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Relativistic stars in de Rham-Gabadadze-Tolley massive gravity

Taishi Katsuragawa,[1] Shin'ichi Nojiri,[1,2] Sergei D. Odintsov,[3,4,5,6] and Masashi Yamazaki[1]

[1]*Department of Physics, Nagoya University, Nagoya 464-8602, Japan*
[2]*Kobayashi-Maskawa Institute for the Origin of Particles and the Universe, Nagoya University, Nagoya 464-8602, Japan*
[3]*Institut de Ciencies de lEspai (IEEC-CSIC), Campus UAB, Carrer de Can Magrans, s/n, 08193 Cerdanyola del Valles, Barcelona, Spain*
[4]*ICREA, Passeig LluAs Companys, 23, 08010 Barcelona, Spain*
[5]*Tomsk State Pedagogical University, 634061 Tomsk, Russia*
[6]*Tomsk State University of Control Systems and Radioelectronics (TUSUR), 634050 Tomsk, Russia*

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We study relativistic stars in the simplest model of the de Rham-Gabadadze-Tolley massive gravity which describes the massive graviton without a ghost propagating mode. We consider the hydrostatic equilibrium and obtain the modified Tolman-Oppenheimer-Volkoff equation and the constraint equation coming from the potential terms in the gravitational action. We give analytical and numerical results for quark and neutron stars and discuss the deviations compared with general relativity and $F(R)$ gravity. It is shown that the theory under investigation leads to a small deviation from general relativity in terms of density profiles and mass-radius relation. Nevertheless, such a deviation may be observable in future astrophysical probes.

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I. INTRODUCTION

Late-time accelerated expansion of the Universe has been confirmed by several independent observations [1–9]. In order to explain the accelerated expansion, we need to include new energy sources, which are known as dark energy (DE). The simplest example of DE is the cosmological constant Λ. Furthermore, by including cold dark matter (CDM) alongside the Standard Model (SM) particles, we obtain the well-known ΛCDM model, which successfully describes the current epoch of the Universe. The ΛCDM model is, however, merely one of the phenomenological models, and from a theoretical point of view, it suffers from several theoretical problems. For example, we need to theoretically explain the very large disagreement between the theoretically estimated values of the cosmological constant and the observational value, and the ratio of ordinary matter and CDM with respect to DE in the current epoch.

Since one may regard that the inclusion of the cosmological constant could be a minimal extension of general relativity, we can also consider the other extensions of general relativity and investigate the possibility that these extended theories could describe the real Universe. As nonminimal extensions of general relativity to explain the current expansion without a cosmological constant, modified gravitational theories have been proposed and investigated well (for a review, see, for instance, [10–14]). In order to establish such a new gravitational theory, it is important to study the cosmological models and compare them with observational data. We should note that modified gravitational theories could also be constrained by astrophysical observations, for example, those of compact stars.

Recently, massive and compact neutron stars whose masses are $M_{\rm NS} \sim 2M_\odot$ ($M_\odot$ is the solar mass) were found [15–18]. It could hardly be understood in the framework of general relativity and hadron physics so far if one uses the stellar matter equations of state, which are comfortable within astrophysics and hadron physics. Thus, there could be two points of view to explain massive neutron stars: one is from the particle physics side, which requires a phenomenological change of the equation of state that is not well justified, and another is the gravitational physics side, using the convenient equation of state for stellar matter. The sizes of the compact objects are determined by the balance between the degeneracy force and gravitational force. In order to explain massive neutron stars, three approaches seem to be reasonable: (i) the repulsive force is stronger than that realized with the standard equations of state, (ii) the attractive force is weaker than that predicted in general relativity, or (iii) we accept both cases (i) and (ii) simultaneously. From the point of view of (i) based on hadron physics, it was suggested that equations of state could be modified by introducing new interactions [19,20]. From the point of view of (ii) based on gravitational physics, it has been suggested that some models of $F(R)$ gravity can explain massive and compact neutron stars [21–23]. In this work, we take on the viewpoint of case (ii) and study if compact objects, quark stars and neutron stars, are realized and how the internal structure of compact objects deviates from that in general relativity if we assume massive gravity coupled with matter, which is described by the standard equations of state.

The de Rham-Gabadadze-Tolley (dRGT) massive gravity [24–26] (for a review, see [27]) is the ghost-free theory with interacting massive spin-2 field. The basic idea of the massive spin-2 field theory was proposed by Fierz and Pauli in Ref. [28], where the consistent free massive spin-2 theory was given by adding a tuned mass term to the free massless spin-2 field theory on flat space-time. It was, however, shown that the Fierz-Pauli theory cannot recover general relativity in the massless limit, due to the so-called van Dam-Veltman-Zakharov (vDVZ) discontinuity [29,30]. After the discovery of the vDVZ discontinuity, in order to avoid this problem, the Fierz-Pauli theory was extended to an interacting theory by replacing the kinetic terms with the Ricci scalar. As a result, the vDVZ discontinuity can be screened by a nonlinear effect called the Vainshtein mechanism [31]. Nevertheless, the nonlinear terms generate a ghost called the Boulware-Deser (BD) ghost [32]. The problem of the BD ghost mode has been discussed for a long time, and the problem has finally been solved in dRGT massive gravity by introducing a new form of mass terms.

dRGT massive gravity is considered to be able to avoid the constraint from the Solar System and terrestrial experiments thanks to the Vainshtein mechanism, where the nonlinear effects hide the extra degree of freedom coupled with the matter source. At the same time, the massive graviton leads to the modification of long-range gravitational force because the gravitational potential is modified to be the Yukawa-type potential, where the scale of modification is characterized by that of the graviton mass. Thus, one may expect that the mass of a massive graviton could be comparable to the cosmological constant, which could explain the accelerated expansion of the Universe without introducing the cosmological constant [33–37].

As we regard the dRGT massive gravity as an alternative theory of gravity, it is interesting to apply this theory to astrophysical phenomena as well as to the accelerated expansion of the Universe. It is very difficult to construct the general framework that quantifies the deviations from the predictions of general relativity in a strong-gravity field because the nonperturbative effects depend on the detail of each theory and parametric treatment is not suitable [38]. Furthermore, it is significant if we could conclude that astrophysical and cosmological applications are compatible with observations in the specific theory of modified gravity. For the above reasons, it is indispensable to study compact objects in dRGT massive gravity in the same way as in $F(R)$ gravity, as an astrophysical test of massive gravity in the strong-gravity regime.

This paper is organized as follows: in Sec. II, we give a brief review of dRGT massive gravity and derive the equations of motion. In Sec. III, we consider the spherically symmetric stellar metric and derive the Tolman-Oppenheimer-Volkoff (TOV) equations in the minimal model of dRGT massive gravity. It is shown that one constraint equation coming from the potential terms in the gravitational action appears. It leads to an explicit difference from the case of general relativity. In Sec. IV, we present the numerical analysis for the quark star and neutron star for some convenient equations of state. In particular, the mass-central density and mass-radius relation are numerically analyzed. In Sec. V, we summarize the obtained results for relativistic stars and discuss the differences in massive gravity between general relativity and $F(R)$ gravity.

