Bergman kernel and its boundary asymptotics

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Chapter 1

Introduction

This thesis focus on the Bergman kernel, a reproducing kernel (determined by the complex structure) of the space of L^2 holomorphic top-degree forms on a complex manifold, and consists mainly of the following two parts[1]. Part I is on the relations between Bergman kernels and potentials (Arakelov-Green function, Evans-Selberg potential, etc.). Part II is on the variation (in particular its asymptotic behaviors) of Bergman kernels at degeneration. Asymptotic behaviors of the variation of Bergman kernels at the limiting case can imply (by plurisubharmonicity and convexity) quantitative relations between Bergman kernels and potentials, which is a direct link from Part II to Part I.

1.1 Background

On a connected complex manifold X, the Bergman kernel for a line bundle L equipped with a Hermitian metric h is defined as

$$B := \sum_j |s_j|_h^2, \tag{1.1}$$

where $\{s_j\}_j$ is a complete orthonormal basis of $H^0(X, L)$. Independent of choices of the basis, the Bergman kernel plays big roles in the study of several complex variables and complex geometry. In this thesis, we only consider the canonical bundle K, which is just the cotangent bundle if X is a Riemann surface.

In 1972, Suita [Su] asked about precise relations between the Bergman kernel B and the so-called logarithmic capacity c, and conjectured that $\pi B \geq c^2$ for potential-theoretically hyperbolic Riemann surfaces. This conjecture has a geometric interpretation that the Gaussian curvature of the metric $c(z)|dz|$ (z being the local coordinate) is bounded from above by -4. The relation between the Suita conjecture and the extension theorem was first

[1] It is organized as follows. Among the total five chapters, Chapter 1 and 2 are the introduction and the preliminaries, respectively. Chapter 3 and Chapters 4 & 5 correspond to Part I and Part II, respectively.

observed by Ohsawa [Oh95], who proved that $750\pi B \geq c^2$. Via tools form several complex variables, many mathematicians contributed to this problem [Siu, B96, Ch, Bł07, GZZ]. The optimal constant version of the Ohsawa-Takegoshi L^2 extension theorem was obtained in [Bł13], which implies that the Suita conjecture holds for all bounded planar domains in $\mathbb{C}$. Moreover, this conjecture was shown to be true in [GZ15] for the Riemann surface setting, i.e., open ones admitting Green functions. Thus, it might be interesting to generalize similar results to potential-theoretically `non`-hyperbolic cases.

As the complex structure deforms, for pseudoconvex domains the variation of Bergman kernels was initially studied by Maitani-Yamaguchi [MY] and generalized to higher dimensional cases by Berndtsson [B06]. For general results on arbitrary dimensional Stein manifolds and complex projective algebraic manifolds, see [B09, T, BP, PT]. These important results indicating semi-positivity properties of the relative canonical bundles recently turn out to have close relations with the L^2 extension theorem [OT, GZ15, Ca, BL, Oh15], the space of Kähler metrics [B09b], etc. For simplicity, let us consider the one-dimensional case, namely a holomorphic family of Riemann surfaces X_λ parametrized by one variable $\lambda \in \mathbb{C}$. The Bergman kernel on X_λ can be written as $B_\lambda = k_\lambda(z) dz \wedge d\bar{z}$ in some local coordinate z for some local function k_λ. Then, the above log-plurisubharmonic variation results imply that $\log k_\lambda(z)$ is plurisubharmonic in (λ, z) and particularly guarantee the following semi-positivity:

$$L_{\lambda,z} := \sqrt{-1}\,\partial_\lambda \bar{\partial}_\lambda \log k_\lambda(z) \geq 0, \tag{1.2}$$

when the fiber X_λ is smooth (see also [Fu, Gr, LY]). (1.2) is a restricted version in the sense that we look at the transversal direction. If some X_{λ_0} is `singular`, a possibly interesting question is to characterize $L_{\lambda,z}$ or $\log k_\lambda(z)$, as λ approaches λ_0.

1.2 Question and Answer

My PhD research goal is: on `Riemann surfaces`,

(A) to find relations between Bergman kernels and potentials (Arakelov-Green function, Evans-Selberg potential, etc.), and

(B) to describe asymptotic behaviors of Bergman kernels near degenerate boundaries.

——States of the art——		——My project——
Hörmander's L^2-estimates/ Kodaira vanishing theorem		
can prove $\Downarrow$ [Ch11, Bł13], difficult!		
Ohsawa-Takegoshi L^2-extension (with optimal constant) $\Longrightarrow$	(hyperbolic) Suita conjecture $\dashrightarrow$	**(A)** (`non-hyperbolic` cases) Suita conjecture
[BL] $\Uparrow$ can prove $\Downarrow$ [GZ15, Ca]		
Berndtsson's plurisubharmonic variations of Bergman kernels	$\dashrightarrow$	**(B)** (at `degeneration`) variations of Bergman kernels

(A) Relations with potentials

It is interesting to generalize the Suita type results to `compact` Riemann surfaces, whose Green functions in the usual sense do not exist. However, in arithmetic algebraic geometry it is known that on a compact manifold the Arakelov-Green function and the Arakelov metric play important roles (with applications to string theory), similarly as the Green function and logarithmic capacity do for a bounded domain. We first dealt with a complex torus $X_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ ($\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$), where explicit formulas for the Bergman kernel, the Arakelov-Green function and the Arakelov metric are given by elliptic functions. In [D14], we numerically found a universal constant independent of the complex structure.

Theorem 1.2.1. *For any complex torus X_τ defined as above, it follows that*

(i) $\alpha\pi B \geq c^2, \alpha \approx 6.2034$ *(c being the Arakelov metric and B being the Bergman kernel);*

(ii) "$=$" *is attainable when* $\operatorname{Im}\tau \approx 1.9192$.

On the other hand, we also aim to generalize the Suita type results to the potential-theoretically `parabolic` case. In [D], for a once-punctured complex torus $X_{\tau,u} := X_\tau \backslash \{u\}$ and a once-punctured complex plane, we compared the Bergman kernel and the fundamental metric by constructing explicitly the Evans-Selberg potential, deriving the fundamental metric and discussing their asymptotic behaviors.

Theorem 1.2.2. *Let $X_{\tau,u}$ be defined as above. Let $B_{\tau,u}$ and $c_{\tau,u}$ be its Bergman kernel and fundamental metric, respectively. In the local coordinate z induced from $\mathbb{C}$, write $B_{\tau,u} = k_{\tau,u}(z)dz \wedge d\bar{z}$ and $c_{\tau,u} = c_{\tau,u}(z)dz \wedge d\bar{z}$. Then, as $z \to u$, it follows that*

$$\frac{\pi k_{\tau,u}(z)}{c_{\tau,u}^2(z)} \sim \frac{\pi \cdot |z-u|^2}{2 \cdot \operatorname{Im}\tau} \to 0^+.$$

Moreover, as $X_{\tau,u}$ degenerates to a singular curve, we obtained the following result.

Theorem 1.2.3. *Under the same assumptions as in Theorem 1.2.2, as* $\operatorname{Im}\tau \to +\infty$, *it follows that*

$$\frac{\pi K_{\tau,u}(z)}{c_{\tau,u}^2(z)} \to 0^+.$$

Either Theorem 1.2.2 or Theorem 1.2.3 implies that the Gaussian curvature the of fundamental metric on $X_{\tau,u}$ can be arbitrarily close to 0^-, which is different from the potential-theoretically hyperbolic case (with an upper bound -4).

Corollary 1.2.1. *The Gaussian curvature of the fundamental metric on a once-punctured complex torus cannot be bounded from above by a negative constant.*

(B) Variations and degenerations

A possibly more interesting question is to study the variation (in particular its asymptotic behaviors) of Bergman kernels at `degeneration`.[2] It is known that the curvature's semi-positivities characterize a certain kind of convexity and are often associated with L^2 estimates and extensions, and our research aims to relate to these abstract objects in a quantitative way. The strict positivity of $L_{\lambda,z}$ relates to the hyperellipticity and Weierstrass points (cf. [B11]). In general, this study is much related to the variation of Hodge structures, especially Schmid's Nilpotent Orbit Theorem (see [De, GrSc, KK, Sc, Zu]), and at least three approaches work for this problem: `elliptic function`, `Taylor expansion` and `pinching coordinate`.

(Elliptic curves with nodes or cusps) In the affine coordinate $(x, y) \in \mathbb{C}^2$, the so-called Legendre family of elliptic curves $X^{(1)}_\lambda := \{y^2 = x(x-1)(x-\lambda)\} \cup \{\infty\}$ gives a general description of genus one compact Riemann surfaces, whose moduli space is $\mathbb{C} \setminus \{0, 1\}$. As λ tends to the moduli space boundary, i.e., $\{0, 1, \infty\}$, $X^{(1)}_\lambda$ degenerates to a singular curve with a node. By using the Weierstrass-$\wp$ function's coordinate parameterization and the elliptic modular lambda function's Taylor expansion, we obtained the four-term asymptotic expansion of $L^{(1)}_{\lambda,z}$ near 0 in [D15, D2]. We observed that $L^{(1)}_{\lambda,z}$ coincides with the Poincaré metric of $\mathbb{C} \setminus \{0, 1\}$ and has hyperbolic growth near 0 (in comparison to the Poincaré metric of a punctured disk). The case of other boundary points 1 and ∞ was studied in [D3].

Theorem 1.2.4. *For $\lambda \in \mathbb{C} \setminus \{0, 1\}$, in the local coordinate z induced from $\mathbb{C}$, write $B^{(1)}_\lambda = k^{(1)}_\lambda(z) dz \wedge d\bar{z}$. Then, as $\lambda \to 0$, it follows that*

$$L^{(1)}_{\lambda,z} = \frac{\sqrt{-1}d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}\left(1 + 2\frac{\log 16}{\log|\lambda|} + 3\left(\frac{\log 16}{\log|\lambda|}\right)^2 + 4\left(\frac{\log 16}{\log|\lambda|}\right)^3 + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^4}\right)\right).$$

Next, we answer the following question: how about the cases of other families of elliptic curves (degenerating to a singular one with a node or a cusp at 0) where the special elliptic function method could not apply? The answer is that we can determine accurately not only the leading term but also the subleading terms by a method based on the Taylor expansions of Abelian differentials (cf. [CMSP, D4]). As a conclusion, it seems that various boundary behaviors of the Bergman kernels on elliptic curves are determined both by the type of singularities and by the complex structure information.

On the one hand, using the above alternative approach we then dealt with another nodal family of curves $X^{(2)}_\lambda := \left\{y^2 = (x-1)(x^2-\lambda)\right\}$, where the leading term asymptotics of $L^{(2)}_{\lambda,z}$ turned out to be exactly the same as that of $L^{(1)}_{\lambda,z}$.

[2] From now on for fixed $l \in \mathbb{Z}^+$, we denote $L^{(l)}_{\lambda,z}$ the curvature form (defined as similarly as in (1.2) for some local coordinate z) of the relative Bergman kernel $B^{(l)}_\lambda = k^{(l)}_\lambda(z) dz \wedge d\bar{z}$ on the curve $X^{(l)}_\lambda$, correspondingly.

On the other hand, for the cusp degeneration case we further considered such family of curves $X_\lambda^{(3)} := \left\{y^2 = x(x^2 - \lambda)\right\}$ with a constant period and thus derived that $L_{\lambda,z}^{(3)} \equiv 0$. Moreover, we found that yet another family of curves $X_\lambda^{(4)} := \left\{y^2 = x(x-\lambda)(x-\lambda^2)\right\}$ with a non-constant period is reducible to the $X_\lambda^{(1)}$ case, i.e.,

$$\frac{\sqrt{-1}d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}$$

becomes the leading term asymptotics of $L_{\lambda,z}^{(4)}$ (also of $L_{\lambda,z}^{(2)}$). A possible explanation for the appearance of hyperbolic growth might be that we can change coordinates (from x to $\lambda \cdot x$) to make the reduction. This interesting connection between the cusp case and the node case in some way strengthens the importance of a Legendre family.

(Hyperelliptic and general curves with nodes) For a family of genus two curves $X_\lambda^{(5)} := \left\{y^2 = x(x-\lambda)(x-1)(x-a)(x-b)\right\}$ degenerating to a singular one with a non-separating node, where a, b, λ are distinct numbers in $\mathbb{C} \setminus \{0, 1\}$ satisfying $1 < |a| < |b|$, we determine the precise coefficient as follows.

Theorem 1.2.5. *In the local coordinate $z = \sqrt{x}$ on $X_\lambda^{(5)}$, write its Bergman kernel as $B_\lambda^{(5)} = k_\lambda^{(5)}(z)dz \wedge d\bar{z}$. Then, as $\lambda \to 0$ for $0 \neq |z| < \sqrt{\frac{c_1}{|c_2|}}$, it follows that*

$$\log k_\lambda^{(5)}(z) = \log \frac{4\pi \cdot |z|^2}{c_1\left|(z^2-1)(z^2-a)(z^2-b)\right|} + \left|\frac{1}{z^2} - \frac{c_2}{c_1}\right|^2 \cdot \frac{c_1}{-\log|\lambda|} + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right),$$

where $c_2 = \mathrm{Im}\left\{\int_a^b \frac{\sqrt{ab}dx}{x\sqrt{(x-1)(x-a)(x-b)}}\right\}$, $c_1 = \pi\,\mathrm{Im}\left\{\tau\left(\frac{1-b}{1-a}\right)\right\}$ and $\tau(\cdot)$ is the inverse function of the modular lambda function.

For general curves near both separating and nonseparating nodes, Habermann-Jost obtained the asymptotic results for the Bergman kernel and its induced L^2 metric on the Teichmuller space by using the pinching-coordinate method in [HJ] (see also [F, Ma, Y]). Without caring precise coefficients our results can serve as alternative proofs to these known works. Nevertheless, our results for hyperelliptic curves are based on a different method and has an advantage that we can explicitly write down the coefficients (especially if one wants to know how the given complex structures relate to them), which usually indicate the geometry of the base varieties and their singularities.

By embedding each $X_\lambda^{(5)}$ to its Jacobian, we observe that the curvature form of the relative Bergman kernel metric on their Jacobians in the transversal direction has hyperbolic growth again (see (5.4) in Chapter 5). This can be regarded as a higher dimensional generalization of the leading term in Theorem 1.2.4.

(Hyperelliptic curves with cusps, Case I) Let $p(x)$ be a polynomial of degree at least 2 with roots of distinct absolute values different from $|\lambda|$ and 0. For a family of hyperelliptic curves $X_\lambda^{(6)} := \{y^2 = x(x^2-\lambda)\cdot p(x)\}$, degenerating to a singular one with a cusp as $\lambda \to 0$, our result on asymptotic behaviors of the Bergman kernel is as follows.

Theorem 1.2.6. *In the local coordinate $z = \sqrt{x}$ on $X_\lambda^{(6)}$, write its Bergman kernel as $B_\lambda^{(6)} = k_\lambda^{(6)}(z)dz \wedge d\bar{z}$. Then, as $\lambda \to 0$ for small $|z| \neq 0$, it holds that*

$$\log k_\lambda^{(6)}(z) = \log \frac{4 + \mathrm{O}(z^4)}{|z^4 \cdot p(z^2)|} + \frac{\mathrm{O}(\lambda^{\frac{1}{4}}) \cdot \mathrm{Re}\left(\sum_{j=2}^{g} z^{2(j-1)}\right)}{1 + \mathrm{O}(z^4)}.$$

We see that both the first two terms are harmonic in λ, and the coefficient of the second term is not necessarily positive. Also, it turns out that the Jacobian varieties of $X_\lambda^{(6)}$ remain being manifolds (i.e., non-degenerate).

(Hyperelliptic curves with cusps, Case II) For a family of genus two curves $X_\lambda^{(7)} := \{y^2 = x(x-\lambda)(x-\lambda^2)(x-a)(x-b)\}$, where a, b, λ are distinct complex numbers in $\mathbb{C}\backslash\{0\}$ satisfying $|a| < |b|$, we determine precise coefficients as follows.

Theorem 1.2.7. *In the local coordinate $z = \sqrt{x}$ on $X_\lambda^{(7)}$, write its Bergman kernel as $B_\lambda^{(7)} = k_\lambda^{(7)}(z)dz \wedge d\bar{z}$. Then, as $\lambda \to 0$ for small $|z| \neq 0$, it holds that*

$$\log k_\lambda^{(7)}(z) = \log \frac{4\pi}{c\,|(z^2-a)(z^2-b)|} + \frac{c}{-\log|\lambda| \cdot |z|^4} + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right),$$

where $c := \pi\,\mathrm{Im}\left\{\tau\left(\frac{b}{a}\right)\right\}$ and $\tau(\cdot)$ is the inverse function of the modular lambda function.

In particular, we found that the curvature form $L_{\lambda,z}^{(7)}$ transversally induces an incomplete metric on the parameter space. Also, both coefficients of the above first two terms depend only on the information away from the cusp, which is not the case for $X_\lambda^{(5)}$. For a family of hyperelliptic curves $X_\lambda^{(8)} := \{y^2 = x(x-\lambda^2)(x-\lambda)\cdot p(x)\}$ (p same as above), the result is more or less the same. For the Jacobian varieties of these curves, hyperbolic growth appears again as $\lambda \to 0$ (see the details in Chapter 5).

(Motivation, again) The following question was raised by Tsuji in Hayama Symposium 2016: Can we recover the singularity information (of the base varieties) from the results on boundary asymptotics (of the Bergman kernels)?

Notice that in its Bergman kernel asymptotic formula, $X_\lambda^{(6)}$ possesses a harmonic second term, which differs from $X_\lambda^{(5)}$ and $X_\lambda^{(7)}$. In order to further distinguish the latter two (with the same subharmonic growth in their positive second terms), we see that the coefficient of the leading term in $\log k_\lambda^{(5)}(z)$ tends to 0, as $z \to 0$, which is not the case for $\log k_\lambda^{(7)}(z)$.

Chapter 2

Preliminaries

2.1 Bergman kernel's variation and the L^2 extension theorem

Let us first recall some basic notations and facts. The (negative) Green function for a bounded domain $D \subset \mathbb{C}$ satisfies the following: [1]

$$\begin{cases} \Delta G_D(\cdot, z) = 2\pi\delta_z \\ G_D(\cdot, z) = 0 \quad on\ \partial D. \end{cases}$$

Define the logarithmic capacity of $\mathbb{C} \setminus D$ with respect to z as

$$c_D(z) := \exp \lim_{\zeta \to z} (G_D(\zeta, z) - \log|\zeta - z|).$$

The Bergman kernel is

$$K_D(z) := \sup\{|f(z)|^2 : \text{f holomorphic in D}, \int_D |f|^2 d\lambda \leq 1\}.$$

In 1972, Suita conjectured that: $\pi K_D \geq c_D^2$, for all $z \in D$ (more generally, on open Riemann surfaces admitting Green functions), which is geometrically interpreted as

$$\operatorname{Curv} c_D|dz| \leq -4,$$

due to the Bergman-Schiffer formula

$$K_D = \frac{1}{\pi}\frac{\partial^2}{\partial z \partial \bar{z}}(\log c_D)\,.$$

For some cases, such as a simply connected domain, the " $=$ " can be achieved; and for an annulus, " $>$ " always holds. However, for other cases, it seems difficult to give an answer to this conjecture by direct computations and we may need a tool. In 2013, Błocki [Bł13] obtained the following beautiful result, implying that Suita conjecture holds for any bounded domain $D \subset \mathbb{C}$. Guan-Zhou [GZ15] later showed that this conjecture, originally stated for open Riemann surfaces admitting Green functions, is true.

[1] As a current, $\log|z|$ is related with the dirac function.

Theorem 2.1.1 (Ohsawa-Takegoshi L^2 extension theorem with optimal constant)**.** *Let Ω be a pseudoconvex domain in $\mathbb{C}^{n-1} \times D$, where D is a bounded domain in $\mathbb{C}$ containing the origin. Then for any holomorphic f in $\Omega' := \Omega \bigcap \{z_n = 0\}$ and φ plurisubharmonic in Ω, one can find a holomorphic extension F of f to Ω such that*

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq \frac{\pi}{(c_D(0))^2} \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda'.$$

As the complex structure changes, the variation of the Bergman kernels was initially studied by Maitani-Yamaguchi [MY], who started from the Bergman-Schiffer formula and the Hopf lemma (regarding the Green function as a defining function), and proved the plurisubharmonicity results concerning the Robin constants and logarithms of the Bergman kernels by using differential geometrical computations.

Theorem 2.1.2 (Maitani-Yamaguchi)**.** *Let Ω be a pseudoconvex domain in $\mathbb{C}_z \times \mathbb{C}_t$ with a smooth boundary. Let $B_t(z)$ be the Bergman kernel function of $\Omega_t := \Omega \cap (\mathbb{C}_z \times \{t\})$. Then $\log B_t(z)$ is a plurisubharmonic function on Ω.*

After that, generalizations of [MY] to higher dimensional cases were made by Berndtsson [B06] using L^2 methods.

Theorem 2.1.3 (Berndtsson)**.** *Let D be a pseudoconvex domain in $\mathbb{C}^n_z \times \mathbb{C}^k_t$, and let Φ be a plurisubharmonic function on D. For each t, set $D_t := D \cap (\mathbb{C}^n_z \times \{t\})$ and $\Phi_t := \Phi|_{D_t}$. Let $B_t(z)$ be the Bergman kernel of the space $A^2(D_t, \Phi_t) := \left\{ f \in \mathcal{O}(D_t) | \int_{D_t} e^{-\Phi_t} |f|^2 < +\infty \right\}$. Then $\log B_t(z)$ is a plurisubharmonic function on D.*

Moreover, Guan-Zhou [GZ15] provided an alternative proof of the log-plurisubharmonic variation of Bergman kernels in a general setting by using the optimal constant version of the Ohsawa-Takegoshi L^2 extension theorem, regarded as a sub-mean-value property of the fiber-wise Bergman kernels. Conversely, Berndtsson-Lempert [BL] showed that the log-plurisubharmonic variation of Bergman kernels can give a proof of rather general versions of the Ohsawa-Takegoshi theorem.

2.2 Arakelov-Green function and Evans-Selberg potential

Let Y be a connected compact Riemann surface of genus $g \geq 1$, whose Bergman kernel in the local coordinate z is written as $K = k(z) dz \wedge d\bar{z}$. The Green function (in the usual sense) does not exist on Y. However, the Arakelov-Green function $g_w(z)$ on Y with a pole w does exist and satisfies the equation [Fa]

$$\frac{\partial^2 g_w(z)}{\partial z \partial \bar{z}} = \frac{\pi}{2} \left(\delta(z - w) - \frac{k(z)}{g} \right).$$

Definition 2.2.1. *With the above notations, the Arakelov metric on Y under the local coordinate w is defined as*

$$c(w)|dw|^2 := \exp \lim_{z \to w} \left(g_w(z) - \log |z - w| \right) |dw|^2.$$

The Arakelov metric has a characterizing property that its Gaussian curvature form is proportional to the Bergman kernel [Ar, J]. Most strikingly, the Arakelov-Green function has a clear meaning in physics, pointed out by Ooguri in his recent famous lecture notes [Oo]. We remark that the Arakelov-Green function is identical to the usual Green function for compact Riemannian manifolds in Riemannian geometry.

In the potential-theoretical sense, all open Riemann surfaces can be classified into two types, namely hyperbolic ones and parabolic ones. The latter case happens if and only if there exists no Green function, or equivalently there exists no non-constant subharmonic function bounded from above. On a potentially-parabolic Riemann surface R, there exist a function φ called an Evans-Selberg potential. Also, it is known that the parabolicity condition is equivalent to the existence of the so-called Evans-Selberg potential, which is a harmonic function with one negative logarithmic pole such that this function tends to $+\infty$ near the boundary [Ev, Se, Ku, Na62] and plays similar roles as a Green function does on a potentially-hyperbolic Riemann surface. Typical examples of parabolic planar domains are the whole complex plane $\mathbb{C}$, finitely-punctured complex planes, and $\mathbb{C} \setminus \mathbb{Z}$. Let us start with the definition of an Evans-Selberg potential [SN, p.351], [SNo, p.114].

Definition 2.2.2. *On an open Riemann surface Σ, an Evans-Selberg potential $E_q(p)$ with a pole $q \in \Sigma$ is a real-valued function satisfying the following conditions:*
(i) For all $p \in \Sigma \setminus \{q\}$, $E_q(p)$ is harmonic with respect to p,
(ii) $E_q(p) - \log |\varphi(p) - \varphi(q)|$ is bounded near q, with φ being the local coordinate,
(iii) $E_q(p) \to +\infty$, as $p \to a_\infty$, the Alexandroff ideal boundary point of Σ.

Moreover, if this potential is symmetric in (p, q) and regarded as a function on $\Sigma \times \Sigma$, then $E(p, q) := E_q(p)$ is called an Evans kernel, whose existence is due to Nakai. Two Evans kernels with the same prescribed singularities at the boundary are up to an additive constant by the maximum principle of subharmonic functions. Importance properties of an Evans kernel are its joint continuity and uniform convergence, implying that it is approximable by Green kernels [Na67].

Proposition 2.2.1 (Nakai)**.** *Let $E(p, q)$ be an Evans kernel on Σ, and $G_t(p, q)$ the negative Green kernel on $\Sigma_t := \{p \in \Sigma \,|\, E(p, q_0) < \log N_t\}$ with a fixed $q_0 \in \Sigma$. Then*

$$E(p, q) = \lim_{t \to +\infty} \left(G_t(p, q) + \log N_t \right) \tag{2.1}$$

uniformly on each compact subset of $\Sigma \times \Sigma$, where $N_t = N(t, q_0)$ is increasing in $t > 0$ and $N_t \nearrow +\infty$, as $t \to +\infty$.

This is useful for computing some explicit formulas of Evans-Selberg potentials. Thus, it seems desirable to determine the Evans kernel by (2.1) as long as explicit formulas of Green kernels are known. Meanwhile, the above Σ_t and N_t are attainable in some special cases. Next, we recall the definition of the so-called fundamental metric, which is a non-compact counterpart of the Arakelov metric. For a potential-theoretically hyperbolic Riemann surface, the fundamental metric is just the Suita metric.

