. $\square$

4.2.4 Proof of Theorem 4.4

In this subsection we show Theorem 4.4. Roughly speaking, our strategy of the proof is to show $\mathrm{Re}((L'/L)(s, \chi)) < 0$ on $\mathrm{Re}(s) = 0$ by (4.4). For this purpose we show the following inequality:

Lemma 4.14. $(\Gamma'/\Gamma)(x)$ *is monotonically increasing on* $x > 0$. *Furthermore, for* $x \in [1 - \frac{1}{1000}, 1]$ *we have*

$$-0.58 < \frac{\Gamma'}{\Gamma}(x) \leq -c_E.$$

Proof. In view of (4.7) it is trivial that $(\Gamma'/\Gamma)(x)$ is monotonically increasing on $x > 0$. This implies that $(\Gamma'/\Gamma)(x) \leq (\Gamma'/\Gamma)(1) = -c_E$ on $x \in [1 - \frac{1}{1000}, 1]$, which is nothing but the second inequality. On the other hand, we have $(\Gamma'/\Gamma)(x) \geq (\Gamma'/\Gamma)(1 - \frac{1}{1000})$ on $x \in [1 - \frac{1}{1000}, 1]$. Computing (4.7) numerically at $s = 1 - \frac{1}{1000}$, we have $(\Gamma'/\Gamma)(1 - \frac{1}{1000}) > -0.5789$. This gives the first inequality. $\square$

Proof of Theorem 4.4. First of all we consider the case $\chi(-1) = 1$. We take any $\delta \in (0, 1/2000)$. We take the contour $\mathcal{C}$ determined by the rectangle with vertices at $-1 \pm 1000i$, $\pm 1000i$ with a small left-semicircular indentation $\delta e^{i\phi}$ $(\phi : 3\pi/2 \to \pi/2)$. We shall prove that $\mathrm{Re}((L'/L)(s, \chi)) < 0$ holds on $s \in \mathcal{C}$. We have already shown in Propositions 4.10 and 4.13 that $\mathrm{Re}((L'/L)(s, \chi)) < 0$ on $\{\sigma \pm 1000i \mid -1 \leq \sigma \leq 0\} \cup \{-1 + it \mid |t| \leq 1000\}$. We consider the case that $s = \sigma + it$ is on the right side of $\mathcal{C}$. We begin with the inequality (4.4). Since $\sigma \leq 0$, we have

$$\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\right) = \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right)\right) - \mathrm{Re}\left(\frac{2}{s}\right) \geq \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right)\right).$$

(4.18) and Lemma 4.14 yield

$$\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)\right) \geq \frac{\Gamma'}{\Gamma}\left(1-\frac{\delta}{2}\right) \geq -0.58.$$

We apply this to (4.4). If $q \geq 7$, we have

$$\mathrm{Re}\left(\frac{L'}{L}(s,\chi)\right) \leq 0.29 - \frac{1}{2}\log\frac{7}{\pi} < -0.110 < 0.$$

As the above discussion, we see $\mathrm{Re}((L'/L)(s,\chi))$ is negative on $s \in \mathcal{C}$. Since $(L'/L)(s,\chi)$ has no poles inside $\mathcal{C}$, the argument principle says $L'(s,\chi)$ has no zeros inside $\mathcal{C}$. Since $\delta \in (0, 1/2000)$ is arbitrary, this implies the result when $\chi(-1) = 1$.

Next we treat the case $\chi(-1) = -1$. We take the contour $\mathcal{C}$ determined by the rectangle with vertices at $-2 \pm 1000i$, $\pm 1000i$. Then we have already shown that $\mathrm{Re}((L'/L)(s,\chi)) < 0$ holds on $s \in \mathcal{C} \setminus [-1000i, 1000i]$. Let $t \in [-1000, 1000]$. Then in the same manner as the case $\chi(-1) = 1$, (4.4) gives

$$\mathrm{Re}\left(\frac{L'}{L}(it,\chi)\right) \leq -\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) - \frac{1}{2}\log\frac{q}{\pi}. \tag{4.27}$$

Since $(\Gamma'/\Gamma)(1/2) = -2\log 2 - c_E$, (4.27) is negative provided $q > 4\pi e^{c_E} = 22.38\ldots$. Thus $\mathrm{Re}((L'/L)(s,\chi)) < 0$ holds on $s \in \mathcal{C}$ if $q \geq 23$. Applying the argument principle and taking the trivial zero $s = -1$ of $L(s,\chi)$ into account, we see that $L'(s,\chi)$ has a unique zero inside $\mathcal{C}$. This completes the proof. □

4.2.5 Proof of Theorems 4.5 and 4.6

In this subsection we show Theorems 4.5 and 4.6.

For short we write the functional equation for $L(s,\chi)$ as $L(s,\chi) = F(s,\chi)L(1-s,\overline{\chi})$, where

$$F(s,\chi) = \varepsilon(\chi)2^s\pi^{s-1}q^{\frac{1}{2}-s}\sin\left(\frac{\pi(s+\kappa)}{2}\right)\Gamma(1-s).$$

Here $\varepsilon(\chi)$ is a constant depending on χ, which satisfies $|\varepsilon(\chi)| = 1$. We also define $G(s,\chi)$ by

$$G(s,\chi) = -\frac{m^s}{\chi(m)\log m}L'(s,\chi).$$

First of all we show

Lemma 4.15. *For $s = \sigma + it$ with $\sigma \geq 2$ and $t \in \mathbb{R}$ we have*

$$|G(s,\chi) - 1| \leq 2\left(1 + \frac{8m}{\sigma}\right)\exp\left(-\frac{\sigma}{2m}\right).$$

Proof. By the Dirichlet series expression for $L(s,\chi)$ we find

$$G(s,\chi) = 1 + \frac{m^s}{\chi(m)\log m}\sum_{n=m+1}^{\infty}\frac{\chi(n)\log n}{n^s}.$$

Thus we have

$$|G(s,\chi)-1| \le \frac{m^\sigma}{\log m}\sum_{n=m+1}^{\infty}\frac{\log n}{n^\sigma}. \tag{4.28}$$

We divide the sum into $n = m+1$ and $n \ge m+2$. The sum over $n \ge m+2$ is estimated as follows:

$$\begin{aligned}\sum_{n=m+2}^{\infty}\frac{\log n}{n^\sigma} &\le \int_{m+1}^{\infty}\frac{\log u}{u^\sigma}du \\ &= \frac{(m+1)^{1-\sigma}\log(m+1)}{\sigma-1} + \frac{(m+1)^{1-\sigma}}{(\sigma-1)^2} \\ &\le \frac{2(m+1)^{1-\sigma}\log(m+1)}{\sigma-1}.\end{aligned}$$

Inserting this into (4.28), we have

$$\begin{aligned}|G(s,\chi)-1| &\le \frac{\log(m+1)}{\log m}\left(\frac{m}{m+1}\right)^\sigma\left(1+2\frac{m+1}{\sigma-1}\right) \\ &\le 2\left(1+\frac{8m}{\sigma}\right)\left(\frac{m}{m+1}\right)^\sigma.\end{aligned} \tag{4.29}$$

Since $\log(1+x) \ge x/2$ on $x \in [0,1]$, we find

$$\left(\frac{m}{m+1}\right)^\sigma = \exp\left(-\sigma\log\left(1+\frac{1}{m}\right)\right) \le \exp\left(-\frac{\sigma}{2m}\right).$$

Applying this to (4.29), we obtain the result. □

By Lemma 4.15 we have

$$|G(s,\chi)-1| \le 4\exp\left(-\frac{\sigma}{2m}\right) \tag{4.30}$$

for $\sigma \ge 10m$. In particular $G(s,\chi)$ has no zeros on $\sigma \ge 8m$.

Let $b_\kappa \in \{1+\kappa, 3+\kappa\}$, $T \ge 2$ and $U \ge 10m$. We apply the Littlewood lemma (see Lemma 2.2 or [Tit39, Section 3.8]) to $G(s,\chi)$ on the rectangle with vertices at $-b_\kappa \pm iT$ and $U \pm iT$. Taking the imaginary part, we have

$$
\begin{aligned}
2\pi &\sum_{\substack{\rho'=\beta'+i\gamma' \\ \beta'>-b_\kappa, -T\le\gamma'\le T}} (\beta' + b_\kappa) \\
&= \int_{-T}^{T} \log|G(-b_\kappa + it, \chi)| dt - \int_{-T}^{T} \log|G(U + it, \chi)| dt \\
&\quad + \int_{-b_\kappa}^{U} \arg G(\sigma + iT, \chi) d\sigma - \int_{-b_\kappa}^{U} \arg G(\sigma - iT, \chi) d\sigma.
\end{aligned} \tag{4.31}
$$

Here we determine the branch of $\log G(s, \chi)$ such that it tends to 0 as $\sigma \to \infty$ and it is holomorphic in $\mathbb{C}\setminus\{\rho'+\lambda \mid L'(\rho', \chi) = 0,\ \lambda \le 0\}$. When there are zeros of $L'(s, \chi)$ on $\mathrm{Im}(s) = \pm T$, we determine $\arg G(\sigma \pm iT) = \lim_{\varepsilon\downarrow 0} \arg G(\sigma \pm i(T + \varepsilon))$. Thanks to (4.30), the second integral on (4.31) tends to 0 as $U \to \infty$. We also note that Theorems 4.1, 4.2, and 4.4 give $\#\{\rho' = \beta'+i\gamma' \mid L'(\rho', \chi) = 0,\ -b_\kappa < \beta' \le 0\} \ll 1$, where the implied constant is absolute. Combining these, we obtain

$$
2\pi \sum_{\substack{\rho'=\beta'+i\gamma' \\ \beta'>0, -T\le\gamma'\le T}} (\beta' + b_\kappa) = I_1 + I_2^+ - I_2^- + O(1), \tag{4.32}
$$

where $I_1 = I_1(b_\kappa, \chi, T)$ and $I_2^\pm = I_2^\pm(b_\kappa, \chi, T)$ are given by

$$
\begin{aligned}
I_1 &= \int_{-T}^{T} \log|G(-b_\kappa + it, \chi)| dt, \\
I_2^\pm &= \int_{-b_\kappa}^{\infty} \arg G(\sigma \pm iT, \chi) d\sigma.
\end{aligned}
$$

We deal with I_1. By the definition we have

$$
I_1 = -2T(b_\kappa \log m + \log\log m) + \int_{-T}^{T} \log|L'(-b_\kappa + it, \chi)| dt. \tag{4.33}
$$

We divide the interval $t \in [-T, T]$ into $|t| \le 20$, $20 < t \le T$, and $-T \le t < -20$. Firstly we consider the case $|t| \le 20$. We have

$$
\log|L'(-b_\kappa + it, \chi)| = \log|L(-b_\kappa + it, \chi)| + \log\left|\frac{L'}{L}(-b_\kappa + it, \chi)\right|.
$$

By the functional equation, the first term on the right is $(1/2 + b_\kappa)\log q + O(1)$ uniformly on $|t| \le 20$. We see from the functional equation together with the discussion in Section 4.2.2 that the second term on the right is $O(\log\log q)$ on $|t| \le 20$. In consequence we obtain

$$
\int_{-20}^{20} \log|L'(-b_\kappa + it, \chi)| dt \ll \log q. \tag{4.34}
$$

Next we deal with the integral over $20 \le t \le T$. By the functional equation we have

$$\begin{aligned}\int_{20}^{T} \log|L'(-b_\kappa+it,\chi)|dt \\ &= \int_{20}^{T} \log|F(-b_\kappa+it,\chi)|dt + \int_{20}^{T} \log\left|\frac{F'}{F}(-b_\kappa+it,\chi)\right| dt \\ &\quad + \int_{20}^{T} \log|L(1+b_\kappa-it,\overline{\chi})|dt \\ &\quad + \int_{20}^{T} \log\left|1-\frac{1}{(F'/F)(-b_\kappa+it,\chi)}\frac{L'}{L}(1+b_\kappa-it,\overline{\chi})\right| dt.\end{aligned} \tag{4.35}$$

By Stirling's formula we have

$$\log|F(-b_\kappa+it,\chi)| = \left(\frac{1}{2}+b_\kappa\right)\log\frac{qt}{2\pi} + O\left(\frac{1}{t}\right),$$

so that

$$\int_{20}^{T} \log|F(-b_\kappa+it,\chi)|dt = \left(\frac{1}{2}+b_\kappa\right)\left(T\log\frac{qT}{2\pi}-T\right) + O(\log(qT)). \tag{4.36}$$

In a similar manner, Stirling's formula for $(\Gamma'/\Gamma)(z)$ gives

$$\frac{F'}{F}(-b_\kappa+it,\chi) = -\log\frac{qt}{2\pi} + O\left(\frac{1}{t}\right).$$

Thus we have

$$\int_{20}^{T} \log\left|\frac{F'}{F}(-b_\kappa+it,\chi)\right| dt = \int_{20}^{T} \log\log\frac{qt}{2\pi}dt + O\left(\int_{20}^{T}\frac{dt}{t\log(qt)}\right). \tag{4.37}$$

Integrating by parts, we see that the first integral on the right turns to

$$\int_{20}^{T} \log\log\frac{qt}{2\pi}dt = T\log\log\frac{qT}{2\pi} - \frac{2\pi}{q}\,\mathrm{Li}\left(\frac{qT}{2\pi}\right) + O(\log\log q).$$

We easily see that the last term on (4.37) is $O(\log\log(qT))$. Combining these, we obtain

$$\int_{20}^{T} \log\left|\frac{F'}{F}(-b_\kappa+it,\chi)\right| dt = T\log\log\frac{qT}{2\pi} - \frac{2\pi}{q}\,\mathrm{Li}\left(\frac{qT}{2\pi}\right) + O(\log\log(qT)). \tag{4.38}$$

We see from the Dirichlet series expression for $\log L(s,\overline{\chi})$ that

$$\int_{20}^{T} \log|L(1+b_\kappa-it,\overline{\chi})| \ll 1. \tag{4.39}$$

We treat the last term on (4.35). Firstly we show the following:

Lemma 4.16. *For $s = \sigma + it$ with $\sigma \le -1$ and $t \ge 20$ we have*

$$\left| \frac{1}{(F'/F)(s,\chi)} \frac{L'}{L}(1-s,\overline{\chi}) \right| \le 2^{\sigma}.$$

Proof. We have

$$\frac{F'}{F}(s,\chi) = -\log\frac{q}{2\pi} + \frac{\pi}{2}\cot\left(\frac{\pi(s+\kappa)}{2}\right) - \frac{\Gamma'}{\Gamma}(1-s). \tag{4.40}$$

Below let $s = \sigma + it$ with $\sigma \le -1$ and $t \ge 1$. It is easy to see that

$$\left| \frac{\pi}{2}\cot\left(\frac{\pi(s+\kappa)}{2}\right) + \frac{\pi i}{2} \right| \le \frac{\pi}{e^{\pi t}-1}. \tag{4.41}$$

We estimate the last term. We start with (4.10). Integration by parts gives

$$\int_0^\infty \frac{u-[u]-\frac{1}{2}}{(u+z)^2}du = -\frac{1}{12z^2} + \int_0^\infty \frac{B_2(u-[u])}{(u+z)^3}du$$

for $z = x+iy$ with $x \ge 1$ and $y \in \mathbb{R}$, where $B_2(X) = X^2 - X + \frac{1}{6}$ is the second Bernoulli polynomial. Estimating it trivially, we find

$$\begin{aligned} \left| \int_0^\infty \frac{u-[u]-\frac{1}{2}}{(u+z)^2}du \right| &\le \frac{1}{12|z|^2} + \frac{1}{6}\int_0^\infty \frac{du}{|u+z|^3} \\ &\le \frac{1}{12|z|^2} + \frac{1}{6}\int_0^\infty \frac{du}{(u^2+|z|^2)^{3/2}} \\ &= \frac{1}{4|z|^2}. \end{aligned}$$

This gives

$$\left| \frac{\Gamma'}{\Gamma}(1-s) - \mathrm{Log}(1-s) \right| \le \frac{1}{2|1-s|} + \frac{1}{4|1-s|^2}. \tag{4.42}$$

Inserting (4.41) and (4.42) into (4.40), we obtain

$$\left| \frac{F'}{F}(s,\chi) + \mathrm{Log}\,\frac{q(1-s)}{2\pi} + \frac{\pi i}{2} \right| \le \frac{\pi}{e^{\pi t}-1} + \frac{1}{2t} + \frac{1}{4t^2}.$$

This implies

$$\left| \frac{F'}{F}(s,\chi) \right| \ge \left| \mathrm{Re}\left(\frac{F'}{F}(s,\chi) \right) \right| \ge \log\frac{qt}{2\pi} - \frac{\pi}{e^{\pi t}-1} - \frac{1}{2t} - \frac{1}{4t^2}.$$

Next we deal with $(L'/L)(1-s,\overline{\chi})$. Using the Dirichlet series expression, we have

$$\left| \frac{L'}{L}(1-s,\overline{\chi}) \right| \le \sum_{n=2}^{\infty} \frac{\log n}{n^{1-\sigma}} \le \frac{\log 2}{2^{1-\sigma}} + \int_2^\infty \frac{\log u}{u^{1-\sigma}}du$$

$$\le \left(1+\frac{3}{2}\log 2\right)2^{\sigma}.$$

Taking $q \ge 3$ into account, we see that

$$\log\frac{qt}{2\pi} - \frac{\pi}{e^{\pi t}-1} - \frac{1}{2t} - \frac{1}{4t^2} \ge 1 + \frac{3}{2}\log 2$$

holds if $t \ge 20$. This completes the proof. □

By Lemma 4.16 we can determine the branch of

$$\log\left(1 - \frac{1}{(F'/F)(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right) \tag{4.43}$$

such that it is holomorphic in a region including $\{\sigma+it \mid \sigma \le -1, t \ge 20\}$ and it tends to 0 as $\sigma \to -\infty$. We apply Cauchy's theorem to (4.43) on the triangle joining $-b_\kappa + 20i$, $-b_\kappa + iT$, and $-T + iT$. Lemma 4.16 says that (4.43) is $O(2^{\sigma})$ on the triangle. This gives

$$\int_{20}^{T} \log\left|1 - \frac{1}{(F'/F)(-b_\kappa+it,\chi)}\frac{L'}{L}(1+b_\kappa-it,\overline{\chi})\right| dt \ll 1. \tag{4.44}$$

Inserting (4.36), (4.38), (4.39), and (4.44) into (4.35), we obtain

$$\begin{aligned}\int_{20}^{T} &\log|L'(-b_\kappa+it,\chi)|dt \\ &= \left(\frac{1}{2}+b_\kappa\right)\left(T\log\frac{qT}{2\pi} - T\right) + T\log\log\frac{qT}{2\pi} - \frac{2\pi}{q}\,\mathrm{Li}\left(\frac{qT}{2\pi}\right) + O(\log(qT)).\end{aligned} \tag{4.45}$$

Since $|L'(\overline{s},\overline{\chi})| = |L'(s,\chi)|$, we obtain the same formula as (4.45) for the integral over $[-T,-20]$. Applying these and (4.34) to (4.33), we reach

$$\begin{aligned} I_1 = 2&\left(\frac{1}{2}+b_\kappa\right)\left(T\log\frac{qT}{2\pi} - T\right) - 2T(b_\kappa\log m + \log\log m) \\ &+ 2T\log\log\frac{qT}{2\pi} - \frac{4\pi}{q}\,\mathrm{Li}\left(\frac{qT}{2\pi}\right) + O(\log(qT)).\end{aligned} \tag{4.46}$$

Next we deal with $I_2^{\pm}$. For this purpose we give the following bounds for $\arg G(\sigma \pm iT, \chi)$:

Proposition 4.17. *For $T \ge 2$ we have*

$$\arg G(\sigma \pm iT,\chi) \ll \begin{cases} \exp(-\sigma/(2m)) & \text{if } 10m \le \sigma, \\ m/\sigma & \text{if } 3 \le \sigma \le 10m, \\ m^{1/2}\log(qT) & \text{if } -5 \le \sigma \le 3, \end{cases} \tag{4.47}$$

where the implied constant is absolute.

