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On the limit of spectral measures associated to a test configuration of a polarized Kähler manifold

By *Tomoyuki Hisamoto* at Nagoya

Abstract. We apply our integral formula of volumes to the family of graded linear series constructed from any test configuration. This solves the conjecture raised by Witt Nyström to the effect that the sequence of spectral measures for the induced $\mathbb{C}^*$-action on the central fiber converges to the canonical measure defined by the associated weak geodesic ray in the space of Kähler metrics. This limit measure coincides with the classical Duistermaat–Heckmann measure if the test configuration is product. As a consequence, we show that the algebraic p-norm of the test configuration is equal to the L^p-norm of tangent vectors on the geodesic ray. Using this result, we give a natural energy theoretic explanation for the lower bound estimate on the Calabi functional by Donaldson, extending the statement to any p-norm ($p \geqslant 1$), and prove an analogous result for Kähler–Einstein metrics.

1. Introduction

Let X be an n-dimensional smooth projective variety and L an ample line bundle over X. In the sequel we also fix a smooth Hermitian metric h on L, which has strictly positive curvature over X. The curvature form defines a Kähler metric in the first Chern class $c_1(L)$. Conversely, any Kähler metric ω in $c_1(L)$ has a Kähler potential φ in each local trivialization neighborhood such that the collection of $e^{-\varphi}$ defines a Hermitian metric with the curvature form $\omega = dd^c\varphi$, uniquely up to multiplication by a constant. We identify h with the collection of weights φ. We denote by $\mathcal{H}$ the set of all $h = e^{-\varphi}$, endowed with the canonical Riemannian metric

$$\|u\|_2 := \left(\int_X u^2 \frac{(dd^c\varphi)^n}{n!}\right)^{\frac{1}{2}}$$

which is defined for any tangent vector u at φ. The space of Kähler metrics is the natural quotient $\mathcal{H}/\mathbb{R}$. There is the canonical K-energy functional $\mathcal{M} : \mathcal{H} \to \mathbb{R}$ such that any constant scalar curvature Kähler metric is characterized as a critical point of this energy. This K-energy

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is known to be convex along any smooth geodesic in $\mathcal{H}$ and it is important to investigate the gradient of the energy at infinity along a given geodesic ray φ_t ($t \in [0, +\infty)$).

As was first demonstrated by Phong–Sturm, it is possible to define geodesic ray (in a *generalized* sense) in $\mathcal{H}$ in terms of certain degenerations of (X, L), which are now called test configurations. A $\mathbb{C}^*$-equivariant flat family of polarized schemes $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ with the property that $(\mathcal{X}_1, \mathcal{L}_1) = (X, L)$ is called a test configuration. We denote the data by $\mathcal{T}$. For each $k \geqslant 1$ let

$$H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}) = \bigoplus_\lambda V_\lambda$$

be the eigenspace decomposition of the induced $\mathbb{C}^*$-action $\rho : \mathbb{C}^* \to \mathrm{Aut}(H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}))$ on the central fiber such that $\rho(\tau)v = \tau^\lambda v$ holds for every $\tau \in \mathbb{C}^*$ and $v \in V_\lambda$. This λ is just an eigenvalue of $A \in \mathrm{End}(H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}))$ which is defined by $\rho(e^t) = \exp tA$. Then we have the asymptotic expansion

$$\frac{\sum_\lambda \lambda \dim V_\lambda}{k \sum_\lambda \dim V_\lambda} = F_0 + F_1 k^{-1} + O(k^{-2}).$$

We call the minus of the coefficient F_1 in the subleading term as the Donaldson–Futaki invariant of $\mathcal{T}$ and denote it by $\mathrm{DF}(\mathcal{T}) := -F_1$. It was first established in [30] that any test configuration $\mathcal{T}$ with fixed metric φ canonically defines a *weak* geodesic ray φ_t emanating from φ, in $\mathcal{H}$ (and therefore in the space of Kähler metrics). Here for the proof of the main theorem we adopt the construction of [33] so that $\varphi_t - F_0$ gives the geodesic ray in [30]. The technical difficulty now arises from the fact that the constructed metric $dd^c\varphi_t$ for each fixed t degenerates in two senses, that is, it is neither smooth nor strictly positive. In fact $\varphi_t(x)$ is not even a C^2-function, but only $C^{1,\alpha}$ ($\alpha < 1$) with respect to the two variables t and x so that $dd^c\varphi_t$ is defined as a closed semipositive current. Therefore, as opposed to the words "geodesic in the space of Kähler metrics", each $dd^c\varphi_t$ is not a Kähler metric, but a degenerate one. This is why we call φ_t *weak* geodesic ray. In this situation it is now conjectured that the Donaldson–Futaki invariant corresponds to $\lim_{t\to\infty} \frac{d}{dt}\mathcal{M}(\varphi_t)$ if the latter one is properly defined for this non-smooth geodesic ray. In this paper we further relate the asymptotic *distribution* of eigenvalues to φ_t and give some application to the estimate for the Donaldson–Futaki invariant. Our main theorem claims that the associated sequence of spectral measures converges to a certain canonical measure defined by φ_t, which coincides with the Duistermaat–Heckman measure of the $\mathbb{C}^*$-action on the central fiber when $\mathcal{X}$ is product. The Monge–Ampère (or Liouville) measure $\mathrm{MA}(\varphi_t)$ is defined for each singular φ_t and equals $(dd^c\varphi_t)^n$ if φ_t is smooth (see Section 2.1).

Theorem 1.1. *Let $\mathcal{T}$ be a test configuration. Then the weak limit of the normalized distribution of eigenvalues is given by the push-forward of the Monge–Ampère measure* $\mathrm{MA}(\varphi_t)$ *to the real line by the tangent vector $\dot{\varphi}_t$. That is, for any $t \geqslant 0$ we have*

$$\lim_{k\to\infty} \frac{n!}{k^n} \sum_\lambda \delta_{\frac{\lambda}{k}} \dim V_\lambda = (\dot{\varphi}_t)_* \mathrm{MA}(\varphi_t).$$

Here $\delta_{\frac{\lambda}{k}}$ denotes the delta function for $\frac{\lambda}{k} \in \mathbb{R}$. In particular, the right hand side measure is independent not only of t but also of φ, and defines the canonical measure.

Theorem 1.1 was first conjectured in [38] and proved for product test configurations in the same paper. The analogous result for geodesic *segments* which are not necessarily associated to a specific test configuration was obtained by Berndtsson ([7]) in a different approach. In that paper he starts from a geodesic segment φ_t $(t \in [0, t_0])$. The collection of Hermitian inner products on the finite-dimensional vector space $H^0(X, L^{\otimes k})$ has natural structure of finite-dimensional homogenous space and conceptually it provides a good approximation to the space $\mathscr{H}$ if one lets $k \to \infty$. In fact the Bergman kernel construction shows that any geodesic segment in $\mathscr{H}$ can be approximated by the sequence of finite-dimensional geodesics, which are determined by the initial point and the corresponding Lie vector field $A_k \in \operatorname{End}(H^0(X, L^{\otimes k}))$. Then the result of [7] states that the spectral measure associated to the eigenvalues of A_k converges to a canonical measure defined by the initial geodesic segment φ_t. In this paper we start from a test configuration and investigate the situation $t_0 \to \infty$ relating it with the degeneration of (X, L).

Recall that the above definition of $\mathrm{DF}(\mathcal{T})$ was motivated by the equivariant Riemann–Roch formula of [1], which can be applied to the product test configuration, and in that case one has the Duistermaat–Heckman measure on the central fiber in the usual way. In terms of geodesic the central fiber corresponds to $t = \infty$ and our canonical measure which is independent of t gives the right generalization to any test configuration. Then Theorem 1.1 can be seen as a part of the *ideal* index theorem for an equivariant family which admits a very singular fiber over the fixed point $0 \in \mathbb{C}$. Taking the p-th moment of the above measure, we may extend the definition of algebraic norm in [16] to any $p \geqslant 1$ and relate it to the L^p-norm of tangent vectors on the weak geodesic ray.

Theorem 1.2. *Let us define the trace-free part of each eigenvalue* λ *as*

$$\bar{\lambda} := \lambda - \frac{\sum_\mu \mu \dim V_\mu}{\sum \dim V_\mu}$$

and for each $p \geqslant 1$ *define the* p*-norm* $\|\mathcal{T}\|_p$ *by*

$$\|\mathcal{T}\|_p := \left(\lim_{k\to\infty} \frac{1}{k^n} \sum_\lambda \left|\frac{\bar{\lambda}}{k}\right|^p \dim V_\lambda\right)^{\frac{1}{p}}.$$

Then the limit exists and

$$\|\mathcal{T}\|_p = \left(\int_X |\dot{\varphi}_t - F_0|^p \frac{\mathrm{MA}(\varphi_t)}{n!}\right)^{\frac{1}{p}}$$

holds.