Definition 2.2.3. *On a potential-theoretically parabolic Riemann surface X, the fundamental metric under the local coordinate z is defined as*

$$c(z)|dz|^2 := \exp \lim_{w\to z} \left(E_w(z) - \log|z-w|\right)|dz|^2,$$

where $E_w(z)$ is an Evans-Selberg potential on X with a pole w.

For general potential-theoretically parabolic Riemann surfaces, it is known [McV] that the Gaussian curvature form of the fundamental metric is

$$-4\frac{\partial^2}{\partial w \partial \bar{w}} \log c(z) = -4\pi k(z)(\leq 0), \tag{2.2}$$

where $k(z)$ is the coefficient of the Bergman kernel $(1,1)$-form under the local coordinate z. In the case of $\mathbb{C}\setminus\{0\}$, it holds that $k(z) \equiv 0$ and then the Gaussian curvature of the fundamental metric is identically equal to 0. Let's then recall some properties on isolated singularities (cf. [B, Ro]).

Proposition 2.2.2 (Removable Singularity Theorem for a harmonic function)**.** *If u is harmonic and bounded on the punctured disc $\{z \in \mathbb{C} : 0 < |z| < 1\}$, then it extends to a harmonic function on the disc.*

Proof. Without loss of generality, we may assume that u is harmonic on the closed disc. According to Poisson Integral Formula, the function defined by

$$v(z) := \int\limits_{|\zeta|=1} u(\zeta)\frac{1-|z|^2}{|\zeta - z|^2}\frac{d\zeta}{2\pi\sqrt{-1}\zeta}, \quad |z|<1$$

and $v(z) := u(z)$ for $|z| = 1$ is a continuous solution of the Dirichlet Problem

$$\begin{cases} \Delta v = 0, & |z| < 1 \\ v = u, & |z| = 1. \end{cases}$$

Therefore, for any $\epsilon > 0$, we consider a new harmonic function $V(z) := v(z) - u(z) + \epsilon \cdot \log|z|$ defined on $0 < |z| < 1$. Obviously $V(z) = 0$ when $|z| = 1$. And since u is bounded near 0, we have $V(z) \to -\infty$ as $|z| \to 0$. By Maximum Principle, we know $V \leq 0$ when $0 < |z| < 1$, which means $v(z) - u(z) + \epsilon \cdot \log|z| \leq 0$. Then letting $\epsilon \to 0$, we get $v(z) \leq u(z)$, when $0 < |z| < 1$. By interchanging the roles of u and v, we will get $u \leq v$. Therefore, $u(z) = v(z)$ when $0 < |z| < 1$ and thus v is an extension of u to the disc. $\square$

A generalized version of the above proposition, for a subharmonic function, can be described as follows (cf. [Ra, thm 3.6.1]).

Proposition 2.2.3 (Removable Singularity Theorem for a subharmonic function)**.** *Let U be an open subset of $\mathbb{C}$, let E be a closed polar set, and let u be a subharmonic function on $U-E$. Suppose that each point of $U \cap E$ has a neighborhood N such that u is bounded from above on $N-E$. Then u has a unique subharmonic extension to the whole of U.*

Now let's go back to a harmonic function, and the following proposition (which can imply Proposition 2.2.2) describes its behavior near each singularity (cf. [ABR, p. 50]).

Proposition 2.2.4 (Bôchner's Theorem)**.** *Let D be a domain in $\mathbb{C}$, let $w \in D$, and let h be a positive harmonic function on $D-\{w\}$. Then $-h$ extends to be a subharmonic function on D, and there exists a harmonic function k on D and a constant $b \geq 0$, such that*

$$h(z) = k(z) - b \log |z-w|, \quad z \in D - \{w\}.$$

2.3 Existence of holomorphic and harmonic objects on Riemann surfaces

It is known that there exists no non-constant subharmonic function which is bounded from above on $\mathbb{C}$, on a finitely-punctured complex plane, or on $\mathbb{C} \setminus \mathbb{Z}$. For the existence of nontrivial L^2 harmonic and holomorphic 1-forms on an open Riemann surface, see [Dod, Oh87].

Base point freeness

Here we recall a basic and known fact saying that on a compact connected Riemann surface of genus $g \geq 1$ the Bergman kernel never vanishes (for the case of a complex torus $X_\tau := \mathbb{C}/\big(\mathbb{Z}+\tau\mathbb{Z}\big)$, a global coordinate can be induced from the complex plane and the Bergman kernel can be locally written as $\frac{1}{\operatorname{Im}\tau} dz \wedge d\bar{z}$, which is positive since $\frac{1}{\operatorname{Im}\tau} > 0$.), which is true because the canonical bundle is base point free (see the following proposition). Without using the Riemann-Roch theorem, we present a simple proof, since it plays a key role in the construction of the Jacobian embeddings.

Proposition 2.3.1. *Let X be a compact connected Riemann surface of genus $g \geq 1$. Then, for each $p \in X$, there exists a holomorphic 1-form s on X, such that $s(p) \neq 0$.*

Proof. Let us assume that $g > 1$. For any $p \in X$, denote m_p the sheaf of germs of holomorphic functions on X vanishing at p. Denote $\mathcal{O}$ the sheaf of germs of holomorphic functions on X. Then it holds that $\mathcal{O}/m_p \cong \mathbb{C}_p$, where $\mathbb{C}_p$ is the skyscraper sheaf. Taking the canonical bundle K on X, one can get the following short exact sequence of sheaves,

$$0 \to K \otimes m_p \xrightarrow{\iota} K \otimes \mathcal{O} \xrightarrow{r} K \otimes \mathbb{C}_p \to 0,$$

where ι and r are inclusion and restriction maps, respectively. And a long exact sequence of cohomology groups can be induced, namely

$$0 \to H^0(X, K \otimes m_p) \xrightarrow{\iota_1} H^0(X, K \otimes \mathcal{O}) \xrightarrow{r_1} H^0(X, K \otimes \mathbb{C}_p) \xrightarrow{\varsigma}$$

$$\xrightarrow{\varsigma} H^1(X, K \otimes m_p) \xrightarrow{\iota_2} H^1(X, K \otimes \mathcal{O}) \xrightarrow{r_2} H^1(X, K \otimes \mathbb{C}_p) \to 0.$$

Firstly, we know that $H^1(X, K \otimes \mathbb{C}_p) = 0$ and $H^0(X, K \otimes \mathbb{C}_p) \cong \mathbb{C}$, since $\mathbb{C}_p$ is the skyscraper sheaf. Secondly, since every holomorphic function on X is a constant function, we know that $H^1(X, K \otimes \mathcal{O}) \cong H^0(X, \mathcal{O})^\star \cong \mathbb{C}$ by Serre Duality theorem (here * denoting the dual of a given space). Again by the Duality theorem, $H^1(X, K \otimes m_p) \cong H^0(X, (m_p)^\star)^\star = H^0(X, \mathcal{O}(-p)^\star)^\star = H^0(X, \mathcal{O}(p))^\star$, where $\mathcal{O}(-p) \cong (m_p)$ is the sheaf associated to the divisor $-p$ (taking value -1 at p and 0 otherwise). Therefore, the long exact sequence can be re-written as

$$0 \to H^0(X, K \otimes m_p) \xrightarrow{\iota_1} H^0(X, K) \xrightarrow{r_1} \mathbb{C} \xrightarrow{\varsigma} H^0(X, \mathcal{O}(p))^\star \xrightarrow{\iota_2} \mathbb{C} \xrightarrow{r_2} 0.$$

By definition, all constant functions (having non-negative orders everywhere) are contained in $\mathcal{O}(p)$. So it holds that $\dim H^0(X, \mathcal{O}(p)) \geq 1$. Next, we claim that $\dim H^0(X, \mathcal{O}(p)) > 1$ is impossible.[2] If not, then there exists a meromorphic function f with a simple pole at p and no other poles. However, a non-constant holomorphic map from X to $\mathbb{P}^1$ must have degree > 1, since $g > 1$. This is a contradiction. Finally, we know that $\dim H^0(X, \mathcal{O}(p)) = 1$ and therefore $H^0(X, \mathcal{O}(p))^\star \cong \mathbb{C}$. Now, the long exact sequence can be re-written as

$$0 \to H^0(X, K \otimes m_p) \xrightarrow{\iota_1} H^0(X, K) \xrightarrow{r_1} \mathbb{C} \xrightarrow{\varsigma} \mathbb{C} \xrightarrow{\iota_2} \mathbb{C} \xrightarrow{r_2} 0.$$

By the exactness, $\mathbb{C} = \ker(r_2) = \text{Image}(\iota_2)$, so ι_2 is surjective. By the fundamental homomorphism theorem, one gets that $\ker(\iota_2) = \{0\}$. By the exactness once more, it holds that $\text{Image}(\varsigma) = \{0\}$. In particular, for $1 \in \mathbb{C}$, ς maps 1 to 0. So, $1 \in \ker(\varsigma) = \text{Image}(r_1)$. Therefore, there exists a $s \in H^0(X, K) \cong \Omega^1(X)$, such that $s(p) = r_1(s) = 1$. $\square$

An even simpler proof by using Riemann-Roch theorem can be found in [Bo]. Simply by removing finite points from the above compact Riemann surface, we will obtain a potentially-parabolic (open) Riemann surface, whose Bergman kernel is the same as the original compact one, since every L^2-holomorphic function can be extended over those finite points. Sometimes, the non-compact version of this classical result is named "Virtanen theorem" [Alh-Sar]. Therefore, it makes sense to take the logarithm of the Bergman kernel (since it is positive) and further consider its variations. The Hodge star operator does not depends on the choice of the basis. Serre duality has a version for non-compact manifolds. The Hausdorffness and the closed-range properties are in some sense equivalent.

[2] The author thanks Prof. R. Kobayashi for clarifying this point.

Parabolicity (potential-theoretical)

Proposition 2.3.2 (Parabolicity of $\mathbb{C}$)**.** *There exists no non-constant subharmonic function which is bounded from above on $\mathbb{C}$.*

Proof. Its proof can be more or less similar to that of Proposition 2.2.2. Suppose there exists a subharmonic function u, which is bounded from above and defined on the whole complex plane $\mathbb{C}$. We are able to show that it must be a constant function. The idea is to define a new function $v := u - \epsilon \log |z|$, outside the unit disc. Finally, taking a limit as ϵ tending to 0, one will prove that the maximum is attainable inside $\mathbb{C}$. By maximum principle, one concludes that u is constant. $\square$

Theorem 2.3.2 and Proposition 2.2.2 naturally yield the following Proposition.

Proposition 2.3.3 (Parabolicity of $\mathbb{C} \setminus \mathbb{Z}$)**.** *There exists no non-constant subharmonic function which is bounded from above on $\mathbb{C} \setminus \mathbb{Z}$.*

Proposition 2.3.4. *For planar domains $D \subset \mathbb{C}$, the following conditions are equivalent:*
(i) D admits no Green function,
(ii) D admits an Evans-Selberg potential,
(iii) $D = \mathbb{C} \setminus E$, for some polar set E,
(iv) The logarithmic capacity of $\mathbb{C} \setminus D$ is 0.

Definition 2.3.1. *A once-punctured complex torus $X_{\tau,u} := X_\tau \backslash \{u\}$ is an open Riemann surface obtained by removing one single point u from a compact complex torus X_τ.*

Proposition 2.3.5. *There exists no non-constant subharmonic function which is bounded from above on $X_\lambda^{(1)} \setminus \{\infty\}$, for $\lambda \in \mathbb{C} \setminus \{0, 1\}$.*

On the one hand, the above theorem on the parabolicity of an once-punctured torus can be easily generalized to arbitrary once-punctured compact Riemann surfaces (since certain harmonic functions can always be constructed), and by Removable Singularity Theorem it can be even generalized to arbitrary finitely-punctured compact Riemann surfaces. On the other hand, the parabolicity could follow straightforwardly from the Removable Singularity Theorem, but the proof (by using Maximum Principle and finding a harmonic function) works for an open Riemann surface X of infinitely many genus, as we will see below. Equivalently, we could say that X admits an Evans-Selberg potential.

Proposition 2.3.6. *There exists no non-constant subharmonic function which is bounded from above on the algebraic curve $X := \left\{(y, x) \in \mathbb{C}^2 \;\middle|\; y^2 = x \prod_{n=1}^{\infty} \left(1 - x^2/n^2\right) \right\}$.*

Sketch of proof. The idea is by contradiction and suppose there exists a subharmonic function u which is bounded from above on X. On X, we define a harmonic function

$$f(y, x) := \begin{cases} 0, & |x| \leq 1 \\ \log |x|, & |x| > 1, \end{cases}$$

and consider $v := u - \epsilon f$. Taking a limit as ϵ tends to 0, one will prove that the maximum is attainable at the "lift" of the unit circle (away from the boundary). By Maximum Principle, one concludes that u is constant, similar as the proof of Proposition 2.3.2. $\square$

2.4 Bergman kernel by Riemann period matrices

It is known that for an elliptic curve $E := \{y^2 = p_\lambda(x)\}$, $p_\lambda(x)$ being a polynomial of x depending on λ of degree 3 or 4, there exists a globally defined basis $\omega := dx/y$ for the Hilbert space of L^2 holomorphic 1-forms. Locally, ω (which depends on λ, x and y) may be locally written under some u-coordinate as $\omega = f_\lambda(u)du$, where $f_\lambda(u)$ is a holomorphic function in $u \in \mathbb{C}$. After normalizing by the L^2 inner product $\omega_0 := C_\lambda^{-0.5}\omega$ will then become an orthonormal basis of that Hilbert space, where

$$C_\lambda := \frac{\sqrt{-1}}{2}\int_E \omega \wedge \overline{\omega} \ > 0 \tag{2.3}$$

is a positive real number depending only on λ. By definition, the Bergman kernel of the canonical bundle on E is just

$$K_\lambda := \omega_0 \wedge \overline{\omega_0} = C_\lambda^{-1} \cdot \omega \wedge \overline{\omega} \overset{locally}{=\!=\!=} C_\lambda^{-1} \cdot |f_\lambda(u)|^2 du \wedge d\bar{u},$$

a $(1,1)$-form with coefficients depending on both λ and the local coordinates. If one makes a change of holomorphic coordinates for the Bergman kernel, then the determinant of the Jacobian (a harmonic term) is multiplied but killed by the $\partial_\lambda\bar{\partial}_\lambda$ operator. In other words,

$$\partial_\lambda\bar{\partial}_\lambda \log C_\lambda^{-1} = \partial_\lambda\bar{\partial}_\lambda \log k_\lambda(\cdot), \tag{2.4}$$

for any local coefficient $k_\lambda(\cdot)$ of the Bergman kernel. (2.4) shows that only C_λ matters for the curvature form, and thus from now on we will focus on the asymptotic behaviors of C_λ as $\lambda \to 0$. The following figure illustrates the degeneration of an elliptic curve.

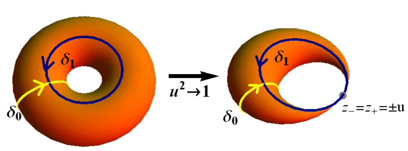

Figure 2.1: Degeneration of a torus (figure from [GJKK])

A Legendre family of elliptic curves

Particularly, if $p_\lambda(x) := x(x-1)(x-\lambda)$, $\lambda \in \mathbb{C} \setminus \{0,1\}$, we get a Legendre family of elliptic curves X_λ. For small $\lambda \to 0$, a double covering of the Riemann sphere can be made by cutting itself from 0 to λ, and from 1 to ∞. Then, we get two cycles δ and γ forming a homologous basis of the elliptic curve, and containing $\{0,\lambda\}$ and $\{\lambda,1\}$, respectively. It is known that

$$C_\lambda = \mathrm{Im}\left(\int_\gamma \omega \cdot \int_\delta \overline{\omega}\right) = \mathrm{Im}\left(\frac{\int_\gamma \omega}{\int_\delta \omega}\right) \cdot \left|\int_\delta \omega\right|^2 > 0. \tag{2.5}$$

Proof of (2.5). Firstly, consider the path integral $p(\cdot) := \int_{z_0}^{\cdot} \omega$ defined on a parallelogram $X_{\lambda,\mathrm{cut}}$ with two sides A and B, which are identified in order to obtain X_λ (regarded as a complex torus). Notice that p is well-defined (independent of paths) and holomorphic, since ω is a holomorphic 1-form and $X_{\lambda,\mathrm{cut}}$ is simply connected. Also, (by Cauchy Integral Theorem) $p(\cdot+A)-p(\cdot) = \int_\delta \omega$ and $p(\cdot+B)-p(\cdot) = \int_\gamma \omega$, implying p is not doubly periodic on X_λ. Moreover, $\frac{\partial p}{\partial u} = f_\lambda(u)$, for some local u-coordinate and thus $\partial p = \omega$. Secondly, we make the following computation

$$d(p \cdot \overline{\omega}) = \partial p \wedge \overline{\omega} + p \cdot \partial(\overline{f_\lambda(u)du}) = \omega \wedge \overline{\omega},$$

and apply the Stokes formula to it on $X_{\lambda,\mathrm{cut}}$. Thirdly, it follows that

$$\int_{\partial X_{\lambda,\mathrm{cut}}} p \cdot \overline{\omega} = \int_{X_{\lambda,\mathrm{cut}}} d(p \cdot \overline{\omega}) = \int_{X_{\lambda,\mathrm{cut}}} \omega \wedge \overline{\omega} = \int_{X_\lambda} \omega \wedge \overline{\omega},$$

which implies that

$$\begin{aligned}
0 < C_\lambda &= \frac{\sqrt{-1}}{2} \int_{\partial X_{\lambda,\mathrm{cut}}} p \cdot \overline{\omega} \\
&= \frac{\sqrt{-1}}{2} \left(\int_\delta p(u)\overline{f_\lambda(u)} d\overline{u} - \int_\delta p(u+B)\overline{f_\lambda(u+B)} d\overline{u} \right. \\
&\qquad \left. + \int_\gamma p(u+A)\overline{f_\lambda(u+A)} d\overline{u} - \int_\gamma p(u)\overline{f_\lambda(u)} d\overline{u} \right) \\
&= \frac{\sqrt{-1}}{2} \left(\int_\delta (p(u) - p(u+B))\, \overline{f_\lambda(u)} d\overline{u} + \int_\gamma (p(u+A) - p(u))\, \overline{f_\lambda(u)} d\overline{u} \right) \\
&= \frac{\sqrt{-1}}{2} \left(\int_\delta \left(- \int_\gamma \omega \right) \overline{f_\lambda(u)} d\overline{u} + \int_\gamma \left(\int_\delta \omega \right) \overline{f_\lambda(u)} d\overline{u} \right) \\
&= \frac{\sqrt{-1}}{2} \left(- \int_\gamma \omega \cdot \int_\delta \overline{\omega} + \int_\gamma \overline{\omega} \cdot \int_\delta \omega \right) = \mathrm{Im} \left(\int_\gamma \omega \cdot \int_\delta \overline{\omega} \right). \quad \square
\end{aligned}$$

To get asymptotic behaviors of the Bergman kernel as $\lambda \to 0$, we may change variables by setting $t = \frac{1}{\lambda}$ and let t (the inverse of λ) tend to ∞. For large t, denote $\tilde{\delta}$ a circle containing points $\{t, \infty\}$ (on a Riemann sphere this is equivalent to say that $-\tilde{\delta}$ contains points $\{0, 1\}$). Similarly, we denote $\tilde{\gamma}$ a circle containing points $\{1, t\}$.

Other families of elliptic curves

Firstly, if $p(x) = (x-1)(x^2-\lambda)$, on the elliptic curve X'_λ ($\lambda \in \mathbb{C} \setminus \{0,1\}$), δ is a big circle centered at the origin containing $-\sqrt{\lambda}$ and $\sqrt{\lambda}$, and γ contains $\sqrt{\lambda}$ and 1. Secondly, if $p(x) = x(x^2-\lambda)$, on the elliptic curve Y_λ ($\lambda \in \mathbb{C} \setminus \{0\}$), δ contains $-\sqrt{\lambda}$ and 0, and γ contains 0 and $\sqrt{\lambda}$. Thirdly, if $p(x) = x(x+\lambda)(x-\lambda^2)$, on the elliptic curve Y'_λ ($\lambda \in \mathbb{C} \setminus \{0\}$), δ contains $-\lambda$ and 0, and γ contains 0 and λ^2. Lastly, the Hodge-Riemann bilinear relations (2.5) hold for all these above cases.

Higher-genus curves

It is known that for the genus $g \geq 2$ hyperelliptic curve $X_\lambda := \{y^2 = P_\lambda(x)\}$, where $P_\lambda(x)$ has distinct roots λ, a_j, and $\lambda \in \mathbb{C} \setminus \{0, \cup_j a_j\}$. There exists a globally defined basis $\omega_1 := dx/y$, $\omega_2 := xdx/y$, ... , $\omega_g := x^{g-1}dx/y$ for the Hilbert space of L^2 holomorphic 1-forms. By definition, the Bergman kernel is the sum of orthogonal base wedging their conjugates. In this case, one needs to orthonormalize the base ω_l by L^2 inner products using the Gram-Schmidt process. There is an equivalent definition of the Bergman kernel (also called canonical kernel), which uses the Riemann period matrix. Any hyperelliptic curve can be obtained as a double covering of the Riemann sphere, cutting itself at the intervals $[0,1]$, $[a,b]$, ... , $[\lambda,\infty]$. We get two types of cycles δ_i and γ_j which forms a homologous basis of the curve, and their intersection number is $\delta_i \cdot \gamma_j = 1$, for $i = j$. Each δ_i is a circle containing the interval $[0,1]$, $[a,b]$, ... , and $[\lambda,\infty]$ respectively. Each γ_j switches from one sheet to another. Then there are two $g \times g$ matrices defined by

$$A(\lambda) \sim \begin{pmatrix} \int_{\delta_1} \omega_1 & \int_{\delta_2} \omega_1 & \cdots & \int_{\delta_g} \omega_1 \\ \int_{\delta_1} \omega_2 & \int_{\delta_2} \omega_2 & \cdots & \int_{\delta_g} \omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\delta_1} \omega_g & \int_{\delta_2} \omega_g & \cdots & \int_{\delta_g} \omega_g \end{pmatrix} =: \{a_{i,j}\}$$

and

$$B(\lambda) \sim \begin{pmatrix} \int_{\gamma_1} \omega_1 & \int_{\gamma_2} \omega_1 & \cdots & \int_{\gamma_g} \omega_1 \\ \int_{\gamma_1} \omega_2 & \int_{\gamma_2} \omega_2 & \cdots & \int_{\gamma_g} \omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\gamma_1} \omega_g & \int_{\gamma_2} \omega_g & \cdots & \int_{\gamma_g} \omega_g \end{pmatrix} =: \{b_{i,j}\}\,.$$

The matrix A is invertible. If one defines a new matrix

$$Z := A^{-1}B,$$

then it can be checked due to the Hodge-Riemann bilinear relation and the Stokes formula that Z is symmetric and has a positive imaginary part, i.e., $\operatorname{Im} Z > 0$. Thus, the Bergman kernel is equivalently defined as

$$K_\lambda := \sum_{i,j=1}^{g} ((\operatorname{Im} Z)^{-1})_{ij}\, \omega_i \wedge \overline{\omega_j}.$$

For fixed $a, b, ...$, it is true that ω_l depends on x, y and λ. Under some $s-$coordinate we may write $\omega = f_\lambda(s)ds$, where $f_\lambda(s)$ is a holomorphic function in $s \in \mathbb{C}$. Also, $f_\lambda(s)$ is a holomorphic function in λ since $\{X_\lambda\}_\lambda$ is a holomorphic family of curves. Thus, the Bergman kernel is a $(1,1)-$form with coefficients depending on both λ and the local coordinates. And after changing coordinates, the coefficients of the Bergman kernel are up to additive harmonic functions with respect to both the parameter λ and the local coordinate. In other words, $\partial_\lambda \bar{\partial}_\lambda \log k_\lambda(\cdot)$ is well-defined. The following figure illustrates one possible degeneration (with a non-separating node) of a genus two curve.

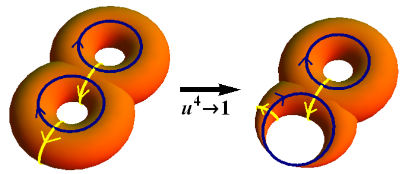

Figure 2.2: Degeneration of a genus two curve (figure from [GJKK])

2.5 General steps and approaches to solve our problems

To solve the problems in Chapter 1, we start from simpler examples such as a family of complex torus or genus-two compact Riemann surface, and then going to higher genus or dimensional cases based on previously obtained results.

Step 1. Genus-one (elliptic function)

From [Ah, p.264], we know that the elliptic modular lambda function $\lambda = \lambda(\tau)$ gives a one-to-one conformal mapping of the region

$$\Omega := \left\{ \tau \in \mathbb{C} \,\middle|\, 0 < \operatorname{Re}\tau < 1,\ |\tau - \frac{1}{2}| > \frac{1}{2},\ \operatorname{Im}\tau > 0 \right\}$$

onto the upper half plane $\mathbb{H}$. Also, this mapping extends continuously to the boundary in such a way that $\tau = \infty$ corresponds to $\lambda = 0$. Let Ω' be the reflection of Ω with respect to the imaginary axis, then Ω and Ω' together correspond to $\mathbb{C} \setminus \{0, 1\}$. In other words, $\operatorname{Im}\tau \to +\infty$ corresponds to $\lambda \to 0$. Since λ is conformal, so is its inverse function $\tau = \lambda^{-1} : \mathbb{C} \setminus \{0, 1\} \to \Omega \cup \Omega'$. Thus, for any fixed $\lambda \in \mathbb{C} \setminus \{0, 1\}$, there exists a complex number $\tau \in \Omega \cup \Omega' \subset \mathbb{H}$. Using 1 and this τ ($\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$) as a lattice one can get a complex torus, denoted by $T_\tau := \mathbb{C}/\big(\mathbb{Z} + \tau\mathbb{Z}\big)$.