In order to show this, we collect consequences of well-known facts. First of all we recall estimates for $G(s, \chi)$.

Lemma 4.18. *For $s = \sigma + it$ with $-10 \le \sigma \le 3$ and $t \in \mathbb{R}$ we have*

$$G(s, \chi) \ll (q\tau)^{20},$$

where $\tau := |t| + 2$ and the implied constant is absolute.

Proof. Cauchy's integral formula gives

$$L'(s, \chi) = \frac{1}{2\pi i} \int_{|w-s|=1} \frac{L(w, \chi)}{(w-s)^2} dw. \tag{4.48}$$

According to [MV06, Corollary 10.10 and Lemma 10.15], $L(s, \chi) \ll (q\tau)^{15}$ holds for $-11 \le \sigma \le 4$ and $t \in \mathbb{R}$. Inserting this into (4.48) and using $m \ll \log q$, we reach the result. □

Next we recall the following formula:

Lemma 4.19. *For $a > 0$ and $b > 0$ we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |a + b\cos\theta| d\theta = \begin{cases} \log \frac{a+\sqrt{a^2-b^2}}{2} & \text{if } a > b, \\ \log(b/2) & \text{if } a \le b. \end{cases}$$

Proof. We calculate the left-hand side as

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \log |a + b\cos\theta| d\theta \\ &\quad = \frac{1}{2\pi} \int_0^{2\pi} \log \left| a + b\frac{e^{i\theta} + e^{-i\theta}}{2} \right| d\theta \\ &\quad = \log\left(\frac{b}{2}\right) + \frac{1}{2\pi} \int_0^{2\pi} \log \left| e^{2i\theta} + \frac{2a}{b} e^{i\theta} + 1 \right| d\theta. \end{aligned} \tag{4.49}$$

We put $\alpha_\pm = -\frac{a}{b} \pm \sqrt{(\frac{a}{b})^2 - 1}$, which are solutions of $X^2 + \frac{2a}{b}X + 1 = 0$. By Jensen's theorem (see Lemma 2.1 or [Tit39, Section 3.61]), (4.49) turns to

$$= \log\left(\frac{b}{2}\right) + \log^+ |\alpha_+| + \log^+ |\alpha_-|,$$

where $\log^+ x = \max\{\log x, 0\}$. We easily check that $|\alpha_+| < 1$ and $|\alpha_-| > 1$ when $a > b$ and that $|\alpha_\pm| = 1$ when $a \le b$. This completes the proof. □

Now we are ready to prove Proposition 4.17. In the proof below $c_1, c_2, \ldots$ are positive constants independent of any parameters.

Proof of Proposition 4.17. We see from $\overline{G(\overline{s},\overline{\chi})} = G(s,\chi)$ that $\arg G(\sigma - iT,\chi) = -\arg G(\sigma + iT,\overline{\chi})$. Thus it suffices to show (4.47) for $\arg G(\sigma + iT,\chi)$ only. Thus we concentrate on $\arg G(\sigma+iT,\chi)$ below. We also note that (4.47) is an immediate consequence of Lemma 4.15 or (4.30) when $\sigma \geq 10m$.

Let $\sigma \in [-10, 10m]$. We put $h := \#\{x \in [\sigma, 10m] \mid \mathrm{Re}(G(\sigma + iT,\chi)) = 0\}$. Then we see that $\arg G(\sigma + iT,\chi) \leq (h + 3/2)\pi$. In order to estimate h, we put

$$H(z,\chi) := \frac{G(z + iT,\chi) + G(z - iT,\overline{\chi})}{2}.$$

For $r > 0$ we denote by $n(r)$ the number of zeros of $H(z,\chi)$ on $|z - 11m| \leq r$, counted with multiplicity. Since $H(z,\chi) = \mathrm{Re}(G(z+iT,\chi))$ for $z \in \mathbb{R}$, we see that $h \leq n(R)$, where

$$R := 11m - \sigma.$$

We see from the above discussion that

$$\arg G(\sigma + iT,\chi) \ll n(R). \tag{4.50}$$

Below we estimate $n(R)$. We take $R_0 > 0$. Then by Jensen's theorem we have

$$\int_0^{R+R_0} \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |H(11m + (R + R_0)e^{i\theta},\chi)| d\theta - \log |H(11m,\chi)|.$$

Since $n(r)$ is nonnegative and monotonically increasing, the left-hand side is bounded below as

$$\int_0^{R+R_0} \frac{n(r)}{r} dr \geq \int_R^{R+R_0} \frac{n(r)}{r} dr \geq n(R) \log \left(1 + \frac{R_0}{R}\right).$$

Combining this with $\log |H(11m,\chi)| = O(1)$, which follows from (4.30), we have

$$n(R) \leq \frac{1}{\log(1 + \frac{R_0}{R})} \left(\frac{1}{2\pi} \int_0^{2\pi} \log |H(11m + (R + R_0)e^{i\theta},\chi)| d\theta + c_1\right). \tag{4.51}$$

First of all we consider the case $3 \leq \sigma \leq 10m$. In this case we restrict R_0 by

$$0 < R_0 \leq \sigma - 2. \tag{4.52}$$

Then we note $11m - (R + R_0) \geq 2$. We see from Lemma 4.15 that

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \log |H(11m + (R + R_0)e^{i\theta},\chi)| d\theta \\ &\qquad \leq \frac{1}{2\pi} \int_0^{2\pi} \log \frac{m}{11m + (R + R_0)\cos\theta} d\theta + c_2. \end{aligned}$$

By Lemma 4.19, this is

$$\leq \log m - \log\left(\frac{11m}{2}\right) + c_2 \leq c_3.$$

We also note that the restriction (4.52) implies $0 < R_0/R \leq c_4$, so that $\log(1 + R_0/R) \gg R_0/R$. Combining these, we obtain

$$n(R) \ll \frac{R}{R_0}.$$

Taking $R_0 = \sigma - 2$, we obtain $n(R) \ll m/\sigma$. This together with (4.50) completes the proof when $3 \leq \sigma \leq 10m$.

Finally we deal with the case $-5 \leq \sigma \leq 3$. In this case we choose $R_0 = 5$. In order to estimate the integral on (4.51), we divide $[0, 2\pi] = \mathcal{I}_1 \cup \mathcal{I}_2$, where

$$\mathcal{I}_1 := \{\theta \in [0, 2\pi] \mid 11m + (R+5)\cos\theta \geq 2\},$$
$$\mathcal{I}_2 := \{\theta \in [0, 2\pi] \mid 11m + (R+5)\cos\theta < 2\}.$$

We take $\theta_0 \in (0, \pi/2)$ such that

$$\cos\theta_0 = \frac{11m - 2}{R+5}.$$

Then we have $\mathcal{I}_1 = [0, \pi - \theta_0] \cup [\pi + \theta_0, 2\pi]$ and $\mathcal{I}_2 = (\pi - \theta_0, \pi + \theta_0)$. Since $\cos\theta_0 = 1 + O(1/m)$ and $\cos\theta_0 = 1 - 2\sin^2(\theta_0/2)$, we see that

$$\theta_0 = O(m^{-1/2}). \tag{4.53}$$

We deal with the integral over $\mathcal{I}_1$. By Lemma 4.15 we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathcal{I}_1} \log|H(11m + (R+5)e^{i\theta}, \chi)|d\theta \\ &\qquad \leq \log m - \frac{1}{2\pi}\int_{\mathcal{I}_1} \log|11m + (R+5)\cos\theta|d\theta + c_5.\end{aligned} \tag{4.54}$$

We see from Lemma 4.19 together with $R + 5 \geq 11m$ that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathcal{I}_1} \log|11m + (R+5)\cos\theta|d\theta \\ &\qquad = \log\frac{R+5}{2} - \frac{1}{2\pi}\int_{\pi-\theta_0}^{\pi+\theta_0} \log|11m + (R+5)\cos\theta|d\theta \\ &\qquad \geq \log m - \frac{\theta_0}{\pi}\log(30m).\end{aligned}$$

Inserting this into (4.54) and using (4.53), we obtain

$$\frac{1}{2\pi}\int_{\mathcal{I}_1} \log|H(11m+(R+5)e^{i\theta},\chi)|d\theta \le c_6.$$

Next we treat the integral over $\mathcal{I}_2$. By Lemma 4.18

$$H(11m+(R+5)e^{i\theta},\chi) \ll (qT')^{20}$$

holds on $\theta \in \mathcal{I}_2$, where $T' := \max\{T, m\}$. This together with (4.53) yields

$$\frac{1}{2\pi}\int_{\mathcal{I}_2} \log|H(11m+(R+5)e^{i\theta},\chi)|d\theta \le c_7 m^{-1/2}\log(qT').$$

Inserting this and $\log(1+5/R) \gg 1/R \gg 1/m$ into (4.51), we obtain

$$n(R) \ll m(m^{-1/2}\log(qT')+1) \ll m^{1/2}\log(qT') \ll m^{1/2}\log(qT). \tag{4.55}$$

Here in the second inequality we used $m \ll \log q$. In the last inequality we also used $\log(qT') \ll \log(q\log q) \ll \log q \ll \log(qT)$ when $T \le m$. Applying (4.55) to (4.50), we reach the result when $-5 \le \sigma \le 3$.

The proof of Proposition 4.17 is completed. □

Proof of Theorem 4.5. Subtracting (4.31) with $b_\kappa = 1+\kappa$ from that with $b_\kappa = 3+\kappa$, we have

$$\begin{aligned} 4\pi N_1(T,\chi) = {} & (I_1(3+\kappa,\chi,T) - I_1(1+\kappa,\chi,T)) \\ & + (I_2^+(3+\kappa,\chi,T) - I_2^+(1+\kappa,\chi,T)) \\ & - (I_2^-(3+\kappa,\chi,T) - I_2^-(1+\kappa,\chi,T)) + O(1). \end{aligned}$$

By (4.46) we have

$$I_1(3+\kappa,\chi,T) - I_1(1+\kappa,\chi,T) = 4T\log\frac{qT}{2\pi m} - 4T + O(\log(qT)).$$

On the other hand Proposition 4.17 gives

$$\begin{aligned} I_2^\pm(3+\kappa,\chi,T) - I_2^\pm(1+\kappa,\chi,T) &= \int_{-3-\kappa}^{-1-\kappa} \arg G(\sigma \pm iT,\chi)d\sigma \\ &\ll m^{1/2}\log(qT). \end{aligned}$$

Combining these, we obtain the result. □

Proof of Theorem 4.6. We start with (4.32). We estimate $I_2^{\pm} = I_2^{\pm}(b_\kappa, \chi, T)$. By Proposition 4.17 we have

$$I_2^{\pm} \ll m \log m + m^{1/2} \log(qT) \ll m^{1/2} \log(qT). \tag{4.56}$$

Here in the last inequality we used $m \ll \log q$. We also note that

$$\begin{aligned} 2\pi &\sum_{\substack{\rho'=\beta'+i\gamma' \\ \beta'>0, -T\le\gamma'\le T}} (\beta' + b_\kappa) \\ &= 2\pi \sum_{\substack{\rho'=\beta'+i\gamma' \\ \beta'>0, -T\le\gamma'\le T}} \left(\beta' - \frac{1}{2}\right) + 2\pi \left(b_\kappa + \frac{1}{2}\right) N_1(T, \chi). \end{aligned}$$

Applying Theorem 4.5, (4.56), and (4.46), we complete the proof. □

4.2.6 Proof of Theorems 4.7, 4.8, and 4.9

In this subsection we show Theorems 4.7–4.9. First of all we investigate the sign of $(L'/L)(s, \chi)$ on $\mathrm{Re}(s) = 1/2$. For convenience we put

$$\mathcal{T} = \mathcal{T}_\chi := \mathbb{R} \setminus \{t \in \mathbb{R} \mid L(1/2 + it, \chi) = 0\}.$$

Lemma 4.20. *Let χ be a non-principal primitive Dirichlet character. Then for $t \in \mathcal{T}$*

$$\mathrm{Re}\left(\frac{L'}{L}\left(\frac{1}{2} + it, \chi\right)\right) < 0 \tag{4.57}$$

holds if one of the following conditions holds:

1. $\kappa = 0$ *and* $q \ge 216$.
2. $\kappa = 0$ *and* $|t| \ge 2$.
3. $\kappa = 1$ *and* $q \ge 10$.
4. $\kappa = 1$ *and* $|t| \ge 3$.

Proof. We begin with (4.5). Since $L(s, \chi) = F(s, \chi) L(1 - s, \overline{\chi})$ and $\overline{L(\overline{s}, \overline{\chi})} = L(s, \chi)$, each zero of $L(s, \chi)$ in $\mathrm{Re}(s) > 1/2$ can be written by $1 - \overline{\rho}$ uniquely, where $\rho = \beta + i\gamma$ is a zero of $L(s, \chi)$ in $0 < \beta < 1/2$. Furthermore, routine calculation gives

$$\frac{\sigma - \beta}{|s - \rho|^2} + \frac{\sigma - (1 - \beta)}{|s - (1 - \overline{\rho})|^2} = (2\sigma - 1) \frac{(\sigma - \frac{1}{2})^2 - (\beta - \frac{1}{2})^2 + (t - \gamma)^2}{|s - \rho|^2 |s - 1 + \overline{\rho}|^2}.$$

Applying these to (4.5), for $s = \sigma + it$ with $L(s,\chi) \neq 0$ we find

$$\operatorname{Re}\left(\frac{L'}{L}(s,\chi)\right) = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa}{2}\right)\right) + \left(\sigma - \frac{1}{2}\right) J(s,\chi), \qquad (4.58)$$

where

$$J(s,\chi) = \sum_{\beta=\frac{1}{2}} \frac{1}{|s-\rho|^2} + 2\sum_{\beta<1/2} \frac{(\sigma-\frac{1}{2})^2 - (\beta-\frac{1}{2})^2 + (t-\gamma)^2}{|s-\rho|^2|s-1+\overline{\rho}|^2}.$$

Thus, for $t \in \mathcal{T}$ we have

$$\operatorname{Re}\left(\frac{L'}{L}\left(\frac{1}{2}+it,\chi\right)\right) = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{\kappa}{2}+\frac{it}{2}\right)\right). \qquad (4.59)$$

We note that the right-hand side is an even function with respect to t. Therefore we concentrate on $t \geq 0$ below. Let $t_0 \in [0,\infty)$. Since the right-hand side of (4.59) is monotonically decreasing on $t \geq 0$ thanks to (4.18), it holds that for $t \in \mathcal{T} \cap [t_0,\infty)$ we have

$$\operatorname{Re}\left(\frac{L'}{L}\left(\frac{1}{2}+it,\chi\right)\right) \leq -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{\kappa}{2}+\frac{it_0}{2}\right)\right). \qquad (4.60)$$

We take $t_0 = 0$. Then we see that (4.57) holds for $t \in \mathcal{T}$ provided

$$q > \pi\exp\left(-\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{\kappa}{2}\right)\right). \qquad (4.61)$$

By [GR00, 8.366.4 and 8.366.5] the right-hand side of (4.61) equals

$$= 8\pi\exp\left(c_E + (-1)^\kappa\frac{\pi}{2}\right) = \begin{cases} 215.3\ldots & \text{if } \kappa = 0, \\ 9.3\ldots & \text{if } \kappa = 1. \end{cases}$$

Thus (4.57) holds if the condition (1) or (3) is satisfied.

We go back to (4.60) and consider the case $\kappa = 0$. In this case $q \geq 5$ holds. We have

$$\log\frac{5}{\pi} > 0.46.$$

On the other hand by numerical computation together with (4.18) and [GR00, 8.366.4] we find

$$\operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\right)\right) = -c_E - \frac{\pi}{2} - 3\log 2 + \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{4})\{(n+\frac{1}{4})^2+1\}}$$

$$> -0.04.$$

Here in the last inequality we discard the sum over $n > 5$ and carry out a numerical calculation. Combining these and (4.60), we see that (4.57) holds if the condition (2) is satisfied.

Finally we treat the case $\kappa = 1$. We note that $\kappa = 1$ implies $q \geq 3$. In a similar manner as the case $\kappa = 0$ we find

$$\log \frac{3}{\pi} > -0.05 \text{ and } \operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{3}{4} + \frac{3i}{2}\right)\right) > 0.37.$$

This together with (4.60) says that (4.57) holds under the condition (4). □

As was mentioned in the remark below Theorem 4.7, (4.2) has already been established in [GS15] for $q < 216$. Thus, when we show Theorems 4.7–4.9, we may assume one of the following conditions:

- $\kappa = 0$ and $q \geq 216$,
- $\kappa = 1$ and $q \geq 23$.

We temporarily fix χ and $T \geq 2$. Let $\rho_0 = 1/2 + i\gamma_0$ be a zero of $L(s, \chi)$ with $-T \leq \gamma_0 \leq T$. Then thanks to (4.58) together with the above assumption on q, there exists $\varepsilon > 0$ such that $\operatorname{Re}((L'/L)(s, \chi))$ is negative on the left semicircle $\{s = \sigma + it \mid |s - \rho_0| = \varepsilon, \sigma \leq 1/2\}$. Thus, considering the discussion in Section 4.2.4, Lemma 4.20 and the above discussion into account, we see that there exists a rectangle $\mathcal{R}$ with vertices $\pm iT$ and $1/2 \pm iT$ having small left semicircles at zeros of $L(s, \chi)$ on $\operatorname{Re}(s) = 0$ and $\operatorname{Re}(s) = 1/2$ such that $\operatorname{Re}((L'/L)(s, \chi))$ is negative on the vertical sides of $\mathcal{R}$. We apply the argument principle to $(L'/L)(s, \chi)$ on $\mathcal{R}$. In consequence we obtain

$$\frac{1}{2\pi}\Delta_{\mathcal{R}} \arg \frac{L'}{L}(s, \chi) = N_1^-(T, \chi) - N^-(T, \chi) - \begin{cases} 1 & \text{if } \kappa = 0, \\ 0 & \text{if } \kappa = 1. \end{cases} \tag{4.62}$$

Here we used the fact that $s = 0$ is a trivial zero of $L(s, \chi)$ if $\kappa = 0$. Based on (4.62), we show Theorems 4.7–4.9 below.

Proof of Theorem 4.7. Since $\operatorname{Re}((L'/L)(s, \chi)) < 0$ on the vertical sides of $\mathcal{R}$, the continuous variation of $\arg(L'/L)(s, \chi)$ along each vertical side is $O(1)$. Next we investigate the horizontal sides. We have

$$\arg \frac{L'}{L}(s, \chi)\Bigg|_{s=1/2+iT}^{s=iT} = \arg L'(s, \chi)\big|_{s=1/2+iT}^{s=iT} - \arg L(s, \chi)\big|_{s=1/2+iT}^{s=iT}. \tag{4.63}$$

The continuous variation of $\arg L'(s,\chi)$ from $s = 1/2 + iT$ to $s = iT$ equals that of $\arg G(s,\chi)$, where the branch of $\arg G(s,\chi)$ is determined in the same manner as in Section 4.2.5. Combining Proposition 4.17, we see that the variation of $\arg L'(s,\chi)$ on (4.63) is $O(m^{1/2}\log(qT))$. On the other hand it is well-known that the last term on (4.63) is $O(\log(qT))$: see [MV06, Lemma 12.8] for example. In summary we see that (4.63) is $O(m^{1/2}\log(qT))$. In the same manner the variation of $\arg(L'/L)(s,\chi)$ from $s = -iT$ to $s = 1/2 - iT$ is $O(m^{1/2}\log(qT))$.