II. THE ACTION AND EQUATION OF MOTION IN DRGT MASSIVE GRAVITY

In this section, we give a brief review of the dRGT massive gravity and derive the equation of motion. The action of the dRGT massive gravity [26] is given by

$$S_{\mathrm{dRGT}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-\det(g)} \times \left[R - 2m_0^2\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{-1}f}\right)\right] + S_{\mathrm{matter}}. \tag{1}$$

Here, $g_{\mu\nu}$ and $f_{\mu\nu}$ are dynamical and reference metrics, respectively, and κ is the gravitational coupling given in terms of the Newton constant of gravitation G, $\kappa^2 = 8\pi G$. In (1), the coefficients β_n and m_0 are free parameters. The matrix $\sqrt{g^{-1}f}$ is defined as the square root of $g^{\mu\rho}f_{\rho\nu}$, that is,

$$\left(\sqrt{g^{-1}f}\right)^{\mu}_{\ \rho}\left(\sqrt{g^{-1}f}\right)^{\rho}_{\ \nu} = g^{\mu\rho}f_{\rho\nu}. \tag{2}$$

For general matrix $\mathbf{X}$, $e_n(\mathbf{X})$'s are defined as polynomials of the eigenvalues of X:

$$\begin{aligned} &e_0(\mathbf{X}) = 1, \qquad e_1(\mathbf{X}) = [\mathbf{X}],\\ &e_2(\mathbf{X}) = \frac{1}{2}([\mathbf{X}]^2 - [\mathbf{X}^2]),\\ &e_3(\mathbf{X}) = \frac{1}{6}([\mathbf{X}]^3 - 3[\mathbf{X}][\mathbf{X}^2] + 2[\mathbf{X}^3]),\\ &e_4(\mathbf{X}) = \frac{1}{24}([\mathbf{X}]^4 - 6[\mathbf{X}]^2[\mathbf{X}^2] + 3[\mathbf{X}^2]^2 \\ &\qquad + 8[\mathbf{X}][\mathbf{X}^3] - 6[\mathbf{X}^4]) = \det(\mathbf{X}),\\ &e_k(\mathbf{X}) = 0 \quad \text{for } k > 4, \end{aligned} \tag{3}$$

where the square brackets denote traces of the matrices, that is, $[X] = X^{\mu}_{\mu}$. For conventional notations in this paper, hereafter, we denote the determinant of a matrix A as $\det(A)$, and $\sqrt{A}$ represents a matrix that is the square root of A.

We should note that a nondynamical tensor is required in order to describe the massive spin-2 field because we cannot construct the potential terms without derivatives only by using $g_{\mu\nu}$. We may consider the invariants that consist of $g_{\mu\nu}$, for example, $g^2_{\mu\nu}$ or g^{μ}_{μ}, but they are constants that correspond to the cosmological constant. We should also note that $e_4(\sqrt{g^{-1}f})$ can be ignored when we study the dynamics because

$$\sqrt{-\det(g)}e_4\left(\sqrt{g^{-1}f}\right) = \sqrt{-\det(f)}, \tag{4}$$

which is nondynamical since $f_{\mu\nu}$ is a nondynamical tensor and does not appear in the equation of motion. By the variation of $g_{\mu\nu}$ in Eq. (1), we obtain the following equation of motion:

$$0 = R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} + \frac{1}{2}m_0^2\sum_{n=0}^{3}(-1)^n\beta_n \times \left[g_{\mu\lambda}Y^\lambda_{(n)\nu}\left(\sqrt{g^{-1}f}\right) + g_{\nu\lambda}Y^\lambda_{(n)\mu}\left(\sqrt{g^{-1}f}\right)\right] - \kappa^2 T_{\mu\nu}. \tag{5}$$

Here, for a matrix $\mathbf{X}$, $Y_n(\mathbf{X})$'s are defined by

$$Y^\lambda_{(n)\nu}(\mathbf{X}) = \sum_{r=0}^{n}(-1)^r(X^{n-r})^\lambda_\nu e_r(\mathbf{X}), \tag{6}$$

or explicitly,

$$\begin{aligned} &Y_0(\mathbf{X}) = \mathbf{1}, \qquad Y_1(\mathbf{X}) = \mathbf{X} - \mathbf{1}[\mathbf{X}], \\ &Y_2(\mathbf{X}) = \mathbf{X}^2 - \mathbf{X}[\mathbf{X}] + \frac{1}{2}\mathbf{1}([\mathbf{X}]^2 - [\mathbf{X}^2]), \\ &Y_3(\mathbf{X}) = \mathbf{X}^3 - \mathbf{X}^2[\mathbf{X}] + \frac{1}{2}\mathbf{X}([\mathbf{X}]^2 - [\mathbf{X}^2]) \\ &\qquad - \frac{1}{6}\mathbf{1}([\mathbf{X}]^3 - 3[\mathbf{X}][\mathbf{X}^2] + 2[\mathbf{X}^3]). \end{aligned} \tag{7}$$

Note that since the e_n's are written in terms of the trace of $g^{-1}f$, the following formula about the variation of the trace could be useful,

$$\delta\mathrm{tr}\left(\left(\sqrt{g^{-1}f}\right)^n\right) = \frac{n}{2}\mathrm{tr}\left(g\left(\sqrt{g^{-1}f}\right)^n\delta g^{-1}\right). \tag{8}$$

Then, we obtain

$$\frac{2}{\sqrt{-\det(g)}}\delta_g\left(\sqrt{-\det(g)}e_n\left(\sqrt{g^{-1}f}\right)\right) = \sum_{r=0}^{n}(-1)^{r+1}\mathrm{tr}\left(g\left(\sqrt{g^{-1}f}\right)^r\delta g^{-1}\right)e_{n-r}\left(\sqrt{g^{-1}f}\right), \tag{9}$$

and the third term in Eq. (5) is symmetrized with respect to the indices μ and ν. If the metrics g and f are diagonal, the matrix $\sqrt{g^{-1}f}$ is symmetric and the equation of motion is written as

$$G_{\mu\nu} + m_0^2 I_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{10}$$

where $G_{\mu\nu}$ is the Einstein tensor, and we define the sum of interaction terms $I_{\mu\nu}$ as follows:

$$I_{\mu\nu} = \sum_{n=0}^{3}(-1)^n\beta_n g_{\mu\lambda}Y^\lambda_{(n)\nu}\left(\sqrt{g^{-1}f}\right). \tag{11}$$

In Eq. (10), $T_{\mu\nu}$ is the energy-momentum tensor, and we assume that matter is minimally coupled to gravity in order to avoid the ghost problem due to nonminimal matter couplings [39,40].

III. MODIFIED TOV EQUATIONS

A. Ansatz

In this section, we study the static and spherical equations of motion with the perfect fluid in hydrostatic equilibrium. It is called the TOV equation in general relativity. At first, we calculate the curvature and the interaction terms for the spherically symmetric case and check how the TOV equation is modified in dRGT massive gravity.