Using Theorem 1.2, we may give an energy theoretic explanation for the lower bound estimate in [16] on the Calabi functional, extending the result to any p-norm $(p \geqslant 1)$. In particular in the Fano case we may justify the idea to obtain the following. Note that when $L = -K_X$, any metric $h = e^{-\varphi}$ can be identified with the positive measure which is described as $e^{-\varphi} \bigwedge_{i=1}^n \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_i$ in each local coordinate. A metric $e^{-\varphi}$ is called a Kähler–Einstein metric if it satisfies the following identity of the measures:

$$(dd^c\varphi)^n = n! e^{-\varphi}.$$

Theorem 1.3. *Let X be a Fano manifold and let $\mathcal{T}$ be a test configuration of $(X, -K_X)$. Then for any smooth Hermitian metric $h = e^{-\varphi}$ on $-K_X$ and exponents $1 \leqslant p, q \leqslant +\infty$ with $1/p + 1/q = 1$ we have*

$$\left\| \frac{n! e^{-\varphi}}{(dd^c\varphi)^n} - 1 \right\|_q \geqslant -\frac{\mathrm{DF}(\mathcal{T})}{\|\mathcal{T}\|_p}.$$

In other word, the deviation from being Kähler–Einstein metric is bounded from below by the Donaldson–Futaki invariant.

Let us briefly explain the outline of our proof of Theorem 1.1. The proof is based on the analytic study for graded linear series, which was exploited in [18]. We apply it to [38]'s family of graded subalgebras

$$W_\lambda = \bigoplus_{k=0}^{\infty} W_{\lambda,k} \subseteq \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}),$$

which is parameterized by $\lambda \in \mathbb{R}$ and constructed from $\mathcal{T}$ as follows. For a given section $s \in H^0(X, L^{\otimes k})$, let us denote its unique invariant extension which is at least meromorphic over $\mathcal{X}$ by $\bar{s}$. We define $W_{\lambda,k}$ as the set of sections s whose invariant extensions $\bar{s}$ have poles along the central fiber $\mathcal{X}_0 = \{t = 0\}$ with order at most $-\lceil \lambda k \rceil$ (here $\lceil \cdot \rceil$ denotes the smallest integer which is greater than $\cdot$). In other words,

$$W_{\lambda,k} := \left\{ s \in H^0(X, L^{\otimes k}) \;\middle|\; t^{-\lceil \lambda k \rceil} \bar{s} \in H^0(\mathcal{X}, \mathcal{L}) \right\}.$$

Then it can be proved algebraically that the limit of spectral measures is given by the Lebesgue–Stieltjes measure of the volume function $\mathrm{vol}(W_\lambda)$ in λ. The main theorem of [18] interprets each volume into the Monge–Ampère measure of associated equilibrium metric $P_{W_\lambda}\varphi$. The Legendre transformation of this family of equilibrium metrics is nothing but the weak geodesic ray φ_t so that we may complete the proof by the recently developed techniques of pluripotential theory. This new approach via the family of graded linear series seems itself interesting and we hope it should be studied more in the future.

2. Analytic description of the volume

2.1. Monge–Ampère operator. In this subsection, we briefly review the definition and basic properties of the Monge–Ampère operator. Let L be a holomorphic line bundle on a projective manifold X. We usually fix a family of local trivialization patches U_α which cover X. A singular Hermitian metric h on L is by definition a family of functions $h_\alpha = e^{-\varphi_\alpha}$ which are defined on corresponding U_α and satisfy the transition rule $\varphi_\beta = \varphi_\alpha - \log|g_{\alpha\beta}|^2$ on $U_\alpha \cap U_\beta$. Here $g_{\alpha\beta}$ are the transition functions of L with respect to the indices α and β. *The weight functions* φ_α are assumed to be locally integrable. If the functions φ_α are smooth, then $\{e^{-\varphi_\alpha}\}_\alpha$ defines a smooth Hermitian metric on L. We usually denote the family $\{\varphi_\alpha\}_\alpha$ by φ and omit the indices of local trivializations. Notice that each $\varphi = \varphi_\alpha$ is only a local function and not globally defined, but the curvature current $\Theta_h = dd^c\varphi$ is globally defined and is semipositive if and only if each function φ is plurisubharmonic (*psh* for short). Here we denote by d^c the real differential operator $\frac{\partial - \bar{\partial}}{4\pi\sqrt{-1}}$. We call such a weight a *psh weight*. The most important example is those of the form $k^{-1}\log(|s_1|^2+\cdots+|s_N|^2)$, defined by some holomorphic sections $s_1,\dots,s_N \in H^0(X, L^{\otimes k})$. Here $|s_i|$ $(1 \leqslant i \leqslant N)$ denotes the absolute value of the corresponding function of each s_i on U_α. We call such weights *(globally) algebraic singular*. More generally, a psh weight φ is said to have a *small unbounded locus* if the pluripolar set $\varphi^{-1}(-\infty)$ is contained in some closed complete pluripolar subset $S \subset X$ (e.g. a proper algebraic subset).

Let n be the dimension of X. The Monge–Ampère operator is defined by

$$\varphi \mapsto (dd^c\varphi)^n$$

when φ is smooth. On the other hand it does not make sense for general φ. The celebrated result of Bedford–Taylor [2] tells us that the right hand side can be defined as a current if φ is at least in the class $L^\infty \cap \mathrm{PSH}(U_\alpha)$. Specifically, by induction on the exponent $q = 1, 2, \dots, n$, it can be defined as

$$\int_{U_\alpha} (dd^c\varphi)^q \wedge \eta := \int_{U_\alpha} \varphi (dd^c\varphi)^{q-1} \wedge dd^c\eta$$

for each test form η. Here $\int$ denotes the canonical pairing of currents and test forms. This is indeed well-defined and defines a closed positive current, because φ is a bounded Borel function and $(dd^c\varphi)^{q-1}$ has measure coefficients by the induction hypothesis. Notice the fact that any closed positive current has measure coefficients.

It is also necessary to consider unbounded psh weights. On the other hand, for our purpose, it is enough to deal with weights with small unbounded loci.

Definition 2.1. Let φ be a psh weight of a singular metric on L. If φ has a small unbounded locus, we define a positive measure $\mathrm{MA}(\varphi)$ on X by

$$\mathrm{MA}(\varphi) := \text{the zero extension of } (dd^c\varphi)^n.$$

Note that the coefficient of $(dd^c\varphi)^n$ is well-defined as a measure on $X \setminus S$.

Actually $(dd^c\varphi)^n$ has a finite mass so that $\mathrm{MA}(\varphi)$ defines a closed positive current on X. For a proof, see [11, Section 1].

Remark 2.1. In [11], the *non-pluripolar* Monge–Ampère product was defined in fact for general psh weights on a compact Kähler manifold. Note that this definition of the Monge–Ampère operator makes the measure $\mathrm{MA}(\varphi)$ to have no mass on any pluripolar set. In other words, $\mathrm{MA}(\varphi)$ ignores the mass which comes from the singularities of φ. For this reason, as a measure-valued function in φ, $\mathrm{MA}(\varphi)$ no longer has the continuous property which holds for bounded psh functions ([21, Theorem 1.11, Proposition 1.12, Theorem 1.15]). For example, if L is ample, it is always possible to find a non-increasing sequence φ_k of smooth psh weights on L with $\varphi_k \to \varphi$, but $\mathrm{MA}(\varphi_k) \to \mathrm{MA}(\varphi)$ fails as soon as $\int \mathrm{MA}(\varphi) < \int \mathrm{MA}(\varphi_k) = L^n$.

We recall a fundamental fact established in [11] which states that the less singular psh weight has the larger Monge–Ampère mass. Recall that given two psh weights φ and φ' on L, φ is said to be *less singular than* φ' if there exists a constant $C > 0$ such that $\varphi' \leqslant \varphi + C$ holds on X. We say that a psh weight has *minimal singularities* if it is minimal with respect to this partial order. When φ is less singular than φ' and φ' is less singular than φ, we say that the two functions have *equivalent singularities*. This defines an equivalence relation.

Theorem 2.1 ([11, Theorem 1.16]). *Let φ and φ' be psh weights with small unbounded loci such that φ is less singular than φ'. Then*

$$\int_X \mathrm{MA}(\varphi') \leqslant \int_X \mathrm{MA}(\varphi)$$

holds.