By the above discussion we conclude that the left-hand side of (4.62) is

$$O\left(m^{1/2}\log(qT)\right)$$

as desired. □

The following proposition is a key point to show Theorems 4.8 and 4.9:

Proposition 4.21. *Let χ be a fixed primitive Dirichlet character satisfying $\kappa = 0$ and $q \geq 216$, or $\kappa = 1$ and $q \geq 23$. Then at least one of the following assertions holds:*

1. *There exists $T_0 = T_0(\chi) > 0$ such that $N^-(T,\chi) > T/2$ for any $T \geq T_0$.*
2. *There exists a sequence $\{T_j\}_{j=1}^{\infty}$ such that $T_j \to \infty$ as $j \to \infty$ and*

$$N_1^-(T_j,\chi) = N^-(T_j,\chi) + \begin{cases} 1 & \text{if } \kappa = 0, \\ 0 & \text{if } \kappa = 1 \end{cases}$$

holds for any $j \in \mathbb{Z}_{\geq 1}$.

Proof. First of all we suppose that there exists a sequence $\{T_j\}_{j=1}^{\infty}$ such that $\mathrm{Re}((L'/L)(\sigma \pm iT_j,\chi)) < 0$ holds for any j and $\sigma \in [0,1/2]$. Then for any j, $\mathrm{Re}((L'/L)(s,\chi))$ is negative on $\mathcal{R}$ with $T = T_j$. This implies that the left-hand side of (4.62) is 0 when $T = T_j$. In this case the assertion (2) in Proposition 4.21 holds.

Next we suppose that $\{T_j\}_{j=1}^{\infty}$ with the above property does not exist. Then for any sufficiently large t there exists $\sigma \in [0,1/2]$ such that $\mathrm{Re}((L'/L)(\sigma+it,\chi))$ or $\mathrm{Re}((L'/L)(\sigma-it,\chi))$ is nonnegative. By Stirling's formula the first two terms on the right-hand side of (4.58) are negative for $s = \sigma + it$ or $s = \sigma - it$. Thus $J(\sigma+it,\chi)$ or $J(\sigma-it,\chi)$ has to be negative. This implies that there exists a zero $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $\beta < 1/2$ satisfying

$$(\beta - \tfrac{1}{2})^2 > (\sigma - \tfrac{1}{2})^2 + (t-\gamma)^2 \text{ or } (\beta - \tfrac{1}{2})^2 > (\sigma - \tfrac{1}{2})^2 + (t+\gamma)^2.$$

This yields $|t-\gamma| < 1/2$ or $|t+\gamma| < 1/2$. We take t as a sufficiently large integer n. Then we see that there exists at least one zero $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $\beta < 1/2$ and $n - 1/2 < |\gamma| < n + 1/2$. In summary we obtain $N^-(T,\chi) \geq T + O_\chi(1)$. This implies the assertion (1) in Proposition 4.21. □

Proof of Theorems 4.8 and 4.9. As was mentioned in the remark below Theorem 4.9, Yıldırım [Yıl96b] has already established the implications (i)$\Longrightarrow$(ii). We suppose (ii). Then we see from the assumption (ii) and Theorem 4.7 that $N^-(T,\chi) = O_\chi(\log T)$. This implies that the assertion (1) in Proposition 4.21 cannot be satisfied. Thus the assertion (2) in Proposition 4.21 holds. Using the assumption (ii) again, we see $N^-(T_j,\chi) = 0$ for any j, which is nothing but (i). □

4.3 Results obtained under the truth of the generalized Riemann hypothesis for $k = 1$

In this section, we extend the results of Akatsuka [Aka12, Theorems 1 and 3], introduced in the previous chapter, to $L'(s,\chi)$.

Throughout this section, we retain some notation defined in the previous section with a slight modification for zeros of $L'(s,\chi)$: We let $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ denote the zeros of $L(s,\chi)$ and $L'(s,\chi)$ in the right half-plane $\mathrm{Re}(s) > 0$. We know that $L(s,\chi)$ has only trivial zeros in $\mathrm{Re}(s) \le 0$ (see Chapter 2 Section 2.3). We remark that zeros of $L'(s,\chi)$ satisfying $\mathrm{Re}(s) \le 0$ can also be regarded as "trivial" zeros (see Theorems 4.1, 4.2, and 4.4 in the previous section or [AS-p, Theorems 1, 2, and 4]). We define $N_1(T,\chi)$ for $T > 0$ as the number of zeros of $L'(s,\chi)$ satisfying $\mathrm{Re}(s) > 0$ and $|\mathrm{Im}(s)| \le T$, counted with multiplicity. Recall also that $\mathrm{Li}(x)$ is as defined in Theorem 2.4.

Our main theorems in this section are as follows:

Theorem 4.22. *Assume that the generalized Riemann hypothesis is true, then for $T \ge 2$, we have*

$$\sum_{\substack{\rho'=\beta'+i\gamma', \\ |\gamma'|\le T}} \left(\beta' - \frac{1}{2}\right) = \frac{T}{\pi}\log\log\frac{qT}{2\pi} + \frac{T}{\pi}\left(\frac{1}{2}\log m - \log\log m\right) - \frac{2}{q}\,\mathrm{Li}\left(\frac{qT}{2\pi}\right)$$
$$+ O\left(m^{1/2}(\log\log(qT))^2 + m\log\log(qT) + m^{1/2}\log q\right),$$

where the sum is counted with multiplicity.

Theorem 4.23. *Assume that the generalized Riemann hypothesis is true, then for $T \ge 2$, we have*

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} + O\left(\frac{m^{1/2}\log(qT)}{(\log\log(qT))^{1/2}} + m^{1/2}\log q\right).$$

Remarks. We mentioned in the previous chapter that in a recent preprint, F. Ge [Ge-p, Theorem 1] showed that we can improve the error term in the estimate on the number of zeros of $\zeta'(s)$ shown by Akatsuka [Aka12, Theorem 3] to

$$O\left(\frac{\log T}{\log\log T}\right).$$

It is expected that we can extend Ge's result to $L'(s,\chi)$. The author is currently working on this topic.

In this section, we first review some basic estimates related to $\log L(s,\chi)$ near the critical line and zero-free regions of $L'(s,\chi)$ in Subsection 4.3.1. In Subsection 4.3.2, we show important lemmas crucial for the proofs of our main theorems and finally prove them in Subsection 4.3.3. For convenience, we use variables s and z as complex numbers, with $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$. Finally, we abbreviate the generalized Riemann hypothesis as GRH.

4.3.1 Preliminaries

4.3.1.1 Bounds related to $\log L(s,\chi)$ near the critical line

In this subsection we give some bounds related to $\log L(s,\chi)$ which can be found in [MV06, Sections 12.1, 13.2, 14.1]. Only for this subsection, we put $\tau := |t| + 4$.

Lemma 4.24. *Assume GRH, then*

$$\log L(\sigma+it,\chi) = O\left(\frac{(\log{(q\tau)})^{2(1-\sigma)}}{(1-\sigma)\log\log{(q\tau)}} + \log\log\log{(q\tau)}\right)$$

holds uniformly for $1/2 + (\log\log{(q\tau)})^{-1} \leq \sigma \leq 3/2$.

Proof. This is straightforward from the inequalities in exercise 6 of [MV06, Section 13.2] (see also page 3 of [MV06-cor] for the corrected exercise 6). □

Lemma 4.25. *Assume GRH, then*

$$\arg L(\sigma+it,\chi) = O\left(\frac{\log{(q\tau)}}{\log\log{(q\tau)}}\right)$$

holds uniformly for $\sigma \geq 1/2$.

Proof. See [Sel46, Section 5] or exercise 11 of [MV06, Section 13.2]. □

With the above lemma and [MV06, Corollary 14.6], we obtain the following estimate on the number of zeros of $L(s,\chi)$ under GRH which is mentioned in Theorem 2.12:

Proposition 4.26. *Assume GRH and let $N(T,\chi)$ denote the number of zeros of $L(s,\chi)$ satisfying* $\mathrm{Re}(s) > 0$ *and* $|\mathrm{Im}(s)| \le T$*, counted with multiplicity. Then for $T \ge 2$,*

$$N(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2\pi} - \frac{T}{\pi} + O\left(\frac{\log(qT)}{\log\log(qT)}\right).$$

Proof. This is a straightforward consequence of [MV06, Corollary 14.6] and [Sel46, Theorem 6] (see exercise 1 of [MV06, Section 14.1]). □

Lemma 4.27.

$$\frac{L'}{L}(\sigma+it,\chi) = \sum_{\substack{\rho=\beta+i\gamma,\\ |\gamma-t|\le 1}} \frac{1}{\sigma+it-\rho} + O(\log(q\tau))$$

holds uniformly for $-1 \le \sigma \le 2$.

Proof. See [MV06, Lemma 12.6]. □

4.3.1.2 Zero-free regions of $L'(s,\chi)$

We begin with a zero-free region of $L'(s,\chi)$ to the right of the critical line.

Proposition 4.28. *$L'(s,\chi)$ has no zeros when*

$$\sigma > 1 + \frac{m}{2}\left(1+\sqrt{1+\frac{4}{m\log m}}\right).$$

Proof. See [Yıl96b, Theorem 2] for $k=1$. □

From the above proposition, it is not difficult to check that $L'(s,\chi)\neq 0$ when $\sigma \ge 1+3m/2$. Next we introduce a zero-free region of $L'(s,\chi)$ to the left of the critical line.

Proposition 4.29. *$L'(s,\chi)$ has no zeros when $\sigma \le 0$ and $|t| \ge 6$. Furthermore, assuming GRH,*

1. *if $\kappa = 0$ and $q \ge 216$, then $L'(s,\chi)$ has a unique zero in* $0 < \mathrm{Re}(s) < 1/2$*;*

2. *if $\kappa = 1$ and $q \ge 23$, then $L'(s,\chi)$ has no zeros in* $0 < \mathrm{Re}(s) < 1/2$*.*

Here

$$\kappa = \begin{cases} 0, & \chi(-1) = 1; \\ 1, & \chi(-1) = -1. \end{cases}$$

Thus under GRH, for any fixed $\epsilon > 0$, there are only possibly finitely many zeros in the region defined by $0 < \sigma < 1/2$ and $|t| \le \epsilon$ for any $L'(s,\chi)$.

Proof. See [AS-p, Theorems 1, 8, and 9] (or see Theorems 4.1, 4.8, and 4.9 in the previous section) and note that $q \ge 3$ in our case. □

4.3.2 Key lemmas

For convenience, we define the function $F(s,\chi)$ and $G(s,\chi)$ as in the previous section:

$$F(s,\chi) := \epsilon(\chi) 2^s \pi^{s-1} q^{\frac{1}{2}-s} \sin\left(\frac{\pi(s+\kappa)}{2}\right)\Gamma(1-s), \tag{4.64}$$

$$G(s,\chi) := -\frac{m^s}{\chi(m)\log m} L'(s,\chi). \tag{4.65}$$

Recall that $\epsilon(\chi)$ is a factor that depends only on χ, satisfying $|\epsilon(\chi)| = 1$ and that from the functional equation for $L(s,\chi)$, we have $L(s,\chi) = F(s,\chi)L(1-s,\overline{\chi})$.

4.3.2.1 Constants σ_1 and t_q

Lemma 4.30. *For $\sigma \geq 2$, we have*

$$|G(\sigma+it,\chi) - 1| \leq 2\left(1+\frac{8m}{\sigma}\right)\left(1+\frac{1}{m}\right)^{-\sigma}$$

and

$$\left|\frac{G}{L}(\sigma+it,\chi) - 1\right| \leq 2\left(1+\frac{8m}{\sigma}\right)\left(1+\frac{1}{m}\right)^{-\sigma}$$

Proof. Let $\sigma \geq 2$. Then from (4.65) and by using the Dirichlet series expression of $L'(s,\chi)$, we can calculate

$$\begin{aligned}
|G(s,\chi) - 1| &= \left| -\frac{m^s}{\chi(m)\log m}\left(-\sum_{n=1}^{\infty}\frac{\chi(n)\log n}{n^s}\right) - 1\right| \\
&= \left|\frac{m^s}{\chi(m)\log m}\sum_{n=m+1}^{\infty}\frac{\chi(n)\log n}{n^s}\right| \leq \frac{m^\sigma}{\log m}\sum_{n=m+1}^{\infty}\frac{\log n}{n^\sigma} \\
&\leq \frac{m^\sigma}{\log m}\frac{\log(m+1)}{(m+1)^\sigma} + \frac{m^\sigma}{\log m}\int_{m+1}^{\infty}\frac{\log x}{x^\sigma}dx \\
&= \frac{m^\sigma}{\log m}\frac{\log(m+1)}{(m+1)^\sigma}\left(1+\frac{m+1}{\sigma-1}+\frac{m+1}{(\sigma-1)^2\log(m+1)}\right) \\
&\leq \frac{m^\sigma}{\log m}\frac{2\log m}{(m+1)^\sigma}\left(1+\frac{4m}{\sigma-1}\right) \leq 2\left(\frac{m}{m+1}\right)^\sigma\left(1+\frac{8m}{\sigma}\right),
\end{aligned}$$

where we have used $m+1 \leq 2m \leq m^2$ and $\sigma-1 \geq \sigma/2$ in the last two inequalities.

By using the Dirichlet series expansion of $(L'/L)(s,\chi)$, with calculation similar to the above, we can show the second inequality in the lemma. □

Applying Stirling's formula of the following form

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \int_0^\infty \frac{[u] - u + \frac{1}{2}}{u+z}du \tag{4.66}$$

$$(-\pi + \delta \le \arg z \le \pi - \delta, \text{ for any } \delta > 0),$$

we can define the holomorphic function

$$\begin{aligned}\log F(s,\chi) := \log\epsilon(\chi) + \left(\frac{1}{2} - s\right)\log\frac{q}{2\pi} + \frac{1}{2}\log\frac{2}{\pi} + \log\sin\frac{\pi}{2}(s+\kappa) \\ + \log\Gamma(1-s)\end{aligned} \tag{4.67}$$

for $\sigma < 1$ and $|t| > 1$, where $0 \le \arg\epsilon(\chi) < 2\pi$ and $\log\sin(\pi(s+\kappa)/2)$ is the holomorphic function on $\sigma < 1, |t| > 1$ satisfying

$$\log\sin\frac{\pi}{2}(s+\kappa) := \begin{cases} \dfrac{(1-s-\kappa)\pi}{2}i - \log 2 - \displaystyle\sum_{n=1}^{\infty}\frac{e^{\pi i(s+\kappa)n}}{n}, & t > 1; \\ \\ \dfrac{(s+\kappa-1)\pi}{2}i - \log 2 - \displaystyle\sum_{n=1}^{\infty}\frac{e^{-\pi i(s+\kappa)n}}{n}, & t < -1. \end{cases}$$

Under the above definitions, we can show the following lemma.

Lemma 4.31. *For $\sigma < 1$ and $\pm t > 1$, we have*

$$\frac{F'}{F}(s,\chi) = -\log(q(1-s)) + \log 2\pi \mp \frac{\pi i}{2} + \frac{1}{2(1-s)} + O\left(\frac{1}{|1-s|^2}\right) + O\left(e^{-\pi|t|}\right),$$

where $-\pi/2 < \arg(1-s) < \pi/2$.

Proof. Applying Stirling's formula (4.66) to $\log\Gamma(z)$ for $\arg z \in (-\pi/2, \pi/2)$, we have

$$\log\Gamma(1-s) = \left(\frac{1}{2} - s\right)\log(1-s) - (1-s) + \frac{1}{2}\log 2\pi + \int_0^\infty \frac{[u] - u + \frac{1}{2}}{u+1-s}du$$

in the region $\sigma < 1$, $|t| > 1$. From (4.67), we can show that

$$\begin{aligned}\log F(s,\chi) = \log\epsilon(\chi) + \frac{\pi}{2}\left(\frac{1}{2} - \kappa\right)i - 1 + \left(\frac{1}{2} - s\right)\left(\log(q(1-s)) - \log 2\pi + \frac{\pi i}{2}\right) \\ + s + \int_0^\infty \frac{[u] - u + \frac{1}{2}}{u+1-s}du - \sum_{n=1}^{\infty}\frac{e^{\pi i(s+\kappa)n}}{n}\end{aligned}$$

holds when $\sigma < 1$ and $t > 1$. Differentiating both sides of the above equation with respect to s, we obtain

$$\frac{F'}{F}(s,\chi) = -\log\left(q(1-s)\right) + \log 2\pi - \frac{\pi i}{2} + \frac{1}{2(1-s)} + O\left(\frac{1}{|1-s|^2}\right) + O\left(e^{-\pi|t|}\right)$$

for $\sigma < 1$ and $t > 1$. We can show similarly for $\sigma < 1$ and $t < -1$. □

Lemma 4.32. *There exists a $\sigma_1 \leq -1$ such that*

$$\left|\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right| < 2^{\sigma}$$

holds for any s with $\sigma \leq \sigma_1$ and $|t| \geq 2$.

Proof. From Lemma 4.31, we know that

$$\frac{F'}{F}(s,\chi) = -\log\left(q(1-s)\right) + O(1)$$

holds when $\sigma < 1$ and $|t| \geq 2$. Hence

$$\left|\frac{F'}{F}(s,\chi)\right| \geq \log\left(q(1-\sigma)\right) - |O(1)|$$

holds in the region $\sigma < 1, |t| \geq 2$. Thus, we can take $\sigma_1' \leq -1$ sufficiently small (i.e. sufficiently large in the negative direction) so that for any s with $\sigma \leq \sigma_1'$ and $|t| \geq 2$, we have

$$\left|\frac{F'}{F}(s,\chi)\right| \geq \frac{1}{2}\log\left(q(1-\sigma)\right) \tag{4.68}$$

for all s in the region $\sigma \leq \sigma_1', |t| \geq 2$.

Next we estimate $(L'/L)(1-s,\overline{\chi})$. In the region $\sigma \leq -1$, $|t| \geq 2$, $(L'/L)(1-s,\overline{\chi})$ can be written as a Dirichlet series, thus we have

$$\begin{aligned}\left|\frac{L'}{L}(1-s,\overline{\chi})\right| &\leq \frac{\log 2}{2^{1-\sigma}} + \sum_{n=3}^{\infty}\frac{\log n}{n^{1-\sigma}} \leq \frac{2^{\sigma}\log 2}{2} + \int_2^{\infty}\frac{\log x}{x^{1-\sigma}}dx \\ &= 2^{\sigma}\left(\frac{\log 2}{2} - \frac{\log 2}{\sigma} + \frac{1}{\sigma^2}\right) \leq 2^{\sigma}\left(1 + \frac{3}{2}\log 2\right).\end{aligned} \tag{4.69}$$

Now combining inequalities (4.68) and (4.69), we have

$$\left|\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right| < 2^{\sigma}\frac{2+3\log 2}{\log\left(q(1-\sigma)\right)}$$

for $\sigma \leq \sigma_1'$ and $|t| \geq 2$. Hence we can find some $\sigma_1 \leq \sigma_1'$ (≤ -1) such that $(2+3\log 2)/\log(q(1-\sigma)) < 1$ holds for any $\sigma \leq \sigma_1$. This implies that

$$\left| \frac{1}{\frac{F'}{F}(s,\chi)} \frac{L'}{L}(1-s,\overline{\chi}) \right| < 2^{\sigma}$$

holds in the region $\sigma \leq \sigma_1, |t| \geq 2$. □

Lemma 4.33. *Assume GRH and fix a σ_1 that satisfies Lemma 4.32. Then there exists a $t_1 > -\sigma_1$ such that*

1. *for any s satisfying $\sigma_1 \leq \sigma \leq 1/2$ and $|t| \geq t_1 - 1$,*

$$\left| \frac{F'}{F}(s,\chi) \right| \geq 1$$

holds and we can take the logarithmic branch of $\log(F'/F)(s,\chi)$ in that region such that it is holomorphic there and $5\pi/6 < \arg(F'/F)(s,\chi) < 7\pi/6$ holds;

2. *for any s satisfying $\sigma_1 \leq \sigma < 1/2$ and $|t| \geq t_1 - 1$,*

$$\frac{L'}{L}(s,\chi) \neq 0$$

holds and we can take the logarithmic branch of $\log(L'/L)(s,\chi)$ in that region such that it is holomorphic there and $\pi/2 < \arg(L'/L)(s,\chi) < 3\pi/2$ holds.