The Weierstrass-$\wp$ function with respect to the lattice $(1, \tau)$ is defined to be

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where the sum ranges over all $\omega = n_1 + n_2\tau$ except 0, and $n_1, n_2 \in \mathbb{Z}$. Letting $e_1 := \wp\left(\frac{1}{2}\right)$, $e_2 := \wp\left(\frac{\tau}{2}\right)$, $e_3 := \wp\left(\frac{1+\tau}{2}\right)$, then according to [Ah, p.277], we know that the Weierstrass-$\wp$ function satisfies

$$\wp'(z)^2 = 4\left(\wp(z) - e_1\right)\left(\wp(z) - e_2\right)\left(\wp(z) - e_3\right).$$

Now change the variables, by setting

$$\begin{cases} x = \frac{\wp(z) - e_2}{e_1 - e_2} \\ y = \frac{\wp'(z)}{2(e_1 - e_2)^{\frac{3}{2}}} \end{cases}.$$

We can check that $y^2 = x\,(x-1)\left(x - \frac{e_3-e_2}{e_1-e_2}\right)$. Then the elliptic modular lambda function $\lambda(\tau) := \frac{e_3-e_2}{e_1-e_2}$, which is conformal, can identify X_λ with a complex torus T_τ. Since the area of the parallelogram obtained from the lattice $(1,\tau)$ is Im τ, the normalized holomorphic 1-form is just $\frac{1}{\sqrt{\text{Im } \tau}}dz$. By definition, the Bergman kernel B_τ on T_τ is written as $\frac{1}{\text{Im } \tau}dz \wedge d\bar{z}$, which means that $k_\lambda(z) = \frac{1}{\text{Im } \tau}$. Taking derivatives, one gets that

$$\begin{aligned} l_{\lambda,z} &:= \frac{\partial^2(\log k_\lambda(z))}{\partial\lambda\partial\bar{\lambda}} = \frac{\partial^2(-\log \text{Im } \tau)}{\partial\lambda\partial\bar{\lambda}} \\ &= -\frac{\partial^2(\log(\frac{\tau-\bar{\tau}}{2\sqrt{-1}}))}{\partial\lambda\partial\bar{\lambda}} = -\frac{\partial(\frac{2\sqrt{-1}}{\tau-\bar{\tau}}\frac{\partial}{\partial\bar{\lambda}}(\frac{\tau-\bar{\tau}}{2\sqrt{-1}}))}{\partial\lambda}. \end{aligned}$$

Since $\tau := \lambda^{-1}$ is holomorphic, implying that $\frac{\partial\tau}{\partial\bar{\lambda}} = 0$, it holds that

$$\begin{aligned} l_{\lambda,z} &= -\frac{\partial(\frac{2\sqrt{-1}}{\tau-\bar{\tau}}\frac{\partial}{\partial\bar{\lambda}}(\frac{-\bar{\tau}}{2\sqrt{-1}}))}{\partial\lambda} = \frac{\partial(\frac{\bar{\tau}'}{\tau-\bar{\tau}})}{\partial\lambda} = \frac{\frac{\partial\bar{\tau}'}{\partial\lambda}\cdot(\tau-\bar{\tau}) - \bar{\tau}'\frac{\partial(\tau-\bar{\tau})}{\partial\lambda}}{(\tau-\bar{\tau})^2} \\ &= \frac{0\cdot(\tau-\bar{\tau}) - \bar{\tau}'\frac{\partial(\tau)}{\partial\lambda}}{(\tau-\bar{\tau})^2} = \frac{-|\tau'|^2}{(\tau-\bar{\tau})^2} = \frac{|\tau'|^2}{4(\text{Im}\,\tau)^2}. \end{aligned}$$

Notice that $(\tau-\bar{\tau})^2 = -4(\text{Im}\,\tau)^2 \le 0$. Next, by the inverse function theorem, it holds that $\tau'(b) = (\lambda^{-1})'(b) = \frac{1}{\lambda'(a)}$, for any $b = \lambda(a)$ (here λ' being the derivative of λ with respect to τ). Therefore, we have

$$l_{\lambda,z} = \frac{|\tau'|^2}{4(\text{Im}\,\tau)^2} = \frac{1}{4(\text{Im}\,\tau\cdot|\lambda'(\tau)|)^2} > 0. \tag{2.6}$$

The above inequality holds due to the fact that the derivative of the elliptic modular lambda function is nowhere vanishing in the domain of definition. Thus $L_{\lambda,z} = \sqrt{-1}\,l_{\lambda,z}\,d\lambda\wedge d\bar{\lambda}$ is a true metric on the moduli space, i.e., $L_{\lambda,z} > 0$, for every $\lambda \in \mathbb{C}\setminus\{0,1\}$. Moreover, it holds that

$$-4\frac{\partial^2}{\partial\lambda\partial\bar{\lambda}}\log\left(\frac{|\tau'|}{2\cdot\text{Im}\,\tau}\right) = -\frac{|\tau'|^2}{(\text{Im}\,\tau)^2}.$$

Next, we introduce two more parameters $\alpha := -\frac{1}{\tau}$ and $\beta := \tau - 1$, both of which have positive imaginary parts as long as $\text{Im}\,\tau > 0$. As $\tau \to 0$ or equivalently as $\lambda(\tau) \to 1$, it follows that $\text{Im}\,\alpha \to +\infty$ and $\lambda(\alpha) \to 0$. By the definition of α, it holds that

$$\text{Im}\,\tau = \text{Im}\left(\frac{-1}{\text{Re}\,\alpha + \sqrt{-1}\,\text{Im}\,\alpha}\right) = \text{Im}\left(\frac{\sqrt{-1}\,\text{Im}\,\alpha - \text{Re}\,\alpha}{|\alpha|^2}\right) = \text{Im}\,\alpha\cdot|\tau|^2. \tag{2.7}$$

Similarly, as $\tau \to 1$ or equivalently as $\lambda(\tau) \to \infty$, it follows that $\beta \to 0$. Since $\text{Im}\,\beta = \text{Im}\,\tau$, we know that $\text{Im}\,\beta \to +\infty$ implies $\lambda(\tau) \to 0$.

Alternatively, one can also use the Taylor expansion of the Abelian differential dx/y to study elliptic curves.

Step 2. Genus-two

Let us consider $X_{a,b,c} := \{y^2 = x(x-1)(x-a)(x-b)(x-c)\}$, which usually represents a genus-two compact Riemann surface (nonsingular), parametrized by three distinct complex numbers $a, b, c \in \mathbb{C} \setminus \{0, 1\}$. However, as a, b, c tend to 0 or 1 or ∞, or towards each other, $X_{a,b,c}$ will become singular. dx/y and xdx/y forms an orthogonal basis for the space of square-integrable holomorphic 1-forms, and after the Gram-Schmidt process we are able to get the Bergman kernel $K_{a,b,c}$. For the Arakelov metric, formulas for genus-two compact Riemann surfaces are known due to Bost, Mestr and Moret-Bailly.

Step 3. Hyperelliptic

As a natural generalization of the genus-two case, a genus $g\,(\geq 2)$ hyperelliptic compact Riemann surface X may be written as $\{y^2 = P(x)\}$, with P being a polynomial of degree > 4. The moduli space of X are of dimension $3g-3$. We can use the Taylor expansions of the Abelian differential for this case.

(Local coordinates near a node) For $X_\lambda^{(10)} := \left\{y^2 = x(x-\lambda)\cdot p(x)\right\}$, which is a family of hyperelliptic curves degenerating to a singular one $X_0^{(10)}$ with a non-separating node, as $\lambda \to 0$, the local coordinate near $x = 0$ can be given by $z = \sqrt{x}$. Thus, it holds that $x = z^2$ and $dx = 2zdz$, which gives that

$$\omega_j = \frac{2z^{2(j-1)}dz}{\sqrt{(z^2-\lambda)\cdot p(z^2)}}.$$

Therefore, in coordinate z the Bergman kernel can be written as

$$\sum_{i,j=1}^{g} (\operatorname{Im}^{-1} Z)_{ij} \frac{4z^{2(i-1)}\cdot \overline{z}^{2(j-1)}}{|(z^2-\lambda)\cdot p(z^2)|} dz\wedge d\bar{z} =: k_\lambda^{(10)}(z) dz\wedge d\bar{z}.$$

In particular, any genus two compact Riemann surfaces is hyperelliptic, and its Bergman kernel near the node $(0,0)$ in coordinate z is written as

$$4\cdot \frac{(\operatorname{Im}^{-1} Z)_{11} + (\operatorname{Im}^{-1} Z)_{12}\overline{z}^2 + (\operatorname{Im}^{-1} Z)_{21}\cdot z^2 + (\operatorname{Im}^{-1} Z)_{22}|z|^4}{|(z^2-1)(z^2-a)(z^2-b)(z^2-\lambda)|} dz\wedge d\bar{z}. \tag{2.8}$$

(Local coordinates near a cusp, Case I) For $X_\lambda^{(6)}$, similarly, it holds that

$$\omega_j = \frac{2z^{2(j-1)}dz}{\sqrt{(z^4-\lambda)\cdot p(z^2)}}.$$

Therefore, in the coordinate z near the cusp $(0,0)$ the Bergman kernel is

$$\sum_{i,j=1}^{g} (\operatorname{Im}^{-1} Z)_{ij} \frac{4z^{2(i-1)}\cdot \overline{z}^{2(j-1)}}{|(z^4-\lambda)\cdot p(z^2)|} dz\wedge d\bar{z} =: k_\lambda^{(6)}(z) dz\wedge d\bar{z}.$$

In the genus two case, the Bergman kernel can be written as

$$4 \cdot \frac{(\mathrm{Im}^{-1} Z)_{11} + (\mathrm{Im}^{-1} Z)_{12}\overline{z}^2 + (\mathrm{Im}^{-1} Z)_{21} \cdot z^2 + (\mathrm{Im}^{-1} Z)_{22}|z|^4}{|(z^4-\lambda)(z^2-a)(z^2-b)|} dz \wedge d\bar{z}. \tag{2.9}$$

(Local coordinates near a cusp, Case II) For $X_\lambda^{(8)}$, similarly, it holds that

$$\omega_j = \frac{2z^{2(j-1)}dz}{\sqrt{(z^2-\lambda)(z^2-\lambda^2)\cdot p(z^2)}}.$$

Therefore, in the coordinate z near the cusp $(0,0)$ the Bergman kernel is

$$\sum_{i,j=1}^{g} (\mathrm{Im}^{-1} Z)_{ij} \frac{4z^{2(i-1)} \cdot \overline{z}^{2(j-1)}}{|(z^2-\lambda)(z^2-\lambda^2)\cdot p(z^2)|} dz \wedge d\bar{z} =: k_\lambda^{(8)}(z) dz \wedge d\bar{z}.$$

In the genus two case, the Bergman kernel can be written as

$$4 \cdot \frac{(\mathrm{Im}^{-1} Z)_{11} + (\mathrm{Im}^{-1} Z)_{12}\overline{z}^2 + (\mathrm{Im}^{-1} Z)_{21} \cdot z^2 + (\mathrm{Im}^{-1} Z)_{22}|z|^4}{|(z^2-\lambda)(z^2-\lambda^2)(z^2-a)(z^2-b)|} dz \wedge d\bar{z}. \tag{2.10}$$

Let us then recall some Taylor expansions. The Taylor expansion of the function $\frac{1}{\sqrt{1-x}}$ for x near $0 \in \mathbb{C}$ says that $\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \mathrm{O}(x^3)$. When $|a| < |s|$, it holds that

$$\frac{1}{\sqrt{s-a}} = \frac{1}{\sqrt{s}} \left\{ 1 + \frac{a}{2s} + \frac{3a^2}{8s^2} + \mathrm{O}(\frac{a^3}{s^3}) \right\}.$$

As $|s| < |a|$, it holds that

$$\frac{1}{\sqrt{s-a}} = \frac{1}{\sqrt{-a}} \cdot \left(1 + \frac{s}{2a} + \frac{3s^2}{8a^2} + \mathrm{O}(\frac{s^3}{a^3}) \right). \tag{2.11}$$

Step 4. General curve

For general curves, we could use the pinching coordinates near a node, namely $\{xy = \lambda\}$, as $\lambda \to 0$. The advantage of a pinching coordinate is that it does not prescribe the complex structure and it works not only for non-separating but also for separating nodes. A complementary method is that any curve can be embedded into its Jacobian variety, and the Bergman kernel is just the pull-back of the flat metric there. By Riemann theta function, the asymptotic behavior is desirable.

Chapter 3

Bergman kernel and potentials

For any complex torus, we compute via elliptic functions the ratio of the Bergman kernel and the Arakelov metric, and obtain a sharp positive lower bound. For a once-punctured complex torus, we compare the Bergman kernel and the fundamental metric, by constructing explicitly the Evans-Selberg potential and discussing its asymptotic behaviors. These works aim to generalize the Suita type results to potential-theoretically non-hyperbolic Riemann surfaces.

3.1 Bergman kernel and Arakelov-Green function: compact Suita conjecture

A complex torus

Since compact Riemann surfaces can be classified by the genus, we will first deal with a complex torus, which is denoted by $X_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, where $\tau \in \mathbb{C}$ and $\operatorname{Im}\tau > 0$. Consider the Hilbert space of L^2-integrable holomorphic 1-forms on X_τ, and we know that the Bergman kernel of X_τ by definition is the (1,1)-form

$$K_{X_\tau}(z) = \frac{1}{\operatorname{Im}\tau} dz \wedge d\bar{z},$$

where z is the local coordinate induced from the complex plane $\mathbb{C}$.

The Arakelov-Green function $g(z,w) : X_\tau \times X_\tau \to \mathbb{R}$ with a pole $w \in X_\tau$ satisfies that

(a) $\frac{\partial^2 g(z,w)}{\partial z \partial \bar{z}} = -\frac{\pi}{2} \cdot \frac{1}{\operatorname{Im}\tau}$ on $X_\tau \setminus \{w\}$;

(b) $g(z,w) = \log|z-w| + O(1)$, as $z \to w$;

(c) $g(w,w) = -\infty$.

And it can be expressed as (see [We])

$$g(z,w) = \log \frac{\|\theta\|(z-w+\frac{1+\tau}{2};\tau)}{\|\eta\|(\tau)}. \tag{3.1}$$

Here $\theta(z;\tau) := \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z)$ is the theta function and

$$\eta(\tau) := q^{\frac{1}{12}} \cdot \prod_{n=1}^{\infty}(1-q^{2n})$$

is the Dedekin-η function, where we denote $\exp(\pi i \tau)$ by q. Here it holds that

$$\|\theta\|(x+iy;\tau) = \sqrt[4]{\mathrm{Im}\ \tau} \cdot e^{-\pi y^2/\mathrm{Im}\ \tau} \cdot |\theta(x+iy;\tau)|,$$

and

$$\|\eta\|(\tau) = \sqrt[4]{\mathrm{Im}\ \tau} \cdot |\eta(\tau)|.$$

Substituting these into (3.1), we will get

$$\begin{aligned}
g(z,w) &= \log \frac{\sqrt[4]{\mathrm{Im}\ \tau} \cdot e^{-\pi\left(\mathrm{Im}\,(z-w+\frac{\tau}{2})\right)^2/\mathrm{Im}\ \tau} \cdot |\theta(z-w+\frac{1+\tau}{2};\tau)|}{\sqrt[4]{\mathrm{Im}\ \tau} \cdot |\eta(\tau)|} \\
&= \log \frac{e^{-\pi\left(\mathrm{Im}\,(z-w+\frac{\tau}{2})\right)^2/\mathrm{Im}\ \tau} \cdot |\theta(z-w+\frac{1+\tau}{2};\tau)|}{|\eta(\tau)|} \\
&= \log \frac{\exp(-\pi\big(\mathrm{Im}\,(z-w+\frac{\tau}{2})\big)^2/\mathrm{Im}\ \tau) \cdot |\theta(z-w+\frac{1+\tau}{2};\tau)|}{|q^{\frac{1}{12}} \cdot \prod_{n=1}^{\infty}(1-q^{2n})|} \\
&= \log \frac{|\theta(z-w+\frac{1+\tau}{2};\tau)|}{|\eta(\tau)|} - \frac{\pi}{\mathrm{Im}\,\tau} \cdot \Big(\mathrm{Im}\,(z-w+\frac{\tau}{2})\Big)^2 = \log\left|\frac{\theta_1(z-w;q)}{\eta(\tau)}\right| - \frac{\pi \cdot (\mathrm{Im}(z-w))^2}{\mathrm{Im}\,\tau},
\end{aligned}$$

where θ_1 is defined in (3.2). From [Mu, p.17], it holds that

$$\theta(z+\frac{1+\tau}{2};\tau) = \exp(-\frac{1}{4}\pi i \tau - \pi i(z+\frac{1}{2})) \sum_{n=-\infty}^{\infty} \exp(\pi i(n+\frac{1}{2})^2\tau + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2}))$$

Applying Jacobi triple product (cf. [Ap, Theorem 14.6]), since $q = \exp(\pi i \tau)$, we know

$$\begin{aligned}
g(z,w) = &\log\left(e^{\frac{-\pi\left(\mathrm{Im}\,(z-w+\frac{\tau}{2})\right)^2}{\mathrm{Im}\ \tau}} \cdot \left| 2q^{\frac{1}{6}} \sin(\pi(z-w)) \prod_{n=1}^{\infty}(1-2\cos(2\pi(z-w))q^{2n}+q^{4n}) \right| \right) \\
&\cdot \left| -\frac{1}{4}\pi i \tau - \pi i(z-w+\frac{1}{2})) \right|.
\end{aligned}$$

By definition, it follows that

$$
\begin{aligned}
c_{X_\tau}(z) &= \exp \lim_{w\to z} \left(g_\tau(z,w) - \log|z-w|\right) \\
&= \lim_{w\to z} \frac{e^{\frac{-\pi\left(\mathrm{Im}\,(z-w+\frac{\tau}{2})\right)^2}{\mathrm{Im}\ \tau}} \cdot |2q^{1/6}\sin(\pi(z-w))\prod_{n=1}^{\infty}(1-2\cos(2\pi(z-w))q^{2n}+q^{4n})e^{(-\frac{1}{4}\pi i\tau-\pi i(z-w+\frac{1}{2}))}|}{|z-w|} \\
&= \exp^{-\pi\left(\mathrm{Im}\,(\frac{\tau}{2})\right)^2/\mathrm{Im}\ \tau} \cdot 2\pi\cdot|q^{1/6}\prod_{n=1}^{\infty}(1-q^{2n})^2\cdot\exp(-\frac{1}{4}\pi i\tau-\pi i(\frac{1}{2}))| \\
&= \exp^{-\frac{\pi}{4}\,\mathrm{Im}\ \tau}\cdot 2\pi\cdot e^{-\frac{\pi}{6}\,\mathrm{Im}\,\tau}\cdot|\prod_{n=1}^{\infty}(1-q^{2n})^2|\cdot\exp^{\frac{\pi}{4}\,\mathrm{Im}\ \tau} \\
&= 2\pi\cdot\exp(-\frac{\pi}{6}\,\mathrm{Im}\,\tau)\cdot|\prod_{n=1}^{\infty}(1-q^{2n})^2| = 2\pi\cdot|\eta(\tau)|^2.
\end{aligned}
$$

To acknowledge the relations between πK_{X_τ} and $c^2_{X_\tau}$, we define their ratio as a new function and get that

$$
\begin{aligned}
F(\tau) :&= \log\frac{\pi K_{X_\tau}}{c^2_{X_\tau}} = \log\frac{\pi\cdot\frac{1}{\mathrm{Im}\ \tau}}{4\pi^2\cdot e^{-\frac{\pi}{3}\,\mathrm{Im}\,\tau}|\prod_{n=1}^{\infty}(1-q^{2n})^4|} \\
&= \log\frac{1}{\mathrm{Im}\ \tau\cdot 4\pi\cdot e^{-\frac{\pi}{3}\,\mathrm{Im}\,\tau}|\prod_{n=1}^{\infty}(1-q^{2n})^4|} = -\log(\mathrm{Im}\ \tau\cdot 4\pi\cdot e^{-\frac{\pi}{3}\,\mathrm{Im}\,\tau}|\prod_{n=1}^{\infty}(1-q^{2n})^4|) \\
&= -\log(\mathrm{Im}\ \tau) - \log(4\pi) + \frac{\pi}{3}\,\mathrm{Im}\,\tau - 4\sum_{n=1}^{\infty}\log|1-q^{2n}|.
\end{aligned}
$$

With the help of computers, we obtained that $F(\tau)\geq -1.8251$ and therefore $\exp F(\tau)\geq 0.1612$, $\forall\,\tau\in\mathbb{C}\,(\mathrm{Im}\,\tau>0)$. This means for any complex torus, the following inequality always holds:

$$
\alpha\pi K_{X_\tau}\geq c^2_{X_\tau},\quad \alpha\approx 6.2034.
$$

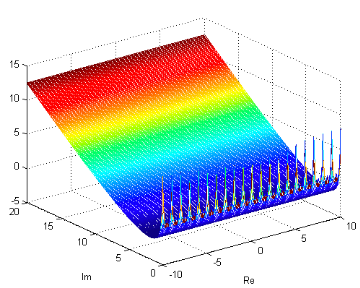

Figure 3.1: The 3D-graph of $F(\tau)$

And the " = " is attained when $\operatorname{Im}\tau \approx 1.9192$ (explicit programs for this numerical optimization attached below). In the proof, the following property is used.

Proposition 3.1.1. *The infinite product* $\prod_{n=1}^{\infty}(1+a_n)$ *converges absolutely if and only if* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

Since the last term of $F(\tau)$ converges and tends to 0 as $\operatorname{Im}\tau \to +\infty$, particularly we obtain the following corollary.

Corollary 3.1.1. *It follows that*

$$\lim_{\operatorname{Im}\tau\to+\infty} \frac{\pi K}{c^2} = +\infty.$$

This result can be read as: For this special torus, the above α can be close to 0.

Remark If we denote

$$\theta_{11}(z;q) := -2q^{1/4}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^{2n})(1-2\cos(2\pi z)q^{2n}+q^{4n}),$$

then the Jacobi triple product will yield that

$$\theta_{11}(z;q) = \exp(\frac{1}{4}\pi i\tau + \pi i(z+\frac{1}{2}))\theta(z+\frac{1+\tau}{2};\tau).$$

Therefore, we could also rewrite

$$g(z,w) = \log \frac{e^{\frac{-\pi\left(Im\,(z-w+\frac{\tau}{2})\right)^2}{Im\,\tau}} \cdot |\theta_{11}(z-w;q)\cdot\exp(-\frac{1}{4}\pi i\tau - \pi i(z-w+\frac{1}{2}))|}{|q^{\frac{1}{12}}\cdot\prod_{n=1}^{\infty}(1-q^{2n})|}.$$

Programs on MATLAB.

To make this paper complete, we attach here the explicit programs, to get the numerical results concerning $F(\tau)$ defined above, running on MATLAB. We first define a function called test(x,y,N) as below:

```
function f = test(x,y,N)

for k = 1:N
   s(k) = log(abs(1-exp(pi*(-y+x*1i))^(2*k)));
end

f = -2*log(y) - log(4*pi) + pi*y/3 - 4*sum(s);
```

Then we make the following computations:

```
function [f,a,b] = myplot(x,y,K,M,N)

if nargin<5,    N = 100;
   if nargin<4, M = 100;
   end;
end

X = linspace(-x,x,M);
Y = linspace(0,y,N);
[X1,Y1] = meshgrid(X,Y);
[m,n] = size(X1);

for i = 1:m
   for j = 1:n
      F(i,j) = test(X1(i,j),Y1(i,j),K);
   end
end

mesh(X1,Y1,F);
[F1,i] = min(F);
[f,j]  = min(F1);
i = i(j);
a = X1(i,j);
b = Y1(i,j);
```

Here $[-x,x]\times[0,y]$ forms the region where we plot the 3D-graph of $F(\tau)$. f represents the minimal value on it for $F(\tau)$, achieved when $\tau = a+bi$. And K, chosen to be large enough, denotes the total times of summation needed to get the desired precision.

To run this program, we may first choose the appropriate x^*, y^* and K^*. Then it suffices to type into the command window:

$$\gg\gg \qquad [f,a,b] = myplot(x^*,y^*,K^*) \qquad \ll\ll$$

Method of Mathematica

The Arakelov-Green function on a complex torus X_τ defined as above with a pole w satisfies the equation

$$\frac{\partial^2 g_w(z)}{\partial z\bar{\partial} z} = \frac{\pi}{2}\left(\delta(z-w) - \frac{1}{\operatorname{Im}\tau}\right),$$

and can be expressed via the theta function as

$$g_w(z) = \log\left|\frac{\theta_1(z-w;q)}{\eta(\tau)}\right| - \frac{\pi\cdot(\operatorname{Im}(z-w))^2}{\operatorname{Im}\tau}.$$

Here $\eta(\tau) = q^{\frac{1}{12}}\cdot\prod_{m=1}^{\infty}(1-q^{2m})$ and

$$\theta_1(z;q) := 2q^{1/4}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^{2n})(1-2\cos(2\pi z)q^{2n}+q^{4n}), \tag{3.2}$$

for $q = \exp(\pi i \tau)$. In this case, it is possible to compare the Bergman kernel and the Arakelov metric. Instead of using programming to numerically approximate the lower bound as in [D14][1], the Gaussian curvature of the Arakelov metric can be computed by *Mathematica* (Version 10.3). The following figure (plotted by *Mathematica*) illustrates that the supremum of an Arakelov-Green function on a torus can be positive.

```
Plot3D[Log [
  Abs [ EllipticTheta[1, x + y I, Exp[-Pi 0.9]]]/
   Abs [DedekindEta [0.9 I]]] - Pi/0.9 y^2, {x, -5, 5}, {y, -1, 1},
 AspectRatio -> 1, PlotRange -> {0.236, 0.229}]
```

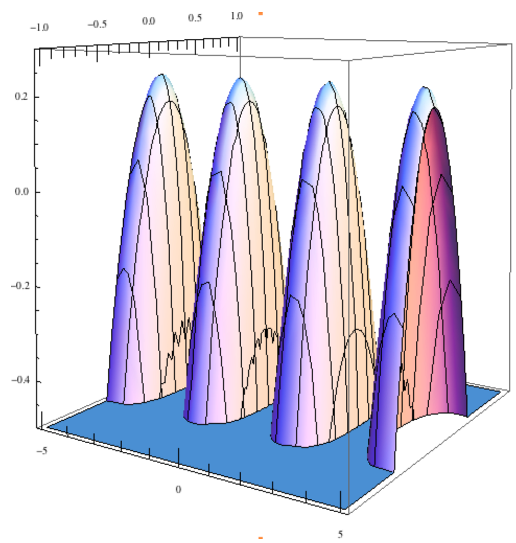

Figure 3.2: Figure of an Arakelov-Green function on a torus

In general, the Arakelov-Green function can be positive at some cases (see Figure 3.2). This is different from the potential-theoretically hyperbolic case where the Green function is always less than 0 in the interior. For the same reason, Berndtsson-Lempert's method cannot be directly applied to the potential-theoretically parabolic case, since an Evans-Selberg potential tends to $+\infty$ near the boundary.