For the dynamical metric $g_{\mu\nu}$ and reference metric $f_{\mu\nu}$, we assume the static and spherically symmetric ansatz in a polar coordinate system,

$$g_{\mu\nu}dx^\mu dx^\nu = -\mathrm{e}^{2\phi}dt^2 + \mathrm{e}^{2\lambda}d\rho^2 + D^2(\rho)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{12}$$

$$f_{\mu\nu}dx^\mu dx^\nu = -h(\rho)dt^2 + h^{-1}(\rho)d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{13}$$

where ρ is the radial coordinate and ϕ and λ are functions of ρ, $\phi = \phi(\rho)$ and $\lambda = \lambda(\rho)$. We note that we do not consider the general class of the reference metric but a specific one that is inspired by the static and spherically symmetric solution in general relativity. $h(\rho)$ is a function of ρ; for example, $h(r) = 1 - \frac{2M}{r}$ for the Schwarzschild-type metric. We also assume that the center of the space-time described by $f_{\mu\nu}$ locates at the center of physical space-time described by $g_{\mu\nu}$ for simplicity.

In order to compare the difference between general relativity and dRGT massive gravity, we change the form of the above ansatz as follows: we define the new variable r so that $D(\rho) = r^2$, which can be solved with respect to ρ, $\rho = \chi(r)$, and we find

$$g_{\mu\nu}dx^\mu dx^\nu = -\mathrm{e}^{2\phi}dt^2 + \mathrm{e}^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{14}$$

$$f_{\mu\nu}dx^\mu dx^\nu = -h(r)dt^2 + h^{-1}(r)(\chi'(r))^2dr^2 + \chi^2(r)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{15}$$

We note that the scalar function $\chi(r)$ corresponds to the degree of freedom of the Stukelberg field. The general coordinate transformation invariance is broken in the massive gravity, but it can be restored by changing the Stukelberg field. In our case, the radial coordinate is chosen so that the dynamical metric identified with physical spacetime is treated in the same procedure as the TOV equation.

For the above ansatz, we obtain the nonvanishing components of the Ricci tensor and the Ricci scalar as follows,

$$R_{tt} = \left[\phi'' + (\phi')^2 - \phi'\lambda' + \frac{2\phi'}{r}\right]e^{2(\phi-\lambda)}, \qquad (16)$$

$$R_{rr} = -\phi'' - (\phi')^2 + \phi'\lambda' + \frac{2\lambda'}{r}, \qquad (17)$$

$$R_{\theta\theta} = -(1 - \lambda' r + \phi' r)e^{-2\lambda} + 1, \qquad (18)$$

$$R_{\varphi\varphi} = \sin^2\theta R_{\theta\theta}, \qquad (19)$$

$$R = 2\left[-\phi'' - (\phi')^2 + \phi'\lambda' + \frac{2\lambda'}{r} - \frac{2\phi'}{r} - \frac{1}{r^2}\right]e^{-2\lambda} + \frac{2}{r^2}, \qquad (20)$$

and the nonvanishing components of the Einstein tensor as follows,

$$G_{tt} = \frac{1}{r^2}e^{2\phi} - \frac{1-2\lambda' r}{r^2}e^{2\phi-2\lambda}, \qquad (21)$$

$$G_{rr} = -\frac{1}{r^2}e^{2\lambda} + \frac{1+2\phi' r}{r^2}, \qquad (22)$$

$$G_{\theta\theta} = r^2\left[\phi'' + (\phi')^2 - \phi'\lambda' - \frac{\lambda'}{r} + \frac{\phi'}{r}\right]e^{-2\lambda}, \qquad (23)$$

$$G_{\phi\phi} = r^2\sin^2\theta\left[\phi'' + (\phi')^2 - \phi'\lambda' - \frac{\lambda'}{r} + \frac{\phi'}{r}\right]e^{-2\lambda}. \qquad (24)$$

B. The minimal model with flat reference metric

Next, we calculate the interaction terms in Eq. (10) to obtain the modified TOV equation in massive gravity. It is, however, not so easy to study all the cases with different parameters and different reference metrics because it is impossible to obtain the general solution for all models in dRGT massive gravity. In this subsection, thus, we specify the parameters β_n and reference metric and study the modified TOV equation.

First, we introduce a minimal model of the dRGT massive gravity where the parameters β_n are chosen as follows:

$$\beta_0 = 3, \qquad \beta_1 = -1, \qquad \beta_2 = 0, \qquad \beta_3 = 0. \qquad (25)$$

Here, we should note that the interaction terms in (1) can be expressed in terms of another variable $K = \sqrt{g^{-1}f} - 1$,

$$\sum_{n=0}^{3}\beta_n e_n\left(\sqrt{g^{-1}f}\right) = \sum_{n=0}^{3}\alpha_n e_n(K). \qquad (26)$$

The parameters α_n are related to β_n by the following relation,

$$\beta_i = (4-i)!\sum_{n=i}^{4}\frac{(-1)^{n+i}}{(4-n)!(n-i)!}\alpha_n. \qquad (27)$$

Then, if we require the flat solution and the recovery of the covariant Fierz-Pauli action in the limit where the gravitational coupling vanishes, the general action of massive gravity with parameters β_n are reduced to a two-parameter family with parameters α_3 and α_4, where the minimal model corresponds to $(\alpha_3, \alpha_4) = (1, 1)$.

So, we find that the interaction terms in the minimal model are given as follows:

$$I_{tt} = g_{tt}(\beta_0 Y^t_{(0)t} - \beta_1 Y^t_{(1)t} + \beta_2 Y^t_{(2)t} - \beta_3 Y^t_{(3)t}) = -e^{2\phi(r)}\left(3 - \frac{2\chi(r)}{r} - \frac{\chi'(r)}{\sqrt{h(r)}}e^{-\lambda(r)}\right), \qquad (28)$$

$$I_{rr} = g_{rr}(\beta_0 Y^r_{(0)r} - \beta_1 Y^r_{(1)r} + \beta_2 Y^r_{(2)r} - \beta_3 Y^r_{(3)r}) = e^{2\lambda(r)}\left(3 - \frac{2\chi(r)}{r} - \sqrt{h(r)}e^{-\phi(r)}\right), \qquad (29)$$

$$I_{\theta\theta} = g_{\theta\theta}(\beta_0 Y^\theta_{(0)\theta} - \beta_1 Y^\theta_{(1)\theta} + \beta_2 Y^\theta_{(2)\theta} - \beta_3 Y^\theta_{(3)\theta}) = r^2\left(3 - \frac{\chi(r)}{r} - \frac{\chi'(r)}{\sqrt{h(r)}}e^{-\lambda(r)} - \sqrt{h(r)}e^{-\phi(r)}\right), \qquad (30)$$

$$I_{\phi\phi} = g_{\phi\phi}(\beta_0 Y^\phi_{(0)\phi} - \beta_1 Y^\phi_{(1)\phi} + \beta_2 Y^\phi_{(2)\phi} - \beta_3 Y^\phi_{(3)\phi}) = r^2\sin^2\theta\left(3 - \frac{\chi(r)}{r} - \frac{\chi'(r)}{\sqrt{h(r)}}e^{-\lambda(r)} - \sqrt{h(r)}e^{-\phi(r)}\right). \qquad (31)$$