Proof. We begin by examining condition (1). Again, from Lemma 4.31, we see that

$$\frac{F'}{F}(s,\chi) = -\log(q(1-s)) + O(1)$$

holds when $\sigma < 1$ and $|t| \geq 2$. Thus for $\sigma_1 \leq \sigma \leq 1/2$ and $|t| \geq 2$, we have

$$\left| \frac{F'}{F}(s,\chi) \right| \geq \log(q|t|) - |O(1)| \geq \log|t| - |O(1)|.$$

Hence, we can find some $t_1' \geq 100$ such that

$$\left| \frac{F'}{F}(s,\chi) \right| \geq 1 \tag{4.70}$$

holds for all s with $\sigma_1 \leq \sigma \leq 1/2$ and $|t| \geq t_1' - 1$. We note that Lemma 4.31 also implies that

$$\frac{F'}{F}(s,\chi) = -\log(q|t|) + O(1)$$

holds when $\sigma_1 \le \sigma \le 1/2$ and $|t| \ge 2 - \sigma_1$. Consequently, we can find some $t_1'' \ge \max\{t_1', 3 - \sigma_1\}$ such that

$$\frac{5\pi}{6} < \arg \frac{F'}{F}(s, \chi) < \frac{7\pi}{6}$$

holds for $\sigma_1 \le \sigma \le 1/2$ and $|t| \ge t_1'' - 1$. Since $(F'/F)(s, \chi)$ is holomorphic, inequality (4.70) tells us that $\log (F'/F)(s, \chi)$ is holomorphic in the region $\sigma_1 \le \sigma \le 1/2, |t| \ge t_1'' - 1$ with this branch.

By the above calculations, we find that t_1'' is a candidate for t_1. Below we examine condition (2) to completely prove the existence of t_1.

Corollary 10.18 of [MV06] allows us to show that

$$\operatorname{Re}\left(\frac{L'}{L}(s, \chi)\right) < -\frac{1}{2} \log \frac{q}{\pi} - \frac{1}{2} \operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa}{2}\right)\right)$$

holds for $\sigma_1 \le \sigma < 1/2$, under GRH. For any small $\delta > 0$, let $|t| > \sigma_1 \tan \delta$. Stirling's formula (4.66) implies

$$\frac{1}{2} \operatorname{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa}{2}\right)\right) = \frac{1}{2} \log \left|\frac{s+\kappa}{2}\right| + O\left(\frac{1}{|s|}\right).$$

Hence we can find some $t_1 \ge t_1''$ large enough so that

$$\operatorname{Re}\left(\frac{L'}{L}(s, \chi)\right) < 0$$

holds for $\sigma_1 \le \sigma < 1/2$ and $|t| \ge t_1 - 1$ and hence $(L'/L)(s, \chi) \ne 0$. Moreover, we can define a branch of $\log (L'/L)(s, \chi)$ so that it is holomorphic in $\sigma_1 \le \sigma < 1/2$, $|t| \ge t_1 - 1$ and

$$\frac{\pi}{2} < \arg \frac{L'}{L}(s, \chi) < \frac{3\pi}{2}$$

holds there. Since this t_1 also satisfies condition (1), the proof is complete. □

Now we fix t_1 which satisfies Lemma 4.33 and take $t_q \in [t_1 + 1, t_1 + 2]$ such that

$$L(\sigma \pm it_q, \chi) \ne 0,\ L'(\sigma \pm it_q, \chi) \ne 0 \tag{4.71}$$

for all $\sigma \in \mathbb{R}$.

Remark. We note that t_q depends on q but it is bounded by a fixed constant that does not depend on q: $t_q \ll t_1 \ll 1$.

4.3.2.2 Bounds related to $\log G(s,\chi)$

In this subsection, we give bounds for $\arg (G/L)(s,\chi)$ and $\arg G(s,\chi)$. We take the logarithmic branches so that $\log L(s,\chi)$ and $\log G(s,\chi)$ tend to 0 as $\sigma \to \infty$ and are holomorphic in $\mathbb{C}\backslash\{z+\lambda \mid L(z,\chi)=0, \lambda \le 0\}$ and $\mathbb{C}\backslash\{z+\lambda \mid L'(z,\chi)=0, \lambda \le 0\}$, respectively. We write

$$-\arg L(\sigma \pm i\tau,\chi) + \arg G(\sigma \pm i\tau,\chi) = \arg \frac{G}{L}(\sigma \pm i\tau,\chi)$$

and take the argument on the right-hand side so that $\log (G/L)(s,\chi)$ tends to 0 as $\sigma \to \infty$ and is holomorphic in $\mathbb{C}\backslash\{z+\lambda \mid (L'/L)(z,\chi)=0 \text{ or } \infty, \lambda \le 0\}$.

Lemma 4.34. *Assume GRH and let $\tau \ge t_q$. Then we have for $1/2 < \sigma \le 10m$,*

$$\arg \frac{G}{L}(\sigma \pm i\tau,\chi) \ll \begin{cases} \dfrac{m}{\sigma} & 3 \le \sigma \le 10m, \\[2ex] \dfrac{m^{1/2}\log\log(q\tau)+m}{\sigma-1/2} & 1/2 < \sigma \le 3. \end{cases}$$

Proof. Let $\tau \ge t_q$ and $1/2 < \sigma \le 10m$. Let

$$u_{G/L} = u_{G/L}(\sigma,\tau;\chi) := \#\left\{u \in [\sigma, 11m] \mid \mathrm{Re}\left(\frac{G}{L}(u \pm i\tau,\chi)\right) = 0\right\},$$

then

$$\left|\arg \frac{G}{L}(\sigma \pm i\tau,\chi)\right| \le \left(u_{G/L}+1\right)\pi.$$

To estimate $u_{G/L}$, we set

$$H_1(z,\chi) := \frac{1}{2}\left(\frac{G}{L}(z \pm i\tau,\chi) + \frac{G}{L}(z \mp i\tau,\overline{\chi})\right)$$

and

$$n_{H_1}(r,\chi) := \#\{z \in \mathbb{C} \mid H_1(z,\chi)=0, |z-11m| \le r\}.$$

Since $H_1(x,\chi) = \mathrm{Re}((G/L)(x \pm i\tau,\chi))$ for $x \in \mathbb{R}$, we have $u_{G/L} \le n_{H_1}(11m-\sigma,\chi)$ for $1/2 < \sigma \le 10m$.

Now we estimate $n_{H_1}(11m-\sigma,\chi)$. We take $\epsilon = \epsilon_{\sigma,\tau} > 0$. It is easy to show that

$$n_{H_1}(11m-\sigma,\chi) \le \frac{1}{\log(1+\epsilon/(11m-\sigma))}\int_0^{11m-\sigma+\epsilon} \frac{n_{H_1}(r,\chi)}{r}dr.$$

Applying Jensen's theorem (cf. Lemma 2.1 or [Tit39, Section 3.61]), we have

$$\int_0^{11m-\sigma+\epsilon} \frac{n_{H_1}(r,\chi)}{r}dr = \frac{1}{2\pi}\int_0^{2\pi} \log|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)|d\theta$$

$$- \log |H_1(11m, \chi)|.$$

Applying the second inequality in Lemma 4.30, we can easily see that $\log |H_1(11m, \chi)| = O(1)$. Therefore

$$\left|\arg \frac{G}{L}(\sigma \pm i\tau, \chi)\right| \leq \frac{1}{\log(1 + \epsilon/(11m - \sigma))} \times \left(\frac{1}{2\pi}\int_0^{2\pi} \log |H_1(11m + (11m - \sigma + \epsilon)e^{i\theta}, \chi)| d\theta + C\right)$$

for some absolute constant $C > 0$.

Now we divide the rest of the proof in two cases:

(a) For $3 \leq \sigma \leq 10m$, we restrict ϵ to satisfy $0 < \epsilon \leq \sigma - 2$. Then $11m + (11m - \sigma + \epsilon)\cos\theta \geq 2$. Applying the second inequality in Lemma 4.30, we can easily obtain

$$|H_1(11m + (11m - \sigma + \epsilon)e^{i\theta}, \chi)| \leq \frac{100m}{11m + (11m - \sigma + \epsilon)\cos\theta}.$$

Recall from Lemma 4.19 that for $c > r > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log |c + r\cos\theta| d\theta = \log \frac{c + \sqrt{c^2 - r^2}}{2} \tag{4.72}$$

holds. By using (4.72), we can easily show that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log |H_1(11m + (11m - \sigma + \epsilon)e^{i\theta}, \chi)| d\theta \\
&\leq \log(100m) - \frac{1}{2\pi}\int_0^{2\pi} \log(11m + (11m - \sigma + \epsilon)\cos\theta) d\theta \\
&= \log(100m) - \log \frac{11m + \sqrt{11m^2 - (11m - \sigma + \epsilon)^2}}{2} \\
&\leq \log(100m) - \log \frac{11m}{2} \ll 1.
\end{aligned}$$

Note that $\epsilon/(11m - \sigma) \leq 10$, thus $\log(1 + \epsilon/(11m - \sigma)) \gg \epsilon/(11m - \sigma)$. Hence

$$\arg \frac{G}{L}(\sigma \pm i\tau, \chi) \ll \frac{11m - \sigma}{\epsilon} \ll \frac{m}{\epsilon}.$$

By taking $\epsilon = \sigma - 2$, we obtain

$$\arg \frac{G}{L}(\sigma \pm i\tau, \chi) \ll \frac{m}{\sigma}.$$

This is the first inequality in Lemma 4.34.

(b) For $1/2 < \sigma \le 3$, we restrict ϵ to satisfy $0 < \epsilon < \sigma - 1/2$ and we divide the interval of integration into

- $\mathcal{I}_1 := \{\theta \in [0, 2\pi] \mid 11m + (11m - \sigma + \epsilon)\cos\theta \ge 2\}$ and
- $\mathcal{I}_2 := \{\theta \in [0, 2\pi] \mid 11m + (11m - \sigma + \epsilon)\cos\theta < 2\}$.

Since $11m + (11m - \sigma + \epsilon)\cos\theta > 1/2$ and $11m - \sigma + \epsilon < 11m$, on $\mathcal{I}_1$, as in the calculation of case (a), we can show that

$$\begin{aligned}\frac{1}{2\pi}\int_{\theta\in\mathcal{I}_1} &\log|H_1(11m + (11m - \sigma + \epsilon)e^{i\theta}, \chi)|d\theta \\ &\le \frac{1}{2\pi}\int_{\theta\in\mathcal{I}_1}\log\frac{100m}{11m + (11m - \sigma + \epsilon)\cos\theta}d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\log\frac{100m}{11m + (11m - \sigma + \epsilon)\cos\theta}d\theta \ll 1.\end{aligned}$$

Now we estimate the integral on $\mathcal{I}_2$. Setting

$$\cos\theta_0 := \frac{11m - 2}{11m - \sigma + \epsilon}$$

for $\theta_0 \in (0, \pi/2)$, we have $\mathcal{I}_2 = (\pi - \theta_0, \pi + \theta_0)$. Applying Lemma 4.27 and Proposition 4.26, and noting that $(L'/L)(x + iy, \chi) = O(1)$ when $x \ge 2$, we have

$$\frac{L'}{L}(x + iy, \chi) = O\left(\frac{\log(q(|y| + 1))}{x - 1/2}\right)$$

for $1/2 < x \le A$, for any fixed $A \ge 2$. Thus,

$$|H_1(11m + (11m - \sigma + \epsilon)e^{i\theta}, \chi)| \le C_1\frac{m^2}{\log m}\frac{\log(q(\tau + 11m))}{11m + (11m - \sigma + \epsilon)\cos\theta - 1/2}$$

for some absolute constant $C_1 > 0$. Hence

$$\begin{aligned}\frac{1}{2\pi}\int_{\theta\in\mathcal{I}_2} &\log|H_1(11m + (11m - \sigma + \epsilon)e^{i\theta}, \chi)|d\theta \\ &\le \frac{1}{2\pi}\int_{\pi-\theta_0}^{\pi+\theta_0}\log\frac{C_1m^2}{\log m}\frac{\log(q(\tau + 11m))}{11m - 1/2 + (11m - \sigma + \epsilon)\cos\theta}d\theta \\ &= \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0}\log\frac{C_1m^2}{\log m}\frac{\log(q(\tau + 11m))}{11m - 1/2 - (11m - \sigma + \epsilon)\cos\theta}d\theta \\ &= \frac{\theta_0}{\pi}\log\frac{C_1m^2\log(q(\tau + 11m))}{\log m}\end{aligned}$$

$$-\frac{1}{2\pi}\int_{-\theta_0}^{\theta_0}\log\left(11m-\frac{1}{2}-(11m-\sigma+\epsilon)\cos\theta\right)d\theta.$$

We note that $\cos\theta_0 = 1+O(1/m)$. By using $1-\cos\theta_0 = 2\sin^2(\theta_0/2)$, we can show

$$\theta_0 \ll \left|\sin^2\frac{\theta_0}{2}\right| \ll \frac{1}{m^{1/2}}.$$

Hence,

$$\begin{aligned}
&\int_{-\theta_0}^{\theta_0}\log\left(11m-\frac{1}{2}-(11m-\sigma+\epsilon)\cos\theta\right)d\theta \\
&\quad=\int_{-\theta_0}^{\theta_0}\log\frac{11m-1/2-(11m-\sigma+\epsilon)\cos\theta}{11m-1/2}d\theta+\int_{-\theta_0}^{\theta_0}\log(11m-1/2)d\theta \\
&\quad=\int_{-\theta_0}^{\theta_0}\log\left(1-\frac{11m-\sigma+\epsilon}{11m-1/2}\cos\theta\right)d\theta+O\left(\frac{\log m}{m^{1/2}}\right)
\end{aligned}$$

Recalling that $\sigma-\epsilon>1/2$ and $\theta_0\in(0,\pi/2)$, we have

$$\int_{-\theta_0}^{\theta_0}\log(1-\cos\theta)d\theta \le \int_{-\theta_0}^{\theta_0}\log\left(1-\frac{11m-\sigma+\epsilon}{11m-1/2}\cos\theta\right)d\theta \le 0.$$

Meanwhile,

$$\begin{aligned}
\int_{-\theta_0}^{\theta_0}\log(1-\cos\theta)d\theta &= \int_{-\theta_0}^{\theta_0}\log\left(2\sin^2\frac{\theta}{2}\right)d\theta = 2\theta_0\log 2+4\int_0^{\theta_0}\log\left(\sin\frac{\theta}{2}\right)d\theta \\
&= 2\theta_0\log 2+4\int_0^{\theta_0}\log\frac{\sin(\theta/2)}{\theta/2}d\theta+4\int_0^{\theta_0}\log\frac{\theta}{2}d\theta \\
&= O(\theta_0)+O\left(\theta_0^3\right)+O\left(\theta_0\log\theta_0^{-1}\right)=O\left(\frac{\log m}{m^{1/2}}\right).
\end{aligned}$$

Therefore when $1/2<\sigma\le 3$, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\log|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)|d\theta \\
&\quad=\frac{1}{2\pi}\left(\int_{\theta\in\mathcal{I}_1}+\int_{\theta\in\mathcal{I}_2}\right)\log|H_1(11m+(11m-\sigma+\epsilon)e^{i\theta},\chi)|d\theta \\
&\quad\ll 1+\frac{\log\log(q(\tau+11m))}{m^{1/2}}+\frac{\log m}{m^{1/2}} \ll 1+\frac{\log\log(q\tau)}{m^{1/2}}.
\end{aligned}$$

Since $0<\epsilon/(11m-\sigma)<1$, we have $\log(1+\epsilon/(11m-\sigma)) \gg \epsilon/m$, thus

$$\arg\frac{G}{L}(\sigma\pm i\tau,\chi) \ll \frac{m}{\epsilon}\left(1+\frac{\log\log(q\tau)}{m^{1/2}}\right).$$

Taking $\epsilon=(\sigma-1/2)/2$, we obtain the second inequality in Lemma 4.34.

□

Lemma 4.35. *Assume GRH and let $A \geq 2$ be fixed. Then there exists a constant $C_0 > 0$ such that*

$$|L'(\sigma + it, \chi)| \leq \exp\left(C_0\left(\frac{(\log q\tau)^{2(1-\sigma)}}{\log\log(q\tau)} + (\log(q\tau))^{1/10}\right)\right)$$

holds for $1/2 - 1/\log\log(q\tau) \leq \sigma \leq A$ and $\tau = |t| + 4$.

Proof. Applying Lemma 4.24 and Cauchy's integral formula, Lemma 4.35 follows. □

Lemma 4.36. *Assume GRH. Then for any $1/2 \leq \sigma \leq 3/4$, we have*

$$\arg G(\sigma \pm i\tau, \chi) = O\Bigg(m^{1/2}(\log\log(q\tau)) \times \left(m^{1/2} + (\log(q\tau))^{1/10} + \frac{(\log(q\tau))^{2(1-\sigma)}}{(\log\log(q\tau))^{3/2}}\right)\Bigg).$$

Proof. The proof is similar to that of Lemma 4.34 but we provide the details for clarity. Let $1/2 \leq \sigma \leq 3/4$ and $\tau > 1$ be large. Put

$$u_G = u_G(\sigma, \tau; \chi) := \#\left\{u \in [\sigma, 1 + 3m/2] \mid \mathrm{Re}\left(G(u \pm i\tau, \chi)\right) = 0\right\},$$

then

$$|\arg G(\sigma \pm i\tau, \chi)| \leq (u_G + 1)\,\pi.$$

To estimate u_G, we set

$$X_1(z, \chi) := \frac{G(z \pm i\tau, \chi) + G(z \mp i\tau, \overline{\chi})}{2}$$

and

$$n_{X_1}(r, \chi) := \#\{z \in \mathbb{C} \mid X_1(z, \chi) = 0, |z - (1 + 3m/2)| \leq r\}.$$

Then we have $u_G \leq n_{X_1}(1 + 3m/2 - \sigma, \chi)$.

Now we estimate $n_{X_1}(1+3m/2-\sigma, \chi)$. For each $\sigma \in [1/2, 3/4]$, we take $\epsilon = \epsilon_{\sigma,\tau}$ satisfying $0 < \epsilon \leq \sigma - 1/2 + (\log\log(q\tau))^{-1}$. It is easy to show that

$$n_{X_1}(1 + 3m/2 - \sigma, \chi) \leq \frac{1 + 3m}{\epsilon} \int_0^{1+3m/2-\sigma+\epsilon} \frac{n_{X_1}(r, \chi)}{r} dr.$$

Applying Jensen's theorem, we have

$$\int_0^{1+3m/2-\sigma+\epsilon} \frac{n_{X_1}(r, \chi)}{r} dr$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|X_1(1+3m/2+(1+3m/2-\sigma+\epsilon)e^{i\theta},\chi)|d\theta$$
$$- \log|X_1(1+3m/2,\chi)|.$$

By using the first inequality in Lemma 4.30, we can easily show

$$\log|X_1(1+3m/2,\chi)| = O(1).$$

As in the proof of Lemma 4.34, we divide the interval of integration into

- $\mathcal{J}_1 := \{\theta \in [0,2\pi] \mid 1+3m/2+(1+3m/2-\sigma+\epsilon)\cos\theta \geq 2\}$ and
- $\mathcal{J}_2 := \{\theta \in [0,2\pi] \mid 1+3m/2+(1+3m/2-\sigma+\epsilon)\cos\theta < 2\}$.