3.2 Explicit formulas of the Evans-Selberg potential

Explicit formulas of the Evans-Selberg potentials are not quite understood, except the case of $\mathbb{C}$ where the logarithmic kernel $\log|p - q|$ becomes a good candidate. In this section, we first provide an explicit formula of the Evans-Selberg potential on the once-punctured complex plane.

[1]The author apologizes for several mistakes contained in [D14].

Theorem 3.2.1. *There exists an Evans-Selberg potential on $\mathbb{C}\setminus\{0\}$ with a pole q given by*

$$E_q(p) := \log\left|\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right|.$$

Since the right hand side is symmetric in (p, q), it is also called an Evans kernel and we have the following corollary.

Corollary 3.2.1. *Let $G_t(p, q)$ be the negative Green kernel on $\left\{z \in \mathbb{C}\,|e^{-2t} < |z| < e^{2t}, t > 0\right\}$. Then it follows that*

$$\lim_{t\to+\infty}\left(G_t(p,q) + \log\left(e^t - e^{-t}\right)\right) = \log\left|\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right|,$$

uniformly on each compact subset of $\mathbb{C}\setminus\{0\}\times\mathbb{C}\setminus\{0\}$.

Proof of Theorem 3.2.1. Without loss of generality assume $q_0 = 1$ in Proposition 2.2, since $E(p, q)$ does not depends on a specific q_0. For any $t > 0$, choose a function $r = r(t) > 0$ (to be determined later) such that $r \searrow 0^+$ as $t \to +\infty$ and the annulus $A_r := \{p \in \mathbb{C}\,\left|r < |p| < \frac{1}{r}\right.\}$ is equal to $\Sigma_t := \{p \in \Sigma\,|E(p, q_0) < \log N_t\}$. By [CH, p.386-388], the negative Green kernel for A_r can be expressed as

$$G_t(p,q) = \mathrm{Re}\left\{\log\left(ip^{-2\pi i\alpha/\log r}\frac{\theta_1(\nu-\alpha;r^2)}{\theta_0(\nu-\alpha;r^2)}\right)\right\},$$

where $\nu := \frac{\log p}{2\pi i}$, $\alpha := \frac{\log q}{2\pi i}$ and thus $G_t(p, q)$ is symmetric in (p, q). Here

$$\theta_1(x;r^2) = -iC_r\cdot r^{\frac{1}{2}}\left(e^{i\pi x} - e^{-i\pi x}\right)\prod_{j=1}^{\infty}\left(1 - r^{4j}e^{2i\pi x}\right)\left(1 - r^{4j}e^{-2i\pi x}\right)$$

and

$$\theta_0(x;r^2) = C_r\cdot\prod_{j=1}^{\infty}\left(1 - r^{4j-2}e^{2i\pi x}\right)\left(1 - r^{4j-2}e^{-2i\pi x}\right)$$

are theta functions with

$$C_r := \prod_{j=1}^{\infty}(1 - r^{4j}).$$

By Proposition 2.2.1, it follows that

$$\begin{aligned}
&E(p,q) = \lim_{t\to\infty}\left(G_t(p,q) + \log N_t\right)\\
&= \lim_{t\to\infty}\left\{\mathrm{Re}\left\{\log\left(ip^{-2\pi i\alpha/\log r}\frac{\theta_1(\nu-\alpha;r^2)}{\theta_0(\nu-\alpha;r^2)}\right)\right\} + \log N_t\right\}\\
&= \lim_{t\to\infty}\left\{\log\left|ip^0\cdot\frac{-ir^{\frac{1}{2}}\left(e^{i\pi(\nu-\alpha)} - e^{-i\pi(\nu-\alpha)}\right)\prod_{j=1}^{\infty}\left(1 - r^{4j}e^{2i\pi(\nu-\alpha)}\right)\left(1 - r^{4j}e^{-2i\pi(\nu-\alpha)}\right)}{\prod_{j=1}^{\infty}\left(1 - r^{4j-2}e^{2i\pi(\nu-\alpha)}\right)\left(1 - r^{4j-2}e^{-2i\pi(\nu-\alpha)}\right)}\right| + \log N_t\right\}\\
&= \lim_{t\to\infty}\left\{\log\left|r^{\frac{1}{2}}\cdot\left(e^{i\pi(\nu-\alpha)} - e^{-i\pi(\nu-\alpha)}\right)\right| + \log N_t\right\}\\
&= \log\left|e^{i\pi(\nu-\alpha)} - e^{-i\pi(\nu-\alpha)}\right| + \lim_{t\to\infty}\left\{\frac{1}{2}\log r + \log N_t\right\} = \log\left|\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right| + \lim_{t\to\infty}\left\{\frac{1}{2}\log r + \log N_t\right\}.
\end{aligned}$$

If we choose N_t and r such that

$$\lim_{t\to\infty}\left\{\frac{1}{2}\log r + \log N_t\right\} = 0, \tag{3.3}$$

then the Evans kernel becomes

$$E(p,q) = \log\left|\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right| = \log\left|\frac{p-q}{\sqrt{pq}}\right|,$$

which is symmetric in (p,q). If a pole q is fixed, then $E_q(p) := E(p,q)$ satisfies all the conditions in Definition 2.2.2 and is indeed an Evans-Selberg potential. In this case, the potential tends to $+\infty$ respectively at the two Alexandroff ideal boundary points, namely 0 and ∞. Finally, setting $N_t := e^t - e^{-t}$ and $r := e^{-2t}$, we know that they satisfy by definition the equality (3.3). Moreover, whenever t is sufficiently large it holds that $A_r = \Sigma_t := \{p \in \Sigma \,|\, E(p,q_0) < \log N_t\}$, i.e.

$$\left\{r < |p| < \frac{1}{r}\right\} = \left\{\left|\sqrt{p} - \frac{1}{\sqrt{p}}\right| < N_t\right\},$$

since N_t is strictly increasing in t. Thus, Theorem 3.2.1 and Corollary 3.2.1 are proved. $\square$

From the result of Theorem 3.2.1, we can construct by hand and obtain the following result (since it satisfies the definition of an Evans-Selberg potential), although we are not sure how to approximate $\mathbb{C}\setminus\{0,1\}$.

Theorem 3.2.2. *An Evans-Selberg potential on $\mathbb{C}\setminus\{0,1\}$ with a pole q is given by*

$$E_q(p) = \log\left|\frac{p-q}{p^{\frac{1}{3}}q^{\frac{1}{3}}(p-1)^{\frac{1}{3}}(q-1)^{\frac{1}{3}}}\right|. \tag{3.4}$$

Infimum of an Evans-Selberg potential's growth orders at the boundary

According to Proposition 2.2.4 (set $D := \mathbb{C} - \{q\}$ and $w := 0 \in D$), there exists a constant $b_0 \geq 0$ such that $E_q(p) - b_0 \log|p|$ is bounded from above near 0. Similarly, there exist a constant $b_\infty \geq 0$ such that $E_q(p) + b_\infty \log|p|$ is bounded from above near ∞. Take $b := \max\{b_0, b_\infty\}$, and then it holds that

$$\overline{\lim_{p\to 0}}\; E_q(p) - b\log|p| < \infty, \overline{\lim_{p\to\infty}}\; E_q(p) + b\log|p| < \infty. \tag{3.5}$$

Theorem 3.2.3. *Let $E_q(p)$ be an Evans-Selberg potential on $\mathbb{C}\setminus\{0\}$ with a pole q. Then, it follows that*

$$\inf\{b \geq 0 \,|\, b \textit{ satisfies } (3.5) \textit{ for } E_q(p)\} = \frac{1}{2}.$$

Proof. Let us prove this by contradiction and denote the left hand side by $b_{\mathbb{C}^*}$. From Theorem 3.2.1 we know that there exists an Evans-Selberg's potential $\phi_{z_0}(z) = \log\left|\sqrt{z/z_0} - \sqrt{z_0/z}\right|$, which means $b_{\mathbb{C}^*} \leq 1/2$. Suppose $b_{\mathbb{C}^*} < 1/2$, then there exist another Evans-Selberg's potential $\varphi_{w_0}(z)$ that satisfies $\overline{\lim\limits_{z\to 0}}\ \varphi_{w_0}(z) - b_{\mathbb{C}^*}\log|z| < \infty$ and $\overline{\lim\limits_{z\to\infty}}\ \varphi_{w_0}(z) + b_{\mathbb{C}^*}\log|z| < \infty$. So $\varphi_{w_0}(z) - \phi_{z_0}(z) < \varphi_{w_0}(z) - b_{\mathbb{C}^*}\log|z| + 1/2\log|z| - \phi_{z_0}(z) < \infty$, as $z \to 0$. Similarly, $\varphi_{w_0}(z) - \phi_{z_0}(z) < \infty$, as $z \to \infty$. Meanwhile since $\phi_{z_0}(z) - \varphi_{w_0}(z)$ is a subharmonic function which is bounded from above on $\mathbb{C}\setminus\{0\}$, then according to Proposition 2.2.3, $\phi_{z_0}(z) - \varphi_{w_0}(z)$ is subharmonic on $\hat{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$, and therefore it is a constant. This is a contradiction to $b_{\mathbb{C}^*} < 1/2$, so $b_{\mathbb{C}^*} = 1/2$. □

For the domain $\mathbb{C}\setminus\{0,1\}$ (also parabolic in the potential-theoretical sense), according to Proposition 2.2.4 there exists a constant $b_0 \geq 0$ such that $E_q(p) - b_0\log|p|$ is bounded from above near 0, for $E_q(p)$ given by (3.4). Similarly, there exist non-negative constants b_1 such that $E_q(p) - b_1\log|p-1|$ is bounded from above near 1, and b_∞ such that $E_q(p) + b_\infty\log|p|$ is bounded from above near ∞. Take $b := \max\{b_0, b_1, b_\infty\}$, and then it holds that

$$\overline{\lim_{p\to 0}} E_q(p) - b\log|p| < \infty, \overline{\lim_{p\to 1}} E_q(p) - b\log|p| < \infty, \overline{\lim_{p\to\infty}}\ E_q(p) + b\log|p| < \infty. \tag{3.6}$$

Theorem 3.2.4. *Let $E_q(p)$ be an Evans-Selberg potential on $\mathbb{C}\setminus\{0,1\}$ with a pole q. Then, it follows that*

$$\inf\{b \geq 0\,|\,b \text{ satisfies } (3.6) \text{ for } E_q(p)\} = \frac{1}{3}.$$

More generally, we should have results for $\hat{\mathbb{C}} - \{z_1, ...z_n\}, (n \geq 1)$, and the right hand side should be $\frac{1}{n}$. We may ask the following question:

Which Evans-Selberg potential can make the infimum equal to $\frac{1}{n}$?

3.3 Suita conjecture for a once-punctured torus

As a typical example of potential-theoretically parabolic Riemann surfaces, $X_{\tau,u}$ admits an Evans-Selberg potential and we have indeed constructed it.

Lemma 3.3.1. *There exists an Evans-Selberg potential on $X_{\tau,u} := X_\tau\setminus\{u\}$ with a pole w given by*

$$E_w^{\tau,u}(z) = \log\left|\frac{\theta_1(z-w;q)}{\theta_1(z-u;q)}\right|,$$

for $z \in X_{\tau,u}\setminus\{w\}$.

Proof. We see that the two terms on the right hand side of (3.1) are responsible for the two terms on the right hand side of (a), respectively. Keeping this in mind, we can construct the Evans-Selberg potential by attaching physics meanings. We regard the potential as an electric flux generated at the pole w and ends at the boundary point u (see [Oo] for physics explanations). Therefore, the Evans-Selberg potential $E_w^{\tau,u}(z)$ with a pole w satisfies that

$$\frac{\partial^2 E_w^{\tau,u}(z)}{\partial z \partial \bar{z}} = \frac{\pi}{2}\left(\delta(z-w) - \delta(z-u)\right),$$

and can be expressed via the theta function as

$$\begin{aligned} E_w^{\tau,u}(z) &= \log\left|\frac{\theta_1(z-w;q)}{\eta(\tau)}\right| - \log\left|\frac{\theta_1(z-u;q)}{\eta(\tau)}\right| = \log\left|\frac{\theta_1(z-w;q)}{\theta_1(z-u;q)}\right| \\ &= \log\left|\frac{\sin(\pi(z-w)) \cdot \prod_{m=1}^{\infty}(1-2\cos(2\pi(z-w))\cdot q^{2m} + q^{4m})}{\sin(\pi(z-u)) \cdot \prod_{m=1}^{\infty}(1-2\cos(2\pi(z-u))\cdot q^{2m} + q^{4m})}\right|. \end{aligned}$$

□

Theorem 3.3.1. *There exists a fundamental metric $c_{\tau,u}$ on $X_{\tau,u} := X_\tau \backslash \{u\}$ under the local coordinate w given by*

$$c_{\tau,u}(w)|dw|^2 = \frac{2\pi \cdot |\eta(\tau)|^3}{|\theta_1(w-u;q)|}|dw|^2.$$

Proof. This can be verified by definition, since

$$\begin{aligned} c_{\tau,u}(w) &= \exp\lim_{z\to w}\left(\log\left|\frac{\theta_1(z-w;q)}{\theta_1(z-u;q)}\right| - \log|z-w|\right) \\ &= \left|\frac{\pi \cdot \prod_{m=1}^{\infty}(1-q^{2m})^2}{\sin(\pi(w-u)) \cdot \prod_{m=1}^{\infty}(1-2\cos(2\pi(w-u))\cdot q^{2m} + q^{4m})}\right| \\ &= \left|\frac{\pi \cdot \eta(\tau)^2}{q^{\frac{1}{6}} \cdot \sin(\pi(w-u)) \cdot \prod_{m=1}^{\infty}(1-2\cos(2\pi(w-u))\cdot q^{2m} + q^{4m})}\right| \\ &= \left|\frac{\pi \cdot \eta(\tau)^2 \cdot 2q^{\frac{1}{4}} \cdot \prod_{m=1}^{\infty}(1-q^{2m})}{q^{\frac{1}{6}} \cdot \theta_1(w-u;q)}\right| = \left|\frac{\pi \cdot \eta(\tau)^2 \cdot 2q^{\frac{1}{4}} \cdot \frac{\eta(\tau)}{q^{\frac{1}{12}}}}{q^{\frac{1}{6}} \cdot \theta_1(w-u;q)}\right| = \left|\frac{2\pi \cdot \eta(\tau)^3}{\theta_1(w-u;q)}\right|. \end{aligned}$$

□

By the second equality above, $c_{\tau,u}$ has the following asymptotic behavior, which will yield Theorem 1.2.2 for any fixed τ.

Corollary 3.3.1. *Under the same assumptions as in Theorem 3.3.1, as $w \to u$, it follows that*

$$c_{\tau,u}(w) \sim \frac{1}{|w-u|} \to +\infty.$$

The fundamental metric and its degenerate case

For the complex plane $\mathbb{C}$ itself, the logarithmic kernel $\log|z-w|$ is an Evans-Selberg potential and thus the fundamental metric under local coordinate w is written as $1\,|dw|^2$, whose Gaussian curvature is 0 by (2.2) since the Bergman kernel is 0. For $\mathbb{C}\setminus\{0\}$, same argument holds for the Gaussian curvature of the fundamental metric, since the Bergman kernel on $\mathbb{C}\setminus\{0\}$ is the same as on $\mathbb{C}$ by the removable singularity theorem for L^2-holomorphic functions. By Theorem 3.2.1, we can compute explicitly the fundamental metric.

Theorem 3.3.2. *The fundamental metric on $\mathbb{C}\setminus\{0\}$ in local coordinate z is given by*

$$|z|^{-1}|dz|^2.$$

Without using the formula (2.2), we also reach that the fundamental metric on $\mathbb{C}\setminus\{0\}$ has Gaussian curvature 0. Moreover, the following asymptotic behaviors hold

$$c(w) \to \begin{cases} 0, & \text{as } w \to \infty \\ +\infty, & \text{as } w \to 0. \end{cases}$$

By studying the asymptotic behaviors of the fundamental metric under degeneration with respect to the complex structure, we will prove Theorem 1.2.3. Relating Theorem 1.2.3 with (2.2), we further get Corollary 1.2.1.

Proof of Theorem 1.2.3. As $\operatorname{Im}\tau \to +\infty$ ($q \equiv \exp(\pi i \tau) \to 0$), it holds that

$$c_{\tau,u}(w) \to \left| \frac{\pi \cdot \prod_{m=1}^{\infty}(1-0^{2m})^2}{\sin(\pi(w-u)) \cdot \prod_{m=1}^{\infty}(1-2\cos(2\pi(w-u))\cdot 0^{2m}+0^{4m})} \right| \to \frac{\pi}{|\sin(\pi(w-u))|}.$$

Therefore, it follows that

$$\frac{\pi K_{\tau,u}(w)}{c^2_{\tau,u}(w)} \to \frac{|\sin(\pi(w-u))|^2}{2\cdot \operatorname{Im}\tau \cdot \pi} \to 0^+,$$

since the denominator is uniformly bounded by 1 for any fixed w. $\square$

On the one hand, at the degenerate case of potential-theoretically hyperbolic Riemann surfaces, we are not sure whether Gaussian curvatures of the Suita metrics are still bounded from above by -4.

On the other hand, for a compact complex torus, the Gaussian curvature of the Arakelov metric is always 0 by the genus reason, although our earlier result in [D14] shows that as $\operatorname{Im}\tau \to +\infty$, it holds that

$$\frac{\pi K_\tau(w)}{c^2_\tau(w)} \to +\infty.$$

Chapter 4

Bergman kernel on degenerate elliptic curves

For a Legendre family of elliptic curves, using two methods (depending on elliptic functions' special properties and Abelian differentials' Taylor expansions) we show that the curvature form of the relative Bergman kernel metric is strictly positive inside the moduli space $\mathbb{C} \setminus \{0, 1\}$ and coincides with the the Poincaré metric there. In particular, the curvature form blows up and has hyperbolic growth near the node 0. For other boundary points 1 and ∞, the asymptotic behaviors are also achieved. For other families of elliptic curves degenerating to singular ones with a node or a cusp, we observe that it is either trivial with a constant period or reducible to the Legendre family case.

4.1 Legendre family: a four-term asymptotic expansion at 0

Let us start from the following question.

Question. *What is the Gaussian curvature of* $L^{(1)}_{\lambda,z}$*?*

After careful computations, the curvature is observed to be identically equal to "-4". Moreover, the result is as follows.

Theorem 4.1.1. *Under the same assumptions as in Theorem 1.1, it follows that* $L^{(1)}_{\lambda,z}$ *is the Poincaré metric of* $\mathbb{C} \setminus \{0, 1\}$.

On the one hand, this result seems to suggest a connection between the Bergman kernel's variation and the moduli space's Poincaré metric. On the other hand, a four-term expansion formula of the Poincaré metric of $\mathbb{C} \setminus \{0, 1\}$ are obtained as a corollary.

Corollary 4.1.1. *Let* $\omega_{0,1}$ *denote the Poincaré metric of* $\mathbb{C} \setminus \{0, 1\}$*. Then, as* $\lambda \to 0$*, it holds that*

$$\omega_{0,1} = \frac{\sqrt{-1}d\lambda \otimes d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2} \left(1 + 2\frac{\log 16}{\log|\lambda|} + 3\left(\frac{\log 16}{\log|\lambda|}\right)^2 + 4\left(\frac{\log 16}{\log|\lambda|}\right)^3 + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^4}\right)\right).$$

The leading term of the above expansion formula implies that near the origin $\omega_{0,1}$ is asymptotically similar to $\omega_{\mathbb{D}^*}$, and the negative second term seems to support that the latter is bigger. Actually, it always holds that $\omega_{0,1} \leq \omega_{\mathbb{D}^*}$, wherever inside $\mathbb{D}^*$ (see [SV]). Last but not least, define $p(\lambda) := -\log(\operatorname{Im}\tau(\lambda))$, where $\tau(\cdot)$ is the inverse function of the elliptic modular lambda function and $\lambda \in \mathbb{C} \setminus \{0,1\}$, we have the following asymptotic expansion formula near 0.

Theorem 4.1.2. *Under the same assumptions as in Corollary 4.1.1, $p(\lambda)$ is a Kähler potential of $\omega_{0,1}$. And as $\lambda \to 0$, it follows that*

$$p(\lambda) = -\log(-\log|\lambda|) + \log\pi + \frac{\log 16}{\log|\lambda|} + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right).$$

We remark that the first three terms in the above right hand side (denoted by $\tilde{p}(\lambda)$) is a Kähler potential that exactly gives rise to the first two terms in the asymptotic expansion in Corollary 4.1.1. From now on, we use the symbol "$\sim$" to denote that the ratio of its both sides tends to 1, as $\lambda \to 0$.

Proof of 1st & 2nd terms in Theorem 1.2.4

Proof. By definition, the Bergman kernel B_τ on X_τ (for its canonical bundle) can be written as $B_\tau = \frac{1}{\operatorname{Im}\tau} dz \wedge d\bar{z}$ in the local coordinate z, which means that $k_\lambda(z) = \frac{1}{\operatorname{Im}\tau}$. Now, we shall analyze the asymptotic behaviors of B_τ as $\operatorname{Im}\tau \to +\infty$ (or equivalently the asymptotic behaviors of B_λ as $\lambda \to 0$). Using $q := \exp(\pi\sqrt{-1}\tau)$, rewrite the elliptic modular lambda function as

$$\lambda(\tau) = 16q - 128q^2 + 704q^3 - 3072q^4 + ... = 16q - 128q^2 + \mathrm{O}(q^3), \tag{4.1}$$

where we write $\mathrm{O}(f(q))$ in place of $g(q)$ if there exists a constant $C \in \mathbb{R}$ such that $\overline{\lim}_{q\to 0} |g(q)/f(q)| = C$. In particular, we have $\mathrm{O}(q^2) = \mathrm{O}(\bar{q}^2) = \mathrm{O}(|q|^2)$. Since $|q| = \exp(-\pi \cdot \operatorname{Im}(\tau))$, i.e.,

$$\operatorname{Im}\tau = \frac{\log|q|}{-\pi}, \tag{4.2}$$

it holds that

$$\log k_\lambda(z) = \log\frac{1}{\operatorname{Im}\tau} = -\log\operatorname{Im}\tau = -\log(-\frac{\log|q|}{\pi}). \tag{4.3}$$

Thus, as $\operatorname{Im}\tau \to +\infty$ ($q \to 0$), we know that

$$|\lambda| = |16q - 128q^2 + \mathrm{O}(q^3)| = |q| \cdot |16 - 128q + \mathrm{O}(q^2)| \sim 16|q| \to 0, \tag{4.4}$$

yielding that $\log|\lambda| = \log|q| + \log|16 - 128q + \mathrm{O}(q^2)|$. Therefore, as $\lambda \to 0$ (or equivalently $\operatorname{Im}\tau \to +\infty$ or $q \to 0$), it follows that

$$\log k_\lambda(z) \sim -\log(-\log|\lambda|),$$

which implies that

$$\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(z) \sim \frac{\sqrt{-1}\,d\lambda\wedge d\bar{\lambda}}{4|\lambda|^2(\log|\lambda|)^2} := l_{\lambda,z}\sqrt{-1}\,d\lambda\wedge d\bar{\lambda},$$

and gives the leading term in Theorem 1.2.4.

In order to get the second term of $l_{\lambda,z}$, we define a new function

$$J_\lambda := l_{\lambda,z} - \frac{1}{4|\lambda|^2(\log|\lambda|)^2}, \tag{4.5}$$

and analyze its asymptotics as $\lambda\to 0$. Substituting (4.2) and (4.6) into (2.6), we get that

$$l_{\lambda,z} = \frac{1}{4(\log|q|)^2\cdot|q|^2\cdot|16-256q+\mathrm{O}(q^2)|^2}.$$

(4.1) also implies that $\lambda'(\tau)=\frac{\partial\lambda}{\partial q}\cdot\frac{\partial q}{\partial\tau}=(16-256q+\mathrm{O}(q^2))\cdot q\cdot\sqrt{-1}\pi$ and thus

$$|\lambda'(\tau)| = |16-256q+\mathrm{O}(q^2)|\cdot|q|\cdot\pi \sim 16\pi|q|\sim\pi|\lambda|. \tag{4.6}$$

Therefore, it holds that

$$\begin{aligned}
&4|q|^2\cdot J_\lambda\\
=&\frac{1}{(\log|q|)^2|16-256q+\mathrm{O}(q^2)|^2}-\frac{1}{|16-128q+\mathrm{O}(q^2)|^2(\log|q|+\log|16-128q+\mathrm{O}(q^2)|)^2}\\
=&\frac{|16-128q+\mathrm{O}(q^2)|^2\left(2(\log|q|)\cdot\log|16-128q+\mathrm{O}(q^2)|+(\log|16-128q+\mathrm{O}(q^2)|)^2\right)}{(\log|q|)^2\cdot|16-256q+\mathrm{O}(q^2)|^2\cdot|16-128q+\mathrm{O}(q^2)|^2\cdot(\log|q|+\log|16-128q+\mathrm{O}(q^2)|)^2}\\
\sim&\frac{|16-128q+\mathrm{O}(q^2)|^2\left(2(\log|q|)\cdot\log 16+(\log 16)^2\right)}{(\log|q|)^2\cdot|16-256q+\mathrm{O}(q^2)|^2\cdot|16-128q+\mathrm{O}(q^2)|^2\cdot(\log|q|+\log|16-128q+\mathrm{O}(q^2)|)^2}\\
\sim&\frac{2\cdot|16-128q+\mathrm{O}(q^2)|^2\cdot\log 16}{(\log|q|)\cdot|16-256q+\mathrm{O}(q^2)|^2\cdot|16-128q+\mathrm{O}(q^2)|^2\cdot(\log|q|)^2}\sim\frac{2\cdot\log 16}{16^2\cdot(\log|q|)^3}.
\end{aligned}$$

As $q\to 0$ (implying $\lambda\to 0$), it follows that

$$J_\lambda\sim\frac{\log 16}{2|\lambda|^2(\log|\lambda|)^3}.$$

Finally combining (4.5), one obtains the second term in Theorem 1.2.4. □

An alternative proof of the first two terms in Theorem 1.2.4, without using the special properties of elliptic functions, is given in [D3]. Let us then generalize Theorem 1.3 (i) in [D15] by proving the following lemma[1], which will be used later in this section.