2.2. Analytic representation of volume. Let X be an n-dimensional smooth complex projective variety and let L be a holomorphic line bundle on X. Graded linear series is by definition a graded $\mathbb{C}$-subalgebra of the section ring

$$W = \bigoplus_{k=0}^{\infty} W_k \subseteq \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}).$$

They appear in many geometric situations. In fact in the present paper we give an application of the analysis of such proper subalgebras to the problem of constant scalar curvature Kähler metric. The volume of graded linear series is the nonnegative real number which measures the size of the graded linear series as follows:

$$\mathrm{vol}(W) := \limsup_{k\to\infty} \frac{\dim W_k}{\frac{k^n}{n!}}.$$

This is finite and in fact the limit of supremum is limit provided $W_k \neq 0$ for sufficiently large k, by the result of [20]. The main result of [18] gives an analytic description of the volume. The analytic counterpart of the volume is the following generalized equilibrium metric, which originates from [3].

Definition 2.2. Let W be a graded linear series of a line bundle L. Fix a smooth Hermitian metric of L and denote it by $h = e^{-\varphi}$, where φ is the weight function defined on a fixed local trivialization neighborhood. We define the equilibrium weight associated to W and φ by

$$P_W\varphi := \sup{}^{*}\left\{ \frac{1}{k}\log|s|^2 \;\middle|\; k \geqslant 1,\ s \in W_k \text{ such that } |s|^2 e^{-k\varphi} \leqslant 1 \right\}.$$

Here $*$ denotes taking the upper semicontinuous regularization of the function. The equilibrium weight $P_W\varphi$ on each local trivialization neighborhood patches together and defines a singular Hermitian metric on L. We call it the equilibrium metric.

As in Section 2.1, we define the Monge–Ampère measure $\mathrm{MA}(P_W\varphi)$ on X.

Theorem 2.2 ([18, Main Theorem]). *Let W be a graded linear series of a line bundle L such that the natural map $X \dashrightarrow \mathbb{P}W_k^*$ is birational onto its image for any sufficiently large k. Then for any fixed smooth Hermitian metric $h = e^{-\varphi}$ we have*

$$\mathrm{vol}(W) = \int_X \mathrm{MA}(P_W\varphi).$$

Note that Theorem 2.2 is valid for general big line bundle which is possibly not ample. We will apply this general formula to the special graded linear series associated to a test configuration of a polarized manifold.

Remark 2.2. With no change of the proof in [18], Theorem 2.2 can also be proved under the assumption W_k is birational onto its image for any sufficiently *divisible* k. For non-complete linear series, the condition $\mathrm{vol}(W) > 0$ only implies that $X \dashrightarrow \mathbb{P}W_k^*$ is generically finite (but not birational in general) onto its image for sufficiently divisible k. For example, when W is defined as the pull-back of $H^0(Y, \mathcal{O}(k))$ by a finite morphism $X \to Y \subseteq \mathbb{P}^N$, $\mathrm{vol}(W) > 0$ holds but W_k never defines a birational map onto its image for any k. For this reason, neither does Theorem 2.2 hold for general W with $\mathrm{vol}(W) > 0$. To be precise, taking a resolution μ_k of the base ideal of W_k and denoting the fixed component of $\mu_k^* W_k$ by F_k, the right hand side in Theorem 2.2 is given by the limit of self-intersection number of line bundles $M_k := \mu_k^* L^{\otimes k} \otimes \mathcal{O}(-F_k)$.

3. Test configuration and associated family of graded linear series

In this section we explain the construction of the family of graded linear series W_λ parametrized by $\lambda \in \mathbb{R}$ from fixed test configuration $(\mathcal{X}, \mathcal{L})$, following the recipe of Witt Nyström's paper [38]. First we introduce the notion of K-stability.

3.1. K-stability.

We start with the following definition.

Definition 3.1 (Definition of test configuration by [15]). Let (X, L) be a polarized manifold. We call the following data a *test configuration* (resp. *semi-test configuration*) for (X, L):

(1) a flat family of schemes with relatively ample (resp. semiample and big) $\mathbb{Q}$-line bundle $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ such that $(\mathcal{X}_1, \mathcal{L}_1) \simeq (X, L)$ holds (the relatively bigness automatically follows from this assumption),

(2) a $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ which makes π equivariant, with respect to the canonical action of $\mathbb{C}^*$ on the target space $\mathbb{C}$.

Remark 3.1. As was pointed out in [23], the above original definition by Donaldson should be a bit modified. For example, if one further assumes $\mathcal{X}$ is normal, then the pathological example in [23] can be removed. On the other hand, the recent paper [36] proposed to consider the class of test configurations whose *norms* $\|\mathcal{T}\|$ are non-zero and this condition seems more natural and appropriate from our viewpoint. In fact Theorem 1.2 gives one evidence. See also [33]. At any rate we do not assume the normality of $\mathcal{X}$ in proving Theorems 1.1 and 1.2.

By the flatness of π, the Hilbert polynomials of $(\mathcal{X}_t, \mathcal{L}_t)$ are independent of $t \in \mathbb{C}$. The $\mathbb{C}^*$-equivariance yields an isomorphism $(\mathcal{X}_t, \mathcal{L}_t) \simeq (X, L)$ for any $t \in \mathbb{C} \setminus \{0\}$. Note that the central fiber $(\mathcal{X}_0, \mathcal{L}_0)$ can be very singular. It is even not reduced in general. A test configuration is said to be *product* if $\mathcal{X} \simeq X \times \mathbb{C}$ and *trivial* if further the action of $\mathbb{C}^*$ on $X \times \mathbb{C}$ is trivial. A test configuration $(\mathcal{X}, \mathcal{L})$ induces the $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k})$ for each $k \geqslant 1$. This action $\rho : \mathbb{C}^* \to \mathrm{Aut}(H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}))$ decomposes the vector space as

$$H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}) = \bigoplus_\lambda V_\lambda$$

such that $\rho(\tau)v = \tau^\lambda v$ holds for any $v \in V_\lambda$ and $\tau \in \mathbb{C}^*$. By the equivariant Riemann–Roch Theorem, the total weight $w(k) = \sum_\lambda \lambda \dim V_\lambda$ is a polynomial of degree $n+1$. Let us denote the coefficients by

$$w(k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}). \tag{3.1}$$

We also write the Hilbert polynomial of (X, L) by

$$N_k := \dim H^0(X, L^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}).$$

The Donaldson–Futaki invariant of a given test configuration is defined to be the minus of the subleading term in the expansion

$$\frac{w(k)}{kN_k} = F_0 + F_1 k^{-1} + O(k^{-2}). \tag{3.2}$$

In other word,

$$\mathrm{DF}(\mathcal{T}) := -F_1 = -\frac{a_0 b_1 - a_1 b_0}{a_0^2}. \tag{3.3}$$

Definition 3.2. A polarization (X, L) is *K-stable* (resp. *K-semistable*) if $\mathrm{DF}(\mathcal{T}) > 0$ (resp. $\mathrm{DF}(\mathcal{T}) \geqslant 0$) holds for any non-trivial test configuration. We say (X, L) is *K-polystable* if it is K-semistable and $\mathrm{DF}(\mathcal{T}) = 0$ holds only for product test configurations.

This notion of K-stability was first introduced in [37]. The above algebraic definition was given in [15]. Note that K-stability is unchanged if one replaces L to $L^{\otimes k}$ since F_1 is so. The equivalence of certain GIT-stability and existence of special metric originate from the Kobayashi–Hitchin correspondence for vector bundles. In the polarized manifolds case, we have the following conjecture.

Conjecture 3.1 (Yau–Tian–Donaldson). *A polarized manifold (X, L) admits a cscK metric if and only if it is K-polystable.*

One direction of the above conjecture was proved in [16,24,25,35]. That is, the existence of cscK metric implies K-polystability of the polarized manifold. See also [5] for the detail study in the Kähler–Einstein case.

The stability of a vector bundle is defined in terms of the *slope* of subbundles and to pursue the analogy to the vector bundle case, [32] studied the special type of test configurations which are defined by subschemes of X, and introduced the slope of a subscheme.

Example 3.1. A pair of an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ and an appropriate $c \in \mathbb{Q}$ define a test configuration as follows. Such a test configuration is called *deformation to the normal cone* with respect to $(\mathcal{J}, c)$: Let $\mathcal{X}$ be the blow-up of $X \times \mathbb{C}$ along $\mathcal{J}$ and let P be the exceptional divisor. The action of $\mathbb{C}^*$ on $X \times \mathbb{C}$ fixes $V(\mathcal{J})$ so that it induces actions on $\mathcal{X}$ and P. We denote the composition of the blow-down $\mathcal{X} \to X \times \mathbb{C}$ and the projection to X by $p : \mathcal{X} \to X$. Let us define the $\mathbb{Q}$-line bundle $\mathcal{L}_c$ on X by $\mathcal{L}_c := p^* L \otimes \mathcal{O}(-cP)$. When $V = V(\mathcal{J})$ is smooth, then P is a compactification of the normal bundle $N_{V/X}$. Let us denote the blow-up along $\mathcal{J}$ by $\mu : X' \to X$ and the exceptional divisor by E. The Seshadri constant of L along $\mathcal{J}$ is defined by

$$\varepsilon(L, \mathcal{J}) := \sup\{c \mid \mu^* L \otimes \mathcal{O}(-cE) \text{ is ample}\}.$$

Then we have the following lemma so that $(\mathcal{X}, \mathcal{L}_c)$ actually defines a test configuration for any sufficiently small c.