Then similarly, applying the first inequality in Lemma 4.30 and (4.72), we can show that

$$\frac{1}{2\pi}\int_{\theta\in\mathcal{J}_1} \log|X_1(1+3m/2+(1+3m/2-\sigma+\epsilon)e^{i\theta},\chi)|d\theta = O(1).$$

Next we estimate the integral on $\mathcal{J}_2$. Setting

$$\cos\theta_0 := \frac{1+3m/2-2}{1+3m/2-\sigma+\epsilon}$$

for $\theta_0 \in (0,\pi/2)$, we have $\mathcal{J}_2 = (\pi-\theta_0, \pi+\theta_0)$ and $\theta_0 = O(m^{-1/2})$. Applying Lemma 4.35, we have

$$|X_1(1+3m/2+(1+3m/2-\sigma+\epsilon)e^{i\theta},\chi)|$$
$$\leq \frac{m^2}{\log m}\exp\left(C_0'\left(\frac{(\log(q\tau))^{-3m-2(1+3m/2-\sigma+\epsilon)\cos\theta}}{\log\log(q\tau)} + (\log(q\tau))^{1/10}\right)\right)$$

for some absolute constant $C_0' > 0$. Thus,

$$\frac{1}{2\pi}\int_{\theta\in\mathcal{J}_2} \log|X_1(1+3m/2+(1+3m/2-\sigma+\epsilon)e^{i\theta},\chi)|d\theta$$
$$\leq \theta_0\left(\log\frac{m^2}{\log m} + C_0'(\log(q\tau))^{1/10}\right)$$
$$+ \frac{C_0'(\log(q\tau))^{-3m}}{2\pi\log\log(q\tau)}\int_{\pi-\theta_0}^{\pi+\theta_0} (\log(q\tau))^{-2(1+3m/2-\sigma+\epsilon)\cos\theta}d\theta$$
$$\leq \theta_0\left(\log\frac{m^2}{\log m} + C_0'(\log(q\tau))^{1/10}\right)$$
$$+ \frac{C_0'(\log(q\tau))^{-3m}}{2\pi\log\log(q\tau)}\int_0^{2\pi} (\log(q\tau))^{-2(1+3m/2-\sigma+\epsilon)\cos\theta}d\theta$$

$$= \theta_0 \left(\log \frac{m^2}{\log m} + C_0'(\log(q\tau))^{1/10} \right)$$
$$+ \frac{C_0'(\log(q\tau))^{-3m}}{\log\log(q\tau)} I_0(2(1 + 3m/2 - \sigma + \epsilon)\log\log(q\tau)),$$

where I_ν is the Bessel function. Since

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}}(1 + o(1)),$$

there exists a constant $C_1' > 0$ such that

$$I_0(2(1 + 3m/2 - \sigma + \epsilon)\log\log(q\tau)) \le C_1' \frac{(\log(q\tau))^{2(1+3m/2-\sigma+\epsilon)}}{(m\log\log(q\tau))^{1/2}}.$$

Hence,

$$\frac{1}{2\pi}\int_{\theta\in\mathcal{J}_2} \log|X_1(1 + 3m/2 + (1 + 3m/2 - \sigma + \epsilon)e^{i\theta}, \chi)|d\theta$$
$$\ll \frac{1}{m^{1/2}}\left((\log(q\tau))^{1/10} + \frac{(\log(q\tau))^{2(1-\sigma+\epsilon)}}{(\log\log(q\tau))^{3/2}} \right).$$

Concluding the above, we have

$$\arg G(\sigma \pm i\tau, \chi) \ll n_{X_1}(1 + 3m/2 - \sigma, \chi)$$
$$\ll \frac{m}{\epsilon}\left(1 + \frac{1}{m^{1/2}}\left((\log(q\tau))^{1/10} + \frac{(\log(q\tau))^{2(1-\sigma+\epsilon)}}{(\log\log(q\tau))^{3/2}} \right)\right).$$

Taking $\epsilon = (\log\log(q\tau))^{-1}$ completes the proof. □

4.3.3 Proof of theorems

4.3.3.1 Evaluation of the main terms

We first prove two propositions which state out the main terms of the equations in our main theorems. We use the functions $F(s, \chi)$ and $G(s, \chi)$ defined in the previous section (see (4.64) and (4.65)).

The following proposition states out the main term of the equation in Theorem 4.22.

Proposition 4.37. *Assume GRH. Take t_q as in* (4.71)*, and set $a_q := 4m$. From Proposition 4.28, we note that $L'(s, \chi) \neq 0$ when $\sigma \ge a_q$. Then for $T \ge t_q$ which satisfies $L(\sigma \pm iT, \chi) \neq 0$ and $L'(\sigma \pm iT, \chi) \neq 0$ for any $\sigma \in \mathbb{R}$, we have*

$$\sum_{\substack{\rho'=\beta'+i\gamma', \\ t_q<\pm\gamma'\le T}} \left(\beta' - \frac{1}{2}\right) = \frac{T}{2\pi}\log\log\frac{qT}{2\pi} + \frac{T}{2\pi}\left(\frac{1}{2}\log m - \log\log m\right) - \frac{1}{q}\operatorname{Li}\left(\frac{qT}{2\pi}\right)$$

$$
\begin{aligned}
&\mp \frac{1}{2\pi}\int_{1/2}^{a_q} (-\arg L(\sigma \pm it_q, \chi) + \arg G(\sigma \pm it_q, \chi))\, d\sigma \\
&\pm \frac{1}{2\pi}\int_{1/2}^{a_q} (-\arg L(\sigma \pm iT, \chi) + \arg G(\sigma \pm iT, \chi))\, d\sigma \\
&+ O(\log\log q) + O(m),
\end{aligned}
$$

where the logarithmic branches are taken as in Subsection 4.3.2.2.

Proof. We first set $a_q := 4m$ and take t_q as in (4.71). We again note that $t_q \ll t_1 \ll 1$. We also take σ_1 which satisfies Lemma 4.32 and fix it. Take $T \geq t_q$ such that $L(\sigma \pm iT, \chi) \neq 0$ and $L'(\sigma \pm iT, \chi) \neq 0$ for all $\sigma \in \mathbb{R}$. Let $\delta \in (0, 1/2]$ and put $b := 1/2 - \delta$.

Applying Littlewood's lemma (cf. Lemma 2.2 or [Tit39, Section 3.8]) to $G(s, \chi)$ on the rectangles with vertices $b \pm it_q$, $a_q \pm it_q$, $a_q \pm iT$, and $b \pm iT$, we obtain

$$
\begin{aligned}
2\pi &\sum_{\substack{\rho'=\beta'+i\gamma',\\ t_q<\pm\gamma'\leq T}} (\beta' - b) \\
&= \int_{t_q}^{T} \log|G(b \pm it, \chi)|dt - \int_{t_q}^{T} \log|G(a_q \pm it, \chi)|dt \\
&\qquad \mp \int_{b}^{a_q} \arg G(\sigma \pm it_q, \chi)d\sigma \pm \int_{b}^{a_q} \arg G(\sigma \pm iT, \chi)d\sigma \\
&=: I_1^{\pm} + I_2^{\pm} \mp \int_{b}^{a_q} \arg G(\sigma \pm it_q, \chi)d\sigma \pm \int_{b}^{a_q} \arg G(\sigma \pm iT, \chi)d\sigma.
\end{aligned}
\tag{4.73}
$$

Applying the first inequality in Lemma 4.30, we can show that $I_2^+ = I_2^- = O(m)$. Below we estimate I_1^+.

$$
\begin{aligned}
I_1^+ &= \int_{t_q}^{T} \log|G(b + it, \chi)|dt = \int_{t_q}^{T} \log\left(\frac{m^b}{\log m}|L'(b + it, \chi)|\right)dt \\
&= \int_{t_q}^{T} \log\frac{m^b}{\log m}dt + \int_{t_q}^{T} \log|L'(b + it, \chi)|dt \\
&= (b\log m - \log\log m)T + \int_{t_q}^{T} \log|F(b + it, \chi)|dt \\
&\qquad + \int_{t_q}^{T} \log\left|\frac{F'}{F}(b + it, \chi)\right|dt + \int_{t_q}^{T} \log|L(1 - b - it, \overline{\chi})|dt \\
&\qquad + \int_{t_q}^{T} \log\left|1 - \frac{1}{\frac{F'}{F}(b + it, \chi)}\frac{L'}{L}(1 - b - it, \overline{\chi})\right|dt + O(t_q\log m) \\
&=: (b\log m - \log\log m)T + I_{12} + I_{13} + I_{14} + I_{15} + O(\log m).
\end{aligned}
\tag{4.74}
$$

Here we recall that $t_q = O(1)$ from our choice of t_q in (4.71).

From (4.67) and Stirling's formula (4.66), we have

$$\begin{aligned} I_{12} = \int_{t_q}^{T} \log |F(b+it,\chi)| dt &= \int_{t_q}^{T} \left(\left(\frac{1}{2} - b \right) \log \frac{qt}{2\pi} + O\left(\frac{1}{t^2} \right) \right) dt \\ &= \left(\frac{1}{2} - b \right) \left(T \log \frac{qT}{2\pi} - T - t_q \log \frac{qt_q}{2\pi} + t_q \right) + O(1). \end{aligned}$$

Lemma 4.31 gives us

$$\log \left| \frac{F'}{F}(b+it,\chi) \right| = \mathrm{Re} \left(\log \frac{F'}{F}(b+it,\chi) \right) = \log\log \frac{q|t|}{2\pi} + O\left(\frac{1}{t^2 \log (q|t|)} \right),$$

thus we have

$$\begin{aligned} I_{13} = \int_{t_q}^{T} \log \left| \frac{F'}{F}(b+it,\chi) \right| dt &= \int_{t_q}^{T} \left(\log\log \frac{qt}{2\pi} + O\left(\frac{1}{t^2 \log (q|t|)} \right) \right) dt \\ &= T \log\log \frac{qT}{2\pi} - t_q \log\log \frac{qt_q}{2\pi} - \int_{t_q}^{T} \frac{1}{\log \frac{qt}{2\pi}} dt + O(1) \\ &= T \log\log \frac{qT}{2\pi} - t_q \log\log \frac{qt_q}{2\pi} - \frac{2\pi}{q} \mathrm{Li} \left(\frac{qT}{2\pi} \right) + O(t_q) \\ &= T \log\log \frac{qT}{2\pi} - \frac{2\pi}{q} \mathrm{Li} \left(\frac{qT}{2\pi} \right) + O(\log\log q). \end{aligned}$$

Next, we estimate I_{14}. We note that $\overline{L(\overline{s},\overline{\chi})} = L(s,\chi)$, hence $|L(1-b-it,\overline{\chi})| = |L(1-b+it,\chi)|$. Take the logarithmic branch of $\log L(s,\chi)$ so that $\log L(s,\chi) = \sum_{n=2}^{\infty} \chi(n)\Lambda(n)(\log n)^{-1} n^{-s}$ holds for $\mathrm{Re}(s) > 1$ and that it is holomorphic in $\mathbb{C} \backslash \{ z + \lambda \mid L(z,\chi) = 0, \lambda \leq 0 \}$. Then applying Cauchy's integral theorem to $\log L(s,\chi)$ on the rectangle with vertices $1-b+it_q$, $a_q + it_q$, $a_q + iT$, $1-b+iT$ and taking the imaginary part, we can show that

$$\begin{aligned} I_{14} = \int_{t_q}^{T} \log |L(1-b-it,\overline{\chi})| dt &= \int_{t_q}^{T} \log |L(1-b+it,\chi)| dt \\ &= \int_{1-b}^{a_q} \arg L(\sigma + it_q, \chi) d\sigma - \int_{1-b}^{a_q} \arg L(\sigma + iT, \chi) d\sigma + O(1). \end{aligned}$$

Finally we estimate I_{15}. Since $L(s,\chi) = F(s,\chi) L(1-s,\overline{\chi})$, we have

$$\frac{1}{\frac{F'}{F}(s,\chi)} \frac{L'}{L}(s,\chi) = 1 - \frac{1}{\frac{F'}{F}(s,\chi)} \frac{L'}{L}(1-s,\overline{\chi}). \tag{4.75}$$

From Lemma 4.33, the function on the left-hand side of (4.75) is holomorphic and has no zeros in $\sigma_1 \leq \sigma < 1/2, |t| \geq t_1 - 1$. From Lemma 4.32, the function on the right-hand side of (4.75) is holomorphic and has no zeros in $\sigma \le \sigma_1, |t| \ge 2$. Thus we can determine

$$\log\left(1-\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right)$$

so that it tends to 0 as $\sigma \to -\infty$ which follows from Lemma 4.32, and that it is holomorphic in $\sigma < 1/2, |t| > t_q - 1 (> t_1 - 1)$. Now we apply Cauchy's integral theorem to it on the trapezoid with vertices $-t_q + it_q$, $b + it_q$, $b + iT$, and $-T + iT$. Lemma 4.32 allows us to show

$$\left(\int_{\sigma_1+iT}^{-T+iT} + \int_{-T+iT}^{-t_q+it_q} + \int_{-t_q+it_q}^{\sigma_1+it_q}\right)\log\left(1-\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right)ds = O(1).$$

Thus taking the imaginary part, we obtain

$$\begin{aligned}
&\int_{t_q}^{T}\log\left|1-\frac{1}{\frac{F'}{F}(b+it,\chi)}\frac{L'}{L}(1-b-it,\overline{\chi})\right|dt\\
&\quad=\int_{\sigma_1}^{b}\arg\left(1-\frac{1}{\frac{F'}{F}(\sigma+iT,\chi)}\frac{L'}{L}(1-\sigma-iT,\overline{\chi})\right)d\sigma\\
&\qquad-\int_{\sigma_1}^{b}\arg\left(1-\frac{1}{\frac{F'}{F}(\sigma+it_q,\chi)}\frac{L'}{L}(1-\sigma-it_q,\overline{\chi})\right)d\sigma+O(1)
\end{aligned}$$

Now we let

$$\log\left(\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(s,\chi)\right)=\log\left(1-\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right)$$

and determine the logarithmic branch of $\log(F'/F)(s,\chi)$ and $\log(L'/L)(s,\chi)$ in the region $\sigma_1 \le \sigma < 1/2, |t| \ge t_q - 1$ as in Lemma 4.33. Note that both of them and the functions on both sides of (4.75) are all continuous with respect to s in $\sigma_1 \le \sigma < 1/2, |t| \ge t_q - 1$. Furthermore, the two regions $\sigma_1 \le \sigma < 1/2, t \ge t_q - 1$ and $\sigma_1 \le \sigma < 1/2, -t \ge t_q - 1$ are connected. Thus we have

$$\arg\left(1-\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right)=-\arg\frac{F'}{F}(s,\chi)+\arg\frac{L'}{L}(s,\chi)+2\pi n_q$$

for some $n_q \in \mathbb{Z}$ that depends only at most on q. From our choice of logarithmic branch, we have $n_q = 0$. Thus,

$$-\frac{2\pi}{3}<\arg\left(1-\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right)<\frac{2\pi}{3} \tag{4.76}$$

for $\sigma_1 \le \sigma < 1/2, |t| \ge t_q - 1$. Therefore we obtain

$$I_{15} = \int_{t_q}^{T} \log \left| 1 - \frac{1}{\frac{F'}{F}(b+it,\chi)} \frac{L'}{L}(1-b-it,\overline{\chi}) \right| dt = O(1).$$

Collecting the above calculations, we have

$$\begin{aligned} I_1^+ &= T \log\log \frac{qT}{2\pi} + (b\log m - \log\log m)\, T - \frac{2\pi}{q} \operatorname{Li}\left(\frac{qT}{2\pi}\right) \\ &+ \left(\frac{1}{2} - b\right)\left(T \log \frac{qT}{2\pi} - T - t_q \log \frac{qt_q}{2\pi} + t_q\right) \\ &+ \int_{1-b}^{a_q} \arg L(\sigma + it_q, \chi) d\sigma - \int_{1-b}^{a_q} \arg L(\sigma + iT, \chi) d\sigma + O(\log\log q). \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} I_1^- &= T \log\log \frac{qT}{2\pi} + (b\log m - \log\log m)\, T - \frac{2\pi}{q} \operatorname{Li}\left(\frac{qT}{2\pi}\right) \\ &+ \left(\frac{1}{2} - b\right)\left(T \log \frac{qT}{2\pi} - T - t_q \log \frac{qt_q}{2\pi} + t_q\right) \\ &- \int_{1-b}^{a_q} \arg L(\sigma - it_q, \chi) d\sigma + \int_{1-b}^{a_q} \arg L(\sigma - iT, \chi) d\sigma + O(\log\log q). \end{aligned}$$

Thus we have

$$\begin{aligned} 2\pi \sum_{\substack{\rho'=\beta'+i\gamma', \\ t_q < \pm\gamma' \le T}} (\beta' - b) &= T \log\log \frac{qT}{2\pi} + (b\log m - \log\log m)\, T - \frac{2\pi}{q} \operatorname{Li}\left(\frac{qT}{2\pi}\right) \\ &+ \left(\frac{1}{2} - b\right)\left(T \log \frac{qT}{2\pi} - T - t_q \log \frac{qt_q}{2\pi} + t_q\right) \\ &\pm \int_{1-b}^{a_q} \arg L(\sigma \pm it_q, \chi) d\sigma \mp \int_{1-b}^{a_q} \arg L(\sigma \pm iT, \chi) d\sigma \\ &\mp \int_{b}^{a_q} \arg G(\sigma \pm it_q, \chi) d\sigma \pm \int_{b}^{a_q} \arg G(\sigma \pm iT, \chi) d\sigma \\ &+ O(\log\log q) + O(m). \end{aligned}$$

Taking $\delta \to 0$, we obtain Proposition 4.37. □

The following proposition states out the main term of the equation in Theorem 4.23.

Proposition 4.38. *Assume GRH. Take t_q as in* (4.71)*. Then for $T \geq 2$ which satisfies $L(\sigma \pm iT, \chi) \neq 0$ and $L'(\sigma \pm iT, \chi) \neq 0$ for all $\sigma \in \mathbb{R}$, we have*

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} - A(t_q,\chi) - B(t_q,\chi) + A(T,\chi) + B(T,\chi) \\ + A(-t_q,\chi) + B(-t_q,\chi) - A(-T,\chi) - B(-T,\chi) + O(m^{1/2}\log q),$$

where

$$A(\tau,\chi) := \frac{1}{2\pi}\arg G\left(\frac{1}{2} + i\tau, \chi\right), \quad B(\tau,\chi) := \frac{1}{2\pi}\arg L\left(\frac{1}{2} + i\tau, \chi\right),$$

and the logarithmic branches are taken as in Subsection 4.3.2.2.