[1] If one drops the lower terms, then the following two-term asymptotic formula holds: $\log k_\lambda(z)\sim -\log(-\log|\lambda|+\log 16)$.

Lemma 4.1.1. *Under the same assumptions as in Theorem 1.2.4, as $\lambda \to 0$, it holds that*

$$k_\lambda(z) = \frac{\pi}{-\log|\lambda| + \log 16 - \frac{\operatorname{Re}\lambda}{2} + \mathrm{O}(\lambda^2)}. \tag{4.7}$$

Proof of Lemma 4.1.1. The preliminary section says that $\frac{1}{k_\lambda(z)} = \operatorname{Im}\tau = \frac{-\log|q|}{\pi}$. As $q \to 0$, it holds that $\frac{1}{k_\lambda(z)} \sim \frac{-\log|\lambda|}{\pi}$. Considering their difference, from (4.1) one gets that

$$\frac{1}{k_\lambda(z)} - \frac{-\log|\lambda|}{\pi} = \frac{1}{\pi}\log\left|\frac{\lambda}{q}\right| = \frac{1}{\pi}\log|16 - 128q + \mathrm{O}(q^2)|.$$

Furthermore, it holds that $|16 - 128q + \mathrm{O}(q^2)|^2 = 16^2 - 32\cdot 128\operatorname{Re} q + \mathrm{O}(q^2)$, which implies that

$$\frac{1}{k_\lambda(z)} - \frac{-\log|\lambda|}{\pi} = \frac{1}{2\pi}\log\left(16^2 - 32\cdot 128\operatorname{Re} q + \mathrm{O}(q^2)\right).$$

The Taylor expansion of $\log t$ at the point $t = 16^2$ says that

$$\log\left(16^2 - 32\cdot 128\operatorname{Re} q + \mathrm{O}(q^2)\right) = \log\left(16^2\right) - 16\operatorname{Re} q + \mathrm{O}(q^2),$$

by which it holds that

$$\frac{1}{k_\lambda(z)} = \frac{-\log|\lambda| + \log 16 - 8\operatorname{Re} q + \mathrm{O}(q^2)}{\pi},$$

as $q \to 0$. Since $\operatorname{Re} q \sim \frac{\operatorname{Re}\lambda}{16}$ and $\mathrm{O}(q^2) = \mathrm{O}(\lambda^2)$, the proof is completed. □

Proof of the third and fourth terms in Theorem 1.2.4

Proof. From Lemma 4.1.1, as $\lambda \to 0$, we know that

$$\log k_\lambda(z) \sim -\log\left(\frac{-\log|\lambda| + \log 16 - \frac{\operatorname{Re}\lambda}{2}}{\pi}\right) := \text{RHS}.$$

Therefore, after some elementary calculations one first gets that

$$\frac{\partial^2(\text{RHS})}{\partial\lambda\partial\bar\lambda} = \frac{1 + \operatorname{Re}\lambda + \frac{1}{4}|\lambda|^2}{4|\lambda|^2\left(-\log|\lambda| + \log 16 - \frac{\operatorname{Re}\lambda}{2}\right)^2}\left(\sim \frac{\partial^2(\log k_\lambda(z))}{\partial\lambda\partial\bar\lambda}\right).$$

Step 1: estimating the third term. Comparing the difference, one gets

$$\begin{aligned}
&\frac{\partial^2(\text{RHS})}{\partial\lambda\partial\bar\lambda} - \frac{1}{4|\lambda|^2(\log|\lambda|)^2} - \frac{\log 16}{2|\lambda|^2(\log|\lambda|)^3}\\
=&\frac{\left(\operatorname{Re}\lambda + \frac{1}{4}|\lambda|^2\right)\cdot(\log|\lambda|)^2 + 2\log|\lambda|(\log 16 - \frac{\operatorname{Re}\lambda}{2}) - (\log 16 - \frac{\operatorname{Re}\lambda}{2})^2}{4|\lambda|^2(\log|\lambda|)^2\left(-\log|\lambda| + \log 16 - \frac{\operatorname{Re}\lambda}{2}\right)^2} - \frac{\log 16}{2|\lambda|^2(\log|\lambda|)^3}\\
\sim&\frac{-(\log 16 - \frac{\operatorname{Re}\lambda}{2})^2\log|\lambda| + 4(\log 16)^2\log|\lambda| - 2(\log 16)^3}{4|\lambda|^2(\log|\lambda|)^3\left(-\log|\lambda| + \log 16 - \frac{\operatorname{Re}\lambda}{2}\right)^2} \sim \frac{3(\log 16)^2}{4|\lambda|^2(\log|\lambda|)^4},
\end{aligned}$$

which means that

$$\frac{\partial^2(\log k_\lambda(z))}{\partial\lambda\partial\bar{\lambda}} = \frac{1}{4|\lambda|^2(\log|\lambda|)^2}\left(1+2\frac{\log 16}{\log|\lambda|}+3\left(\frac{\log 16}{\log|\lambda|}\right)^2+\mathrm{O}\left(\frac{1}{(\log|\lambda|)^3}\right)\right).$$

Step 2: estimating the fourth term. Similarly as above, we know that

$$\begin{aligned}
&\frac{\partial^2(\mathrm{RHS})}{\partial\lambda\partial\bar{\lambda}} - \frac{1}{4|\lambda|^2(\log|\lambda|)^2} - \frac{\log 16}{2|\lambda|^2(\log|\lambda|)^3} - \frac{3(\log 16)^2}{4|\lambda|^2(\log|\lambda|)^4}\\
\sim&\frac{3\log|\lambda|(\log 16)^2-2(\log 16)^3}{4|\lambda|^2(\log|\lambda|)^3\left(-\log|\lambda|+\log 16-\frac{\operatorname{Re}\lambda}{2}\right)^2} - \frac{3(\log 16)^2}{4|\lambda|^2(\log|\lambda|)^4}\\
=&\frac{-2(\log|\lambda|)(\log 16)^3-3(\log 16)^2\left(-2(\log|\lambda|)(\log 16-\frac{\operatorname{Re}\lambda}{2})+(\log 16-\frac{\operatorname{Re}\lambda}{2})^2\right)}{4|\lambda|^2(\log|\lambda|)^4\left(-\log|\lambda|+\log 16-\frac{\operatorname{Re}\lambda}{2}\right)^2}\\
\sim&\frac{4\log|\lambda|(\log 16)^3-3(\log 16)^4}{4|\lambda|^2(\log|\lambda|)^4\left(-\log|\lambda|+\log 16-\frac{\operatorname{Re}\lambda}{2}\right)^2}\sim\frac{(\log 16)^3}{|\lambda|^2(\log|\lambda|)^5},
\end{aligned}$$

as $\lambda\to 0$, which finishes the full proof of Theorem 1.2.4. $\square$

Remark As $\lambda\to 0$, we do not know why our results on the asymptotic behaviors of Bergman kernels depend only on $|\lambda|$. Moreover, we will see in the next section that the positivity of the above third term contributes to the completeness argument in the proof of Theorem 4.1.1. It turns out that the subleading terms in the asymptotic expansion contain more "logarithmic" information, slowing down the growth order at infinity of the left hand side. As we can see, even though the 2nd and the 4th terms tend to $-\infty$, the left hand side of the above formula, which is mainly affected by the leading term, still tends to $+\infty$. It is expected that each term should have certain geometrical interpretations.

Proof of Theorem 4.1.1

Proof of Theorem 4.1.1. We first compute the Gaussian curvature of the Kähler metric $L_{\lambda,z}$ on $\mathbb{C}\setminus\{0,1\}$. From the preliminary section, it is known that

$$L_{\lambda,z}=\frac{\sqrt{-1}\cdot|\tau'|^2}{4(\operatorname{Im}\tau)^2}\,d\lambda\wedge d\bar{\lambda}=:\sqrt{-1}(J_\lambda)^2\,d\lambda\wedge d\bar{\lambda}.$$

Therefore, it follows that

$$\frac{-4\partial^2\log(J_\lambda)}{\partial\lambda\partial\bar{\lambda}}=\frac{-4\partial^2\log(\frac{|\tau'|}{2\cdot\operatorname{Im}\tau})}{\partial\lambda\partial\bar{\lambda}}=\frac{-4\partial^2\log(|\tau'|)}{\partial\lambda\partial\bar{\lambda}}+\frac{4\partial^2\log(2\cdot\operatorname{Im}\tau)}{\partial\lambda\partial\bar{\lambda}}.$$

Since $\tau(\cdot)$, the inverse function of the elliptic modular function, is also conformal, it holds that $\log(|\tau'|)$ is harmonic with respect to λ. So, we get

$$\frac{-4\partial^2\log(J_\lambda)}{\partial\lambda\partial\bar{\lambda}}=\frac{4\partial^2\log(2\cdot\operatorname{Im}\tau)}{\partial\lambda\partial\bar{\lambda}}=-\frac{|\tau'|^2}{(\operatorname{Im}\tau)^2}.$$

Furthermore, it follows that

$$\mathrm{Curv}(L_{\lambda,z}) = \frac{\frac{-4\partial^2 \log(J_\lambda)}{\partial\lambda\partial\bar{\lambda}}}{(J_\lambda)^2} = \frac{-\frac{|\tau'|^2}{(\operatorname{Im}\tau)^2}}{(\frac{|\tau'|}{2\cdot\operatorname{Im}\tau})^2} \equiv -4.$$

To prove that $L_{\lambda,z}$ is complete at 0, we use our asymptotic result in Theorem 1.2.4. Since the subleading terms are all incomplete near 0, the sum of the first and second terms becomes a complete metric (with a non-constant curvature) on $\mathbb{D}^*$, denoted by $\omega'_{\mathbb{D}^*}$. Then, due to the positivity of the third term we get that $L_{\lambda,z} > \omega'_{\mathbb{D}^*}$, which guarantees the completeness of $L_{\lambda,z}$ at 0. For the completeness at other boundary points 1 and ∞, we proceed with the behaviors of the elliptic modular lambda function under the composition with inverse or translation mappings (cf. [K-R, D3]). □

Corollary 4.1.1 follows from Theorem 4.1.1 and Theorem 1.2.4.

Proof of Theorem 4.1.2

Proof. Previous computations in the preliminary section show that

$$0 < \frac{1}{4(\operatorname{Im}\tau \cdot |\lambda'(\tau)|)^2} = \frac{\partial^2(\log(k_\lambda(z)))}{\partial\lambda\partial\bar{\lambda}} = \frac{\partial^2(p(\lambda))}{\partial\lambda\partial\bar{\lambda}}.$$

From Theorem 4.1.1, it follows that $p(\lambda)$ is a Kähler potential of $\omega_{0,1}$. First, let us consider its leading term of the asymptotic expansion. By (4.1) and (4.2), as $\lambda \to 0$, it can be seen that

$$p(\lambda) \sim -\log(-\log|\lambda|) =: p_1(\lambda).$$

Actually, $p_1(\lambda)$ is the potential's leading term near $\lambda = 0$ and satisfies

$$\frac{\partial^2(p_1(\lambda))}{\partial\lambda\partial\bar{\lambda}} = \frac{1}{4|\lambda|^2(\log|\lambda|)^2}.$$

By (4.7) in order to get the second term, we use $p(\lambda)$ to subtract $p_1(\lambda)$ and analyze their difference function, namely

$$\begin{aligned} p(\lambda) - p_1(\lambda) &\sim -\log(-\log|\lambda| + \log 16) + \log\pi + \log(-\log|\lambda|) \\ &= \log\left(\frac{1}{-\log|\lambda| + \log 16}\right) + \log\pi + \log(-\log|\lambda|) \\ &= \log\pi - \log\left(1 - \frac{\log 16}{\log|\lambda|}\right) \sim \log\pi + \frac{\log 16}{\log|\lambda|} \sim \log\pi =: p_2(\lambda). \end{aligned}$$

The second to last similarity relation holds due to the Taylor expansion of $\log(1+t)$ at 0. Similarly, we see that the third term is just $\frac{\log 16}{\log|\lambda|} =: p_3(\lambda)$. □

Remark We now verify our claim after Theorem 4.1.2 that $\tilde{p}(\lambda)$ is a Kähler potential that exactly gives rise to the first two terms in the asymptotic expansion in Corollary 4.1.1. To check this, we make the following computations.

$$\partial\left(\frac{1}{\log|\lambda|}\right)=\frac{-d\lambda}{2\lambda(\log|\lambda|)^2},\qquad \bar{\partial}\left(\frac{1}{\log|\lambda|}\right)=\frac{-d\bar{\lambda}}{2\bar{\lambda}(\log|\lambda|)^2},$$

$$\begin{aligned}\partial\bar{\partial}\left(\frac{1}{\log|\lambda|}\right)&=\partial\left(\frac{-d\bar{\lambda}}{2\bar{\lambda}(\log|\lambda|)^2}\right)=\frac{2\partial\left(\bar{\lambda}(\log|\lambda|)^2\right)\wedge d\bar{\lambda}}{4\bar{\lambda}^2(\log|\lambda|)^4}\\ =&\frac{2\bar{\lambda}\partial\left((\log|\lambda|)^2\right)\wedge d\bar{\lambda}}{4\bar{\lambda}^2(\log|\lambda|)^4}=\frac{2\bar{\lambda}2(\log|\lambda|)\frac{d\lambda}{2\lambda}\wedge d\bar{\lambda}}{4\bar{\lambda}^2(\log|\lambda|)^4}=\frac{d\lambda\wedge d\bar{\lambda}}{2|\lambda|^2(\log|\lambda|)^3}.\end{aligned}$$

Thus, since $p_2(\lambda)$ is a constant, it holds that

$$\frac{\partial^2(p_2(\lambda)+p_3(\lambda))}{\partial\lambda\partial\bar{\lambda}}=\frac{\log 16}{2|\lambda|^3(\log|\lambda|)^2}.$$

4.2 Legendre family: asymptotic formulas at 1 and ∞

In this section, explicit asymptotic formulas of the relative Bergman kernel metric for a Legendre family of elliptic curves near the moduli space boundary points 1 and ∞ are obtained respectively. These asymptotic behaviors also characterize the Poincaré hyperbolic metric and its Kähler potential on $\mathbb{C}\setminus\{0,1\}$.

Theorem 4.2.1. *Under the same assumptions as in Theorem 1.2.4, it follows that*

(i) *as* $\lambda\to 1$, $\log k_\lambda^{(1)}(z)\sim\log(-\log|\lambda-1|)$,

(ii) *and as* $\lambda\to\infty$, $\log k_\lambda^{(1)}(z)\sim\log(\log|\lambda|)$.

In particular, both the right hand sides of (i) and (ii) tend to $+\infty$. Rather than taking immediate second-order partial derivatives, we make more careful computations on the curvature forms and derive the following theorem.

Theorem 4.2.2. *Under the same assumptions as in Theorem 1.2.4, it follows that*

(i) *as* $\lambda\to 1$,

$$L_{\lambda,z}\sim\frac{\sqrt{-1}}{4|\lambda-1|^2(\log|\lambda-1|)^2}d\lambda\wedge d\bar{\lambda},$$

(ii) *and as* $\lambda\to\infty$,

$$L_{\lambda,z}\sim\frac{\sqrt{-1}}{4|\lambda|^2(\log|\lambda|)^2}d\lambda\wedge d\bar{\lambda}.$$

Notice that the right hand sides of (i) and (ii) tend to $+\infty$ and 0^+, respectively. And this is different from the potentials in Theorem 4.2.1 which have the same limit. The proofs of Theorems 4.2.1 and 4.2.2 are mainly due to the elliptic modular lambda function's special properties (in particular its behavior under the composition with inverse or translation mappings). By Theorem 4.1.1, we get the following corollary.

Corollary 4.2.1. *Let $\omega_{0,1}$ denote the Poincaré hyperbolic metric on $\lambda \in \mathbb{C} \setminus \{0,1\}$ with a Kähler potential $p(\lambda) := -\log(\operatorname{Im}\tau(\lambda))$, where $\tau(\cdot)$ is the inverse of the elliptic modular lambda function. Then, it follows that*

(i) *as $\lambda \to 1$,*

$$p(\lambda) \sim \log(-\log|\lambda - 1|),$$

$$\omega_{0,1} \sim \frac{\sqrt{-1}}{4|\lambda-1|^2(\log|\lambda-1|)^2} d\lambda \otimes d\bar{\lambda},$$

(ii) *and as $\lambda \to \infty$,*

$$p(\lambda) \sim \log(\log|\lambda|),$$

$$\omega_{0,1} \sim \frac{\sqrt{-1}}{4|\lambda|^2(\log|\lambda|)^2} d\lambda \otimes d\bar{\lambda}.$$

We remark that our result agrees in the limiting case $\lambda \to 1$ inside $\mathbb{D}^*$ with the fact that $\omega_{0,1} \le \omega_{\mathbb{D}^*}$ (see e.g. [SV, H]).

Proof of Theorem 4.2.1

Combining our results in [D15] and introducing two new parameters $\alpha := -\frac{1}{\tau}$ and $\beta := \tau - 1$, we will prove new results. We shall use the following well-known properties of the elliptic modular lambda function (see [Ah, p.279–280]):

(A) As $\operatorname{Im}\alpha \to +\infty$, it holds that

$$\lambda(\alpha) \sim 16e^{\pi\sqrt{-1}\alpha} \to 0,$$

which means that $\log\lambda(\alpha) \sim \pi\sqrt{-1}\alpha$.

(B) $\lambda(-\frac{1}{\tau}) = 1 - \lambda(\tau)$.

(C) $\lambda(\beta+1) = \frac{\lambda(\beta)}{\lambda(\beta)-1} = 1 + \frac{1}{\lambda(\beta)-1}$ ($\implies \lambda(\beta) - 1 = \frac{1}{\lambda(\beta+1)-1}$).

Proof of Theorem 4.2.1. Claim (i). As $\tau \to 0$ ($\Longleftrightarrow \operatorname{Im}\alpha \to +\infty$), since $\log k_{\lambda(\tau)}(z) = -\log\operatorname{Im}\tau$, we know by (2) that $\log k_{\lambda(\tau)}(z) \sim -\log\operatorname{Im}\alpha + \log|\alpha|^2$. Theorem 1.3 (i) in [D15] says that as $\operatorname{Im}\alpha \to +\infty$, one has

$$-\log\operatorname{Im}\alpha \sim -\log(-\log|\lambda(\alpha)|),$$

which yields as $\tau \to 0$ that

$$\log k_{\lambda(\tau)}(z) \sim -\log(-\log|\lambda(\alpha)|) + 2\log|\alpha|.$$

On the other hand by Property (A) we know that

$$\pi|\alpha| \sim |\log\lambda(\alpha)| = |\log|\lambda(\alpha)| + \sqrt{-1}\arg(\alpha)| \sim |\log|\lambda(\alpha)|| = -\log|\lambda(\alpha)|,$$

as $\operatorname{Im}\alpha \to +\infty$ ($\Longleftrightarrow \lambda(\alpha) \to 0$), which gives that $\log|\alpha| \sim \log(-\log|\lambda(\alpha)|)$. Therefore, by Property (B) for the Bergman kernel we have proved that

$$\begin{aligned}\log k_{\lambda(\tau)}(z) &\sim -\log(-\log|\lambda(\alpha)|) + 2\log(-\log|\lambda(\alpha)|)\\ &= \log(-\log|\lambda(\alpha)|) = \log(-\log|\lambda(\tau)-1|) \to +\infty,\end{aligned}$$

as $\lambda(\alpha) \to 0$ ($\Longleftrightarrow \lambda(\tau) \to 1$).

Claim (ii). It follows from Claim (i) that as $\beta \to 0$ ($\Longleftrightarrow \tau \to 1$),

$$-\log\operatorname{Im}\beta \sim \log(-\log|\lambda(\beta)-1|),$$

which implies by Property (C) that

$$\begin{aligned}\log k_{\lambda(\tau)}(z) &= -\log\operatorname{Im}\tau = -\log\operatorname{Im}\beta\\ &\sim \log(-\log|\lambda(\beta)-1|) = \log(\log|\lambda(\beta+1)-1|)\\ &= \log(\log|\lambda(\tau)-1|) \sim \log(\log|\lambda(\tau)|) \to +\infty,\end{aligned}$$

as $\lambda(\tau) \to \infty$. □

Proof of Theorem 4.2.2

Proof. Claim (i). From Property (B), one knows that $\lambda'(\alpha)\cdot\frac{\partial\alpha}{\partial\tau} = -\lambda'(\tau)$, which implies

$$|\lambda'(\tau)| = \frac{|\lambda'(\alpha)|}{|\tau|^2}.$$

By equalities (2.6) and (2.7), as $\operatorname{Im}\alpha \to +\infty$, it holds that

$$\begin{aligned}\frac{\partial^2(\log k_{\lambda(\tau)}(z))}{\partial\lambda\partial\bar{\lambda}} &= \frac{1}{4(\operatorname{Im}\tau\cdot|\lambda'(\tau)|)^2} = \frac{1}{4(\operatorname{Im}\alpha\cdot|\tau|^2\cdot\frac{|\lambda'(\alpha)|}{|\tau|^2})^2}\\ &= \frac{1}{4(\operatorname{Im}\alpha\cdot|\lambda'(\alpha)|)^2} = \frac{\partial^2(\log k_{\lambda(\alpha)}(z))}{\partial\lambda\partial\bar{\lambda}}.\end{aligned}$$

Theorem 1.3 (ii) in [D15] says that

$$\frac{\partial^2(\log k_{\lambda(\alpha)}(z))}{\partial\lambda\partial\bar{\lambda}} \sim \frac{1}{4|\lambda(\alpha)|^2(\log|\lambda(\alpha)|)^2},$$

as $\operatorname{Im}\alpha \to +\infty (\Longleftrightarrow \tau \to 0)$, which yields as $\lambda(\tau) \to 1$ that

$$\frac{\partial^2(\log k_{\lambda(\tau)}(z))}{\partial\lambda\partial\bar{\lambda}} \sim \frac{1}{4|\lambda(\tau)-1|^2(\log|\lambda(\tau)-1|)^2} \to +\infty.$$

Claim (ii). By Property (B) we get that

$$\lambda'(\beta)\cdot\frac{\partial\beta}{\partial\tau}\cdot(\lambda(\tau)-1)+(\lambda(\beta)-1)\cdot\lambda'(\tau)=0.$$

This means $\lambda'(\tau)=\frac{\lambda'(\beta)\cdot(\lambda(\tau)-1)}{-\lambda(\beta)+1}$ and therefore

$$|\lambda'(\tau)|=\frac{|\lambda'(\beta)|\cdot|(\lambda(\tau)-1)|}{|\lambda(\beta)-1|}.$$

By (2.6) again, it follows that

$$\begin{aligned}\frac{\partial^2(\log k_{\lambda(\tau)}(z))}{\partial\lambda\partial\bar{\lambda}} &= \frac{1}{4\left(\operatorname{Im}\tau\cdot|\lambda'(\tau)|\right)^2}\\ &= \frac{|\lambda(\beta)-1|^2}{4\left(\operatorname{Im}\beta\cdot|\lambda'(\beta)|\cdot|\lambda(\tau)-1|\right)^2} = \frac{\partial^2(\log k_{\lambda(\beta)}(z))}{\partial\lambda\partial\bar{\lambda}}\cdot\frac{|\lambda(\beta)-1|^2}{|\lambda(\tau)-1|^2}.\end{aligned}$$

By Claim (i), as $\lambda(\beta)\to 1$ it holds that

$$\frac{\partial^2(\log k_{\lambda(\beta)}(z))}{\partial\lambda\partial\bar{\lambda}} \sim \frac{1}{4\left(|\lambda(\beta)-1|\cdot\log|\lambda(\beta)-1|\right)^2},$$

which means that

$$\begin{aligned}\frac{\partial^2(\log k_{\lambda(\tau)}(z))}{\partial\lambda\partial\bar{\lambda}} &\sim \frac{1}{4\left(|\lambda(\beta)-1|\cdot\log|\lambda(\beta)-1|\right)^2}\cdot\frac{|\lambda(\beta)-1|^2}{|\lambda(\tau)-1|^2}\\ &= \frac{1}{4(\log|\lambda(\beta)-1|)^2\cdot|\lambda(\tau)-1|^2} = \frac{1}{4(-\log|\lambda(\tau)-1|)^2\cdot|\lambda(\tau)-1|^2}\\ &\sim \frac{1}{4(\log|\lambda(\tau)|)^2\cdot|\lambda(\tau)|^2} \to 0^+,\end{aligned}$$

as $\lambda(\tau)\to\infty$. The proof is thus finished. □

At last, the following table indicates how the relative Bergman kernel and its curvature form change as the parameter varies. Here τ is the inverse function of the elliptic modular lambda function. As we can see, all the three cases have different asymptotic behaviors.