Now, we derive the equation of motion (10) in the minimal model. For the matter field, we consider a perfect fluid with the following energy-momentum tensor,

$$T_{\mu\nu} = \mathrm{diag}(e^{2\phi}\rho, e^{2\lambda}P, r^2P, r^2\sin^2\theta P). \qquad (32)$$

Then, one obtains (t,t), (r,r), and (θ,θ), (φ,φ) components as follows:

$$-8\pi G\rho(r) = -\frac{1}{r^2} + \frac{1-2\lambda'(r)r}{r^2}e^{-2\lambda(r)} + m_0^2\left(3 - \frac{2\chi(r)}{r} - \frac{\chi'(r)}{\sqrt{h(r)}}e^{-\lambda(r)}\right), \qquad (33)$$

$$8\pi GP(r) = -\frac{1}{r^2} + \frac{1+2\phi'(r)r}{r^2}e^{-2\lambda(r)} + m_0^2\left(3 - \frac{2\chi(r)}{r} - \sqrt{h(r)}e^{-\phi(r)}\right), \tag{34}$$

$$8\pi GP(r) = e^{-2\lambda(r)}\left(\phi'' + (\phi')^2 - \phi'\lambda' - \frac{\lambda'}{r} + \frac{\phi'}{r}\right) + m_0^2\left(3 - \frac{\chi(r)}{r} - \frac{\chi'(r)}{\sqrt{h(r)}}e^{-\lambda(r)} - \sqrt{h(r)}e^{-\phi(r)}\right). \tag{35}$$

We should note that the (θ, θ) or (φ, φ) component of the field equation plays a crucial role in contrast to general relativity. In the static and spherically symmetric case, the Einstein equation leads to two nontrivial equations, (t, t) and (r, r) components. However, in massive gravity, we need to take the (θ, θ) or (φ, φ) component into account because the degrees of freedom increase by introducing the second metric $f_{\mu\nu}$.

Next, we fix the reference metric $f_{\mu\nu}$. For simplicity, we assume that $h(r) = 1$ in the reference metric $f_{\mu\nu}$, that is, we consider the Minkowski metric as the reference one with an extra arbitrary function $\chi(r)$:

$$f_{\mu\nu}dx^\mu dx^\nu = -dt^2 + (\chi'(r))^2 dr^2 + \chi^2(r)(d\theta^2 + \sin^2 d\varphi^2), \quad h(r) = 1. \tag{36}$$

Note that the choice of reference metric requires special attention for cosmological applications. If one considers the simple Friedmann-Robertson-Walker (FRW) ansatz for $g_{\mu\nu}$ with the Minkowski metric for $f_{\mu\nu}$, one cannot obtain nontrivial flat FRW cosmology [33,34]. However, in this case we choose the flat reference metric because it is better to limit the number of free parameters for the numerical calculation later. As a result, in our model, the free parameter is only the graviton mass. Furthermore, we fix the graviton mass by choosing the mass to be the cosmological scale.

In this case, the (t, t), (r, r), and $(\theta, \theta), (\varphi, \varphi)$ components of the equation of motion are given by

$$-8\pi G\rho(r) = -\frac{1}{r^2} + \frac{1-2\lambda'(r)r}{r^2}e^{-2\lambda(r)} + m_0^2\left(3 - \frac{2\chi(r)}{r} - \chi'(r)e^{-\lambda(r)}\right). \tag{37}$$

$$8\pi GP(r) = -\frac{1}{r^2} + \frac{1+2\phi'(r)r}{r^2}e^{-2\lambda(r)} + m_0^2\left(3 - \frac{2\chi(r)}{r} - e^{-\phi(r)}\right), \tag{38}$$

$$8\pi GP(r) = e^{-2\lambda(r)}\left(\phi'' + (\phi')^2 - \phi'\lambda' - \frac{\lambda'}{r} + \frac{\phi'}{r}\right) + m_0^2\left(3 - \frac{\chi(r)}{r} - \chi'(r)e^{-\lambda(r)} - e^{-\phi(r)}\right). \tag{39}$$

Finally, we change the variables and rewrite the field equations (37) and (38). Let us define the variable $M(r)$, which is called the mass parameter, as follows:

$$e^{-2\lambda(r)} = 1 - \frac{2GM(r)}{r}, \tag{40}$$

because we expect that external space-time is described by the asymptotically Schwarzschild metric. Differentiating the above relation (40) with respect to r, we obtain the following equation,

$$-\frac{2G}{r^2}M'(r) = -\frac{1}{r^2} + e^{-2\lambda}(1 - 2r\lambda')\frac{1}{r^2}. \tag{41}$$

Equation (37) can be rewritten in terms of the mass parameter $M(r)$ as

$$\begin{aligned}\frac{2G}{r^2}M'(r) &= 8\pi G\rho(r) + m_0^2\left[3 - \frac{2\chi(r)}{r} - \chi'(r)\left(1 - \frac{2GM(r)}{r}\right)^{1/2}\right]\\ GM'(r) &= 4\pi G\rho(r)r^2 + \frac{1}{2}m_0^2 r^2\left[3 - \frac{2\chi(r)}{r} - \chi'(r)\left(1 - \frac{2GM(r)}{r}\right)^{1/2}\right].\end{aligned} \tag{42}$$

On the other hand, when we operate the covariant derivative on Eq. (10), we find that

$$\nabla_\mu(G^{\mu\nu} + m_0^2 I^{\mu\nu}) = \nabla_\mu T^{\mu\nu}. \tag{43}$$

In general relativity with $I^{\mu\nu} = 0$, $\nabla_\mu T^{\mu\nu} = 0$ is automatically derived from the Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$. Therefore, it is reasonable that $T_{\mu\nu}$ is assumed to be separately conserved, and the conservation law $\nabla^\mu T_{\mu\nu} = 0$ gives

$$\phi' = -(P+\rho)^{-1}P', \qquad \phi = -\int (P+\rho)^{-1}P' dr. \tag{44}$$

Equation (38) can be rewritten in terms of the energy-density ρ and the pressure P as

$$8\pi GP(r) = -\frac{1}{r^2} + \frac{1}{r^2}[1 - 2(P+\rho)^{-1}P'r]\left(1 - \frac{2GM(r)}{r}\right) + m_0^2\left(3 - \frac{2\chi(r)}{r} - \mathrm{e}^{\int (P+\rho)^{-1}P'dr}\right). \tag{45}$$

Furthermore, Eq. (39) is given by

$$\begin{aligned} 8\pi GP(r) = &\left[-((P+\rho)^{-1}P')' + ((P+\rho)^{-1}P')^2 - \frac{1}{r}(P+\rho)^{-1}P'\right]\left(1 - \frac{2GM(r)}{r}\right) \\ &-\frac{1}{2}\left[(P+\rho)^{-1}P' - \frac{1}{r}\right]\left(1 - \frac{2GM(r)}{r}\right)' + m_0^2\left(3 - \frac{\chi(r)}{r} - \chi'(r)\left(1 - \frac{2GM(r)}{r}\right)^{1/2} - \mathrm{e}^{\int (P+\rho)^{-1}P'dr}\right). \end{aligned} \tag{46}$$