Lemma 3.1 ([32, Lemma 4.19]). *For any* $0 < c < \varepsilon(L, \mathcal{J})$, $\mathcal{L}_c$ *is a* π*-ample* $\mathbb{Q}$*-line bundle.*

The slope theory of [32] was further developed in [26]. Consider a flag of ideal sheaves $\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_{N-1} \subseteq \mathcal{O}_X$ and fix $c \in \mathbb{Q}_{>0}$. Let us take the blow up $\mathcal{X}$ of $X \times \mathbb{C}$ along the $\mathbb{C}^*$-invariant ideal sheaf

$$\mathcal{J} := \mathcal{J}_0 + t\mathcal{J}_1 + \cdots + t^{N-1}\mathcal{J}_{N-1} + (t^N)$$

and denote the exceptional divisor by P. Let us denote the projection map by $p : \mathcal{X} \to X$. Then $\mathcal{X}$ naturally admits a line bundle $\mathcal{L} := p^*L \otimes \mathcal{O}(-cP)$. In the paper [26], Odaka derived the intersection number formula of the Donaldson–Futaki invariant for this type of semi-test configuration defined by flag ideals. The point is that any test configuration whose total space $\mathcal{X}$ is normal can be dominated by the above type of *semi*-test configuration.

Proposition 3.1 ([26, Proposition 3.10]). *For an arbitrary normal test configuration* $\mathcal{T}$*, there exist a flag of ideal sheaves* $\mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_{N-1} \subseteq \mathcal{O}_X$ *and a rational number* $c \in \mathbb{Q}_{>0}$ *such that* $\mathcal{T}' = (\mathcal{X}', \mathcal{L}')$ *defined by the flag is a semi-test configuration which dominates* $\mathcal{T}$ *by a morphism* $f : \mathcal{X}' \to \mathcal{X}$ *with* $\mathcal{L}' = f^*\mathcal{L}$*. Moreover,* $\mathrm{DF}(\mathcal{T}') = \mathrm{DF}(\mathcal{T})$ *holds.*

3.2. The associated family of graded linear series. Let us denote the $\mathbb{C}^*$-action on the test configuration $(\mathcal{X}, \mathcal{L})$ by $\rho : \mathbb{C}^* \to \mathrm{Aut}(\mathcal{X}, \mathcal{L})$. For any $s \in H^0(X, L^{\otimes k})$, it naturally defines an invariant section $\bar{s} \in H^0(\mathcal{X}_{t \neq 0}, \mathcal{L}^{\otimes k})$ by

$$\bar{s}(\rho(\tau)x) := \rho(\tau)s(x) \quad (\tau \in \mathbb{C}^*,\ x \in \mathcal{X}_{t \neq 0}).$$

If we set t as the parameter of underlying space $\mathbb{C}$, for any $\lambda \in \mathbb{Z}$, $t^{-\lambda}\bar{s}$ defines a meromorphic section of $\mathcal{L}^{\otimes k}$ over $\mathcal{X}$. We then introduce the following filtration to measure the order of these meromorphic sections along the central fiber.

Definition 3.3. Fix a test configuration $(\mathcal{X}, \mathcal{L})$. For each $\lambda \in \mathbb{R}$, we define the subspace of $H^0(X, L^{\otimes k})$ by

$$\mathcal{F}_\lambda H^0(X, L^{\otimes k}) := \{s \in H^0(X, L^{\otimes k}) \mid t^{-\lceil \lambda \rceil}\bar{s} \in H^0(\mathcal{X}, \mathcal{L})\}. \tag{3.4}$$

By definition, we have $(\rho(\tau)s)(x) = \rho(\tau)s(\rho^{-1}(\tau)(x))$ so it holds

$$(\rho(\tau)\bar{s})(x) = \rho(\tau)\bar{s}(\rho^{-1}(\tau))(x) = \bar{s}(x),$$

i.e. $\bar{s}$ is invariant under the $\mathbb{C}^*$-action. On the other hand, regarding t as the section of $\mathcal{O}_{\mathcal{X}}$, we have

$$(\rho(\tau)t)(x) = \rho(\tau)t(\rho^{-1}(\tau)x) = \rho(\tau)(\tau^{-1}t(x)) = \tau^{-1}t(x).$$

Therefore $t^{-\lceil \lambda \rceil}\bar{s}$ is an eigenvector of weight $\lceil \lambda \rceil$ with respect to the $\mathbb{C}^*$-action. Note that the filtration is multiplicative, i.e.

$$\mathcal{F}_\lambda H^0(X, L^{\otimes k}) \cdot \mathcal{F}_{\lambda'} H^0(X, L^{k'}) \subset \mathcal{F}_{\lambda+\lambda'} H^0(X, L^{k+k'})$$

holds for any $\lambda, \lambda' \in \mathbb{R}$ and $k, k' \geqslant 0$. The relation to the weight of the action on the central fiber is given by the following proposition.

Proposition 3.2. *Let us denote the weight decomposition of the $\mathbb{C}^*$-action by*

$$H^0(\mathcal{X}_0, \mathcal{L}_{\mathcal{X}_0}) = \bigoplus_{\lambda} V_\lambda.$$

Then, for any $\lambda \in \mathbb{R}$, we have

$$\dim \mathcal{F}_\lambda H^0(X, L^{\otimes k}) = \sum_{\lambda' \geqslant \lambda} \dim V_{\lambda'}. \tag{3.5}$$

Note that every weight is actually an integer so that each side of (3.5) is unchanged if one replaces λ by $\lceil \lambda \rceil$. A fundamental fact established in [30] is that this filtration is actually linearly bounded in the following sense.

Lemma 3.2 ([30, Lemma 4]). *For any test configuration $(\mathcal{X}, \mathcal{L})$ there exists a constant $C > 0$ such that for any $k \geqslant 1$ and λ with $\dim V_\lambda > 0$, one has $|\lambda| \leqslant Ck$.*

In other words, there exists a constant $C > 0$ such that

$$\mathcal{F}_{-Ck} H^0(X, L^{\otimes k}) = H^0(X, L^{\otimes k}) \quad \text{and} \quad \mathcal{F}_{Ck} H^0(X, L^{\otimes k}) = \{0\}$$

hold for every $k \geqslant 1$.

Definition 3.4. We set

$$\lambda_0 := \sup\{\lambda \mid \mathcal{F}_{\lambda k} H^0(X, L^{\otimes k}) = H^0(X, L^{\otimes k}) \text{ for any } k \geqslant 1\},$$
$$\lambda_c := \inf\{\lambda \mid \mathcal{F}_{\lambda k} H^0(X, L^{\otimes k}) = \{0\} \text{ for any } k \geqslant 1\}.$$

By Lemma 3.2, λ_0 and λ_c are both finite. Lemma 3.2 indicates us to consider the graded linear series

$$W_\lambda = \bigoplus_{k=0}^{\infty} W_{\lambda,k} := \bigoplus_{k=0}^{\infty} \mathcal{F}_{\lambda k} H^0(X, L^{\otimes k}). \tag{3.6}$$

For each $\lambda \in \mathbb{R}$. It was shown in [36] that this family contains enough information about the original test configuration. A result of [38] in fact gives the explicit formula for b_0.

Theorem 3.1 (Reformulation of [38, Corollary 6.6]). *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration. Then the quantity b_0 is obtained by the Lebesgue–Stieltjes integral of λ with respect to* $\mathrm{vol}(W_\lambda)$. *That is,*

$$n! b_0 = -\int_{-\infty}^{\infty} \lambda d(\mathrm{vol}(W_\lambda)).$$

Theorem 3.1 actually follows from [38, Corollary 6.6] by change of variables in integration. See also the proof of [10, Theorem 1.10]. Note that the concave function $G[\mathcal{T}]$ on the Okounkov body $\Delta(L)$ in [38] is determined by the property

$$G[\mathcal{T}]^{-1}([\lambda, \infty)) = \Delta(W_\lambda),$$

where $\Delta(W_\lambda) \subset \mathbb{R}^n$ is the Okounkov body of W_λ in the sense of [22, Definition 1.15] and that $n!$ times the Euclidean volume $\mathrm{vol}(\Delta(W_\lambda))$ equals $\mathrm{vol}(W_\lambda)$. Here, however, we give a self-contained proof of the above theorem, for the reader's convenience. The proof is essentially the same but we do not use Okounkov body as in [38] or [10].