As the parameter τ tends to	As the parameter λ tends to	relative Bergman kernel $\log k_\lambda(z)$	the curvature form $\sqrt{-1}\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(z)$
∞	0	$\to -\infty$	$\to +\infty$
0	1	$\to +\infty$	$\to +\infty$
1	∞	$\to +\infty$	$\to 0^+$

4.3 Legendre family: the leading asymptotics at 0 by Taylor expansion

For a Legendre family of elliptic curves, the two-term asymptotic expansion of the relative Bergman kernel metric near the degenerate boundary is shown by an approach based on Abelian differentials' Taylor expansions and elliptic curves' periods. Namely, the horizontal curvature form has hyperbolic growth with an explicit second term at the node. The proofs do not depend on special elliptic functions. To re-prove first two terms in Theorem 1.2.4, the following lemma is needed.

Lemma 4.3.1. *Let C_λ be defined as in* (2.3). *Then as $\lambda \to 0$, it holds that*

$$\log C_\lambda^{-1} \sim -\log\left(-\log|\lambda|\right).$$

Proof of Lemma 4.3.1. The numerator and the denominator in (2.5) will be estimated respectively (cf. [CMSP]), and we make it precise so as to give the second term asymptotic expansion later on. The Taylor expansion of the function $(\sqrt{1-\lambda})^{-1}$ at 0 says that $(\sqrt{1-\lambda})^{-1} = 1 + \frac{\lambda}{2} + \frac{3\lambda^2}{8} + \mathrm{O}(\lambda^3)$. For $|x| > 1$, it holds that $(\sqrt{1-x^{-1}})^{-1} = 1 + \frac{1}{2x} + \frac{3}{8x^2} + \mathrm{O}(x^{-3})$, which means that

$$\left(\sqrt{x(x-1)}\right)^{-1} = \frac{1}{x} + \frac{1}{2x^2} + \frac{3}{8x^3} + \mathrm{O}(\frac{1}{x^4}).$$

Ⓐ The Numerator. By the construction of $\tilde{\gamma}$ we know that

$$\begin{aligned}
&\int_{\tilde{\gamma}} \omega = -2\int_1^t \omega = -2\int_1^2 \omega - 2\int_2^t \omega \\
=& -2\int_1^2 \omega - 2\int_2^t \frac{dx}{\sqrt{x(x-1)(x-t)}} \\
=& -2\int_1^2 \omega - 2\int_2^t \left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{(x-1)}} - \frac{1}{\sqrt{x}}\right)\frac{dx}{\sqrt{x(x-t)}} \\
=& -2\int_1^2 \omega - 2\int_2^t \left(\frac{1}{\sqrt{x}}\right)\frac{dx}{\sqrt{x(x-t)}} - 2\int_2^t \left(\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x}}\right)\frac{dx}{\sqrt{x(x-t)}} \\
=& -2\int_1^2 \omega - 2\int_2^t \frac{dx}{x\sqrt{x-t}} - 2\int_2^t \left(\frac{\sqrt{x}-\sqrt{x-1}}{\sqrt{x-1}\cdot\sqrt{x}}\right)\frac{dx}{\sqrt{x(x-t)}} \\
=& -2\int_1^2 \omega - 2\int_2^t \frac{dx}{x\sqrt{x-t}} - 2\int_2^t \left(\frac{1}{\sqrt{x-1}\cdot\sqrt{x}\cdot\left(\sqrt{x}+\sqrt{x-1}\right)}\right)\frac{dx}{\sqrt{x(x-t)}}.
\end{aligned}$$

Denote $I := \int_2^t \frac{dx}{x\sqrt{x-t}}$ and $J := \int_2^t \left(\frac{1}{\sqrt{x-1}\cdot\sqrt{x}\cdot\left(\sqrt{x}+\sqrt{x-1}\right)}\right)\frac{dx}{\sqrt{x(x-t)}}$. Then it follows that

$$\int_{\tilde{\gamma}} \omega = -2\int_1^2 \omega - 2I - 2J = -2\left(\int_1^2 \omega + I + J\right). \tag{4.8}$$

Firstly, let's look at $\int_1^2 \omega$. As $t \to \infty$, it holds that

$$\begin{aligned}\int_1^2 \omega = \int_1^2 \frac{dx}{\sqrt{x(x-1)(x-t)}} &\sim \int_1^2 \frac{dx}{\sqrt{-t}\cdot\sqrt{x(x-1)}} \\ =&\frac{1}{\sqrt{-t}} \log\left(2\sqrt{x(x-1)}+2x-1\right)\Big|_1^2 \\ =&\frac{1}{\sqrt{-t}} \log\left(2\sqrt{2}+4-1\right) = \frac{\log(2\sqrt{2}+3)}{\sqrt{-t}}.\end{aligned}$$

Secondly, let's look at I. As $t \to \infty$, an elementary computation shows that

$$\begin{aligned}I =& \frac{2}{\sqrt{t}} \arctan\sqrt{\frac{x-t}{t}}\Bigg|_2^t = \frac{2}{\sqrt{t}}\arccos\frac{1}{\sqrt{1+\frac{x-t}{t}}}\Bigg|_2^t \\ =& \frac{2}{\sqrt{t}}\arccos\frac{1}{\sqrt{\frac{x}{t}}}\Bigg|_2^t = \frac{2}{\sqrt{t}}\arccos\sqrt{\frac{t}{x}}\Bigg|_2^t \\ =& \frac{2}{\sqrt{t}}\sqrt{-1}\log\left(\sqrt{\frac{t}{x}}+\sqrt{\frac{t}{x}-1}\right)\Bigg|_2^t = \frac{-2}{\sqrt{t}}\sqrt{-1}\log\left(\sqrt{\frac{t}{2}}+\sqrt{\frac{t}{2}-1}\right) \\ =&\frac{-2}{\sqrt{t}}\sqrt{-1}\log\left(\sqrt{\frac{t}{2}}\left(1+\sqrt{1-\frac{2}{t}}\right)\right) = \frac{-2}{\sqrt{t}}\sqrt{-1}\left(\frac{1}{2}\log\frac{t}{2}+\log\left(1+\sqrt{1-\frac{2}{t}}\right)\right) \\ \sim&\frac{-2}{\sqrt{t}}\sqrt{-1}\left(\frac{1}{2}\log\frac{t}{2}+\log 2\right) \sim \frac{\log t}{\sqrt{-t}}.\end{aligned}$$

Thirdly, let's look at J. On the one hand, $x \geq 2$ implies that $\left(1\cdot\sqrt{x}\cdot(\sqrt{2}+1)\right)^{-1} \geq \left(\sqrt{x-1}\cdot\sqrt{x}\cdot\left(\sqrt{x}+\sqrt{x-1}\right)\right)^{-1}$. Substituting it into J, we will get its bound from one direction, i.e.,

$$\int_2^t \left(\frac{1}{1\cdot\sqrt{x}\cdot(\sqrt{2}+1)}\right)\frac{dx}{\sqrt{x(x-t)}} = (\sqrt{2}-1)\int_2^t \frac{dx}{x\sqrt{(x-t)}} = (\sqrt{2}-1)I.$$

On the other hand, $x-1 < x$ implies that

$$\left(\sqrt{x-1}\cdot\sqrt{x}\cdot\left(\sqrt{x}+\sqrt{x-1}\right)\right)^{-1} > \left(\sqrt{x}\cdot\sqrt{x}\cdot\left(\sqrt{x}+\sqrt{x}\right)\right)^{-1}.$$

After similar substitutions, we will get the other bound for J, namely

$$\begin{aligned}
&\int_2^t \left(\frac{1}{\sqrt{x}\cdot\sqrt{x}\cdot(\sqrt{x}+\sqrt{x})}\right)\frac{dx}{\sqrt{x(x-t)}} = \frac{1}{2}\int_2^t \frac{dx}{x^2\sqrt{x-t}} \\
=&\frac{1}{2}\left(\frac{-1\cdot\sqrt{x-t}}{-t\cdot 1\cdot x}\bigg|_2^t - \frac{2\cdot 2-3}{2\cdot(-t)}\int_2^t \frac{dx}{x\sqrt{x-t}}\right) \\
=&\frac{1}{2}\left(\frac{\sqrt{x-t}}{t\cdot x}\bigg|_2^t + \frac{1}{2\cdot t}\int_2^t \frac{dx}{x\sqrt{x-t}}\right) = \frac{1}{2}\left(-\frac{\sqrt{2-t}}{2t}+\frac{I}{2\cdot t}\right).
\end{aligned}$$

This will imply another bound for $I+J$ which is

$$\left(1+\frac{1}{4\cdot t}\right)\cdot I - \frac{\sqrt{2-t}}{4t} \sim I.$$

Combining these two sides estimates, we know[2] that there exists a positive real number $C \in \left[1,\sqrt{2}\right]$, such that

$$I+J \sim C\cdot I,$$

as $t \to \infty$. Now by (4.8), the numerator has the following asymptotic behavior.

$$\begin{aligned}
\int_{\tilde{\gamma}}\omega = -2\left(\int_1^2 \omega + I + J\right) &\sim -2\left(\frac{\log(2\sqrt{2}+3)}{\sqrt{-t}} + C\cdot I\right) \\
\sim -2\left(\frac{\log(2\sqrt{2}+3)}{\sqrt{-t}} + C\cdot\frac{\log t}{\sqrt{-t}}\right) &\sim \frac{-2C\cdot\log t}{\sqrt{-t}}.
\end{aligned}$$

Finally, let us change variables by setting $s=\frac{1}{x}$ and $t=\frac{1}{\lambda}$. We know $dx=-s^{-2}ds$. Then as $\lambda\to 0$, it follows that

$$\begin{aligned}
\frac{2C\cdot\log\lambda\cdot\sqrt{\lambda}}{\sqrt{-1}} &\sim -2\int_1^t \frac{dx}{\sqrt{x(x-1)(x-t)}} = -2\int_1^\lambda \frac{-s^{-2}ds}{\sqrt{\frac{1}{s}(\frac{1}{s}-1)(\frac{1}{s}-\frac{1}{\lambda})}} \\
&= 2\int_1^\lambda \frac{ds}{\sqrt{s(s-1)(\frac{s}{\lambda}-1)}} = \int_\gamma \omega\cdot\sqrt{\lambda}.
\end{aligned}$$

This means that as $\lambda\to 0$, it holds that

$$\int_\gamma \omega \sim -2C\cdot\log\lambda\cdot\sqrt{-1}. \tag{4.9}$$

[2] This is not a precise argument, and up to here what we could only know is that the quotient of $I+J$ and I is bounded by some constant C. However, in the next section when proving the second term we give a precise proof which determines this C to be 1.

Ⓑ The Denominator. By the construction of $-\tilde{\delta}$, which contains points $\{0, 1\}$, we know that when t is large (as $t \to \infty$) compared to the radius R it holds that

$$\begin{aligned}\int_{-\tilde{\delta}} \omega &= \int_{\tilde{\delta}} \frac{dx}{\sqrt{x(x-1)(x-t)}} \sim \int_{-\tilde{\delta}} \frac{dx}{\sqrt{x \cdot x \cdot (-t)}} \\ &= \frac{1}{\sqrt{-t}} \int_{-\tilde{\delta}} \frac{dx}{x} \sim \frac{1}{\sqrt{-t}} \cdot 2\pi\sqrt{-1} = \frac{2\pi}{\sqrt{t}}.\end{aligned}$$

We then make the same change of variables by setting $s = \frac{1}{x}$, $t = \frac{1}{\lambda}$. For small λ, $-\delta$ is a circle containing points $\{1, \infty\}$ (on a Riemann sphere this is equivalent to say that δ contains only points $\{0, \lambda\}$). As $\lambda \to 0$, it holds that

$$2\pi \cdot \sqrt{\lambda} \sim \int_{-\delta} \frac{-s^{-2} ds}{\sqrt{\frac{1}{s}(\frac{1}{s}-1)(\frac{1}{s}-\frac{1}{\lambda})}} = \sqrt{\lambda} \cdot \int_{\delta} \frac{ds}{\sqrt{s(s-1)(s-\lambda)}},$$

which means that

$$\int_{\delta} \frac{ds}{\sqrt{s(s-1)(s-\lambda)}} \sim 2\pi. \tag{4.10}$$

Combining (4.9) and (4.10) as $\lambda \to 0$, we know that

$$\frac{\int_{\gamma} \omega}{\int_{\delta} \omega} \sim \frac{-2C \cdot \log \lambda \cdot \sqrt{-1}}{2\pi} = \frac{C \cdot \log \lambda}{\pi\sqrt{-1}}.$$

Moreover, it holds that

$$\operatorname{Im}\left(\frac{C \cdot \log \lambda}{\pi\sqrt{-1}}\right) = \operatorname{Im}\left(-\frac{C\sqrt{-1}}{\pi} \cdot \log|\lambda|\right) = -\frac{C \cdot \log|\lambda|}{\pi}.$$

Therefore, as $\lambda \to 0$ we will get

$$\begin{aligned}\log C_{\lambda}^{-1} &= \log \frac{1}{\operatorname{Im}\left(\frac{\int_{\gamma}\omega}{\int_{\delta}\omega}\right) \cdot \left|\int_{\delta}\omega\right|^2} \sim \log \frac{1}{\operatorname{Im}\left(-\frac{C \cdot \log \lambda}{\pi\sqrt{-1}}\right)} - 2\log(2\pi) \\ &= \log \frac{1}{-\frac{C \cdot \log|\lambda|}{\pi}} - 2\log(2\pi) \sim \log\left(\frac{1}{-\log|\lambda|}\right).\end{aligned}$$

Thus, Lemma 4.3.1 is proved. □

By using Lemma 4.3.1 and (2.4), we get the following asymptotic behaviors

$$\begin{aligned}
\partial_\lambda\bar{\partial}_\lambda \log k_\lambda(\cdot) \sim &- \partial_\lambda\bar{\partial}_\lambda \log\left(-\log|\lambda|\right) = -\partial_\lambda\left(\frac{\bar{\partial}_\lambda(\log|\lambda|)}{\log|\lambda|}\right) \\
&= -\frac{\partial_\lambda\bar{\partial}_\lambda(\log|\lambda|) - \partial_\lambda(\log|\lambda|)\wedge\bar{\partial}_\lambda(\log|\lambda|)}{(\log|\lambda|)^2} = \frac{\partial_\lambda(\log|\lambda|)\wedge\bar{\partial}_\lambda(\log|\lambda|)}{(\log|\lambda|)^2} \\
&= \frac{\partial_\lambda(\log|\lambda|^2)\wedge\bar{\partial}_\lambda(\log|\lambda|^2)}{4(\log|\lambda|)^2} = \frac{\frac{\partial_\lambda(|\lambda|^2)}{|\lambda|^2}\wedge\frac{\bar{\partial}_\lambda(|\lambda|^2)}{|\lambda|^2}}{4(\log|\lambda|)^2} = \frac{\partial_\lambda(|\lambda|^2)\wedge\bar{\partial}_\lambda(|\lambda|^2)}{4|\lambda|^4(\log|\lambda|)^2} \\
&= \frac{\bar{\lambda}d\lambda\wedge\lambda d\bar{\lambda}}{4|\lambda|^4(\log|\lambda|)^2} = \frac{d\lambda\wedge d\bar{\lambda}}{4|\lambda|^2(\log|\lambda|)^2},
\end{aligned}$$

as $\lambda \to 0$. In this way, the hyperbolic growth of the leading term in Theorem 1.2.4 can thus be shown. For the leading term asymptotics of the period, see [CMSP], where it is remarked that the sub-leading terms of the period is holomorphic with respect to λ.

4.4 Legendre family: the two-term asymptotics at 0 by Taylor expansion

Roughly speaking, the proof of Lemma 4.3.1 actually shows that there exists a positive real number $C \in [1, \sqrt{2}]$, such that as $\lambda \to 0$ it holds that

$$\tau(\lambda) \sim \frac{C \cdot \log\lambda}{\sqrt{-1}\pi},$$

where $\tau(\lambda) := \frac{\int_\gamma \omega}{\int_\delta \omega}$ is the period of $X_\lambda^{(1)}$, $\lambda \in \mathbb{C}\setminus\{0,1\}$. In this section, we will show that the above constant C can actually be chosen as "1". Moreover, we will use the same method to determine the precise second term, which requires little tricks.

Two-term asymptotic expansion of the period

In [Ah, p.280], two-term asymptotic expansion of the elliptic modular lambda function is essentially studied where the Weierstrass-$\wp$ function is used to investigate the period of $X_\lambda^{(1)}$. However, we reprove this fact by using the Taylor expansions of Abelian differentials.

Proposition 4.4.1. *Let $\tau(\lambda)$ denote the period of the elliptic curve X_λ, $\lambda \in \mathbb{C}\setminus\{0,1\}$. Then as $\lambda \to 0$, it holds that*

$$\tau(\lambda) \sim \frac{\log\lambda - \log 16}{\sqrt{-1}\pi}.$$

Proof of Proposition 4.4.1. Let us go back to (4.8) in the proof of Lemma 4.3.1, where we separate the numerator into a proper term $-2\int_1^2 \omega$ and an improper term $-2\int_2^t \omega$. Now, for any real number $\epsilon > 1$, we re-separate the numerator as $-2\int_1^s \omega$ and $-2\int_s^t \omega$. Similarly, as $t \to \infty$, it holds that

$$\begin{aligned}\int_1^\epsilon \omega = \int_1^\epsilon \frac{dx}{\sqrt{x(x-1)(x-t)}} &\sim \int_1^\epsilon \frac{dx}{\sqrt{-t}\cdot\sqrt{x(x-1)}}\\ =&\frac{1}{\sqrt{-t}}\log\left(2\sqrt{x(x-1)}+2x-1\right)\Big|_1^\epsilon\\ =&\frac{1}{\sqrt{-t}}\log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)\end{aligned}$$

and

$$\int_\epsilon^t \frac{dx}{x\sqrt{x-t}} = \frac{2}{\sqrt{t}}\arctan\sqrt{\frac{x-t}{t}}\Bigg|_\epsilon^t = \frac{-2}{\sqrt{t}}\sqrt{-1}\log\left(\sqrt{\frac{t}{\epsilon}}+\sqrt{\frac{t}{\epsilon}-1}\right)$$

$$=\frac{-2}{\sqrt{t}}\sqrt{-1}\log\left(\sqrt{\frac{t}{\epsilon}}\left(1+\sqrt{1-\frac{\epsilon}{t}}\right)\right) = \frac{-2}{\sqrt{t}}\sqrt{-1}\left(\frac{1}{2}\log\frac{t}{\epsilon}+\log\left(1+\sqrt{1-\frac{\epsilon}{t}}\right)\right)$$

$$=\frac{-2}{\sqrt{t}}\sqrt{-1}\left(\frac{1}{2}\log\frac{t}{\epsilon}+\log\left(2-\frac{1}{2}\cdot\frac{2\epsilon}{t}-\frac{1}{8}\cdot\frac{4\epsilon^2}{t^2}+\mathrm{O}(\frac{\epsilon^3}{t^3})\right)\right)$$

$$\sim\frac{-2}{\sqrt{t}}\sqrt{-1}\left(\frac{1}{2}\log\frac{t}{\epsilon}+\log 2\right) = \frac{1}{\sqrt{-t}}\left(\log t+2\log 2-\log\epsilon\right).$$

Also, $x\geq\epsilon$ implies that

$$\left(\sqrt{\epsilon-1}\cdot\sqrt{x}\cdot(\sqrt{\epsilon}+\sqrt{\epsilon-1})\right)^{-1}\geq\left(\sqrt{x-1}\cdot\sqrt{x}\cdot\left(\sqrt{x}+\sqrt{x-1}\right)\right)^{-1}>0.$$

Therefore, we will get that

$$\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\cdot\frac{1}{\sqrt{x}} = \left(\frac{1}{\sqrt{\epsilon-1}\cdot\sqrt{x}\cdot(\sqrt{\epsilon}+\sqrt{\epsilon-1})}\right)+\frac{1}{\sqrt{x}}\geq\frac{1}{\sqrt{x-1}}>\frac{1}{\sqrt{x}}.$$

Thus, $\int_\epsilon^t\omega=\int_\epsilon^t\frac{dx}{\sqrt{x(x-1)(x-t)}}$ can be squeezed by two terms A and B, namely

$$A:=\int_\epsilon^t\frac{dx}{x\sqrt{(x-t)}}\sim\frac{1}{\sqrt{-t}}\left(\log t+2\log 2-\log\epsilon\right),$$

and

$$B:=\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\cdot A\sim\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\cdot\frac{1}{\sqrt{-t}}\left(\log t+2\log 2-\log\epsilon\right).$$

Finally, the numerator can be squeezed by $-2(A+\int_1^\epsilon\omega)$ and $-2(B+\int_1^\epsilon\omega)$. On the one hand, it holds that

$$\begin{aligned}-2\left(A+\int_1^\epsilon\omega\right)&\sim\frac{-2}{\sqrt{-t}}\left(\log t+2\log 2-\log\epsilon\right)+\frac{-2}{\sqrt{-t}}\log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)\\&\sim\frac{-2}{\sqrt{-t}}\left(\left(\log t+2\log 2\right)+\log\left(2\sqrt{\frac{\epsilon-1}{\epsilon}}+2-\frac{1}{\epsilon}\right)\right).\end{aligned}$$

On the other hand, it holds that

$$\begin{aligned}
&-2\left(B+\int_1^{\epsilon}\omega\right)\sim -2\left(\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\cdot\frac{1}{\sqrt{-t}}\left(\log t+2\log 2-\log\epsilon\right)+\frac{\log\epsilon}{\sqrt{-t}}\right)\\
=&\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\cdot\frac{-2}{\sqrt{-t}}\left(\log t+2\log 2-\log\epsilon\right)+\frac{-2}{\sqrt{-t}}\log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)\\
=&\frac{-2}{\sqrt{-t}}\left(\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\left(\log t+2\log 2-\log\epsilon\right)+\log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)\right)\\
=&\frac{-2}{\sqrt{-t}}\left(\left(\frac{\sqrt{\epsilon}}{\sqrt{\epsilon-1}}\right)\left(\log t+2\log 2\right)+\log\left(2\sqrt{\epsilon(\epsilon-1)}+2\epsilon-1\right)-\frac{\sqrt{\epsilon}\cdot\log\epsilon}{\sqrt{\epsilon-1}}\right).
\end{aligned}$$

Since ϵ is arbitrary, after taking the limit $\epsilon\to\infty$ we will see that the numerator is asymptotic to

$$\frac{-2}{\sqrt{-t}}\left(\log t+2\log 2+\log(2+2)\right)=\frac{-2}{\sqrt{-t}}\left(\log t+\log 16\right),$$

as $t\to\infty$. Similarly, we change variables again. Taking the inverse of t as $\lambda\to 0$, we get

$$2\sqrt{-1}\cdot(-\log\lambda+\log 16)\cdot\sqrt{\lambda}\sim-\int_{\tilde{\gamma}}\omega\cdot\sqrt{\lambda},$$

which implies that the numerator is asymptotic to $-2\sqrt{-1}\left(\log\lambda-\log 16\right)$, as $\lambda\to 0$. Comparing it with the denominator, we finally obtain that

$$\tau(\lambda)=\frac{\int_\gamma\omega}{\int_\delta\omega}\sim\frac{2\sqrt{-1}\left(\log\lambda-\log 16\right)}{-2\pi}=\frac{\log\lambda-\log 16}{\pi\sqrt{-1}},$$

as $\lambda\to 0$. This finishes the proof of Proposition 4.4.1. □

Combining Proposition 4.4.1 and (2.5), we can get the following two-term asymptotic expansion for $\log C_\lambda^{-1}$.

Lemma 4.4.1. *Let C_λ be defined as in (2.3). Then as $\lambda\to 0$, it holds that*

$$\log C_\lambda^{-1}\sim-\log\left(-\log|\lambda|+\log 16\right)+C'.$$

An alternative proof of Theorem 4.4.1

A two-term restricted version of Theorem 1.2.4 is stated as follows.

Theorem 4.4.1. *Under the same assumptions as in Theorem 1.2.4, as $\lambda\to 0$, it follows that*

$$L^{(1)}_{\lambda,z}=\frac{\sqrt{-1}d\lambda\wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}\left(1+2\frac{\log 16}{\log|\lambda|}+\mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right)\right).$$

An alternative proof of Theorem 4.4.1. By (2.4) and Lemma 4.4.1, we first make the following computations on the horizontal curvature form of the Bergman kernel as $\lambda \to 0$.

$$
\begin{aligned}
\partial_\lambda \bar{\partial}_\lambda \log k_\lambda(\cdot) &= \partial_\lambda \bar{\partial}_\lambda \log C_\lambda^{-1} \sim -\partial_\lambda \bar{\partial}_\lambda \log\left(-\log|\lambda| + \log 16\right) \\
&= -\partial_\lambda \left(\frac{\bar{\partial}_\lambda (\log|\lambda|)}{\log|\lambda| - \log 16} \right) = -\frac{-\partial_\lambda (\log|\lambda| - \log 16) \wedge \bar{\partial}_\lambda (\log|\lambda|)}{(\log|\lambda| - \log 16)^2} \\
&= \frac{\partial_\lambda (\log|\lambda|) \wedge \bar{\partial}_\lambda (\log|\lambda|)}{(\log|\lambda| - \log 16)^2} = \frac{d\lambda \wedge d\bar{\lambda}}{4|\lambda|^2 (\log|\lambda| - \log 16)^2}.
\end{aligned}
$$

By the leading term asymptotic expansion for the above left hand side, in order to determine the second term we consider a new difference function h, namely

$$
h(\lambda) := \frac{1}{4|\lambda|^2 (\log|\lambda| - \log 16)^2} - \frac{1}{4|\lambda|^2 (\log|\lambda|)^2}.
$$

Making further computations

$$
\partial(\log|\lambda|) = \frac{1}{2}\partial(\log(\lambda\bar{\lambda})) = \frac{\bar{\lambda} d\lambda}{2|\lambda|^2} = \frac{d\lambda}{2\lambda} \quad \text{and} \quad \bar{\partial}(\log|\lambda|) = \frac{d\bar{\lambda}}{2\bar{\lambda}},
$$

we will know that

$$
\begin{aligned}
h(\lambda) &= \frac{(\log|\lambda|)^2 - (\log|\lambda| - \log 16)^2}{4|\lambda|^2 (\log|\lambda| - \log 16)^2 (\log|\lambda|)^2} \\
&= \frac{2\log|\lambda| \cdot \log 16 - (\log 16)^2}{4|\lambda|^2 (\log|\lambda| - \log 16)^2 (\log|\lambda|)^2} = \frac{2\log|\lambda| \cdot \log 16 - (\log 16)^2}{4|\lambda|^2 (\log|\lambda| - \log 16)^2 (\log|\lambda|)^2} \\
&\sim \frac{2(\log 16)}{4|\lambda|^2 (\log|\lambda| - \log 16)^2 (\log|\lambda|)} \sim \frac{(\log 16)}{2|\lambda|^2 (\log|\lambda|)^3},
\end{aligned}
$$

which implies that

$$
l^{(1)}_{\lambda,z} \sim \frac{1}{4|\lambda|^2 (\log|\lambda|)^2} + h(\lambda) \sim \frac{1}{4|\lambda|^2 (\log|\lambda|)^2} + \frac{(\log 16)}{2|\lambda|^2 (\log|\lambda|)^3},
$$

as $\lambda \to 0$. Thus, the proof is completed. □

We finally remark that the importance of this alternative approach is that it works for higher genus cases, where properties of special elliptic functions could not be applied to.