Additionally, the interaction term $I_{\mu\nu}$ has to be separately conserved, $\nabla_\mu I^{\mu\nu} = 0$, because $\nabla_\mu G^{\mu\nu} = 0$ and $\nabla_\mu T^{\mu\nu} = 0$. Note that we can express the interaction terms as $I^{\mu\nu} = X^\mu_\lambda g^{\lambda\nu}$, where X^μ_λ is defined as

$$X^\mu_\lambda = \mathrm{diag}\left(3 - \frac{2\chi(r)}{r} - \chi'(r)\mathrm{e}^{-\lambda(r)}, 3 - \frac{2\chi(r)}{r} - \mathrm{e}^{-\phi(r)}, 3 - \frac{\chi(r)}{r} - \chi'(r)\mathrm{e}^{-\lambda(r)} - \mathrm{e}^{-\phi(r)}, 3 - \frac{\chi(r)}{r} - \chi'(r)\mathrm{e}^{-\lambda(r)} - \mathrm{e}^{-\phi(r)}\right). \tag{47}$$

Thus, the constraint is written as $\nabla_\mu X^\mu_\lambda = 0$, and a nontrivial relation is given by $\nabla_\mu X^\mu_r = 0$ as

$$0 = \left(\frac{2}{r} + \phi'(r)\right)\chi'(r)\mathrm{e}^{-\lambda} - \frac{2\chi'(r)}{r} = \left(\frac{2}{r} - (P+\rho)^{-1}P'\right)\left(1 - \frac{2GM(r)}{r}\right)^{1/2} - \frac{2}{r}. \tag{48}$$

Now, we introduce the dimensionless variables defined by

$$M \to mM_\odot, \qquad r \to r_g r, \qquad \rho \to \tilde{\rho}M_\odot/r_g^3, \qquad P \to pM_\odot/r_g^3, \qquad m_0 \to \alpha M_\odot. \tag{49}$$

Here $M_\odot$ is the solar mass and $r_g = GM_\odot$. After the short calculation, Eqs. (42), (45), and (46) are rewritten as

$$m'(r) = 4\pi\tilde{\rho}(r)r^2 + \frac{1}{2}\alpha^2(r_g M_\odot)^2 r^2\left[3 - \frac{2\chi(r)}{r} - \chi'(r)\left(1 - \frac{2m(r)}{r}\right)^{1/2}\right], \tag{50}$$

$$8\pi p(r) = -\frac{1}{r^2} + \frac{1}{r^2}[1 - 2(p+\tilde{\rho})^{-1}p'r]\left(1 - \frac{2m(r)}{r}\right) + \alpha^2(r_g M_\odot)^2\left(3 - \frac{2\chi(r)}{r} - \mathrm{e}^{\int (p+\tilde{\rho})^{-1}p'dr}\right), \tag{51}$$

$$\begin{aligned} 8\pi p(r) = &\left[-((p+\tilde{\rho})^{-1}p')' + ((p+\tilde{\rho})^{-1}p')^2 - \frac{1}{r}(p+\tilde{\rho})^{-1}p'\right]\left(1 - \frac{2m(r)}{r}\right) - \frac{1}{2}\left[(p+\tilde{\rho})^{-1}p' - \frac{1}{r}\right]\left(1 - \frac{2m(r)}{r}\right)' \\ &+ \alpha^2(r_g M_\odot)^2\left(3 - \frac{\chi(r)}{r} - \chi'(r)\left(1 - \frac{2m(r)}{r}\right)^{1/2} - \mathrm{e}^{\int (p+\tilde{\rho})^{-1}p'dr}\right). \end{aligned} \tag{52}$$

And, the constraint (48) is rewritten as

$$0 = \left(-(p+\tilde{\rho})^{-1}p' + \frac{2}{r}\right)\left(1 - \frac{2m(r)}{r}\right)^{1/2} - \frac{2}{r}. \tag{53}$$

For $m_0 = 0$, Eqs. (50) and (51) reduce to ordinary TOV equations consistently,

$$\begin{aligned} m'(r) &= 4\pi\tilde{\rho}(r)r^2, \\ p'(r) &= \frac{4\pi p(r)r^3 + m(r)}{r(r - 2m(r))}(p(r) + \tilde{\rho}(r)). \end{aligned} \tag{54}$$

We should note that the constraint equation (53) does not appear in the above equations because the interaction terms do not appear in the action (1). We also note that, in Eq. (38), $\mathrm{e}^{-\phi}$ appears from the mass term, which is a unique property in massive gravity. Since ϕ' can be expressed as a function of ρ and P in Eq. (44), the integrations of ρ and P appear in the modified TOV equation. As we will see in the next subsection, the TOV equation in massive gravity becomes a second-order differential equation because of the integration.

C. Constraint and field equations

In the previous subsection, we saw that one constraint equation comes from the conservation law for the interaction terms. In this subsection, we substitute the constraint into two field equations and complete their further deformation into a form suitable for numerical calculation.

At first, the field equations are written as

$$m'(r) = 4\pi\tilde{\rho}(r)r^2 + \frac{1}{2}\alpha^2(r_g M_\odot)^2 r^2\left[3 - \frac{2\chi(r)}{r} - \chi'(r)\left(1 - \frac{2m(r)}{r}\right)^{1/2}\right], \tag{55}$$

$$8\pi p(r) = -\frac{1}{r^2} + \frac{1}{r^2}(1 - 2qr)\left(1 - \frac{2m(r)}{r}\right) + \alpha^2(r_g M_\odot)^2\left(3 - \frac{2\chi(r)}{r} - \mathrm{e}^{\int qdr}\right), \tag{56}$$

$$\begin{aligned}8\pi p(r) = &\left[q\left(q - \frac{1}{r}\right) - q'\right]\left(1 - \frac{2m(r)}{r}\right) - \frac{1}{2}\left(q - \frac{1}{r}\right)\left(1 - \frac{2m(r)}{r}\right)' \\ &+ \alpha^2(r_g M_\odot)^2\left(3 - \frac{\chi(r)}{r} - \chi'(r)\left(1 - \frac{2m(r)}{r}\right)^{1/2} - \mathrm{e}^{\int qdr}\right),\end{aligned} \tag{57}$$

and the constraint is

$$0 = \left(\frac{2}{r} - q\right)\left(1 - \frac{2m(r)}{r}\right)^{1/2} - \frac{2}{r}. \tag{58}$$

Here, we define a new variable q as

$$q \equiv (p + \tilde{\rho})^{-1}p', \qquad \tilde{\rho} = \frac{p'}{q} - p. \tag{59}$$