Set the counting function of weights as

$$f(\lambda) = f_k(\lambda) := \sum_{\lambda' \geqslant \lambda} \dim V_{\lambda'} = \dim \mathcal{F}_\lambda H^0(X, L^{\otimes k}). \tag{3.7}$$

It is easy to show that $f_k(\lambda)$ is actually a left-continuous and non-increasing function. Hence the Lebesgue–Stieltjes integral makes sense and

$$w(k) := \sum_\lambda \lambda \dim V_\lambda = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}) = -\int_{-\infty}^{\infty} \lambda df(\lambda) = -\int_{-\infty}^{\infty} k\lambda df(k\lambda)$$

hold for any k. For any small $\varepsilon > 0$, integration by parts yields

$$-\int_{-\infty}^{\infty} k\lambda df(k\lambda) = -[k\lambda f(k\lambda)]_{\lambda_0-\varepsilon}^{\infty} + \int_{\lambda_0-\varepsilon}^{\infty} kf(k\lambda)d\lambda.$$

By the definition of the volume we have

$$\limsup_{k\to\infty} \frac{f(k\lambda)}{\frac{k^n}{n!}} = \mathrm{vol}(W_\lambda).$$

If $\mathrm{vol}(W_\lambda) > 0$, the limit of supremum is in fact limit for k sufficiently divisible, by [20, Theorem 4] (or by the proofs of [14, Theorem 3.10, Corollary 3.11 and Lemma 3.2]). Therefore the Dominated Convergence Theorem concludes

$$n!b_0 = (\lambda_0 - \varepsilon)L^n + \int_{\lambda_0-\varepsilon}^{\infty} \mathrm{vol}(W_\lambda)d\lambda.$$

Thus we obtain Theorem 3.1.

We remark that one advantage to consider such graded linear series is to avoid the difficulty coming from the singularity of the central fiber $\mathcal{X}_0$. On the other hand, we have to treat with the difficulty coming from the non-completeness of linear series in this setting.

Example 3.2. Let $(\mathcal{X}, \mathcal{L})$ be the test configuration defined by an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ and $c \in \mathbb{Q}$ as in Example 3.1. Then the associated W_λ are computed to be

$$W_{\lambda,k} = \begin{cases} H^0(X, L^{\otimes k}), & \lambda \leqslant -c, \\ H^0(X, L^{\otimes k} \otimes \mathcal{J}^{\lceil \lambda k \rceil + ck}), & -c < \lambda \leqslant 0, \\ \{0\}, & \lambda > 0, \end{cases}$$

for any k. As a result, we have

$$n!b_0 = -cL^n + \int_0^c (\mu^* L \otimes \mathcal{O}(-\lambda E))^n d\lambda.$$

4. Study of weak geodesic rays

In this section we apply Theorem 2.2 to each W_λ constructed from the test configuration to study the associated weak geodesic ray.

4.1. Construction of weak geodesic. One of the guiding principles to the existence problem of constant scalar curvature Kähler metrics is to study the Riemannian geometry on the space of Kähler metrics in the first Chern class of L. A result of Phong and Sturm (in [30]) gives a milestone in this direction. They showed that a test configuration canonically defines a weak geodesic ray emanating from any fixed point φ in the space of Kähler metrics. This builds a bridge between the algebraic definition of K-stability and the analytic setting where the cscK metrics live. Later it was shown by [33] that one can also define the same weak geodesic via the associated family of graded linear series $\{W_\lambda\}$. Let us now recall their construction. Throughout this subsection we fix a smooth strictly psh weight φ. It will be shown that φ and the family of graded linear series $\{W_\lambda\}$ canonically define the weak geodesic emanating from φ.

Recall that a family of psh weights ψ_t $(a < t < b)$ is called a *weak geodesic* (resp. *weak sub-geodesic*) if

$$\Psi(x,\tau) := \psi_{-\log|\tau|}(x) \quad (\tau \in \mathbb{C},\, e^{-b} < |\tau| < e^{-a})$$

is locally bounded plurisubharmonic and satisfies the Monge–Ampère equation

$$\mathrm{MA}(\Psi) = 0 \quad (\text{resp.} \geqslant 0).$$

Here we consider $\Psi(x,\tau)$ as the function of $(n+1)$ variables and the Monge–Ampère operator is defined in the manner of Bedford–Taylor. (The product space is not compact but Bedford–Taylor's Monge–Ampère operator is well-defined for those plurisubharmonic weights just as was explained in Section 2.) Note also that $\mathrm{MA}(\Psi) \geqslant 0$ is automatically satisfied since Ψ is plurisubharmonic. When each $dd^c\varphi_t$ is a smooth Kähler metric, there is the canonical Riemannian metric which is defined for a tangent vector u at φ_t by

$$\|u\|^2 := \int_X u^2 \frac{\mathrm{MA}(\varphi_t)}{n!}.$$

By [34], it is known that $\mathrm{MA}(\Psi) = 0$ if and only if the geodesic curvature for this metric is zero.

First note that given a test configuration $(\mathcal{X}, \mathcal{L})$, the associated family $\{W_\lambda\}$ defines the family of equilibrium weights $P_{W_\lambda}\varphi$. From now on let us set

$$\psi_\lambda := P_{W_\lambda}\varphi.$$

The first easy observation is that ψ_λ is decreasing with respect to λ. As a consequence of Demailly's Bergman approximation argument and Lemma 3.2, we have

$$\psi_\lambda = \varphi \quad \text{if } \lambda < \lambda_0 \quad \text{and} \quad \lambda_c = \inf\{\lambda \mid \psi_\lambda = -\infty\}.$$

Further by the multiplicativity of $\mathcal{F}_\lambda H^0(X, L^{\otimes k})$ one can see that ψ_λ is concave with respect to λ. The main result of [33] states that the Legendre transform of ψ_λ defines a weak geodesic ray.

Theorem 4.1 ([33, Theorems 1.1, 1.2 and 9.2]). *Set the Legendre transform of ψ_λ by*

$$\varphi_t := \sup{}^*\{\psi_\lambda + t\lambda \mid \lambda \in \mathbb{R}\} \quad \text{for } t \in [0, +\infty). \tag{4.1}$$

Then φ_t defines a weak geodesic ray emanating from φ. Moreover, $\varphi_t - F_0$ coincides with the weak geodesic ray constructed in [30]*, which is known to have $C^{1,\alpha}$-regularity (for an arbitrary* $0 < \alpha < 1$*) in two variables t and x by the main result of* [31].

It is immediate to show that φ_t is a bounded psh weight emanating from φ and that it is convex with respect to t. The geodecity is derived from the maximality of $P_{W_\lambda}\varphi$:

$$\psi_\lambda = \varphi \quad \text{a.e. with respect to } \mathrm{MA}(\psi_\lambda). \tag{4.2}$$

One of the technical points in [33] is to show (4.2). Such a property is due to the fact that ψ_λ is defined as the upper envelopes of sufficiently many algebraic weights.

Note that the inverse Legendre transform maps φ_t to ψ_λ by

$$\psi_\lambda = \inf_{t \geqslant 0}\{\varphi_t - t\lambda\} \tag{4.3}$$

which holds for *every* point of X. This is a consequence of Kiselman's minimum principle for plurisubharmonic functions (see [33, Remark 6.4]). Therefore the two curves contain an equivalent information. Moreover, by the $C^{1,\alpha}$-regularity in t and x, the above infimum is always achieved and therefore ψ_λ is a continuous function in x and λ. The time derivative $\dot{\varphi_t}(x)$ is defined for every $x \in X$. We identify this derivative with the tangent vector of the weak geodesic. Then the gradient map relation

$$-\psi_\lambda(x) + \varphi_t(x) = t\lambda \tag{4.4}$$

holds everywhere if one sets $\lambda := \dot{\varphi}_t(x)$.

4.2. Proof of Theorem 1.1. We prove Theorem 1.1. It was shown in [38] that the push-forward of the Lebesgue measure by the concave function $G[\mathcal{T}]$ on the Okounkov body $\Delta(L)$ gives the weak limit, namely,

$$\lim_{k\to\infty} \frac{n!}{k^n} \sum_\lambda \delta_{\frac{k}{\lambda}} \dim V_\lambda = n! G[\mathcal{T}]_*(d\lambda|_{\Delta(L)}). \tag{4.5}$$

Recall that $G[\mathcal{T}]$ is characterized by the property

$$G[\mathcal{T}]^{-1}([\lambda, \infty)) = \Delta(W_\lambda),$$

where $\Delta(W_\lambda) \subseteq \mathbb{R}^n$ is the Okounkov body of W_λ in the sense of [22, Definition 1.15] and $n!$ times the Euclidean volume $\mathrm{vol}(\Delta(W_\lambda))$ gives $\mathrm{vol}(W_\lambda)$. Therefore it is easy to observe that the right hand side of (4.5) equals $-d(\mathrm{vol}(W_\lambda))$.