Proof. Take $a_q, \sigma_1, t_q, T, \delta, b$ as in the beginning of the proof of Proposition 4.37. Let $b' := 1/2 - \delta/2$. Replacing b by b' in (4.73), we have

$$2\pi \sum_{\substack{\rho'=\beta'+i\gamma', \\ t_q<\pm\gamma'\leq T}} (\beta' - b') = \int_{t_q}^{T} \log|G(b' \pm it, \chi)|dt - \int_{t_q}^{T} \log|G(a_q \pm it, \chi)|dt \\ \mp \int_{b'}^{a_q} \arg G(\sigma \pm it_q, \chi)d\sigma \pm \int_{b'}^{a_q} \arg G(\sigma \pm iT, \chi)d\sigma.$$

Subtracting these from (4.73), we obtain

$$\delta\pi \sum_{\substack{\rho'=\beta'+i\gamma', \\ t_q<\pm\gamma'\leq T}} 1 = \int_{t_q}^{T} \left(\log|G(b \pm it, \chi)| - \log|G(b' \pm it, \chi)|\right) dt \\ \mp \int_{b}^{b'} \arg G(\sigma \pm it_q, \chi)d\sigma \pm \int_{b}^{b'} \arg G(\sigma \pm iT, \chi)d\sigma \\ =: J_1^{\pm} \mp \int_{b}^{b'} \arg G(\sigma \pm it_q, \chi)d\sigma \pm \int_{b}^{b'} \arg G(\sigma \pm iT, \chi)d\sigma.$$

We estimate $J_1^{\pm}$. From (4.74), we have

$$J_1^{+} = \int_{t_q}^{T} \left(\log|G(b + it, \chi)| - \log|G(b' + it, \chi)|\right) dt \\ = (b - b')(T - t_q)\log m + \int_{t_q}^{T} (\log|F(b + it, \chi)| - \log|F(b' + it, \chi)|)dt \\ + \int_{t_q}^{T} \left(\log\left|\frac{F'}{F}(b + it, \chi)\right| - \log\left|\frac{F'}{F}(b' + it, \chi)\right|\right) dt$$

$$+\int_{t_q}^{T} \left(\log|L(1-b-it,\overline{\chi})| - \log|L(1-b'-it,\overline{\chi})|\right) dt$$

$$+\int_{t_q}^{T} \left(\log\left|1-\frac{1}{\frac{F'}{F}(b+it,\chi)}\frac{L'}{L}(1-b-it,\overline{\chi})\right| \right.$$

$$\left. -\log\left|1-\frac{1}{\frac{F'}{F}(b'+it,\chi)}\frac{L'}{L}(1-b'-it,\overline{\chi})\right|\right) dt$$

$$=: (b-b')(T-t_q)\log m + J_{12} + J_{13} + J_{14} + J_{15}.$$

Applying Cauchy's theorem to $\log F(s,\chi)$ on the rectangle C with vertices $b+it_q$, $b'+it_q$, $b'+iT$, $b+iT$, and taking the imaginary part, we have

$$J_{12} = \int_b^{b'} \arg F(\sigma+it_q,\chi)d\sigma - \int_b^{b'} \arg F(\sigma+iT,\chi)d\sigma.$$

From (4.67), we can show that

$$J_{12} = \left(T\log\frac{qT}{2\pi} - T\right)\frac{\delta}{2} - \left(t_q\log\frac{qt_q}{2\pi} - t_q\right)\frac{\delta}{2} + O(\delta)$$

Next, we take the logarithmic branch of $\log(F'/F)(s,\chi)$ as in condition (1) of Lemma 4.33. Applying Cauchy's integral theorem to $\log(F'/F)(s,\chi)$ on C taking the imaginary part, we have

$$J_{13} = \int_b^{b'} \arg\frac{F'}{F}(\sigma+it_q,\chi)d\sigma - \int_b^{b'} \arg\frac{F'}{F}(\sigma+iT,\chi)d\sigma = O(\delta)$$

To estimate J_{14}, we define a branch of $\log L(s,\chi)$ as in the estimation of I_{14} in the proof of Proposition 4.37 and apply Cauchy's integral theorem on the rectangle with vertices $1-b'+it_q$, $1-b+it_q$, $1-b+iT$, $1-b'+iT$. Taking the imaginary part we obtain

$$J_{14} = -\int_{1-b'}^{1-b} \arg L(\sigma+it_q,\chi)d\sigma + \int_{1-b'}^{1-b} \arg L(\sigma+iT,\chi)d\sigma.$$

Finally, we define a branch of

$$\log\left(1-\frac{1}{\frac{F'}{F}(s,\chi)}\frac{L'}{L}(1-s,\overline{\chi})\right)$$

as in the estimation of I_{15} in the proof of Proposition 4.37 and apply Cauchy's integral theorem to it on C. Taking the imaginary part, we have

$$J_{15} = \int_b^{b'} \arg\left(1-\frac{1}{\frac{F'}{F}(\sigma+it_q,\chi)}\frac{L'}{L}(1-\sigma-it_q,\overline{\chi})\right)d\sigma$$

$$
- \int_b^{b'} \arg\left(1 - \frac{1}{\frac{F'}{F}(\sigma + iT, \chi)} \frac{L'}{L}(1 - \sigma - iT, \overline{\chi})\right) d\sigma
$$

$$
= O(\delta)
$$

by (4.76). Then we estimate J_1^- similarly.

We then obtain

$$
\begin{aligned}
\delta\pi \sum_{\substack{\rho'=\beta'+i\gamma', \\ t_q<\pm\gamma'\leq T}} 1 = &-(T - t_q)\frac{\delta}{2}\log m + \left(T \log\frac{qT}{2\pi} - T\right)\frac{\delta}{2} - \left(t_q \log\frac{qt_q}{2\pi} - t_q\right)\frac{\delta}{2} \\
&\mp \int_{1-b'}^{1-b} \arg L(\sigma \pm it_q, \chi) d\sigma \pm \int_{1-b'}^{1-b} \arg L(\sigma \pm iT, \chi) d\sigma \\
&\mp \int_b^{b'} \arg G(\sigma \pm it_q, \chi) d\sigma \pm \int_b^{b'} \arg G(\sigma \pm iT, \chi) d\sigma + O(\delta).
\end{aligned}
$$

Taking the limit $\delta \to 0$ and applying the mean value theorem, for $\tau = \pm t_q$ and $\tau = \pm T$ we have

$$
\lim_{\delta\to 0} \frac{1}{\pi\delta} \int_{1-b'}^{1-b} \arg L(\sigma + i\tau, \chi) d\sigma = B(\tau, \chi)
$$

and

$$
\lim_{\delta\to 0} \frac{1}{\pi\delta} \int_b^{b'} \arg G(\sigma + i\tau, \chi) d\sigma = A(\tau, \chi)
$$

by noting that $b = 1/2 - \delta$ and $b' = 1/2 - \delta/2$. Hence,

$$
\begin{aligned}
N_1(T, \chi) - N_1(t_q, \chi) = &\frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} - \left(\frac{t_q}{\pi}\log\frac{qt_q}{2m\pi} - \frac{t_q}{\pi}\right) \\
&- A(t_q, \chi) - B(t_q, \chi) + A(T, \chi) + B(T, \chi) \\
&+ A(-t_q, \chi) + B(-t_q, \chi) - A(-T, \chi) - B(-T, \chi) + O(1).
\end{aligned}
$$

Referring to [AS-p, Theorem 5] (see Theorem 4.5), we see that

$$
N_1(t_q, \chi) = \frac{t_q}{\pi}\log\frac{qt_q}{2m\pi} - \frac{t_q}{\pi} + O\left(m^{1/2}\log(qt_q)\right). \tag{4.77}
$$

Hence,

$$
\begin{aligned}
N_1(T, \chi) = &\frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} - A(t_q, \chi) - B(t_q, \chi) + A(T, \chi) + B(T, \chi) \\
&+ A(-t_q, \chi) + B(-t_q, \chi) - A(-T, \chi) - B(-T, \chi) + O(m^{1/2}\log q).
\end{aligned}
$$

If $2 \leq T < t_q$, then $N_1(T, \chi) \leq N_1(t_q, \chi) = O(m^{1/2}\log q)$, which can be included in the error term. Thus the proof is complete. □

4.3.3.2 Completion of the proofs

We begin with the proof of Theorem 4.22. Referring to [AS-p, Theorem 6] (see Theorem 4.6), we have

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ |\gamma'|\le t_q}} \left(\beta' - \frac{1}{2}\right) \ll m^{1/2}\log q. \tag{4.78}$$

This also implies that when $2 \le T < t_q$,

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ |\gamma'|\le T}} \left(\beta' - \frac{1}{2}\right) \ll m^{1/2}\log q.$$

Next, we estimate

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ t_q<|\gamma'|\le T}} \left(\beta' - \frac{1}{2}\right).$$

We divide the proof in two cases.

Case 1: For $T \ge t_q$ which satisfies $L(\sigma \pm iT, \chi) \ne 0, L'(\sigma \pm iT, \chi) \ne 0$ for all $\sigma \in \mathbb{R}$.

In this case, we apply Proposition 4.37 and provoke Lemmas 4.25, 4.34, and 4.36 to obtain the error term.

We apply Lemmas 4.25, 4.36, and 4.34 to obtain

$$\int_{1/2}^{1/2+(\log(q\tau))^{-1}} \arg L(\sigma \pm i\tau, \chi)d\sigma \ll 1,$$

$$\int_{3}^{a_q} \arg \frac{G}{L}(\sigma \pm i\tau, \chi)d\sigma \ll m\log m$$

for $\tau \ge t_q$, and

$$\int_{1/2}^{1/2+(\log(qt_q))^{-1}} \arg G(\sigma \pm it_q, \chi)d\sigma \ll \frac{m^{1/2}}{(\log\log q)^{1/2}},$$

$$\int_{1/2}^{1/2+(\log(qT))^{-1}} \arg G(\sigma \pm iT, \chi)d\sigma \ll \frac{m^{1/2}}{(\log\log(qT))^{1/2}},$$

$$\int_{1/2+(\log(qt_q))^{-1}}^{3} \arg \frac{G}{L}(\sigma \pm it_q, \chi)d\sigma \ll m^{1/2}(\log\log q)^2 + m\log\log q,$$

$$\int_{1/2+(\log (qT))^{-1}}^{3} \arg \frac{G}{L}(\sigma \pm iT,\chi)d\sigma \ll m^{1/2}(\log\log (qT))^2 + m\log\log (qT).$$

Inserting the above estimates into the formula given in Proposition 4.37 and adding this to (4.78), we obtain the equation in Theorem 4.22 for Case 1.

Case 2: For $T \geq t_q$ such that any of $L(\sigma + iT,\chi) \neq 0$, $L(\sigma - iT,\chi) \neq 0$, $L'(\sigma + iT,\chi) \neq 0$, or $L'(\sigma - iT,\chi) \neq 0$ is *not* satisfied for some $\sigma \in \mathbb{R}$.

In this case, first we look for some small $0 < \epsilon < (\log\log (qT))^{-1}$ such that $L(\sigma \pm i(T \pm \epsilon),\chi) \neq 0, L'(\sigma \pm i(T \pm \epsilon),\chi) \neq 0$ holds for all $\sigma \in \mathbb{R}$ and apply the method of Case 1, so we obtain

$$\begin{aligned}\sum_{\substack{\rho'=\beta'+i\gamma',\\ |\gamma'|\leq T\pm\epsilon}} \left(\beta' - \frac{1}{2}\right) = \frac{(T\pm\epsilon)}{\pi}\log\log\frac{q(T\pm\epsilon)}{2\pi} \\ + \frac{T\pm\epsilon}{\pi}\left(\frac{1}{2}\log m - \log\log m\right) \\ - \frac{2}{q}\,\mathrm{Li}\left(\frac{q(T\pm\epsilon)}{2\pi}\right) \\ + O\left(m^{1/2}(\log\log (qT))^2 + m\log\log (qT) + m^{1/2}\log q\right).\end{aligned}$$

Noting that

$$\sum_{\substack{\rho'=\beta'+i\gamma',\\ t_q-1<|\gamma'|\leq T-\epsilon}} \left(\beta' - \frac{1}{2}\right) \leq \sum_{\substack{\rho'=\beta'+i\gamma',\\ t_q-1<|\gamma'|\leq T}} \left(\beta' - \frac{1}{2}\right) \leq \sum_{\substack{\rho'=\beta'+i\gamma',\\ t_q-1<|\gamma'|\leq T+\epsilon}} \left(\beta' - \frac{1}{2}\right)$$

together with (4.78), we easily show that the equation in Theorem 4.22 also holds for this case. □

To complete the proof of Theorem 4.23, as in the proof of Theorem 4.22, we also consider two cases. In the first case, for $T \geq 2$ which satisfies $L(\sigma \pm iT,\chi) \neq 0, L'(\sigma \pm iT,\chi) \neq 0$ for all $\sigma \in \mathbb{R}$, the error terms are estimated as follows: From Lemma 4.36, we have

$$\arg G\left(\frac{1}{2} \pm it_q,\chi\right) = O\left(\frac{m^{1/2}\log q}{(\log\log q)^{1/2}}\right)$$

and

$$\arg G\left(\frac{1}{2} \pm iT,\chi\right) = O\left(\frac{m^{1/2}\log (qT)}{(\log\log (qT))^{1/2}}\right).$$

From Lemma 4.25, we have

$$\arg L\left(\frac{1}{2} \pm it_q,\chi\right) = O\left(\frac{\log q}{\log\log q}\right), \quad \arg L\left(\frac{1}{2} \pm iT,\chi\right) = O\left(\frac{\log (qT)}{\log\log (qT)}\right).$$

Therefore,

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} + O\left(\frac{m^{1/2}\log(qT)}{(\log\log(qT))^{1/2}}\right) + O(m^{1/2}\log q)$$

for this case.

In the second case, we consider for $T \geq 2$ such that any of $L(\sigma + iT, \chi) \neq 0$, $L(\sigma - iT, \chi) \neq 0$, $L'(\sigma + iT, \chi) \neq 0$, or $L'(\sigma - iT, \chi) \neq 0$ is *not* satisfied for some $\sigma \in \mathbb{R}$. Similar to the proof of Theorem 4.22, we look for some small $0 < \epsilon < (\log(qT))^{-1}$ such that $L(\sigma \pm i(T \pm \epsilon), \chi) \neq 0, L'(\sigma \pm i(T \pm \epsilon), \chi) \neq 0$ holds for all $\sigma \in \mathbb{R}$. Applying the method of the first case we obtain

$$\begin{aligned} N_1(T \pm \epsilon, \chi) = &\frac{T \pm \epsilon}{\pi}\log\frac{q(T \pm \epsilon)}{2m\pi} - \frac{T \pm \epsilon}{\pi} + O\left(\frac{m^{1/2}\log(qT)}{(\log\log(qT))^{1/2}}\right) \\ &+ O(m^{1/2}\log q). \end{aligned} \tag{4.79}$$

Noting the inequalities

$$N_1(T - \epsilon, \chi) \leq N_1(T, \chi) \leq N_1(T - \epsilon, \chi) + (N_1(T + \epsilon, \chi) - N_1(T - \epsilon, \chi)),$$

from (4.79) we can easily deduce

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi} - \frac{T}{\pi} + O\left(\frac{m^{1/2}\log(qT)}{(\log\log(qT))^{1/2}}\right) + O(m^{1/2}\log q)$$

for this case. $\square$

Chapter 5

Further research: An ergodic value distribution of zeta functions and L-functions

In this chapter, we introduce the author's collaborative work [LS-p] with J. Lee on a certain mean-value of meromorphic functions by using specific ergodic transformations, which we call affine Boolean transformations. Birkhoff's ergodic theorem is used to transform the mean-value into a computable integral which allows us to completely determine the mean-value of this ergodic type. As examples, we introduce some applications to zeta functions and L-functions. We also prove an equivalence of the Lindelöf hypothesis of the Riemann zeta function in terms of its certain ergodic value distribution associated with affine Boolean transformations.

5.1 Introduction

In [LW09], M. Lifshitz and M. Weber investigated the value distribution of the *Riemann zeta function* $\zeta(s)$ by using the Cauchy random walk. They proved that almost surely

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \zeta\left(\frac{1}{2}+iS_n\right) = 1 + o\left(\frac{(\log N)^b}{N^{1/2}}\right)$$

holds for any $b > 2$ where $\{S_n\}_{n=1}^{\infty}$ is the Cauchy random walk. This result implies that most of the values of $\zeta(s)$ on the critical line are quite small. Analogous to [LW09], T. Srichan investigated the value distributions of Dirichlet L-functions and Hurwitz zeta functions by using the Cauchy random walk in [Sri15].

The first approach to investigate the ergodic value distribution of $\zeta(s)$ was done by J. Steuding. In [Ste12], he studied the ergodic value distribution of $\zeta(s)$ on vertical lines under the Boolean transformation.

We are interested in studying the ergodic value distribution of a larger class of meromorphic functions which includes but is not limited to the Selberg class (of zeta functions and L-functions) and their derivatives, on vertical lines under more general Boolean transformations, which we shall call *affine Boolean transformation* $T_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ given by

$$T_{\alpha,\beta}(x) := \begin{cases} \frac{\alpha}{2}\left(\frac{x+\beta}{\alpha} - \frac{\alpha}{x-\beta}\right), & x \neq \beta; \\ \beta, & x = \beta \end{cases} \tag{5.1}$$

for an $\alpha > 0$ and a $\beta \in \mathbb{R}$. Below is our main theorem. For a given $c \in \mathbb{R}$, we shall denote by $\mathbb{H}_c$ and $\mathbb{L}_c$ the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > c\}$ and the line $\{z \in \mathbb{C} \mid \mathrm{Re}(z) = c\}$.

Theorem 5.1. *Let f be a meromorphic function on $\mathbb{H}_c$ satisfying the following conditions.*

1. *There exists an $M > 0$ and a $c' > c$ such that for any $t \in \mathbb{R}$, we have*

$$|f(\{\sigma + it \mid \sigma > c'\})| \leq M.$$

2. *There exists a non-increasing continuous function $\nu : (c, \infty) \to \mathbb{R}$ such that if σ is sufficiently near c then $\nu(\sigma) \leq 1 + c - \sigma$, and that for any small $\epsilon > 0$, $f(\sigma + it) \ll_{f,\epsilon} |t|^{\nu(\sigma)+\epsilon}$ as $|t| \to \infty$.*

3. *f has at most one pole of order m in $\mathbb{H}_c$ at $s = s_0 = \sigma_0 + it_0$, that is, we can write its Laurent expansion near $s = s_0$ as*

$$\frac{a_{-m}}{(s-s_0)^m} + \frac{a_{-(m-1)}}{(s-s_0)^{m-1}} + \cdots + \frac{a_{-1}}{s-s_0} + a_0 + \sum_{n=1}^{\infty} a_n (s-s_0)^n \tag{5.2}$$

for $m \geq 0$, where we set $m = 0$ if f has no pole in $\mathbb{H}_c$.

Then for any $s \in \mathbb{H}_c \backslash \mathbb{L}_{\sigma_0}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f\left(s + iT_{\alpha,\beta}^n x\right) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2 + (\tau-\beta)^2} d\tau \tag{5.3}$$

for almost all $x \in \mathbb{R}$.