Equation (58) is the constraint for $\nabla_\mu I^{\mu\nu} = 0$ and can be rewritten as

$$\left(1 - \frac{2m(r)}{r}\right)^{1/2} = \left(1 - \frac{1}{2}qr\right)^{-1}, \tag{60}$$

$$m(r) = \frac{1}{2}r - \frac{1}{2}r\left(1 - \frac{1}{2}q(r)r\right)^{-2}, \tag{61}$$

$$m'(r) = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{1}{2}q(r)r\right)^{-2} - \frac{1}{2}r\left(1 - \frac{1}{2}q(r)r\right)^{-3}(q'r + q). \tag{62}$$

Thus, by substituting Eq. (58) into Eqs. (55), (56), and (57), we obtain

$$\begin{aligned}&\frac{1}{2} - \frac{1}{2}\left(1 - \frac{1}{2}qr\right)^{-2} - \frac{1}{2}r\left(1 - \frac{1}{2}qr\right)^{-3}(q'r + q) \\ &\quad = 4\pi\left(\frac{p'}{q} - p\right)r^2 + \frac{1}{2}\alpha^2(r_g M_\odot)^2 r^2\left[3 - \frac{2\chi(r)}{r} - \chi'(r)\left(1 - \frac{1}{2}qr\right)^{-1}\right] \\ &8\pi p(r) = 8\pi\frac{p'}{q} - \frac{1}{r^2} + \frac{1}{r^2}\left(1 - \frac{1}{2}q(r)r\right)^{-2} + \frac{1}{r}\left(1 - \frac{1}{2}q(r)r\right)^{-3}(q'r + q) \\ &\quad + \alpha^2(r_g M_\odot)^2\left[3 - \frac{2\chi(r)}{r} - \chi'(r)\left(1 - \frac{1}{2}qr\right)^{-1}\right],\end{aligned} \tag{63}$$

$$8\pi p(r) = -\frac{1}{r^2} + \frac{1}{r^2}(1 - 2qr)\left(1 - \frac{1}{2}qr\right)^{-2} + \alpha^2(r_g M_\odot)^2\left(3 - \frac{2\chi(r)}{r} - \mathrm{e}^{\int qdr}\right), \tag{64}$$

$$\begin{aligned}8\pi p(r) = &\left[q\left(q - \frac{1}{r}\right) - q'\right]\left(1 - \frac{1}{2}qr\right)^{-2} - \frac{1}{2}\left(q - \frac{1}{r}\right)(q'r + q)\left(1 - \frac{1}{2}qr\right)^{-3} \\ &+ \alpha^2(r_g M_\odot)^2\left[3 - \frac{\chi(r)}{r} - \chi'(r)\left(1 - \frac{1}{2}qr\right)^{-1} - \mathrm{e}^{\int qdr}\right].\end{aligned} \tag{65}$$

D. Consistency check

In the last section, we derived the three field equations where the mass parameter $m(r)$ is eliminated by substituting the constraint. Here, we have three arbitrary functions; χ, p, and $\tilde{\rho}$. On the other hand, we have three field equations and use one equation of state later. So, the system appears overconstrained, and we need to check that the two equations are identical.

From Eq. (63) and Eq. (65), we can eliminate χ' and obtain

$$-8\pi\frac{p'}{q} = -\frac{1}{r^2} + \left(\frac{1}{r^2} - q\left(q - \frac{1}{r}\right) + q'\right)\left(1 - \frac{1}{2}qr\right)^{-2} + \frac{1}{2}\left(q + \frac{1}{r}\right)(q'r + q)\left(1 - \frac{1}{2}qr\right)^{-3} + \alpha^2(r_g M_\odot)^2\left[-\frac{\chi(r)}{r} + e^{\int q dr}\right]. \tag{66}$$

Furthermore, we use Eq. (64) and eliminate χ:

$$8\pi p(r) + 16\pi\frac{p'}{q} = \frac{1}{r^2} - \frac{1}{r^2}(2q'r^2 - 2q^2r^2 + 4qr + 1)\left(1 - \frac{1}{2}qr\right)^{-2} - \frac{1}{r}(1 + qr)(q'r + q)\left(1 - \frac{1}{2}qr\right)^{-3} + 3\alpha^2(r_g M_\odot)^2\left[1 - e^{\int q dr}\right]. \tag{67}$$

On the other hand, from Eq. (64), we obtain

$$\alpha^2(r_g M_\odot)^2\chi' = -4\pi p' r - 4\pi p - \frac{1}{2r^2}\left(1 - \frac{1}{2}qr\right)^{-2}(1 - 2qr) + \frac{1}{2r}\left(1 - \frac{1}{2}qr\right)^{-3}(q'r + q)(1 - 2qr) - \frac{1}{r}\left(1 - \frac{1}{2}qr\right)^{-2}(q'r + q) + \frac{1}{2r^2} + \frac{1}{2}\alpha^2(r_g M_\odot)^2\left[3 - (1 + qr)e^{\int q dr}\right], \tag{68}$$

and substitute it into Eq. (63)

$$8\pi p = 8\pi\frac{p'}{q} - \frac{1}{r^2} + \frac{1}{r^2}\left(1 - \frac{1}{2}q(r)r\right)^{-2} + \frac{1}{r}\left(1 - \frac{1}{2}q(r)r\right)^{-3}(q'r + q) + 3\alpha^2(r_g M_\odot)^2 + 8\pi p - \frac{1}{r^2}\left(1 - \frac{1}{2}qr\right)^{-2}(1 - 2qr) + \frac{1}{r^2} - \alpha^2(r_g M_\odot)^2\left[3 - e^{\int q dr}\right] + \left(4\pi p' r + 4\pi p + \frac{1}{2r^2}\left(1 - \frac{1}{2}qr\right)^{-2}(1 - 2qr) - \frac{1}{2r}\left(1 - \frac{1}{2}qr\right)^{-3}(q'r + q)(1 - 2qr) + \frac{1}{r}\left(1 - \frac{1}{2}qr\right)^{-2}(q'r + q) - \frac{1}{2r^2} - \frac{1}{2}\alpha^2(r_g M_\odot)^2\left[3 - (1 + qr)e^{\int q dr}\right]\right)\left(1 - \frac{1}{2}qr\right)^{-1}. \tag{69}$$

After tedious calculation, we obtain the following equation,

$$8\pi p + 16\pi\frac{p'}{q} = \frac{1}{r^2} - \frac{1}{r^2}(2q'r^2 - 2q^2r^2 + 4qr + 1)\left(1 - \frac{1}{2}qr\right)^{-2} - \frac{1}{r}\left(1 - \frac{1}{2}qr\right)^{-3}(q'r + q)(1 + qr) + 3\alpha^2(r_g M_\odot)^2\left[1 - e^{\int q dr}\right]. \tag{70}$$

Equations (67) and (70) are degenerate, and the number of functions and that of equations are identical.

Finally, we find that we need to solve the following equation,

$$8\pi pq + 16\pi p' = \frac{q}{r^2} - \frac{1}{r^2}(2qq'r^2 - 2q^3r^2 + 4q^2r + q)\left(1 - \frac{1}{2}qr\right)^{-2} - \frac{q}{r}\left(1 - \frac{1}{2}qr\right)^{-3}(q'r + q)(1 + qr) + 3\alpha^2(r_g M_\odot)^2\left[q - qe^{\int q dr}\right]. \tag{71}$$

For the equation of state that relates the energy density $\rho(r)$ with the pressure $p(r)$, Eq. (71) is expressed as the differential equation with respect to $p(r)$. By solving the differential equation, we can find the r-dependence of the pressure $p(r)$.