Next we apply Theorem 2.2 to W_λ. This is possible thanks to the lemma of Boucksom–Chen ([10]). They in fact proved that the linear series W_λ *contains an ample series* for $\lambda < \lambda_c$ in their terminology and as a corollary we have:

Lemma 4.1 (Corollary of [10, Lemma 1.6]). *If $\lambda < \lambda_c$, the natural map $X \dashrightarrow \mathbb{P}W^*_{\lambda,k}$ is birational onto its image for any k sufficiently divisible.*

Now we may reduce the proof of Theorem 1.1 to showing

$$-d\int_X \mathrm{MA}(\psi_\lambda) = (\dot{\varphi}_t)_* \mathrm{MA}(\varphi_t). \tag{4.6}$$

By the main result of [31], φ_t has the $C^{1,\alpha}$-regularity so that we can apply the following.

Proposition 4.1 ([7, Proposition 2.2]). *Let φ_t be a weak geodesic ray in the space of Kähler metrics and assume that $\varphi_t(x)$ is of C^1-class in two variables t and x. Then for any compactly supported C^1-function f, the value of the integral*

$$\int_X f(\dot{\varphi}_t)\,\mathrm{MA}(\varphi_t)$$

is independent of t.

In other words the right hand side of (4.6) is independent of t and the proof is reduced to the case $t = 0$. Then by basic measure theory we conclude Theorem 1.1 if for any $\lambda \in \mathbb{R}$

$$\int_X \mathrm{MA}(\psi_\lambda) = \int_{\{\dot{\varphi}_0 \geqslant \lambda\}} \mathrm{MA}(\varphi) \tag{4.7}$$

holds. It is sufficient to show

$$\int_{\{\dot{\varphi}_0 > \lambda\}} \mathrm{MA}(\varphi) \leqslant \int_X \mathrm{MA}(\psi_\lambda) \leqslant \int_{\{\dot{\varphi}_0 \geqslant \lambda\}} \mathrm{MA}(\varphi) \tag{4.8}$$

for any $\lambda \in \mathbb{R}$. Indeed from Theorem 2.1 the first inequality of (4.8) yields

$$\int_{\{\dot{\varphi}_0 > \lambda + \varepsilon\}} \mathrm{MA}(\varphi) \leqslant \int_X \mathrm{MA}(\psi_{\lambda+\varepsilon}) \leqslant \int_X \mathrm{MA}(\psi_\lambda)$$

for any $\varepsilon > 0$ so that letting $\varepsilon \to 0$ we have

$$\int_{\{\dot{\varphi}_0 \geqslant \lambda\}} \mathrm{MA}(\varphi) \leqslant \int_X \mathrm{MA}(\psi_\lambda).$$

Combining with the second inequality of (4.8), we obtain (4.7).

The following lemma is directly deduced from the definition of φ_t.

Lemma 4.2. *For every point in X, $\dot{\varphi}_0 \geqslant \lambda$ holds if and only if $\psi_\lambda = \varphi$. In particular,*

$$\int_{\{\dot{\varphi}_0 \geqslant \lambda\}} \mathrm{MA}(\varphi) = \int_{\{\psi_\lambda = \varphi\}} \mathrm{MA}(\varphi)$$

holds.

Proof. Let x be a point of X. If $\psi_\lambda(x) = \varphi(x)$, then

$$\dot{\varphi}_0(x) := \inf_{t>0} \frac{\varphi_t(x) - \varphi(x)}{t} \geqslant \frac{\psi_\lambda(x) + t\lambda - \varphi(x)}{t} \geqslant \lambda.$$

On the other hand, for any fixed $x \in X$ the convexity of φ_t with respect to t yields

$$\psi_\lambda(x) = \inf_{t \geqslant 0}\{\varphi_t(x) - t\lambda\} \geqslant \inf_{t \geqslant 0}\{t\dot{\varphi}_0(x) + \varphi(x) - t\lambda\}.$$

Then by the Legendre relation (4.3), the assumption $\dot{\varphi}_0(x) \geqslant \lambda$ implies

$$\inf_{t \geqslant 0}\{t\dot{\varphi}_0(x) + \varphi(x) - t\lambda\} \geqslant \varphi(x). \qquad \square$$

In the case of Example 3.1, the result of [3] yields much stronger conclusion that ψ_λ has $C^{1,1}$-regularity on the bounded locus and

$$\mathrm{MA}(\psi_\lambda) = 1_{\{\psi_\lambda = \varphi\}}\,\mathrm{MA}(\varphi)$$

holds. We need to give a proof of (4.8) without such a regularity of ψ_λ. Our argument is essentially the same as the proof the comparison principle for the Monge–Ampère operator (see e.g. [21, Theorem 1.16]). First note that the set $\{\dot\varphi_0 > \lambda\}$ is open (thanks to the regularity result of [31]) and contained in $\{\psi_\lambda = \varphi\}$. Therefore by the locality of the Monge–Ampère product we have

$$\int_{\{\dot\varphi_0>\lambda\}} \mathrm{MA}(\psi_\lambda) = \int_{\{\dot\varphi_0>\lambda\}} \mathrm{MA}(\varphi).$$

Then we obtain the one side inequality of (4.8),

$$\int_X \mathrm{MA}(\psi_\lambda) \geqslant \int_{\{\dot\varphi_0>\lambda\}} \mathrm{MA}(\varphi).$$

Let us take any $\varepsilon > 0$ to prove the converse inequality. Thanks to the maximality (4.2) we have

$$\int_X \mathrm{MA}(\psi_\lambda) = \int_{\{\psi_\lambda>\varphi-\varepsilon\}} \mathrm{MA}(\psi_\lambda).$$

Since ψ_λ is continuous, the set $\{\psi_\lambda > \varphi - \varepsilon\}$ is open. For this reason we use again the locality of the Monge–Ampère product to obtain

$$\int_{\{\psi_\lambda>\varphi-\varepsilon\}} \mathrm{MA}(\psi_\lambda) = \int_{\{\psi_\lambda>\varphi-\varepsilon\}} \mathrm{MA}(\max\{\psi_\lambda, \varphi-\varepsilon\}).$$

The right hand side equals

$$L^n - \int_{\{\psi_\lambda\leqslant\varphi-\varepsilon\}} \mathrm{MA}(\max\{\psi_\lambda, \varphi-\varepsilon\})$$

by Theorem 2.1. Therefore we obtain

$$\int_X \mathrm{MA}(\psi_\lambda) \leqslant L^n - \int_{\{\psi_\lambda<\varphi-\varepsilon\}} \mathrm{MA}(\max\{\psi_\lambda, \varphi-\varepsilon\}) = L^n - \int_{\{\psi_\lambda<\varphi-\varepsilon\}} \mathrm{MA}(\varphi).$$

If $\varepsilon > 0$ tends to 0, then the set $\{\psi_\lambda < \varphi - \varepsilon\}$ converges to $\{\dot\varphi_0 < \lambda\}$ hence

$$\int_X \mathrm{MA}(\psi_\lambda) \leqslant \int_{\{\dot\varphi_0\geqslant\lambda\}} \mathrm{MA}(\varphi).$$

This ends the proof.

Remark 4.1. It is well known that the classical Duistermaat–Heckmann measure for a Hamiltonian action has piecewise polynomial density. One can show that in our singular setting the canonical measure has at least piecewise continuous density in $\lambda < \lambda_c$. That is, there is a piecewise continuous function $f(\lambda)$ such that

$$-d\,\mathrm{vol}(W_\lambda) = f(\lambda)d\lambda$$

holds for $\lambda < \lambda_c$. In fact, as Example 3.2, one can compute $\mathrm{vol}(W_\lambda)$ for a general test configuration using Proposition 3.1 so that $\mathrm{vol}(W_\lambda) = \mathrm{vol}(\mu^* L \otimes \mathcal{O}(-E_\lambda))$ holds for some effective divisor E_λ. By [22] or by [12], the left hand side has continuous derivative which is expressed by certain restricted volumes. By the same argument one can use Proposition 3.1 to prove Lemma 4.1 provided $\mathcal{X}$ is normal but we omit the detail since using [10, Lemma 1.6] is rather simple and valid for arbitrary test configurations.