4.5 Another nodal family of degenerate elliptic curves

For another nodal-type degenerate family of elliptic curves $X_\lambda^{(2)}$, we study its Bergman kernel by analyzing the Taylor expansions of Abelian differentials.

Theorem 4.5.1. *In the local coordinate z on $X_\lambda^{(2)}$, write its Bergman kernel as $B_\lambda^{(2)} = k_\lambda^{(2)}(z)dz \wedge d\bar{z}$, for $\lambda \in \mathbb{C} \setminus \{0,1\}$. Then as $\lambda \to 0$, it holds that*

$$L_{\lambda,z}^{(2)} \sim \frac{\sqrt{-1}d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}.$$

Proof of Theorem 4.5.1. By the construction of γ, we know as $\lambda \to 0$ that

$$\begin{aligned}\int_\gamma \omega =& -2\int_{\sqrt{\lambda}}^1 \omega = -2\int_{\sqrt{\lambda}}^1 \frac{dx}{\sqrt{(x-1)(x^2-\lambda)}} \\ \sim& -2\int_{\sqrt{\lambda}}^1 \frac{dx}{x\sqrt{x-1}} = -2\cdot 2\sqrt{-1}\log\left(\sqrt{\frac{1}{x}} + \sqrt{\frac{1}{x}-1}\right)\Bigg|_{\sqrt{\lambda}}^1 \\ =& 4\sqrt{-1}\log\left(\sqrt{\frac{1}{\sqrt{\lambda}}} + \sqrt{\frac{1}{\sqrt{\lambda}}-1}\right) \sim 4\sqrt{-1}\log\left(\sqrt{\frac{1}{\sqrt{\lambda}}}\right) = -\sqrt{-1}\log\lambda.\end{aligned}$$

Since δ is a big circle containing $\sqrt{\lambda}$ and $-\sqrt{\lambda}$, on $X_\lambda^{(2)}$ it is equivalent to say that $-\delta$ contains only 1 and ∞. We then make changes of variables by setting $s = \frac{1}{x}$, $t = \frac{1}{\lambda}$ and denote the corresponding big circle by $-\tilde{\delta}$ which contains 1 and 0. Then, it follows that

$$\begin{aligned}\int_\delta \omega =& \int_\delta \frac{dx}{\sqrt{(x-1)(x^2-\lambda)}} = -\int_{-\tilde{\delta}} \frac{-s^{-2}ds}{\sqrt{(\frac{1}{s}-1)(\frac{1}{s^2}-\frac{1}{t})}} \\ =& \int_{-\tilde{\delta}} \frac{ds}{\sqrt{s(s-1)(\frac{s^2}{t}-1)}} = \int_{-\tilde{\delta}} \frac{\sqrt{t}\cdot ds}{\sqrt{s(s-1)(s^2-t)}} \sim \int_{-\tilde{\delta}} \frac{\sqrt{t}\cdot ds}{s\sqrt{-t}} = 2\pi,\end{aligned}$$

as $\lambda \to 0$ ($t \to \infty$). Therefore, we know that

$$\frac{\int_\gamma \omega}{\int_{\tilde{\delta}} \omega} \sim \frac{-\sqrt{-1}\log\lambda}{2\pi}.$$

By (2.4) and (2.5) we have finished the proof of Theorem 4.5.1. □

4.6 Two families of degenerate elliptic curves with cusps

For the cusp degeneration case, it seems that the type of singularities determines various boundary behaviors: either trivial with a constant period or reducible to the case of $X_\lambda^{(1)}$.

Theorem 4.6.1. *In the local coordinate z on $X_\lambda^{(3)}$, write its Bergman kernel as $B_\lambda^{(3)} = k_\lambda^{(3)}(z)dz \wedge d\bar{z}$, for $\lambda \in \mathbb{C} \setminus \{0\}$. Then as $\lambda \to 0$, it holds that $L_{\lambda,z}^{(3)} \equiv 0$.*

Proof of Theorem 4.6.1. We will show that the period of the elliptic curve $X_\lambda^{(3)}$ is constant, for all $\lambda \in \mathbb{C} \setminus \{0\}$. For Y_λ, the two cycles can be chosen such that δ is a big circle centered at the origin which contains $-\sqrt{\lambda}$ and 0, and γ satisfies that

$$\int_\gamma \omega = -2\int_0^{\sqrt{\lambda}} \omega = -2\int_0^{\sqrt{\lambda}} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}}.$$

Since ω is holomorphic along δ, by Cauchy Integral Theorem, we can choose a path which is homologous to δ and consists of two circles c_1 and c_2 (of radius r, centered at $-\sqrt{\lambda}$ and 0, respectively) and two straight lines l_1 and l_2 connecting almost $-\sqrt{\lambda}+r$ and $-r$. Near the point $-\sqrt{\lambda}$, using polar coordinate we can denote x by $-\sqrt{\lambda}+re^{\sqrt{-1}\theta}$, $\theta \in [\beta, 2\pi-\beta]$, for small $\beta > 0$. Then,

$$\begin{aligned}\int_{c_1} \omega &= \int_\beta^{2\pi-\beta} \frac{re^{\sqrt{-1}\theta}\sqrt{-1}d\theta}{\sqrt{(-\sqrt{\lambda}+\epsilon e^{\sqrt{-1}\theta}-1)(-\sqrt{\lambda}+\epsilon e^{\sqrt{-1}\theta}-\sqrt{\lambda})re^{\sqrt{-1}\theta}}} \\ &= \int_\beta^{2\pi-\beta} \frac{\sqrt{re^{\sqrt{-1}\theta}}\sqrt{-1}d\theta}{\sqrt{(-\sqrt{\lambda}+re^{\sqrt{-1}\theta}-1)(-\sqrt{\lambda}+re^{\sqrt{-1}\theta}-\sqrt{\lambda})}} \to 0,\end{aligned}$$

as $r \to 0$. Similarly, we can get $\int_{c_2} \omega \to 0$. Since r is arbitrary and ω changes the sign when switching between l_1 and l_2, we know that

$$\begin{aligned}\int_\delta \omega &= \int_{c_1} \omega + \int_{c_2} \omega + \int_{l_1} \omega + \int_{l_2} \omega \\ &= \int_{l_1} \omega + \int_{l_2} \omega = -2\int_{-\sqrt{\lambda}}^0 \omega = -2\int_{-\sqrt{\lambda}}^0 \frac{dx}{\sqrt{x(x^2-\lambda)}}.\end{aligned}$$

After making changes of variables by setting $s = -x$, we know that

$$\int_\delta \omega = -2\int_{\sqrt{\lambda}}^0 \frac{-ds}{\sqrt{-s(s^2-\lambda)}} = -2\int_{\sqrt{\lambda}}^0 \frac{\sqrt{-1}ds}{\sqrt{s(s^2-\lambda)}} = 2\int_0^{\sqrt{\lambda}} \frac{\sqrt{-1}ds}{\sqrt{s(s^2-\lambda)}},$$

which implies

$$\frac{\int_\gamma \omega}{\int_\delta \omega} = \frac{-2\int_0^{\sqrt{\lambda}} \omega}{2\sqrt{-1}\int_0^{\sqrt{\lambda}} \omega} \equiv \sqrt{-1}.$$

Although their ratio is constant, we still can determine the asymptotics of the numerator and the denominator, first up to an multiplier and then with a precise constant. Take $\int_\gamma \omega$ for example, we observe that $x+\sqrt{\lambda}$ is bounded on γ by $\sqrt{\lambda}$ and $2\sqrt{\lambda}$ which have the same order of growth. The antiderivative of the remaining term can be written down, namely

$$\int_0^{\sqrt{\lambda}} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})}} = \log\left(2\sqrt{x(x-\sqrt{\lambda})}+2x-\sqrt{\lambda}\right)\Big|_0^{\sqrt{\lambda}} = \pi\sqrt{-1}.$$

Therefore, we conclude that both the numerator and the denominator have the order of growth $\mathrm{O}\left(\lambda^{-\frac{1}{4}}\right)$. Now, we will determine precisely the numerator and the denominator as below and see their relations with a Legendre family.

$$
\begin{aligned}
\int_\delta \omega = -2\int_{-\sqrt{\lambda}}^{0} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}} &\overset{q=x+\sqrt{\lambda}}{=\!=\!=} -2\int_0^{\sqrt{\lambda}} \frac{dq}{\sqrt{q(q-\sqrt{\lambda})(q-2\sqrt{\lambda})}} \\
\overset{q=v\cdot\sqrt{\lambda}}{=\!=\!=} -2\int_0^1 \frac{\sqrt{\lambda}\cdot dv}{\sqrt{v\cdot\sqrt{\lambda}\cdot(v\cdot\sqrt{\lambda}-\sqrt{\lambda})\cdot(v\cdot\sqrt{\lambda}-2\sqrt{\lambda})}} \\
= \frac{-2}{\sqrt{\sqrt{\lambda}}}\int_0^1 \frac{dv}{\sqrt{v(v-1)(v-2)}} := \frac{\alpha}{\sqrt{\sqrt{\lambda}}}.
\end{aligned}
$$

We know that this constant α is one period (the denominator part) of a Legendre family of elliptic curve $X_2 := \left\{y^2 = x(x-1)(x-2)\right\}$. Also, we know that $\left|\int_\delta \omega\right|^2 = |\alpha|^2 \cdot |\lambda|^{0.5}$. Finally by (2.5) and (2.4), for all λ, it follows that

$$
\frac{\partial^2(\log k_\lambda(\cdot))}{\partial\lambda\partial\bar{\lambda}} = \frac{\partial^2\left\{\mathrm{Im}\left(\sqrt{-1}\right)\cdot|\alpha|^2\cdot|\lambda|^{0.5}\right\}^{-1}}{\partial\lambda\partial\bar{\lambda}} \equiv 0. \qquad \square
$$

For the elliptic curve $X_\lambda^{(4)}$ with a non-constant period, we estimate the numerator and the denominator, obtained from two cycle δ containing 0 and λ^2, and γ containing λ^2 and λ, respectively.

Theorem 4.6.2. *In the local coordinate z on $X_\lambda^{(4)}$, write its Bergman kernel as $B_\lambda^{(4)} = k_\lambda^{(4)}(z)dz\wedge d\bar{z}$, for $\lambda \in \mathbb{C}\setminus\{0\}$. Then, as $\lambda \to 0$, it holds that*

$$
L_{\lambda,z}^{(4)} \sim \frac{\sqrt{-1}d\lambda\wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}.
$$

Proof of Theorem 4.6.2. Firstly, as $\lambda \to 0$, it follows that

$$
\begin{aligned}
\int_\delta \omega = -2\int_0^{\lambda^2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} &\overset{q=x-\lambda^2}{=\!=\!=} \int_{-\lambda^2}^{0} \frac{-2dq}{\sqrt{q(q+\lambda^2)(q+\lambda^2-\lambda)}} \\
\overset{q=-\lambda^2\cdot v}{=\!=\!=} -2\int_1^0 \frac{-\lambda^2 dv}{\sqrt{-\lambda^2 v(-\lambda^2 v+\lambda^2)(-\lambda^2 v+\lambda^2-\lambda)}} \\
= \frac{-2}{-\lambda\sqrt{-1}}\int_0^1 \frac{dv}{\sqrt{v(v-1)(v-1+\frac{1}{\lambda})}} \sim \frac{1}{-\lambda\sqrt{-1}}\int_{\tilde{\gamma}} \frac{dv}{v\sqrt{-1+\frac{1}{\lambda}}} \\
= \frac{1}{-\lambda\sqrt{-1}}\frac{2\pi\sqrt{-1}}{\sqrt{-1+\frac{1}{\lambda}}} \sim \frac{2\pi}{-\sqrt{\lambda}}.
\end{aligned}
$$

Secondly, it holds that

$$
\begin{aligned}
&-2\int_0^{\lambda}\frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \overset{q=x-\lambda}{=\!=\!=} -2\int_{-\lambda}^{0}\frac{dq}{\sqrt{q(q+\lambda)(q+\lambda-\lambda^2)}}\\
&\overset{q=-\lambda\cdot v}{=\!=\!=} -2\int_1^0\frac{-\lambda dv}{\sqrt{-\lambda v\cdot(-\lambda)(v-1)\cdot(-\lambda)(v-1+\lambda)}}\\
&=-\frac{-2}{\sqrt{-\lambda}}\int_0^1\frac{dv}{\sqrt{v(v-1)(v-1+\lambda)}} \overset{u=1-v}{=\!=\!=} \frac{2}{\sqrt{-\lambda}}\int_1^0\frac{-du}{\sqrt{-u(-u+1)(-u+\lambda)}}\\
&=\frac{2}{-\sqrt{\lambda}}\int_0^1\frac{du}{\sqrt{u(u-1)(u-\lambda)}} \underset{t=\frac{1}{\lambda}}{\overset{u=\frac{1}{s}}{=\!=\!=}} \frac{2}{-\sqrt{\lambda}}\int_{\infty}^1\frac{-s^{-2}ds}{\sqrt{\frac{1}{s}(\frac{1}{s}-1)(\frac{1}{s}-\frac{1}{t})}}\\
&=\frac{2}{\lambda}\int_1^{\infty}\frac{ds}{\sqrt{s(s-1)(s-t)}} \sim \frac{2}{\lambda}\int_1^{\infty}\frac{ds}{s\sqrt{s-t}} = \frac{4\sqrt{-1}}{\lambda\sqrt{t}}\log\left(\sqrt{\frac{t}{x}}+\sqrt{\frac{t}{x}-1}\right)\Bigg|_1^{\infty}\\
&=\frac{4\sqrt{-1}}{\sqrt{\lambda}}\left(\log\sqrt{-1}-\log\left(\sqrt{t}+\sqrt{t-1}\right)\right) \sim \frac{2\sqrt{-1}\log\lambda}{\sqrt{\lambda}},
\end{aligned}
$$

as $\lambda \to 0$, and therefore we know that

$$
\int_{\gamma}\omega = -2\int_0^{\lambda}\frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} + 2\int_0^{\lambda^2}\frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \sim \frac{2\sqrt{-1}\log\lambda}{\sqrt{\lambda}}.
$$

Finally, $\tau \sim \frac{\log\lambda}{\sqrt{-1}\pi} > 0$ and $\operatorname{Im}\tau \sim \frac{-\log|\lambda|}{\pi}$. By (2.5) it follows that,

$$
C_{\lambda} \sim \frac{-4\pi}{|\lambda|}\cdot\log|\lambda|,
$$

as $\lambda \to 0$. By (2.4), we know that hyperbolic growth appears again for $L_{\lambda}^{(4)}$. $\square$

Chapter 5

Bergman kernel on degenerate hyperelliptic curves

Near degenerate boundaries with nodes or cusps, we estimate asymptotic behaviors of the relative Bergman kernel metrics for a holomorphic family of hyperelliptic curves, with applications to their Jacobians. Specifically, the curvature form tends near a node to an incomplete metric on the parameter space, but tends near a certain cusp to 0. These results are different from the elliptic curve case and the type of singularities determines various boundary asymptotics. For the genus-two case particularly, asymptotic formulas with precise coefficients involving the complex structure information are written down explicitly.

5.1 Non-separating node: genus-two curves with precise coefficients

We start with the following two lemmas by analyzing the asymptotics of the matrices A and B on $X_\lambda^{(5)}$, respectively.

Lemma 5.1.1. *Under the same assumptions as in Theorem 1.2.5, as $\lambda \to 0$, it holds that*

$$A(\lambda) \sim \begin{pmatrix} \frac{-2\pi}{\sqrt{ab}} & 0 \\ 0 & \tilde{C}_{ab} \end{pmatrix},$$

where $\tilde{C}_{ab} := -2\int_1^a \frac{dx}{\sqrt{(x-1)(x-a)(x-b)}}$.

Proof of Lemma 5.1.1. We estimate all the four elements one by one. Firstly, $a_{11} = \int_{\delta_1} \omega_1$, where δ_1 only contains 0 and λ. Changing variables by setting $t = \frac{1}{\lambda}(\to \infty)$ and $s = \frac{1}{x}$. As $\lambda \to 0$, we will get that

$$
\begin{aligned}
a_{11} &\xlongequal[t=\frac{1}{\lambda}]{s=\frac{1}{x}} \int_{\tilde{\delta}_1} \frac{-s^{-2}ds}{\sqrt{\frac{1}{s}(\frac{1}{s}-\frac{1}{t})(\frac{1}{s}-1)(\frac{1}{s}-a)(\frac{1}{s}-b)}} = -\int_{-\tilde{\delta}_1} \frac{-\sqrt{s}\sqrt{t}ds}{\sqrt{(s-1)(s-\frac{1}{a})(s-\frac{1}{b})(s-t)}\sqrt{ab}} \\
&= \int_{-\tilde{\delta}_1} \frac{\sqrt{s}ds}{\sqrt{-ab(s-1)(s-\frac{1}{a})(s-\frac{1}{b})}} \left(1+\frac{s}{2t}+\mathrm{O}(\frac{s^2}{t^2})\right) \\
&= \int_{big} \frac{ds}{-s\sqrt{-ab}} \left(1+\frac{s}{2t}+\mathrm{O}(\frac{s^2}{t^2})\right)\left(1+\frac{1}{2s}+\mathrm{O}(\frac{1}{s^2})\right)\left(1+\frac{1}{2as}+\mathrm{O}(\frac{1}{s^2})\right)\left(1+\frac{1}{2bs}+\mathrm{O}(\frac{1}{s^2})\right) \\
&= \int_{big} \frac{ds}{-s\sqrt{-ab}} \left(1+\frac{s}{2t}+\mathrm{O}(\frac{s^2}{t^2})\right)\left(1+\mathrm{O}(\frac{1}{s})\right) \\
&= \frac{1}{-\sqrt{-ab}} \int_{big} \left(1+\mathrm{O}(\frac{1}{t})\right)\frac{ds}{s} = \frac{2\pi}{-\sqrt{ab}}\left(1+\mathrm{O}(\frac{1}{t})\right) \sim \frac{2\pi}{-\sqrt{ab}}.
\end{aligned}
$$

where $\tilde{\delta}_1$ contains $\{t,\infty\}$, $-\tilde{\delta}_1$ contains $\{0,1,\frac{1}{a},\frac{1}{b}\}$ and big is a big circle containing $\{0,1,\frac{1}{a},\frac{1}{b}\}$. Here $1<|s|<|t|$. Secondly, we look at $a_{21}=\int_{\delta_1}\omega_2$ and similarly it holds that

$$
\begin{aligned}
a_{21} &= \int_{big} \frac{ds}{-s^2\sqrt{-ab}} \left(1+\frac{s}{2t}+\mathrm{O}(\frac{s^2}{t^2})\right)\left(1+\frac{1}{2s}+\mathrm{O}(\frac{1}{s^2})\right)\left(1+\frac{1}{2as}+\mathrm{O}(\frac{1}{s^2})\right)\left(1+\frac{1}{2bs}+\mathrm{O}(\frac{1}{s^2})\right) \\
&= \int_{big} \frac{ds}{-s^2\sqrt{-ab}} \left(1+\frac{s}{2t}+\mathrm{O}(\frac{s^2}{t^2})\right)\left(1+\mathrm{O}(\frac{1}{s})\right) \\
&= \frac{1}{-\sqrt{-ab}} \int_{big} \left(\frac{1}{2t}+\mathrm{O}(\frac{1}{t^2})\right)\frac{ds}{s} = \frac{2\pi}{-\sqrt{ab}}\left(\frac{1}{2t}+\mathrm{O}(\frac{1}{t^2})\right) \xrightarrow{t\to\infty} 0.
\end{aligned}
$$

Thirdly, let δ_2 contain $\{0,\lambda,1,a\}$. Then, it holds that

$$
\begin{aligned}
a_{12} &= \int_{\delta_2} \frac{dx}{\sqrt{x(x-1)(x-a)(x-b)}} \cdot \frac{1}{\sqrt{x}} \left(1+\frac{\lambda}{2x}+\mathrm{O}(\frac{\lambda^2}{x^2})\right) \\
&= \int_{\delta_2} \frac{dx}{x\sqrt{(x-1)(x-a)(x-b)}} \left(1+\frac{\lambda}{2x}+\mathrm{O}(\frac{\lambda^2}{x^2})\right) \sim \int_{\delta_2} \frac{dx}{x\sqrt{(x-1)(x-a)(x-b)}} \\
&= \int_{\delta_2} \frac{dx}{x\sqrt{(x-b)}} \cdot \frac{1}{\sqrt{x}}\left(1+\frac{1}{2x}+\mathrm{O}(\frac{1}{x^2})\right) \cdot \frac{1}{\sqrt{x}}\left(1+\frac{a}{2x}+\mathrm{O}(\frac{1}{x^2})\right) \\
&= \int_{\delta_2} \frac{dx}{x^2\sqrt{(x-b)}} \cdot \left(1+\mathrm{O}(\frac{1}{x})\right),
\end{aligned}
$$

where $|a|<|x|<|b|$. Since δ_2 doesn't contain b, $\frac{1}{\sqrt{x-b}}$ is holomorphic and therefore bounded on δ_2 by $C\in\mathbb{C}$. Then, it holds that

$$
a_{12} \sim C\int_{\delta_2} \frac{dx}{x^2} \cdot \left(1+\mathrm{O}(\frac{1}{x})\right) = 0.
$$

Lastly, we deal with $a_{22} = \int_{\delta_2} \omega_2$. Similarly, it holds that

$$\begin{aligned}
a_{22} =& \int_{\delta_2} \frac{xdx}{\sqrt{x(x-1)(x-a)(x-b)}} \cdot \frac{1}{\sqrt{x}} \left(1 + \frac{\lambda}{2x} + \mathrm{O}(\frac{\lambda^2}{x^2})\right) \\
=& \int_{\delta_2} \frac{dx}{\sqrt{(x-1)(x-a)(x-b)}} \left(1 + \frac{\lambda}{2x} + \mathrm{O}(\frac{\lambda^2}{x^2})\right) \sim \int_{\delta_2} \frac{dx}{\sqrt{(x-1)(x-a)(x-b)}} := \tilde{C}_{a,b},
\end{aligned}$$

where δ_2 contain $\{1, a\}$. Thus, we finish the proof of Lemma 5.1.1. $\square$

Lemma 5.1.2. *Under the same assumptions as in Theorem 1.2.5, as $\lambda \to 0$, it holds that*

$$B(\lambda) \sim \begin{pmatrix} \frac{-2\log\lambda}{\sqrt{-ab}} & \tilde{C}'_{ab} \\ \frac{-2}{\sqrt{-ab}} & \tilde{C}''_{ab} \end{pmatrix},$$

where $\tilde{C}'_{ab} := -2\int_a^b \frac{dx}{x\sqrt{(x-1)(x-a)(x-b)}}$ *and* $\tilde{C}''_{ab} := -2\int_a^b \frac{dx}{\sqrt{(x-1)(x-a)(x-b)}}$.