IV. NUMERICAL ANALYSIS FOR RELATIVISTIC STARS

A. Quark stars

In the last section, the fundamental equations and the dynamical variables have been formulated to investigate a relativistic star in dRGT massive gravity. In this section, we study the quark star and neutron star by numerical simulations. Such stars have been studied in $F(R)$ gravity [21–23,41,42]. We compare our results in massive gravity with those in $F(R)$ gravity as well as in general relativity.

The methodology in our numerical simulations is discussed below. In our study, we solve the third order ordinary differential equation with three boundary conditions. We set the first conditions as a value of central density by hand because we need relations of some parameters for a certain central density region. Also, we set the second condition as $p'(r=0)=0$. Finally, we set the third condition to be that the radius of the star becomes identical with that in general relativity for the choice of central density. The radius of the star $r=r_0$ is defined by the condition $p(r_0)=0$.

For solving the equation numerically, we treat the problem as an initial-value problem at the center $r=0$. We set two of the initial conditions as a value of central density and $p'(r=0)=0$ as before. But we should choose the value of $p''(r=0)$ such that the last boundary condition is satisfied. So we optimize the value of $p''(r=0)$ (shooting method).

We should note that the only free parameter in our model is the graviton mass, and we assume $m_0=\sqrt{\Lambda}$ because we use dRGT massive gravity. Also, the graviton mass is constrained by observation in the Solar System and should be small [43–48].

At first, we study the quark star, using the equation of state called the MIT bag model [42]. The equation of state is given by

$$p = c(\rho - 4B). \tag{72}$$

Here, c depends on the mass of strange quark m_s, and $c=0.28$ if we choose $m_s=250$ [MeV]. B is called a bag constant, which we fix as $B=60\ \mathrm{MeV/fm^3}$. Then, we search $\rho(r)$, which satisfies $p(r)=0$. Note that the equation of state is linear; thus, we stop the calculation if $p(r)$ gets negative and define $r=r_0$ as the point between $p(r)>0$ and $p(r)<0$.

We plot the $m-\tilde{\rho}$, $m-r_{\mathrm{max}}$ relations in general relativity (blue line) and massive gravity (orange line) (see Figs. 1 and 2). The total mass of the quark star is smaller than that in general relativity.

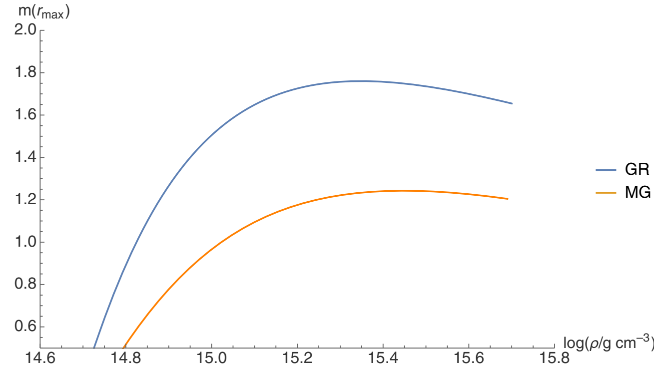

FIG. 1. This figure shows the mass-central density relations for quark star in general relativity and massive gravity.

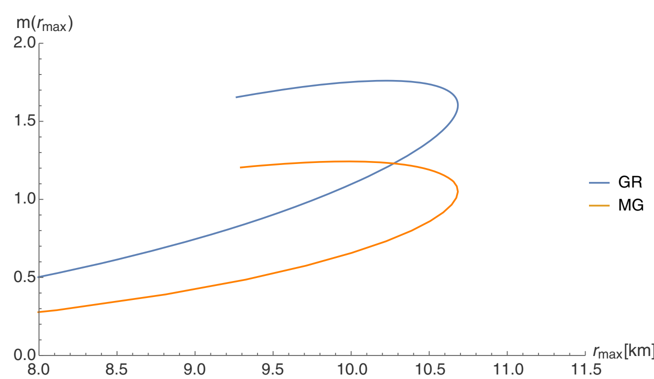

FIG. 2. This figure shows the mass-radius relations for quark star in general relativity and massive gravity.

B. Neutron stars

Next, we study the neutron stars, using the equation of state called the SLy model [49]. The equation of state is given by

$$\xi = \log(\rho/\mathrm{g\ cm^{-3}}),$$
$$\zeta = \log(P/\mathrm{dyn\ cm^{-2}}),$$
$$f_0(x) = \frac{1}{e^x+1}, \tag{73}$$

$$\zeta = \frac{6.22+6.121\xi+0.006004\xi^3}{1+0.16345\xi} f_0(6.50(\xi-11.8440)) + (17.24+1.065\xi)f_0(6.54(11.8421-\xi)) + (-22.003+1.5552\xi)f_0(9.3(14.19-\xi)) + (23.73-1.508\xi)f_0(1.79(15.13-\xi)). \tag{74}$$

In this case, we utilize a function fitted to numerical points in the SLy model (see Fig. 3), which is valid if $\rho \geq 10^5$ [g/cm^3]. We should note that the equation of state

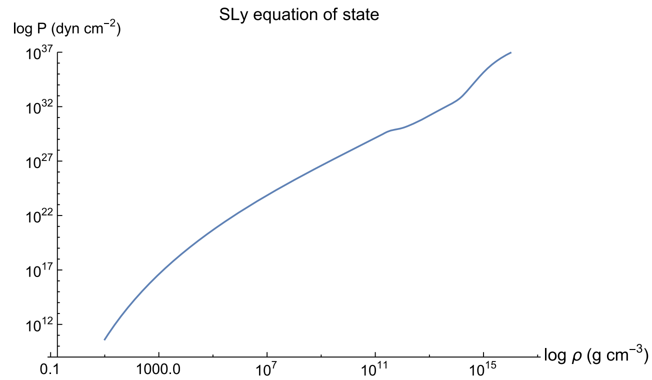

FIG. 3. SLy equation of state.

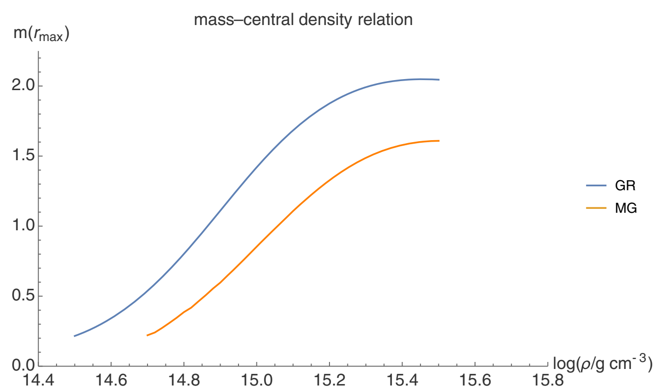

FIG. 4. This figure shows the mass-central density relations for neutron star in general relativity and massive gravity.

includes a logarithmic function, and the radius is ill defined if $p(r) < 0$. Therefore, in this case, we define the radius $r = r_0$ as $p(r_0) < 10^{-11}$ and remove the ill-defined points.