We denote the right-hand side of the above formula by $l_{\alpha,\beta}(s)$. If f has no pole in $\mathbb{H}_c$,

$$l_{\alpha,\beta}(s) = f(s + \alpha + i\beta) \tag{5.4}$$

for all $s \in \mathbb{H}_c$*. If* f *has a pole at* $s = s_0 = \sigma_0 + it_0$*,*

$$l_{\alpha,\beta}(s) = \begin{cases} f(s+\alpha+i\beta) + B_m(s_0), & c < \mathrm{Re}(s) < \sigma_0, s \neq s_0 - \alpha - i\beta; \\ \displaystyle\sum_{n=0}^{m} \frac{a_{-n}}{(-2\alpha)^n}, & c < \mathrm{Re}(s) < \sigma_0, s = s_0 - \alpha - i\beta; \\ f(s+\alpha+i\beta), & \mathrm{Re}(s) > \sigma_0; \end{cases} \tag{5.5}$$

where

$$B_m(s_0) = \sum_{n=1}^{m} \frac{a_{-n}}{i^n\left(\beta + i\alpha - i(s-s_0)\right)^n} - \sum_{n=1}^{m} \frac{a_{-n}}{i^n\left(\beta - i\alpha - i(s-s_0)\right)^n}.$$

Moreover when $m = 1$*, we can extend the result in* (5.3) *to the line* $\mathbb{L}_{\sigma_0}$ *by setting*

$$l_{\alpha,\beta}(\sigma_0 + it) = f(\sigma_0 + \alpha + i(t+\beta)) - \frac{a_{-1}\alpha}{\alpha^2 + (t_0 - t - \beta)^2} \tag{5.6}$$

for any $t \in \mathbb{R}$*.*

In the next section, we first give a few examples as applications of our main theorem, Theorem 5.1, to the Riemann zeta function, Dirichlet L-functions, Dedekind zeta functions, Hurwitz zeta functions, and their derivatives. We will briefly review some basics of ergodic theory and see an ergodic property of affine Boolean transformations in Section 5.3. In Section 5.4, we will complete the proof of Theorem 5.1.

5.2 Some applications to zeta functions and L-functions

In the following examples, we write $f^{(0)}$ to express f itself and we define

$$A_k(s) := \frac{(-1)^k k!}{i^{k+1}} \left(\frac{1}{\left(\beta + i\alpha - i(s-1)\right)^{k+1}} - \frac{1}{\left(\beta - i\alpha - i(s-1)\right)^{k+1}} \right)$$

for any non-negative integer k.

Example 5.2 (The Riemann zeta function)**.** *For any* $k \geq 0$ *and* $s \in \mathbb{H}_{-1/2} \backslash \mathbb{L}_1$*, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \zeta^{(k)}\left(s + iT^n_{\alpha,\beta}x\right) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{\zeta^{(k)}(s+i\tau)}{\alpha^2 + (\tau-\beta)^2} d\tau$$

for almost all $x \in \mathbb{R}$*.*

Denoting the right-hand side of the above formula by $l^{(k)}_{\alpha,\beta}(s)$, we have

$$l^{(k)}_{\alpha,\beta}(s) = \begin{cases} \zeta^{(k)}(s+\alpha+i\beta) + A_k(s), & -1/2 < \mathrm{Re}(s) < 1, s \neq 1-\alpha-i\beta; \\ (-1)^k\gamma_k - \dfrac{k!}{(2\alpha)^{k+1}}, & -1/2 < \mathrm{Re}(s) < 1, s = 1-\alpha-i\beta; \\ \zeta^{(k)}(s+\alpha+i\beta), & \mathrm{Re}(s) > 1; \end{cases}$$

where

$$\gamma_k := \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1}\right).$$

If $k = 0$, we can extend the result to the line $\mathbb{L}_1$ by setting

$$l^{(0)}_{\alpha,\beta}(1+it) = \zeta^{(0)}(1+\alpha+i(t+\beta)) - \frac{\alpha}{\alpha^2+(t+\beta)^2}.$$

Remark that Steuding showed Example 5.2 when $k = 0$, $\alpha = 1$, and $\beta = 0$ thus Example 5.2 is a generalization of [Ste12, Theorem 1.1].

Proof of Example 5.2. We first note that for any $k \geq 0$, $\zeta^{(k)}(s)$ has an absolute convergent Dirichlet series expression when $\mathrm{Re}(s) > 1$. Thus condition (1) of Theorem 5.1 is satisfied for any $c' > 1$. From the Laurent expansion of $\zeta(s)$ near its pole $s = 1$ (see [Bri55, Theorem]), we can deduce the Laurent expansion of $\zeta^{(k)}(s)$ for any $k \geq 0$ near $s = 1$:

$$\zeta^{(k)}(s) = \frac{(-1)^k k!}{(s-1)^{k+1}} + (-1)^k\gamma_k + \sum_{n=k}^{\infty}\frac{(-1)^{n+1}\gamma_{n+1}}{(n-k+1)!}(s-1)^{n-k+1}.$$

Thus for $k \geq 0$, $\zeta^{(k)}(s)$ has a pole of order $k+1$ at $s = 1$. Moreover, we can show by using [Tit86, pp. 95–96][2] that

$$\zeta^{(k)}(\sigma+it) \ll_{k,\epsilon} |t|^{\mu(\sigma)+\epsilon} \tag{5.7}$$

holds with

$$\mu(\sigma) \leq \begin{cases} 0 & \text{if } \sigma > 1; \\ (1-\sigma)/2 & \text{if } 0 \leq \sigma \leq 1; \\ 1/2-\sigma & \text{if } \sigma < 0; \end{cases}$$

for any $k \geq 0$. Therefore we can apply Theorem 5.1 with $c = -1/2$, $s_0 = 1$, and $m = k+1$ to $\zeta^{(k)}(s)$. □

[2]Phragmén-Lindelöf theorem, as introduced in Lemma 1.1, is used here.

We can also show that this ergodic mean-value is related to the Lindelöf hypothesis. We first show that the Lindelöf hypothesis can be rewritten in terms of $\zeta^{(k)}(s)$.

Theorem 5.3. *Let $k \in \mathbb{N}$. The Lindelöf hypothesis: For any $\epsilon > 0$,*

$$\zeta\left(\frac{1}{2}+it\right) \ll_\epsilon |t|^\epsilon \quad \text{as } |t| \to \infty$$

holds if only if, for any $\epsilon > 0$,

$$\zeta^{(k)}\left(\frac{1}{2}+it\right) \ll_{k,\epsilon} |t|^\epsilon \quad \text{as } |t| \to \infty$$

holds.

The above theorem implies that we can restate the Lindelöf hypothesis as:

$$\text{For any } \epsilon > 0,\ \zeta^{(k)}\left(\frac{1}{2}+it\right) \ll_{k,\epsilon} |t|^\epsilon \quad \text{as } |t| \to \infty$$

for any non-negative integer k.

Proof of Theorem 5.3. Suppose that the Lindelöf hypothesis is true. Thus by using the functional equation for $\zeta(s)$,

$$\zeta(\sigma+it) \ll_\epsilon |t|^{\mu(\sigma)+\epsilon/2}$$

holds for any $\epsilon > 0$ with

$$\mu(\sigma) \le \begin{cases} 0 & \text{if } \sigma \ge 1/2; \\ 1/2-\sigma & \text{if } \sigma < 1/2. \end{cases}$$

Then by Cauchy's integral theorem, for any $k \in \mathbb{N}$ we have

$$\zeta^{(k)}\left(\frac{1}{2}+it\right) = \frac{k!}{2\pi i}\int_{\gamma_r} \frac{\zeta(z)}{(z-1/2-it)^{k+1}}dz,$$

where $\gamma_r := \{z \in \mathbb{C} \mid |z-1/2-it| = r\}$. Taking $r = \epsilon/2$,

$$\begin{aligned}\left|\zeta^{(k)}\left(\frac{1}{2}+it\right)\right| &\ll_k \int_{\gamma_r} \frac{|\zeta(z)|}{|z-1/2-it|^{k+1}}|dz| \ll_{k,\epsilon} |t+\epsilon/2|^{\mu(1/2-\epsilon/2)+\epsilon/2} \\ &\ll_\epsilon |t|^{\mu(1/2-\epsilon/2)+\epsilon/2} \le |t|^{1/2-(1/2-\epsilon/2)+\epsilon/2} = |t|^\epsilon.\end{aligned}$$

Now suppose that for some $k \in \mathbb{N}$,

$$\zeta^{(k)}\left(\frac{1}{2}+it\right) \ll_{k,\epsilon} |t|^{\epsilon}$$

holds for any $\epsilon > 0$. Then $\zeta^{(k)}(\sigma+it) \ll_{k,\epsilon} |t|^{\mu(\sigma)+\epsilon}$ for $\sigma \geq 1/2$. Note that

$$\left|\zeta^{(k-1)}\left(\frac{1}{2}+it\right)\right| \leq \left|\zeta^{(k-1)}(3+it)\right| + \int_{1/2+it}^{3+it} \left|\zeta^{(k)}(z)\right| |dz| \ll_{k,\epsilon} |t|^{\epsilon}.$$

This implies

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll_{\epsilon} |t|^{\epsilon}.$$

□

We can then reformulate the Lindelöf hypothesis in terms of ergodic value distribution of $\zeta^{(k)}(s)$ on vertical lines under affine Boolean transformations as follows:

Theorem 5.4. *Let k be a non-negative integer. The Lindelöf hypothesis is true if and only if, there exist $\alpha > 0$, $\beta \in \mathbb{R}$ such that for any $l \in \mathbb{N}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left|\zeta^{(k)}\left(1/2 + iT_{\alpha,\beta}^{n} x\right)\right|^{2l} \tag{5.8}$$

exists for almost all $x \in \mathbb{R}$.

Proof of Theorem 5.4. From Theorem 5.3, we can restate the Lindelöf hypothesis as

$$\zeta^{(k)}\left(\frac{1}{2}+it\right) \ll_{k,\epsilon} |t|^{\epsilon} \quad \text{as } |t| \to \infty \tag{5.9}$$

for any non-negative integer k. We then show that the hypothesis in the form (5.9) is equivalent to the existence of the limit in (5.8).

Replacing the function $\zeta(s)$ by $\zeta^{(k)}(s)$ in the proof of Theorem 4.1 in [Ste12], we can easily show the necessary condition for the Lindelöf hypothesis (in the form (5.9)).

To show the sufficient condition for the Lindelöf hypothesis, we note that

$$\zeta^{(k)}(s) = (-1)^{k-1} \int_{1}^{\infty} \frac{[x]-x+1/2}{x^{s+1}} (\log x)^{k-1} \left(-s\log x + k\right) dx + \frac{(-1)^k k!}{(s-1)^{k+1}}$$

so that $|\zeta^{(k+1)}(1/2+it)| < C_k|t|$ holds for any $|t| \geq 1$ for some $C_k > 0$ which may depend only on k. Further, for $\tau \geq 1$,

$$\frac{1}{\alpha^2+(\tau-\beta)^2} = \frac{1}{\tau^2(1+(\alpha/\tau)^2+2|\beta|/\tau+(\beta/\tau)^2)} \geq C_{\alpha,\beta}\frac{1}{\tau^2} \geq C_{\alpha,\beta}\frac{1}{1+\tau^2}$$

for some $C_{\alpha,\beta} > 0$ that depends only on α and β. Then again we can replace the function $\zeta(s)$ by $\zeta^{(k)}(s)$ in the proof of Theorem 4.1 in [Ste12] to obtain the sufficient condition for the Lindelöf hypothesis (in the form (5.9)). This completes our proof of Theorem 5.4. $\square$

Example 5.5 (Dirichlet L-functions)**.** *Let $L(s,\chi)$ be the Dirichlet L-function associated with Dirichlet character χ.*

(i) If χ is non-principal, for any $s \in \mathbb{H}_{-1/2}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} L^{(k)}\left(s + iT^n_{\alpha,\beta}x, \chi\right) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{L^{(k)}(s+i\tau,\chi)}{\alpha^2 + (\tau-\beta)^2} d\tau$$
$$= L^{(k)}(s+\alpha+i\beta,\chi)$$

for almost all $x \in \mathbb{R}$.

(ii) If $\chi = \chi_0$ is principal, for any $s \in \mathbb{H}_{-1/2} \backslash \mathbb{L}_1$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} L^{(k)}\left(s + iT^n_{\alpha,\beta}x, \chi_0\right) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{L^{(k)}(s+i\tau,\chi_0)}{\alpha^2 + (\tau-\beta)^2} d\tau$$

for almost all $x \in \mathbb{R}$. Denoting the right-hand side of the above formula by $l^{(k)}_{\alpha,\beta}(s,\chi_0)$, we have

$$l^{(k)}_{\alpha,\beta}(s,\chi_0) = \begin{cases} L^{(k)}(s+\alpha+i\beta,\chi_0) + \gamma_{-1}(\chi_0)A_k(s), \\ \qquad\qquad -1/2 < \mathrm{Re}(s) < 1, s \neq 1-\alpha-i\beta; \\ \gamma_k(\chi_0) - \dfrac{k!\gamma_{-1}(\chi_0)}{(2\alpha)^{k+1}}, \\ \qquad\qquad -1/2 < \mathrm{Re}(s) < 1, s = 1-\alpha-i\beta; \\ L^{(k)}(s+\alpha+i\beta,\chi_0), \\ \qquad\qquad \mathrm{Re}(s) > 1; \end{cases}$$

where $\gamma_{-1}(\chi_0)$, $\gamma_k(\chi_0)$'s are constants that depend only on χ_0. They are coefficients of the Laurent expansion of $L^{(k)}(s,\chi_0)$ near $s=1$. If $k=0$, we can also show the result on $\mathbb{L}_1$ by setting

$$l^{(0)}_{\alpha,\beta}(1+it,\chi_0) = L^{(0)}(1+\alpha+i(t+\beta),\chi_0) - \frac{\alpha\gamma_{-1}(\chi_0)}{\alpha^2+(t+\beta)^2}.$$

Proof of Example 5.5. As in the proof of Example 5.2, for any non-negative integer k, $L^{(k)}(s,\chi)$ has an absolute convergent Dirichlet series expression when $\mathrm{Re}(s) > 1$.

Referring to [Red82, Lemma 2], we know that $L^{(k)}(s,\chi)$ also satisfies an inequality similar to (5.7).

If χ is non-principal, $L^{(k)}(s,\chi)$ is entire for all $k \geq 0$. Thus $L^{(k)}(s,\chi)$ satisfies (5.4) of Theorem 5.1 for all $s \in \mathbb{H}_{-1/2}$.

Otherwise (that is, when $\chi = \chi_0$), $L^{(k)}(s,\chi_0)$ ($k \geq 1$) has a pole of order $k+1$ at $s = 1$. Hence we can also apply Theorem 5.1 with $c = -1/2$, $s_0 = 1$, and $m = k+1$ to $L^{(k)}(s,\chi_0)$ with the Laurent coefficients as discussed in [IK99, Theorem 2]. □

Example 5.6 (Dedekind zeta functions)**.** *Let $\zeta_{\mathbb{K}}(s)$ be the Dedekind zeta function of a number field $\mathbb{K}$ over $\mathbb{Q}$ of degree $d_{\mathbb{K}}$. Then for any $k \geq 0$ and $s \in \mathbb{H}_{1/2-1/d_{\mathbb{K}}} \backslash \mathbb{L}_1$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \zeta_{\mathbb{K}}^{(k)}\left(s + iT_{\alpha,\beta}^n x\right) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{\zeta_{\mathbb{K}}^{(k)}(s+i\tau)}{\alpha^2 + (\tau-\beta)^2} d\tau$$

for almost all $x \in \mathbb{R}$.

Denoting the right-hand side of the above formula by $l_{\mathbb{K}_{\alpha,\beta}}^{(k)}(s)$, we have

$$l_{\mathbb{K}_{\alpha,\beta}}^{(k)}(s) = \begin{cases} \zeta_{\mathbb{K}}^{(k)}(s+\alpha+i\beta) + \gamma_{-1}(\mathbb{K}) A_k(s), \\ \qquad\qquad 1/2 - 1/d_{\mathbb{K}} < \mathrm{Re}(s) < 1, s \neq 1-\alpha-i\beta; \\ k!\gamma_k(\mathbb{K}) - \dfrac{k!\gamma_{-1}(\mathbb{K})}{(2\alpha)^{k+1}}, \\ \qquad\qquad 1/2 - 1/d_{\mathbb{K}} < \mathrm{Re}(s) < 1, s = 1-\alpha-i\beta; \\ \zeta_{\mathbb{K}}^{(k)}(s+\alpha+i\beta), \\ \qquad\qquad \mathrm{Re}(s) > 1; \end{cases}$$

where $\gamma_{-1}(\mathbb{K})$, $\gamma_k(\mathbb{K})$'s are constants that depend only on $\mathbb{K}$. They are coefficients of the Laurent expansion of $\zeta_{\mathbb{K}}^{(k)}(s)$ near $s = 1$. If $k = 0$, we can also show the result on $\mathbb{L}_1$ by setting

$$l_{\mathbb{K}_{\alpha,\beta}}^{(0)}(1+it) = \zeta_{\mathbb{K}}^{(0)}(1+\alpha+i(t+\beta)) - \frac{\alpha\gamma_{-1}(\mathbb{K})}{\alpha^2+(t+\beta)^2}.$$

Proof of Example 5.6. We refer to [Ste03, Theorem 2] for the bound of the form (5.7) and to [HIKW04, pp. 496–497] for the Laurent coefficients of $\zeta_{\mathbb{K}}^{(k)}(s)$ near its pole at $s = 1$. The rest of the proof proceeds as in the proof of Example 5.2 with $c = 1/2 - 1/d_{\mathbb{K}}$, $s_0 = 1$, and $m = k+1$. □

Remark. We can also show results analogous to Theorems 5.3 and 5.4 for Dirichlet L-functions associated with primitive Dirichlet characters and Dedekind zeta functions, if we formulate the *extended Lindelöf hypothesis* as:

$$\text{For any } \epsilon > 0,\ f\left(\frac{1}{2}+it\right) \ll_{f,\epsilon} |t|^{\epsilon} \quad \text{as } |t| \to \infty$$

for these functions (f is any of these zeta functions and L-functions). We do not discuss this further but we remark that we can show these analogous results by using methods similar to the methods used in proving Theorems 5.3 and 5.4.

Example 5.7 (Hurwitz zeta functions)**.** *For non-negative integer k, $0 < \mathfrak{a} \leq 1$, and any s satisfying* $\mathrm{Re}(s) > -1/2$ *and* $\mathrm{Re}(s) \neq 1$*, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \zeta^{(k)}(s + iT_{\alpha,\beta}^n x, \mathfrak{a}) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{\zeta^{(k)}(s + i\tau, \mathfrak{a})}{\alpha^2 + (\tau - \beta)^2} d\tau$$

for almost all x in $\mathbb{R}$.

Denoting the right-hand side of the above formula by $l_{\alpha,\beta}^{(k)}(s, \mathfrak{a})$, we have

$$l_{\alpha,\beta}^{(k)}(s, \mathfrak{a}) = \begin{cases} \zeta^{(k)}(s + \alpha + i\beta, \mathfrak{a}) + A_k(s), \\ \qquad\qquad -1/2 < \mathrm{Re}(s) < 1, s \neq 1 - \alpha - i\beta; \\ k!\gamma_k(\mathfrak{a}) - \dfrac{k!}{(2\alpha)^{k+1}}, \\ \qquad\qquad -1/2 < \mathrm{Re}(s) < 1, s = 1 - \alpha - i\beta; \\ \zeta^{(k)}(s + \alpha + i\beta, \mathfrak{a}), \\ \qquad\qquad \mathrm{Re}(s) > 1; \end{cases}$$

where

$$\gamma_k(\mathfrak{a}) := \frac{(-1)^k}{k!} \lim_{N\to\infty} \left(\sum_{n=0}^{N} \frac{\log^k(n + \mathfrak{a})}{n + \mathfrak{a}} - \frac{\log^{k+1}(N + \mathfrak{a})}{k + 1} \right)$$

is a coefficient of the Laurent expansion of $\zeta_{\mathbb{K}}^{(k)}(s)$ near $s = 1$. If $k = 0$, we can also show the result on $\mathbb{L}_1$ by setting

$$l_{\alpha,\beta}^{(0)}(1 + it, \mathfrak{a}) = \zeta^{(0)}(1 + \alpha + i(t + \beta), \mathfrak{a}) - \frac{\alpha}{\alpha^2 + (t + \beta)^2}.$$

Proof of Example 5.7. The proof also follows that of Example 5.2 where we put $c = -1/2$, $s_0 = 1$, and $m = k + 1$. Here, we refer to [Red82, Lemma 2] for the bound of the form (5.7) and to [Ber72, Theorem 1] for the Laurent coefficients of $\zeta^{(k)}(s, \mathfrak{a})$ near its pole at $s = 1$. □

5.3 Affine Boolean transformations

In this section, we will show the ergodicity of $T_{\alpha,\beta}$ defined in (5.1) with respect to a proper measure. To state our main theorem, let us recall some basic notation.