4.3. Norms on the weak geodesic ray. We conclude this paper by discussing some consequences of Theorem 1.1, which are concerned with the p-norm of test configuration.

Definition 4.1. Fix any test configuration $(\mathcal{X}, \mathcal{L})$ of a polarized manifold L. Let

$$H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes k}) = \bigoplus_{\lambda} V_\lambda$$

be the weight decomposition of the induced $\mathbb{C}^*$-action. Define the trace-free part of each eigenvalue λ as

$$\bar{\lambda} := \lambda - \frac{1}{N_k} \sum_{\mu} \mu \dim V_\mu$$

and introduce the p-norms ($p \in \mathbb{Z}_{\geqslant 0}$) of the test configuration by

$$Q_p := \lim_{k \to \infty} \frac{1}{k^n} \sum_{\lambda} \left(\frac{\lambda}{k} \right)^p \dim V_\lambda$$

and

$$N_p := \lim_{k \to \infty} \frac{1}{k^n} \sum_{\lambda} \left(\frac{\bar{\lambda}}{k} \right)^p \dim V_\lambda .$$

Especially in the case $p = 2$ we denote Q_2 and N_2 by Q and $\|\mathcal{T}\|^2 = \|\mathcal{T}\|_2^2$. Note that the limits exist since the summations on the right hand side can be described by the appropriate Hilbert polynomial.

It is easy to see that $Q_1 = b_0$, $N_1 = 0$, $N_2 = Q_2 - \frac{{b_0}^2}{a_0}$, and

$$\frac{1}{N_k} \sum_{\lambda} \frac{\lambda}{k} \to F_0 = \frac{b_0}{a_0}.$$

These norms are introduced by [16] and played the important role in their result for the lower bound of the Calabi functional. We can obtain the geometric meanings of these norms in word of weak geodesic ray.

Theorem 4.2. *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration and let φ_t be the weak geodesic associated to $(\mathcal{X}, \mathcal{L})$. Then we have*

$$Q_p = \int_X (\dot{\varphi}_t)^p \frac{\mathrm{MA}(\varphi_t)}{n!}$$

and

$$N_p = \int_X (\dot{\varphi}_t - F_0)^p \frac{\mathrm{MA}(\varphi_t)}{n!}.$$

Proof. By the same argument in the proof of Theorem 3.1, we obtain

$$n! Q_p = -\int_{-\infty}^{\infty} \lambda^p \, d \operatorname{vol}(W_\lambda).$$

This can also be obtained from the result of [38] if one notes the volume characterization of the concave function $G[\mathcal{T}]$ in [38]. Taking the p-th moment of the two measures in Theorem 1.1, we deduce the claim. The formulas for N_p can be proved in the same way. □

Let us examine Theorem 4.2. In the case $p = 0$ it only states that $n!a_0 = \int_X \mathrm{MA}(\varphi_t)$ and this can be easily seen from the definition of Bedford–Taylor's Monge–Ampère product. The case $p = 1$ yields

$$n!b_0 = \int_X \dot{\varphi}_t \,\mathrm{MA}(\varphi_t).$$

In other words, the Aubin–Mabuchi energy functional along the weak geodesic is given by

$$\mathcal{E}(\varphi_t, \varphi) := \int_0^1 dt \int_X \dot{\varphi}_t \,\mathrm{MA}(\varphi_t) = n!b_0 t.$$

(For the definition of the Aubin–Mabuchi energy of a singular Hermitian metric, see [11].) This is a result well known to the experts. For example, the proof of [5] in the Fano case works exactly the same way to yield that along the weak geodesic b_0 gives the gradient of the Aubin–Mabuchi energy. We have reproved it in the viewpoint of the associated family of graded linear series. It is conjectured that the gradient of the K-energy at infinity corresponds to the Donaldson–Futaki invariant. This supports the variational approach to the existence problem. The most interesting case is $p = 2$. This might be a new result and yields a part of Theorem 1.2. In particular, we obtain the following.

Corollary 4.1. *For any test configuration, the norm $\|\mathcal{T}\|$ is zero if and only if the associated weak geodesic ray φ_t is $\varphi + F_0 t$.*

Only the case where the exponent p is even was treated in [16] to assure the positivity of the norm, but now we may define the positive norm for odd p integrating the function $|\lambda|^p$, in place of λ^p, by each measure. In particular, we can see that the limit

$$\|\mathcal{T}\|_p^p := \lim_{k\to\infty} \frac{1}{k^n} \sum_{\lambda} \left(\frac{|\bar{\lambda}|}{k}\right)^p \dim V_\lambda,$$

which cannot necessarily be described by a Hilbert polynomial, exists and coincides with the L^p-norm of the tangent vector. Thus Theorem 1.2 was proved. Letting $p \to +\infty$, we obtain

$$\|\mathcal{T}\|_\infty := \lim_{p\to\infty} \|\mathcal{T}\|_p = \sup_X |\dot{\varphi}_t - F_0|. \tag{4.9}$$

In particular, the right hand side is independent of t and φ.

Let us remark some relation with [16] and prove Theorem 1.3. Let us denote the scalar curvature of the Kähler metric $dd^c\varphi$ by S_φ and its mean value by $\hat{S}$. The main result of [16] states that

$$(Q_p)^{\frac{1}{p}} \|S_\varphi\|_{L^q} \geqslant b_1$$

and

$$\|\mathcal{T}\|_p \cdot \|S_\varphi - \hat{S}\|_{L^q} \geqslant -\mathrm{DF}(\mathcal{T}) \tag{4.10}$$

hold for any even p and the conjugate q which satisfies $1/p + 1/q = 1$. As a result one can see that the existence of a constant scalar curvature Kähler metric implies K-semistability. In view of (4.10), [36] suggested the stronger notion of K-stability which implies

$$\mathrm{DF}(\mathcal{T}) \geqslant \delta \|\mathcal{T}\|_p \tag{4.11}$$

for some uniform constant $\delta > 0$. One of the motivation of this definition is to show the existence of constant scalar curvature Kähler metrics. At the same time, the above condition excludes the pathological example raised in [23]. Corollary 4.1 supports the validity of [36]'s suggestion since the gradient of the K-energy along the trivial ray $\varphi + F_0 t$ is zero.

Let us give an energy theoretic explanation for (4.10). Thanks to Theorem 1.2, we can apply the Hölder inequality to obtain

$$\left(\int_X |\dot\varphi_0 - F_0|^p \frac{\mathrm{MA}(\varphi)}{n!}\right)^{\frac{1}{p}} \left(\int_X |S_\varphi - \hat S|^q \frac{\mathrm{MA}(\varphi)}{n!}\right)^{\frac{1}{q}} \geqslant \int_X (\dot\varphi_0 - F_0)(S_\varphi - \hat S)\frac{\mathrm{MA}(\varphi)}{n!} \tag{4.12}$$

for any pair (p, q) with $1/p + 1/q = 1$. Then the right hand side is the minus of the gradient of K-energy along the weak geodesic ray. The definition of the gradient for singular φ_t is not so clear but if it was well-defined, it should be increasing with respect to t. Moreover the limit gradient should be smaller just as much as the multiplicity of the central fiber than the Donaldson–Futaki invariant. (See also [27–29].) Assuming these points we have

$$\int_X (\dot\varphi_0 - F_0)(S_\varphi - \hat S)\frac{\mathrm{MA}(\varphi)}{n!} \geqslant -\mathrm{DF}(\mathcal{T}). \tag{4.13}$$

Notice that (4.13) implies (4.10) for *any* $1 \leqslant p \leqslant +\infty$. One of the proof of (4.13) following the above line will be given in our preparing note in collaboration with Robert Berman and David Witt Nyström. In the present paper we may give a complete proof of the analogous result for the Kähler–Einstein metrics of Fano manifolds. The point is that in the Fano case we may replace the K-energy to the Ding functional

$$\varphi_t \mapsto -\log \int_X e^{-\varphi_t}$$

to obtain the corresponding result. Here we assume $L = -K_X$ so that the singular Hermitian metric $e^{-\varphi_t}$ can be identified with the positive measure $e^{-\varphi_t} \bigwedge_{i=1}^n \frac{\sqrt{-1}}{2} dz_i \wedge dz_i$. Convexity of the Ding functional along any weak geodesic ray was established in [8] and the relation between the gradient of the Ding functional and $\mathrm{DF}(\mathcal{T})$ was shown in [5].