We plot the $m - \tilde{\rho}$, $m - r_{\rm max}$ relations (see Figs. 4 and 5), where we interpolate lines between plotted points after we remove the ill-behaved points. The region of total mass is narrow compared with the case in general relativity; that is, a massive neutron star cannot be realized.

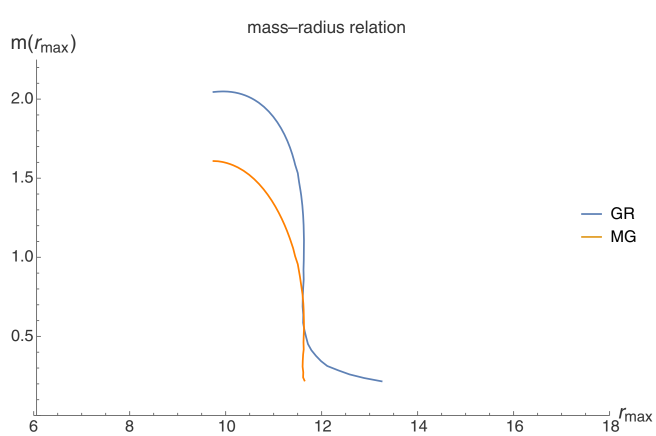

FIG. 5. This figure shows the mass-radius relations for neutron star in general relativity and massive gravity.

V. SUMMARY AND DISCUSSION

We have investigated the relativistic stars in the dRGT massive gravity, which is a nonlinear theory of massive gravitons. We assumed the perfect fluid in hydrostatic equilibrium with the standard equation of state, and after we specified the parameters and reference metric and assumed that the graviton mass is comparable with the cosmological constant, we derived the mass-central density and mass-radius profile for quark stars and neutron stars numerically.

We have concluded that the TOV equation is corrected by the term that is proportional to the graviton mass, which results from the potential term of massive gravitons in the action, and one constraint equation appears if we assume conservation of the energy-momentum tensor. The correction is very small if we consider the light graviton mass against a massive object. The mass-radius relation is more constrained than that in general relativity, and we have shown that the maximal mass gets smaller for quark stars and neutron stars. Therefore, the compact massive neutron star cannot be explained in this specific version of dRGT massive gravity, which is different from $F(R)$ gravity [21,22,41]. However, the results of our work do not indicate that dRGT massive gravity should be excluded by observation. Indeed, we have assumed the standard equations of state to study the maximal mass as well as a particular minimal version of massive gravity. It could be that the massive neutron star could be possible if we choose different equations of state or consider a more complicated version of massive gravity with more parameters.

Although the massive neutron star cannot be realized, we can distinguish dRGT massive gravity from general relativity. The mass-central density and mass-radius relation for the quark star show that the deviation from general relativity is very small. However, for the neutron star, quite an important difference appears between the cases of a small radius and a large one. This may be evidence to quantify the difference from general relativity in the strong-gravity regime. This deviation is considered to be derived from the constraint equation (58), which relates the energy density and pressure inside the neutron star. At the same time, the effect of the mass term is very small, and it is given by the ratio between the graviton mass and neutron star mass $\sim\mathcal{O}(\alpha^2)$. The mass term affects the geometry outside the star and causes the accelerated expansion of the Universe at large scales. Note that the constraint does not depend on the mass of the graviton.

On the other hand, neutron stars in $F(R)$ gravity have been studied in order to test it in astrophysical phenomenology. In previous studies, the gravitational action was assumed to be $F(R) = R + h(R)$, where the function $f(R)$ corresponds to the deviation from general relativity. The mass-central density and mass-radius relation were studied as well in our work, and it has been shown that, for a specific function of $f(R)$, the massive neutron star whose mass $M \sim 2M_{\odot}$ is realized.

In this work, we did not study all models of dRGT massive gravity because we fixed the free parameters β_n and the reference metric $f_{\mu\nu}$. Thus, we have two simple ways to generalize our work. First, one can change the parameters α_3 and α_4 in the two-parameter family of dRGT massive gravity. It means that the potential of the massive graviton is changed and it would lead to a different mass-radius relation. Second, we can change the reference metric from Minkowski to other ones. As we mentioned, we have considered the specific class of the reference metric and formulated the modified TOV equation. A flat reference metric brings a simple but nontrivial modification to the TOV equation. The reference metric may be chosen in a more general form, so that it may not be the solution of equations of motion in general relativity. Regarding the choice of reference metric, we can also generalize our study to the massive bigravity theory, which describes the interaction between the massless spin-2 field, corresponding to usual massless graviton, and massive spin-2 field [50–52] [for the $F(R)$ gravity extension of massive gravity, see [53–55]]. In bigravity, the reference metric is dynamical and determined by the equation of motion; thus, we need not fix the reference metric by hand.

Finally, we make some remarks about compact stars in dRGT massive gravity. The compact stars usually have a maximal mass, and then gravitational collapse to a black hole is inevitable. If the horizon is formed, the interaction terms diverge at the horizon because of $\sqrt{g^{-1}f}$. This singularity is not removable; therefore, the naked singularity appears. However, we do not know what happens after gravitational collapse, precisely.

In massive gravity, the basic equation forms as $G_{\mu\nu} + m_0^2 I_{\mu\nu} = \kappa^2 T_{\mu\nu}$. For example, in the case of the Schwarzschild metric, the Einstein tensor is equal to zero, $G_{\mu\nu} = 0$. If the energy-momentum tensor $T_{\mu\nu} = 0$, the interaction terms $I_{\mu\nu}$ should be zero although these terms include a nontrivial effect caused by $\sqrt{g^{-1}f}$, except for the case that $f_{\mu\nu}$ is proportional to the Schwarzschild metric. For general space-time with a horizon, if the interaction terms diverge at the horizon, $G_{\mu\nu}$ and/or $T_{\mu\nu}$ should diverge, although the strength of the gravitational force, curvature, is finite at the horizon; thus, the energy-density and pressure are also finite. From the viewpoint of analogy to classical mechanics, the mass term is a potential of the metric, and the solution cannot arrive at the horizon if the potential goes to positive infinity at the horizon. For the above reasons, the metric with a horizon may not be a solution.

dRGT massive gravity has the cutoff scale Λ_3, where the ghost mode caused by the higher derivatives is suppressed. Λ_3 depends on the graviton mass m_0, and the cutoff scale is low if the graviton mass is small. In this work, we have assumed $m_0 = \sqrt{\Lambda}$; then, $\Lambda_3 \sim 1000$ [km], which is larger than the scale of compact stars. Heavier massive gravitons make the cutoff scale higher, but it spoils the motivation to explain the late-time acceleration as modified gravity. If we accept the cosmological constant and heavy graviton mass, we can explain relativistic stars, and our numerical results do not change drastically because the correction is proportional to the ratio between the graviton mass and the massive star, and it is still small.

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