We denote by $\mathcal{B}$ and ν the *Borel σ-algebra* on $\mathbb{R}$ and the *Lebesgue measure* on $\mathcal{B}$. For a given $\alpha > 0, \beta \in \mathbb{R}$, let us define the function $\mu_{\alpha,\beta}$ by

$$\mu_{\alpha,\beta}(A) := \frac{\alpha}{\pi} \int_A \frac{d\tau}{\alpha^2 + (\tau - \beta)^2}$$

for any $A \in \mathcal{B}$. One can easily check that $\mu_{\alpha,\beta}$ is a probability on $\mathcal{B}$ and

$$\mu_{\alpha,\beta}(A) = \frac{\alpha}{\pi} \int_A \frac{d\tau}{\alpha^2 + (\tau - \beta)^2} \leq \int_A \frac{d\tau}{\alpha\pi} = \frac{1}{\alpha\pi}\nu(A) \tag{5.10}$$

for any $A \in \mathcal{B}$. In particular, this implies that $\mu_{\alpha,\beta}(A) = 0$ if $\nu(A) = 0$.

Theorem 5.8. *For given $\alpha > 0, \beta \in \mathbb{R}$, $T_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ is measure preserving with respect to $\mu_{\alpha,\beta}$, that is, for any $A \in \mathcal{B}$, we have*

$$\mu_{\alpha,\beta}(T_{\alpha,\beta}^{-1}(A)) = \mu_{\alpha,\beta}(A).$$

Moreover, it is ergodic, that is, if $T_{\alpha,\beta}^{-1}(A) = A$, then either $\mu_{\alpha,\beta}(A)$ or $\mu_{\alpha,\beta}(X \backslash A)$ is 0.

Applying Birkhoff's ergodic theorem, we have an ergodic mean-value of an integrable function. Let us denote by $T_{\alpha,\beta}^n$ the *n-th iteration* of $T_{\alpha,\beta}$, that is,

$$T_{\alpha,\beta}^n := \underbrace{T_{\alpha,\beta} \circ T_{\alpha,\beta} \circ \cdots \circ T_{\alpha,\beta}}_{n \text{ times}}.$$

Corollary 5.9. *If $f : \mathbb{R} \to \mathbb{R}$ is integrable with respect to $\mu_{\alpha,\beta}$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T_{\alpha,\beta}^n x = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(\tau)d\tau}{\alpha^2 + (\tau - \beta)^2} \tag{5.11}$$

for almost all $x \in \mathbb{R}$.

See [EW11, Theorem 2.30] for the proof of Birkhoff's ergodic theorem. Corollary 5.9 follows immediately from Birkhoff's ergodic theorem and Theorem 5.8.

Birkhoff's ergodic theorem describes the relation between the space average of a function and the time average along the orbit. In the next section, we will apply Corollary 5.9 to transform a mean-value of ergodic type into a computable integral.

In the rest of this section, we complete the proof of Theorem 5.8. We first recall the famous result given by R. Adler and B. Weiss.

Lemma 5.10. *The Boolean transformation $T_{1,0}$ is measure preserving with respect to ν. Moreover, it is ergodic.*

See [AW73, Theorem and Main Theorem] for the proof of Lemma 5.10.

Proof of Theorem 5.8. We first check that $T := T_{1,0}$ is measure preserving and ergodic with respect to $\mu := \mu_{1,0}$. Let us denote by χ_A the *indicator function* of $A \subset \mathbb{R}$. It follows from a simple calculation that

$$\begin{aligned}\mu(T^{-1}(A)) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\chi_A(T(\tau))d\tau}{1+\tau^2} \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\chi_A(T(\tau))dT(\tau)}{1+T(\tau)^2} = \mu(A)\end{aligned}$$

for any $A \in \mathcal{B}$. Thus T is measure preserving with respect to μ. If $T^{-1}(A) = A$, it follows from Lemma 5.10 that either $\nu(A)$ or $\nu(X\backslash A)$ is 0. Hence, by (5.10), either $\mu(A)$ or $\mu(X\backslash A)$ must be 0.

Next, let us consider the general case. Defining the affine transformation $\phi_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ by $\phi_{\alpha,\beta}(x) := \alpha x + \beta$, we can easily check that

$$T_{\alpha,\beta} = \phi_{\alpha,\beta} \circ T \circ \phi_{\alpha,\beta}^{-1}$$

and

$$\mu_{\alpha,\beta}(A) = \mu(\phi_{\alpha,\beta}^{-1}(A)).$$

Since T is measure preserving with respect to μ, we have

$$\begin{aligned}\mu_{\alpha,\beta}(T_{\alpha,\beta}^{-1}(A)) &= \mu(\phi_{\alpha,\beta}^{-1}(T_{\alpha,\beta}^{-1}(A)) \\ &= \mu(\phi_{\alpha,\beta}^{-1}(\phi_{\alpha,\beta}(T^{-1}(\phi_{\alpha,\beta}^{-1}(A)))) \\ &= \mu(T^{-1}(\phi_{\alpha,\beta}^{-1}(A))) \\ &= \mu(\phi_{\alpha,\beta}^{-1}(A)) = \mu_{\alpha,\beta}(A).\end{aligned}$$

Moreover, if $T_{\alpha,\beta}^{-1}(A) = A$, we have

$$T^{-1}(\phi_{\alpha,\beta}^{-1}(A)) = \phi_{\alpha,\beta}^{-1}(T_{\alpha,\beta}^{-1}(A)) = \phi_{\alpha,\beta}^{-1}(A).$$

Since T is ergodic with respect to μ, either $\mu_{\alpha,\beta}(A) = \mu(\phi_{\alpha,\beta}^{-1}(A))$ or $\mu_{\alpha,\beta}(X\backslash A) = \mu(X\backslash\phi_{\alpha,\beta}^{-1}(A))$ is 0. □

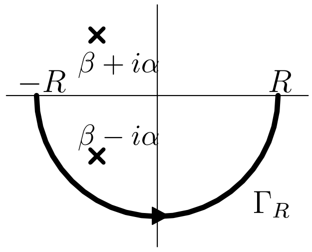
5.4 Proof of the main theorem

Proof of Theorem 5.1. It follows from Corollary 5.9 that (5.3) holds. For the case $m = 1$, we set the values of the integrand to be the principal value on the line $\mathbb{L}_{\sigma_0}$ and since this is integrable, as we shall see below in Case 3 of the evaluation of

$l_{\alpha,\beta}$, we can now apply Corollary 5.9 for all $s \in \mathbb{H}_c$. In the rest of this section, we evaluate $l_{\alpha,\beta}$ to complete the proof of Theorem 5.1.

Suppose that f has no pole in $\mathbb{H}_c$. The poles of the integrand in $l_{\alpha,\beta}$ in $\mathbb{H}_c$ are coming only from the zeros of $\alpha^2 + (\tau - \beta)^2$. For any $s = \sigma + it \in \mathbb{H}_c$, we consider the counterclockwise oriented semicircle Γ_R for a sufficiently large $R > |s| + \alpha + |\beta|$ as in Figure 5.1. Then applying Cauchy's integral theorem, we have

Figure 5.1:



$$
\begin{aligned}
l_{\alpha,\beta}(s) &= \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2 + (\tau-\beta)^2} d\tau \\
&= \frac{\alpha}{\pi} \left(\lim_{R \to \infty} \int_{\Gamma_R} \frac{f(s+i\tau)}{\alpha^2 + (\tau-\beta)^2} d\tau - 2\pi i \operatorname{Res}_{\tau=\beta-i\alpha} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} \right).
\end{aligned}
$$

Note that we can find a $\sigma' \in (c, \sigma)$ sufficiently near c. Setting $\epsilon = (\sigma' - c)/2$, we have

$$
\begin{aligned}
\int_{\Gamma_R} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} d\tau &= \int_{\pi}^{2\pi} \frac{f(s+iRe^{i\theta})}{\alpha^2 + (Re^{i\theta}-\beta)^2} iRe^{i\theta} d\theta \\
&\ll_{\alpha,\beta} \frac{1}{R} \left(\int_{\pi}^{5\pi/4} + \int_{5\pi/4}^{7\pi/4} + \int_{7\pi/4}^{2\pi} \right) |f(s+iRe^{i\theta})| d\theta \\
&\ll \frac{1}{R} \Bigg(\max_{\theta \in [\pi, 5\pi/4] \cup [7\pi/4, 2\pi]} |f(\sigma - R\sin\theta + i(t + R\cos\theta))| \\
&\qquad + \max_{\theta \in [5\pi/4, 7\pi/4]} |f(\sigma - R\sin\theta + i(t+R\cos\theta))| \Bigg) \\
&\le \frac{1}{R} \left(\max_{\theta \in [\pi, 5\pi/4] \cup [7\pi/4, 2\pi]} |t + R\cos\theta|^{\nu(\sigma - R\sin\theta)+\epsilon} + M \right) \\
&\le \frac{1}{R} \left(\max_{\theta \in [\pi, 5\pi/4] \cup [7\pi/4, 2\pi]} |t + R\cos\theta|^{1+c-\sigma'+\epsilon} + M \right) \\
&\ll_{f,\epsilon} R^{c-\sigma'+\epsilon} \left(\frac{|t|}{R} + 1 \right)^{1+c-\sigma'+\epsilon} + \frac{M}{R},
\end{aligned}
$$

thus the integral on Γ_R vanishes as R tends to ∞. By simple calculations, we find that

$$\begin{aligned}\mathrm{Res}_{\tau=\beta-i\alpha}\frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} &= \lim_{\tau\to\beta-i\alpha}(\tau-\beta+i\alpha)\times\frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} \\ &= \frac{f(s+\alpha+i\beta)}{-2\alpha i}.\end{aligned} \tag{5.12}$$

Hence we obtain (5.4) for all $s\in\mathbb{H}_c$.

Suppose that f has a pole at $s=s_0=\sigma_0+it_0$ and $\sigma_0>c$. Now for $s=\sigma+it\in\mathbb{H}_c$, the integrand has three simple poles: $\tau=\beta+i\alpha$, $\tau=\beta-i\alpha$, and $\tau=i(s-s_0)$. Here we divide the proof into three cases according to the condition whether the pole $\tau=i(s-s_0)$ is below ($c<\sigma<\sigma_0$, see Figures 5.2 and 5.3), above ($\sigma>\sigma_0$, see Figure 5.4), or on the real line ($\sigma=\sigma_0$, see Figure 5.5) of the τ-plane.

<u>Case 1:</u> $\mathrm{Im}(i(s-s_0))<0$.

We first consider when $i(s-s_0)\neq\beta-i\alpha$ and set Γ_R as in Figure 5.2.

Figure 5.2:

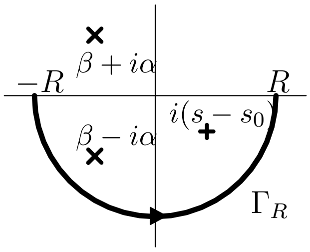
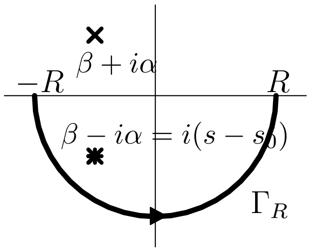


Figure 5.3:



Again by applying Cauchy's integral theorem, we can show that

$$\begin{aligned}l_{\alpha,\beta}(s) &= \frac{\alpha}{\pi}\int_{\mathbb{R}}\frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}d\tau \\ &= -2\alpha i\left(\mathrm{Res}_{\tau=\beta-i\alpha}\frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}+\mathrm{Res}_{\tau=i(s-s_0)}\frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}\right).\end{aligned}$$

Substituting (5.12) into the above, we obtain

$$l_{\alpha,\beta}(s)=f(s+\alpha+i\beta)-2\alpha i\,\mathrm{Res}_{\tau=i(s-s_0)}\frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}.$$

From the Laurent expansion (5.2) of f, we can calculate

$$\frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}=\frac{i}{2\alpha}\left(\sum_{n=0}^{\infty}\frac{(\tau-i(s-s_0))^n}{(\beta+i\alpha-i(s-s_0))^{n+1}}-\sum_{n=0}^{\infty}\frac{(\tau-i(s-s_0))^n}{(\beta-i\alpha-i(s-s_0))^{n+1}}\right)$$

$$\times \sum_{n=-m}^{\infty} a_n i^n (\tau - i(s-s_0))^n.$$

Thus

$$-2\alpha i \operatorname{Res}_{\tau=i(s-s_0)} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}$$
$$= \sum_{n=1}^{m} \frac{a_{-n}}{i^n \left(\beta + i\alpha - i(s-s_0)\right)^n} - \sum_{n=1}^{m} \frac{a_{-n}}{i^n \left(\beta - i\alpha - i(s-s_0)\right)^n}.$$

Thus combining the above calculations and setting

$$B_m(s_0) := \sum_{n=1}^{m} \frac{a_{-n}}{i^n \left(\beta + i\alpha - i(s-s_0)\right)^n} - \sum_{n=1}^{m} \frac{a_{-n}}{i^n \left(\beta - i\alpha - i(s-s_0)\right)^n},$$

we obtain

$$l_{\alpha,\beta}(s) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} d\tau = f(s+\alpha+i\beta) + B_m(s_0).$$

This is the first equation of (5.5).

Now suppose that $i(s-s_0) = \beta - i\alpha$ (see Figure 5.3). This case only appears when $\sigma_0 - \alpha > c$. By calculations similar to the above, we have

$$l_{\alpha,\beta}(s) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} d\tau = -2\alpha i \operatorname{Res}_{\substack{\tau=\beta-i\alpha \\ =i(s-s_0)}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}.$$

We consider the Laurent expansion of the integrand near $\tau = \beta - i\alpha = i(s-s_0)$:

$$\frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} = \left(\sum_{n=-m}^{\infty} a_n i^n (\tau - \beta + i\alpha)^n \right)$$
$$\times \frac{1}{\tau-\beta+i\alpha} \times \frac{1}{-2\alpha i} \times \frac{1}{1-\frac{\tau-\beta+i\alpha}{2\alpha i}}$$
$$= \frac{1}{-2\alpha i} \times \frac{1}{\tau-\beta+i\alpha} \times \frac{1}{1-\frac{\tau-\beta+i\alpha}{2\alpha i}} \sum_{n=-m}^{\infty} a_n i^n (\tau-\beta+i\alpha)^n$$
$$= \frac{1}{-2\alpha i} \frac{1}{\tau-\beta+i\alpha} \sum_{n=0}^{\infty} \left(\frac{\tau-\beta+i\alpha}{2\alpha i} \right)^n \sum_{n=-m}^{\infty} a_n i^n (\tau-\beta+i\alpha)^n.$$

Hence,

$$\operatorname{Res}_{\substack{\tau=\beta-i\alpha \\ =i(s-s_0)}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} = \frac{1}{-2\alpha i} \sum_{n=0}^{m} \frac{a_{-n}}{(-2\alpha)^n}$$

and so we obtain the second equation of (5.5).

Case 2: $\mathrm{Im}(i(s-s_0)) > 0$.

In this case, the integrand of $l_{\alpha,\beta}(s)$ has only one pole in the lower half-plane (see Figure 5.4). Thus by a method similar to the case when f has no pole in $\mathbb{H}_c$,

Figure 5.4:

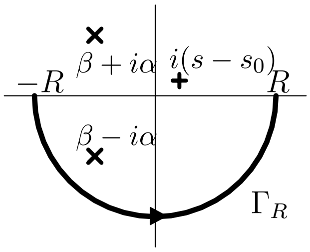
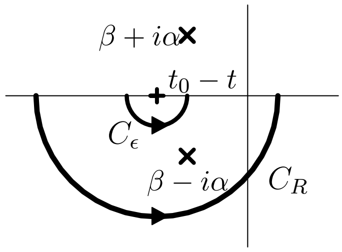


Figure 5.5:



we can show that

$$\int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}d\tau = -2\pi i \,\mathrm{Res}_{\tau=\beta-i\alpha} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2} \overset{(5.12)}{=} -2\pi i \times \frac{f(s+\alpha+i\beta)}{-2\alpha i}$$
$$= \frac{\pi}{\alpha} f(s+\alpha+i\beta).$$

Thus

$$l_{\alpha,\beta}(s) = \frac{\alpha}{\pi}\int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}d\tau = f(s+\alpha+i\beta).$$

This is the third equation of (5.5).

Case 3: $\mathrm{Im}(i(s-s_0)) = 0$ (only for the case $m=1$).

Since $\mathrm{Im}(i(s-s_0)) = 0$ $(s_0 = \sigma_0 + it_0)$, s satisfies $\mathrm{Re}(s) = \sigma_0$ in this case. For convenience, we write $s = \sigma_0 + it$. In this case, we take the principal value of the integrand as in [Ste12, p. 367] and so we obtain

$$\begin{aligned}\int_{\mathbb{R}} \frac{f(\sigma_0+i(t+\tau))}{\alpha^2+(\tau-\beta)^2}d\tau = \lim_{\substack{R\to\infty \\ \epsilon\to 0^+}} \left(\int_{C_R} - \int_{C_\epsilon}\right) \frac{f(\sigma_0+i(t+\tau))}{\alpha^2+(\tau-\beta)^2}d\tau \\ - 2\pi i \,\mathrm{Res}_{\tau=\beta-i\alpha} \frac{f(\sigma_0+i(t+\tau))}{\alpha^2+(\tau-\beta)^2},\end{aligned} \tag{5.13}$$

where C_R and C_ϵ are the counterclockwise oriented semicircles of radius R $(R > 1+|s|+\alpha+|\beta|)$ and ϵ centered at $\tau = t_0 - t$ located in the lower half of the τ-plane (see Figure 5.5).

As the other cases, the integral along C_R vanishes as R tends to ∞. On the other hand, the integral along C_ϵ is evaluated as

$$\begin{aligned}\int_{C_\epsilon} \frac{f(\sigma_0 + i(t+\tau))}{\alpha^2 + (\tau - \beta)^2} d\tau &= \int_\pi^{2\pi} \frac{f(\sigma_0 + i(t_0 + \epsilon e^{i\theta}))}{\alpha^2 + (t_0 - t + \epsilon e^{i\theta} - \beta)^2} i\epsilon e^{i\theta} d\theta \\ &= \int_\pi^{2\pi} \left(\frac{a_{-1}}{i\epsilon e^{i\theta}} + O(1) \right) \frac{i\epsilon e^{i\theta}}{\alpha^2 + (t_0 - t + \epsilon e^{i\theta} - \beta)^2} d\theta\end{aligned}$$

hence

$$\lim_{\epsilon \to 0^+} \int_{C_\epsilon} \frac{f(\sigma_0 + i(t+\tau))}{\alpha^2 + (\tau - \beta)^2} d\tau = \frac{a_{-1}\pi}{\alpha^2 + (t_0 - t - \beta)^2}.$$

Again from (5.12),

$$\operatorname{Res}_{\tau = \beta - i\alpha} \frac{f(\sigma_0 + i(t+\tau))}{\alpha^2 + (\tau - \beta)^2} = \frac{f(\sigma_0 + \alpha + i(t+\beta))}{-2\alpha i}.$$

These imply that (5.6) holds. □

Remark that the method used in Case 3 in the proof does not work if $m > 1$.

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