Theorem 4.3 ([8, Theorem 1.1]). *Assume $L = -K_X$. For any weak sub-geodesic ψ_t in the space of Kähler metrics, the function $-\log \int_X e^{-\psi_t}$ is convex in t.*

Theorem 4.4 (Direct consequence of [5, Theorem 1.3]). *Let X be a Fano manifold and let $(\mathcal{X}, \mathcal{L})$ be a test configuration of the polarization $(X, -K_X)$. Denote the associated weak geodesic ray emanating from smooth φ by φ_t. Then along the geodesic, the gradient of the Ding functional at infinity is bounded from above by the Donaldson–Futaki invariant. Precisely,*

$$\frac{d}{dt}\bigg|_{t=\infty} \left(-\log \int_X e^{-(\varphi_t - F_0 t)} - \mathcal{E}(\varphi_t - F_0 t)\right) \leqslant \mathrm{DF}(\mathcal{T})$$

holds.

Note that our normalization of φ_t following [33] makes $\varphi_t - F_0 t$ to be the corresponding geodesic ray in [30] and [5]. As a corollary of these results we obtain

$$\int_X (\dot\varphi_0 - F_0)\left(e^{-\varphi} - \frac{\mathrm{MA}(\varphi)}{n!}\right) \leqslant \mathrm{DF}(\mathcal{T}) \tag{4.14}$$

with some appropriate normalization for φ, and then the same argument as (4.12) yields

$$\|\mathcal{T}\|_p \left\| \frac{n!e^{-\varphi}}{\mathrm{MA}(\varphi)} - 1 \right\|_{L^q} \geqslant -\mathrm{DF}(\mathcal{T}) \tag{4.15}$$

for any $1 \leqslant p \leqslant +\infty$. Thus we proved Theorem 1.3 strictly. This can be seen as the analogue of the Donaldson's result in the Fano case.

Finally we remark that the strong K-stability condition (4.11) follows from the analytic condition

$$\int_X (\dot{\varphi}_0 - F_0)(S_{\varphi_t} - \hat{S}) \frac{\mathrm{MA}(\varphi_t)}{n!} \leqslant -\delta \|\dot{\varphi}_0 - F_0\|, \tag{4.16}$$

in case S_{φ_t} is well-defined. It is interesting to ask whether this condition implies the properness of the K-energy.

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References

[1] *M. F. Atiyah* and *R. Bott*, The moment map and equivariant cohomology, Topology **23** (1984), no. 1, 1–28.
[2] *E. Bedford* and *A. Taylor*, The Dirichlet problem for a complex Monge–Ampère equation, Invent. Math. **37** (1976), no. 1, 1–44.
[3] *R. J. Berman*, Bergman kernels and equilibrium measures for ample line bundles, preprint 2007, `http://arxiv.org/abs/0704.1640`.
[4] *R. J. Berman*, Bergman kernels and equilibrium measures for line bundles over projective manifolds, Amer. J. Math. **131** (2009), no. 5, 1485–1524.
[5] *R. J. Berman*, K-polystability of Q-Fano varieties admitting Kahler–Einstein metrics, preprint 2012, `http://arxiv.org/abs/1205.6214`.
[6] *R. J. Berman*, *S. Boucksom*, *V. Guedj* and *A. Zeriahi*, A variational approach to complex Monge–Ampère equations, preprint 2009, `http://arxiv.org/abs/0907.4490`.
[7] *B. Berndtsson*, Probability measures related to geodesics in the space of Kähler metrics, preprint 2009, `http://arxiv.org/abs/0907.1806`.
[8] *B. Berndtsson*, A Brunn–Minkowski type inequality for Fano manifolds and the Bando–Mabuchi uniqueness theorem, preprint 2011, `http://arxiv.org/abs/1103.0923`.
[9] *S. Boucksom*, On the volume of a line bundle, Internat. J. Math. **13** (2002), no. 10, 1043–1063.
[10] *S. Boucksom* and *H. Chen*, Okounkov bodies of filtered linear series, Compos. Math. **147** (2011), no. 4, 1205–1229.
[11] *S. Boucksom*, *P. Eyssidieux*, *V. Guedj* and *A. Zeriahi*, Monge–Ampère equations in big cohomology classes, Acta Math. **205** (2010), no. 2, 199–262.

[12] *S. Boucksom*, *C. Favre* and *M. Jonsson*, Differentiability of volumes of divisors and a problem of Teissier, J. Algebraic Geom. **18** (2009), no. 2, 279–308.

[13] *X. Chen* and *Y. Tang*, Test configuration and geodesic rays, in: Géométrie différentielle, physique mathématique, mathématiques et société (I). Volume en l'honneur de Jean Pierre Bourguignon, Astérisque **321**, Société Mathématique de France, Paris (2008), 139–167.

[14] *L. Di Biagio* and *G. Pacienza*, Restricted volumes of effective divisors, preprint 2012, `http://arxiv.org/abs/1207.1204`.

[15] *S. K. Donaldson*, Scalar curvature and stability of toric varieties, J. Differential Geom. **62** (2002), no. 2, 289–349.

[16] *S. K. Donaldson*, Lower bounds on the Calabi functional, J. Differential Geom. **70** (2005), no. 3, 453–472.

[17] *T. Fujita*, Approximating Zariski decomposition of big line bundles, Kodai Math. J. **17** (1994), no. 1, 1–3.

[18] *T. Hisamoto*, On the volume of graded linear series and Monge–Ampère mass, Math. Z. **275** (2013), no. 1–2, 233–243.

[19] *S. Y. Jow*, Okounkov bodies and restricted volumes along very general curves, Adv. Math. **223** (2010), no. 4, 1356–1371.

[20] *K. Kaveh* and *A. G. Khovanskii*, Newton convex bodies, semigroups of integral points, graded algebras and intersection theory, Ann. of Math. (2) **176** (2012), 925–978.

[21] *S. Kołodziej*, The complex Monge–Ampère equation and pluripotential theory, Mem. Amer. Math. Soc. **178** (2005), no. 840.

[22] *R. Lazarsfeld* and *M. Mustaţă*, Convex bodies associated to linear series, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 5, 783–835.

[23] *C. Li* and *C. Xu*, Special test configurations and K-stability of Fano varieties, preprint 2013, `http://arxiv.org/abs/1111.5398`, Ann. of Math. (2), to appear.

[24] *T. Mabuchi*, K-stability of constant scalar curvature polarization, preprint 2008, `http://arxiv.org/abs/0812.4093`.

[25] *T. Mabuchi*, A stronger concept of K-stability, preprint 2009, `http://arxiv.org/abs/0910.4617`.

[26] *Y. Odaka*, A generalization of Ross–Thomas' slope theory, Osaka J. Math. **50** (2013), no. 1, 171–185.

[27] *S. T. Paul* and *G. Tian*, CM stability and the generalized Futaki invariant I, preprint 2008, `http://arxiv.org/abs/math/0605278`.

[28] *S. T. Paul* and *G. Tian*, CM stability and the generalized Futaki invariant II, in: From probability to geometry II. Volume in honor of the 60th birthday of Jean-Michel Bismut, Société Mathématique de France, Astérisque **328**, Paris (2009), 339–354.

[29] *D. H. Phong*, *J. Ross* and *J. Sturm*, Deligne pairings and the Knudsen–Mumford expansion, J. Differential Geom. **78** (2008), no. 3, 475–496.

[30] *D. H. Phong* and *J. Sturm*, Test configurations for K-stability and geodesic rays, J. Symplectic Geom. **5** (2007), no. 2, 221–247.

[31] *D. H. Phong* and *J. Sturm*, Regularity of geodesic rays and Monge–Ampère equations, Proc. Amer. Math. Soc. **138** (2010), no. 10, 3637–3650.

[32] *J. Ross* and *R. Thomas*, An obstruction to the existence of constant scalar curvature Kähler metrics, J. Differential Geom. **72** (2006), no. 3, 429–466.

[33] *J. Ross* and *D. Witt Nyström*, Analytic test configurations and geodesic rays, preprint 2013, `http://arxiv.org/abs/1101.1612`, J. Symplectic Geom., to appear.

[34] *S. Semmes*, Complex Monge–Ampère and symplectic manifolds, Amer. J. Math. **114** (1992), no. 3, 495–550.

[35] *J. Stoppa*, K-stability of constant scalar curvature Kähler manifolds, Adv. Math. **221** (2009), no. 4, 1397–1408.

[36] *G. Székelyhidi*, Filtrations and test-configurations, preprint 2013, `http://arxiv.org/abs/1111.4986`.

[37] *G. Tian*, Kähler–Einstein metrics with positive scalar curvature, Invent. Math. **130** (1997), no. 1, 1–37.

[38] *D. Witt Nyström*, Test configurations and Okounkov bodies, Compos. Math. **148** (2012), 1736–1756.

Tomoyuki Hisamoto, Graduate School of Mathematics, Nagoya University,
464-8602, Furou-chou, Chikusa-ku, Nagoya, Aichi, Japan
e-mail: hisamoto.tomoyuki@gmail.com

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