Proof of Lemma 5.1.2 . Again, all the four elements are estimated one by one. Firstly as $t \to 0$, we make the following computations.

$$\int_1^t \frac{ds}{s\sqrt{s-t}} = \frac{2}{\sqrt{t}}\sqrt{-1}\log\left(\sqrt{t} + \sqrt{t-1}\right)\Big|_1^t \sim \frac{\sqrt{-1}}{\sqrt{t}}\log t. \tag{5.1}$$

$$\int_1^t \frac{ds}{s^2\sqrt{s-t}} = \frac{\sqrt{s-t}}{ts}\Big|_1^t + \frac{1}{2t}\int_1^t \frac{ds}{s\sqrt{s-t}} = \frac{-\sqrt{1-t}}{t} + \frac{\sqrt{-1}}{2t\sqrt{t}}\log t \sim -\frac{\sqrt{-1}}{\sqrt{t}}. \tag{5.2}$$

In particular, (5.1) yields the boundedness of

$$\int_1^t \frac{\sqrt{t}}{s\sqrt{s-t}} \cdot \mathrm{O}(\frac{1}{s})ds.$$

Then, for b_{11} it follows that

$$\begin{aligned}
b_{11} =& \int_{\gamma_1} \omega_1 = -2\int_\lambda^1 \frac{dx}{\sqrt{x(x-\lambda)(x-1)(x-a)(x-b)}} \\
=& -2\int_t^1 \frac{-\sqrt{s}\sqrt{t}ds}{\sqrt{(s-1)(s-\frac{1}{a})(s-\frac{1}{b})(s-t)}\sqrt{ab}} \\
=& -2\int_1^t \frac{\sqrt{s}\sqrt{t}ds}{\sqrt{s-t}\sqrt{ab}}\frac{1}{s\sqrt{s}}\left(1 + \frac{1}{2s} + \mathrm{O}(\frac{1}{s^2})\right)\left(1 + \frac{1}{2as} + \mathrm{O}(\frac{1}{s^2})\right)\left(1 + \frac{1}{2bs} + \mathrm{O}(\frac{1}{s^2})\right) \\
=& -2\int_1^t \frac{\sqrt{s}\sqrt{t}ds}{\sqrt{s-t}\sqrt{ab}}\frac{1}{s\sqrt{s}}\left(1 + \mathrm{O}(\frac{1}{s})\right) = -\frac{2}{\sqrt{ab}}\int_1^t \frac{\sqrt{t}ds}{s\sqrt{s-t}}\left(1 + \mathrm{O}(\frac{1}{s})\right) \\
\sim& -\frac{2}{\sqrt{ab}}\int_1^t \frac{\sqrt{t}ds}{s\sqrt{s-t}} \sim \frac{-2\log\lambda}{\sqrt{-ab}},
\end{aligned}$$

where $1 < |s| < |t|$. Secondly, by a similar argument(see also [CMSP, D3]), we can get that

$$\begin{aligned}
b_{21} = \int_{\gamma_1} \omega_2 &= -2 \int_\lambda^1 \frac{x dx}{\sqrt{x(x-\lambda)(x-1)(x-a)(x-b)}} \\
&= -2 \int_t^1 \frac{-\sqrt{s}\sqrt{t} ds}{s\sqrt{(s-1)(s-\frac{1}{a})(s-\frac{1}{b})(s-t)}\sqrt{ab}} \\
&= -2 \int_1^t \frac{\sqrt{t} ds}{\sqrt{s(s-t)}\sqrt{ab}} \frac{1}{s\sqrt{s}} \left(1 + \mathrm{O}(\frac{1}{s})\right)\left(1 + \frac{1}{2as} + \mathrm{O}(\frac{1}{s^2})\right)\left(1 + \frac{1}{2bs} + \mathrm{O}(\frac{1}{s^2})\right) \\
&= -2 \int_1^t \frac{\sqrt{t} ds}{s^2\sqrt{s-t}\sqrt{ab}} \left(1 + \mathrm{O}(\frac{1}{s})\right) \sim -\frac{2\sqrt{t}}{\sqrt{ab}} \int_1^t \frac{ds}{s^2\sqrt{s-t}} \sim \frac{-2}{\sqrt{-ab}}.
\end{aligned}$$

Thirdly, it follows that

$$\begin{aligned}
b_{12} = \int_{\gamma_2} \omega_1 &= \int_{\gamma_2} \frac{dx}{\sqrt{x(x-\lambda)(x-1)(x-a)(x-b)}} \\
&= \int_{\gamma_2} \frac{dx}{\sqrt{x(x-1)(x-a)(x-b)}} \cdot \frac{1}{\sqrt{x}} \left(1 + \frac{\lambda}{2x} + \mathrm{O}(\frac{\lambda^2}{x^2})\right) \\
&\sim \int_{\gamma_2} \frac{dx}{x\sqrt{(x-1)(x-a)(x-b)}} = -2 \int_a^b \frac{dx}{x\sqrt{(x-1)(x-a)(x-b)}} =: \tilde{C}'_{ab},
\end{aligned}$$

where $|a| < |x| < |b|$. Lastly, it holds that,

$$\begin{aligned}
b_{22} = \int_{\gamma_2} \omega_1 &= \int_{\gamma_2} \frac{x dx}{\sqrt{x(x-\lambda)(x-1)(x-a)(x-b)}} \\
&= \int_{\gamma_2} \frac{x dx}{\sqrt{x(x-1)(x-a)(x-b)}} \cdot \frac{1}{\sqrt{x}} \left(1 + \frac{\lambda}{2x} + \mathrm{O}(\frac{\lambda^2}{x^2})\right) \\
&\sim \int_{\gamma_2} \frac{dx}{\sqrt{(x-1)(x-a)(x-b)}} = -2 \int_a^b \frac{dx}{\sqrt{(x-1)(x-a)(x-b)}} =: \tilde{C}''_{ab},
\end{aligned}$$

and this finishes the proof of Lemma 5.1.2. $\square$

Combining Lemmas 5.1.1 and 5.1.2, we will prove the following lemma, which leads to the asymptotics for the Bergman kernels by combining (2.8).

Lemma 5.1.3. *Let $Z_\lambda^{(5)}$ denote the period matrix of $X_\lambda^{(5)}$. Then, as $\lambda \to 0$, it holds that*

$$(\operatorname{Im} Z_\lambda^{(5)})^{-1} \sim \frac{\pi}{-c_1 \log|\lambda| - c_2^2} \begin{pmatrix} c_1 & -c_2 \\ -c_2 & -\log|\lambda| \end{pmatrix}.$$

Proof of Lemma 5.1.3. By Lemma 5.1.1, we know that

$$A^{-1} \sim \begin{pmatrix} \frac{-2\pi}{\sqrt{ab}} & 0 \\ 0 & \tilde{C}_{ab} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{-\sqrt{ab}}{2\pi} & 0 \\ 0 & \tilde{C}_{ab}^{-1} \end{pmatrix},$$

as $\lambda \to 0$. Therefore, it follows that

$$Z = A^{-1}B \sim \begin{pmatrix} \frac{-\sqrt{ab}}{2\pi} & 0 \\ 0 & \tilde{C}_{ab}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \frac{-2\log\lambda}{\sqrt{-ab}} & \tilde{C}'_{ab} \\ \frac{-2}{\sqrt{-ab}} & \tilde{C}''_{ab} \end{pmatrix} = \begin{pmatrix} \frac{-\sqrt{-1}}{\pi}\log\lambda & \frac{-\tilde{C}'_{ab}\sqrt{ab}}{2\pi} \\ \frac{-2}{\tilde{C}_{ab}\sqrt{-ab}} & \frac{\tilde{C}''_{ab}}{\tilde{C}_{ab}} \end{pmatrix}.$$

Since Z is symmetric, this implies that

$$\sqrt{-1}ab\tilde{C}_{ab}\tilde{C}'_{ab} = 4\pi,$$

namely

$$\sqrt{-1}ab\int_1^a \frac{dx}{\sqrt{(x-1)(x-a)(x-b)}} \cdot \int_a^b \frac{dx}{x\sqrt{(x-1)(x-a)(x-b)}} = \pi.$$

Moreover, as $\lambda \to 0$ we know that,

$$\operatorname{Im} Z \sim \begin{pmatrix} -\frac{\log|\lambda|}{\pi} & \frac{-1}{2\pi}\operatorname{Im}\left\{\tilde{C}'_{ab}\sqrt{ab}\right\} \\ \frac{-1}{2\pi}\operatorname{Im}\left\{\tilde{C}'_{ab}\sqrt{ab}\right\} & \operatorname{Im}\left\{\frac{\tilde{C}''_{ab}}{\tilde{C}_{ab}}\right\} \end{pmatrix} =: \frac{1}{\pi}\begin{pmatrix} -\log|\lambda| & c_2 \\ c_2 & c_1 \end{pmatrix},$$

which proves Lemma 5.1.3. Here

$$c_2 = \frac{-1}{2}\operatorname{Im}\left\{\tilde{C}'_{ab}\sqrt{ab}\right\} = \operatorname{Im}\left\{\int_a^b \frac{\sqrt{ab}dx}{x\sqrt{(x-1)(x-a)(x-b)}}\right\} \in \mathbb{R},$$

$$c_1 = \pi\operatorname{Im}\left\{\frac{\tilde{C}''_{ab}}{\tilde{C}_{ab}}\right\} = \pi\operatorname{Im}\left\{\frac{\int_a^b \frac{dx}{\sqrt{(x-1)(x-a)(x-b)}}}{\int_1^a \frac{dx}{\sqrt{(x-1)(x-a)(x-b)}}}\right\} = \pi\operatorname{Im}\left\{\tau\left(\frac{1-b}{1-a}\right)\right\} > 0, \tag{5.3}$$

and $\tau(\cdot)$ is the inverse of the elliptic modular lambda function. We could also derive that $c_1 > 0$, due to the fact that $\operatorname{Im} Z$ positive definite. Also, it holds that $-c_1\log|\lambda| - c_2^2 = |-c_1\log|\lambda| - c_2^2| > 0$. □

Proof of Theorem 1.2.5. By (2.8), we know that near the node $(0,0)$, the coefficient of the Bergman kernel in the local coordinate $z = \sqrt{x}$ is given by

$$\begin{aligned} k_\lambda(z) &= \sum_{i,j=1}^{2} (\operatorname{Im}^{-1} Z)_{ij} \frac{4z^{2(2-i)} \cdot \overline{z}^{2(2-j)}}{|(z^2-1)(z^2-a)(z^2-b)(z^2-\lambda)|} \\ &= 4 \cdot \frac{(\operatorname{Im}^{-1} Z)_{11} + (\operatorname{Im}^{-1} Z)_{12}\overline{z}^2 + (\operatorname{Im}^{-1} Z)_{21} \cdot z^2 + (\operatorname{Im}^{-1} Z)_{22}|z|^4}{|(z^2-1)(z^2-a)(z^2-b)(z^2-\lambda)|} \\ &\sim 4 \cdot \frac{c_1 - c_2\overline{z}^2 - c_2 \cdot z^2 - \log|\lambda| \cdot |z|^4}{|(z^2-1)(z^2-a)(z^2-b)(z^2-\lambda)|} \cdot \frac{\pi}{-c_1\log|\lambda| - c_2^2}, \end{aligned}$$

as $\lambda \to 0$. It is not hard to see that the leading term asymptotic expansion of $k_\lambda(z)$ is

$$\frac{4\pi|z|^2}{c_1\,|(z^2-1)(z^2-a)(z^2-b)|}.$$

Subtracting the leading term from $k_\lambda(z)$, we determine the two-term asymptotic expansion as follows. As $\lambda \to 0$, it holds that

$$k_\lambda(z) \sim \left\{ \frac{|z|^2}{c_1} + \left|1 - \frac{c_2}{c_1}z^2\right|^2 \cdot \frac{1}{-\log|\lambda| \cdot |z|^2} \right\} \cdot \frac{4\pi}{|(z^2-1)(z^2-a)(z^2-b)|}.$$

Taking the logarithm, as $\lambda \to 0$, we will know that

$$\begin{aligned}\log k_\lambda(z) &\sim \log \frac{4\pi \cdot |z|^2}{|(z^2-1)(z^2-a)(z^2-b)|} + \log \left\{ \frac{1}{c_1} + \left|\frac{1}{z^2} - \frac{c_2}{c_1}\right|^2 \cdot \frac{1}{-\log|\lambda|} \right\} \\ &= \log \frac{4\pi \cdot |z|^2}{c_1\,|(z^2-1)(z^2-a)(z^2-b)|} + \left|\frac{1}{z^2} - \frac{c_2}{c_1}\right|^2 \cdot \frac{c_1}{-\log|\lambda|} + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right). \qquad \square\end{aligned}$$

Notice that the coefficients in front of $(-\log|\lambda|)^{-1}$ is strictly positive, for small $|z| \neq 0$, and $\log k_\lambda^{(5)}(z)$ has Lelong number zero at the origin. Moreover, we further obtain the curvature form of the relative Bergman kernel metric on $X_\lambda^{(5)}$.

Theorem 5.1.1. *Under the same assumptions as in Theorem 1.2.5, as $\lambda \to 0$, it follows that*

$$\begin{aligned}\partial\bar\partial \log k_\lambda^{(5)}(z) \sim &\left|\frac{1}{z^2} - \frac{c_2}{c_1}\right|^2 \frac{c_1 \cdot d\lambda \wedge d\bar\lambda}{2|\lambda|^2(-\log|\lambda|)^3} + \frac{(c_2 - c_1 z^{-2}) \cdot \bar z^{-3} d\lambda \wedge d\bar z}{\lambda(-\log|\lambda|)^2} \\ &+ \frac{(c_2 - c_1 \bar z^{-2}) \cdot z^{-3} dz \wedge d\bar\lambda}{\bar\lambda(-\log|\lambda|)^2} + \frac{4c_1 dz \wedge d\bar z}{|z|^6(-\log|\lambda|)}.\end{aligned}$$

5.2 Non-separating node: hyperelliptic curves, general curves and Jacobians

The results in Section 5.1 can be generalized[1] to a family of hyperelliptic curves $X_\lambda^{(10)} := \{y^2 = x(x-\lambda)p(x)\}$, with a non-separating nodal degeneration as $\lambda \to 0$, where $p(x)$ is a polynomial of degree at least 3 with roots a_j of distinct absolute values.

Theorem 5.2.1. *In the local coordinate $z = \sqrt{x}$ on $X_\lambda^{(10)}$, write its Bergman kernel as $B_\lambda^{(10)} = k_\lambda^{(10)}(z) dz \wedge d\bar z$, $\lambda \in \mathbb{C} \setminus \{0, \cup_j a_j\}$. Then, as $\lambda \to 0$ for $|z| \neq 0$ small, it holds that*

$$L_{\lambda,z}^{(10)} \sim \frac{C(z) \cdot d\lambda \wedge d\bar\lambda}{|\lambda|^2(-\log|\lambda|)^3},$$

where $C(z) > 0$ is a function of z depending on $p(x)$.

[1] After a preliminary version of this work was presented as TSIMF in January 2016, the author was kindly informed by Prof. Z. Huang about the paper [HJ].

After analyzing the asymptotics of the matrices A and B respectively, we can get the asymptotics of the period matrix.

Lemma 5.2.1. *For $X_\lambda^{(10)}$, as $\lambda \to 0$, it holds that*

$$A(\lambda) \sim \begin{pmatrix} \frac{-2\pi}{\sqrt{a_1 a_2 \dots}} & 0 & \dots & 0 \\ 0 & \alpha_{22} & \dots & \alpha_{2,g} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{g,2} & \dots & \alpha_{g,g} \end{pmatrix} \text{ and } B(\lambda) \sim \begin{pmatrix} \frac{-2\log\lambda}{\sqrt{-a_1 a_2 \dots}} & \beta_{12} & \dots & \beta_{1,g} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2,g} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{g,1} & \beta_{g,2} & \dots & \beta_{g,g} \end{pmatrix},$$

where α_{ij} and β_{ij} are constants depending on $p(x)$.

Then, this yields the following lemma on the asymptotics of the period matrix.

Lemma 5.2.2. *Let $Z_\lambda^{(10)}$ denote the period matrix of $X_\lambda^{(10)}$. Then as $\lambda \to 0$, it holds that*

$$\operatorname{Im} Z(\lambda) \sim \begin{pmatrix} \frac{-\log|\lambda|}{\pi} & C_{12} & \dots & C_{1g} \\ C_{21} & C_{22} & \dots & C_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ C_{g,1} & C_{g,2} & \dots & C_{g,g} \end{pmatrix},$$

where $C_{i,j}$ are real-valued constants depending on $p(x)$.

Proof of Theorem 5.2.1. By Lemma 5.2.1, we know that

$$A^{-1} \sim \begin{pmatrix} \frac{\sqrt{a_a a_2 \dots}}{-2\pi} & 0 & \dots & 0 \\ 0 & \widetilde{\alpha}_{22} & \dots & \widetilde{\alpha}_{2,g} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \widetilde{\alpha}_{g,2} & \dots & \widetilde{\alpha}_{g,g} \end{pmatrix},$$

as $\lambda \to 0$. Therefore, it follows that

$$Z = A^{-1}B \sim \begin{pmatrix} \frac{-\sqrt{-1}\log\lambda}{\pi} & \widetilde{\beta}_{12} & \dots & \widetilde{\beta}_{1,g} \\ \widetilde{\beta}_{21} & \widetilde{\beta}_{22} & \dots & \widetilde{\beta}_{2,g} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\beta}_{g,1} & \widetilde{\beta}_{g,2} & \dots & \widetilde{\beta}_{g,g} \end{pmatrix},$$

where $\widetilde{\beta}_{i,j} = \widetilde{\beta}_{j,i}$ are constants for all $(i,j) \neq (1,1)$, since Z is symmetric. Also, we have

$$(\operatorname{Im} Z)^{-1} \sim \begin{pmatrix} \frac{\pi}{-\log|\lambda|} & \frac{\widetilde{C}_{12}}{-\log|\lambda|} & \dots & \frac{\widetilde{C}_{1g}}{-\log|\lambda|} \\ \frac{\widetilde{C}_{21}}{-\log|\lambda|} & \widetilde{C}_{22} & \dots & \widetilde{C}_{2,g} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\widetilde{C}_{g1}}{-\log|\lambda|} & \widetilde{C}_{g,2} & \dots & \widetilde{C}_{g,g} \end{pmatrix},$$

where $C_{ij} = C_{ji}$ and $C_{kg} = C_{gk}(2 \leq i, j, k \leq g)$ are real numbers. We know that the coefficient of the Bergman kernel in the local coordinate $z = \sqrt{x}$ is given by

$$k_\lambda(z) = \sum_{i,j=1}^{g} (\mathrm{Im}^{-1} Z)_{ij} \frac{4z^{2(i-1)} \cdot \overline{z}^{2(j-1)}}{|(z^2-\lambda)\cdot p(z^2)|}$$

$$= \frac{4\left\{ \sum\limits_{i,j=2}^{g} \widetilde{C}_{ij} z^{2(i-1)} \overline{z}^{2(j-1)} + \frac{\sum_{i=2}^{g} \widetilde{C}_{1i}\left(z^{2(g-i)} + \overline{z}^{2(g-j)}\right) + \pi}{-\log|\lambda|} \right\}}{|(z^2-\lambda)\cdot p(z^2)|} + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right).$$

As $\lambda \to 0$, it follows that

$$\log k_\lambda(z) = \log \frac{4 \sum\limits_{i,j=2}^{g} \widetilde{C}_{ij} z^{2(i-1)} \overline{z}^{2(j-1)}}{|(z^2-\lambda)\cdot p(z^2)|} + \frac{\pi + 2\,\mathrm{Re}\left\{ \sum\limits_{i=2}^{g} \widetilde{C}_{1i} z^{2(g-i)} \right\}}{-\log|\lambda| \cdot \sum\limits_{i,j=2}^{g} \widetilde{C}_{ij} z^{2(i-1)} \overline{z}^{2(j-1)}} + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right),$$

for $0 \neq |z|$ is small, which means that $\log k_\lambda(z) = C + \mathrm{O}(\frac{1}{\log|\lambda|})$. □

Remark We remark that the Bergman kernel on the Jacobian of $X_\lambda^{(10)}$ (of genus g) can be written as $(\det \mathrm{Im}\, Z(\lambda))^{-1} dW \wedge d\overline{W} =: K_\lambda(W) dW \wedge d\overline{W}$, in the local coordinate $W \in \mathbb{C}^g$. Thus, it holds that $\log K_\lambda(W) \sim -\log(-\log|\lambda|)$, as $\lambda \to 0$, where $\det \mathrm{Im}\, Z(\lambda) \to +\infty$. We observe that all a_j, which determine the complex structure of $X_\lambda^{(10)}$, play no role in the final result for the Jacobian. Therefore, as $\lambda \to 0$ for $|W| \neq 0$ small, it holds that

$$\partial_\lambda \bar{\partial}_\lambda \log K_\lambda(W) \sim \frac{d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}. \tag{5.4}$$

Without caring precise coefficients, the leading term in our Theorem 5.1.1 can be interpreted as special cases of Proposition 3.2 (ii) in [HJ] for $m = 1$. One can also make changes of variables from our family to a pinching-coordinate family. Comparing the results of Habermann- Jost [HJ] and ours, there seems to be no essential difference for the Bergman kernels near non-separating nodes on general degenerate curves and hyperelliptic degenerate curves. However, a big difference exists in degeneration between genus-one and higher-genus curves, probably due to the Uniformization Theorem.

5.3 Cusp I: genus-two curves with precise coefficients

For a family of genus two curves $X_\lambda^{(8)} := \{y^2 = x(x^2-\lambda)(x-a)(x-b)\}$ degenerating to a singular one $X_0^{(8)}$ ($\equiv X_0^{(7)}$) with a cusp as $\lambda \to 0$, where a, b, λ are distinct complex numbers in $\mathbb{C} \setminus \{0\}$ satisfying $|a| < |b|$, we determine the precise coefficients as follows.

Theorem 5.3.1. *In the local coordinate $z = \sqrt{x}$ on $X_\lambda^{(8)}$, write its Bergman kernel as $B_\lambda^{(8)} = k_\lambda^{(8)}(z) dz \wedge d\bar{z}$. Then, as $\lambda \to 0$ for small $|z| \neq 0$, it follows that*

$$\log k_\lambda(z) = \log \frac{4\left(c + |z|^4\right)}{c|z^4(z^2-a)(z^2-b)|} - \frac{(z^2+\overline{z}^2)\operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}}{c+|z|^4} + \mathrm{O}\left(\lambda^{\frac{1}{2}}\right),$$

where $c := \operatorname{Im}\left\{\tau\left(\frac{b}{a}\right)\right\}$ and $c' := \dfrac{\sqrt{ab}\int_a^b \frac{dx}{x\sqrt{x(x-a)(x-b)}}}{-\int_0^1 \frac{du}{\sqrt{u(u-1)(u-2)}}}$.

To prove the above theorem, we need to prove the following Lemma 5.3.1 and Lemma 5.3.2, by analyzing asymptotics of two matrices $A_\lambda^{(8)} := \left(\int_{\delta_j} \omega_i\right)_{ij}$ and $B_\lambda^{(8)} := \left(\int_{\gamma_j} \omega_i\right)_{ij}$ (where δ_j and ω_i are as above), respectively.

Lemma 5.3.1. *Under the same assumptions as in Theorem 5.3.1, as $\lambda \to 0$, it holds that*

$$A_\lambda^{(8)} \sim \begin{pmatrix} \frac{C_{(1,2)}}{-\sqrt{ab\sqrt{\lambda}}} & 0 \\ \frac{C'_{(1,2)}\sqrt{\sqrt{\lambda}}}{-\sqrt{ab}} & C_{a,b} \end{pmatrix},$$

where $C_{a,b} := \int_0^a \frac{-2dx}{\sqrt{x(x-a)(x-b)}}$, $C_{(1,2)} := \int_0^1 \frac{-2du}{\sqrt{u(u-1)(u-2)}}$ and $C'_{(1,2)} := \int_0^1 \frac{-2(u-1)du}{\sqrt{u(u-1)(u-2)}}$.

Proof of Lemma 5.3.1. We estimate all the four elements one by one. Firstly, let δ_1 contain only $-\sqrt{\lambda}$ and 0. By Cauchy Integral Theorem, we can get that

$$a_{11} = \int_{\delta_1} \omega_1 = -2\int_{-\sqrt{\lambda}}^0 \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}},$$

where $x - \sqrt{\lambda}$ is bounded by $-\sqrt{\lambda}$ and $-2\sqrt{\lambda}$. Then, there exists a real number $C \in [\frac{1}{\sqrt{2}}, 1]$ such that

$$\begin{aligned}
a_{11} &= \frac{-2C}{\sqrt{-\sqrt{\lambda}}} \int_{-\sqrt{\lambda}}^0 \frac{dx}{\sqrt{x(x+\sqrt{\lambda})(x-a)(x-b)}} \\
&= \frac{C}{\sqrt{-\sqrt{\lambda}}} \int_{\tilde{\delta_1}} \frac{dx}{\sqrt{x(x+\sqrt{\lambda})}} \frac{1}{\sqrt{-a}} \left(1 + \frac{x}{2a} + \mathrm{O}(\frac{x^2}{a^2})\right) \frac{1}{\sqrt{-b}} \left(1 + \frac{x}{2b} + \mathrm{O}(\frac{x^2}{b^2})\right) \\
&= \frac{C}{-\sqrt{-ab\sqrt{\lambda}}} \int_{\tilde{\delta_1}} \frac{dx}{\sqrt{x(x+\sqrt{\lambda})}} (1 + \mathrm{O}(x)) \\
&= \frac{C}{-\sqrt{-ab\sqrt{\lambda}}} \int_{\tilde{\delta_1}} \frac{dx}{\sqrt{x}} \frac{1}{\sqrt{x}} \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) (1 + \mathrm{O}(x)) \\
&\sim \frac{C}{-\sqrt{-ab\sqrt{\lambda}}} \int_{\tilde{\delta_1}} \frac{dx}{x} (1 + \mathrm{O}(x)) = \frac{2\pi C}{-\sqrt{ab\sqrt{\lambda}}},
\end{aligned}$$

where $\tilde{\delta_1}$ contains only $-\sqrt{\lambda}$ and 0 which satisfies $|\lambda| < |x| < |a|$. Notice that the Taylor expansions are different since a and b are not contained in δ_1, while $-\sqrt{\lambda}$ and 0 are. Furthermore, we will determine this constant C precisely by using Taylor expansions and elliptic functions.

$$
\begin{aligned}
a_{11} &= -2\int_{-\sqrt{\lambda}}^{0}\omega_1 = -2\int_{-\sqrt{\lambda}}^{0}\frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}}\\
&= -2\int_{-\sqrt{\lambda}}^{0}\frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}}\frac{1}{\sqrt{-a}}\frac{1}{\sqrt{-b}}\left(1+\frac{x}{2a}+\mathrm{O}(x^2)\right)\left(1+\frac{x}{2b}+\mathrm{O}(x^2)\right)\\
&= \frac{-2}{-\sqrt{ab}}\int_{-\sqrt{\lambda}}^{0}\frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}}(1+\mathrm{O}(x))\\
&\overset{s=x+\sqrt{\lambda}}{=\!=\!=} \frac{-2}{-\sqrt{ab}}\int_{0}^{\sqrt{\lambda}}\frac{ds}{\sqrt{s(s-\sqrt{\lambda})(s-2\sqrt{\lambda})}}\left(1+\mathrm{O}(s-\sqrt{\lambda})\right)\\
&\overset{s=u\cdot\sqrt{\lambda}}{=\!=\!=} \frac{-2}{-\sqrt{ab}}\int_{0}^{1}\frac{\sqrt{\lambda}\cdot du}{\sqrt{u\cdot\sqrt{\lambda}\cdot(u\cdot\sqrt{\lambda}-\sqrt{\lambda})\cdot(u\cdot\sqrt{\lambda}-2\sqrt{\lambda})}}(1+\mathrm{O}(\lambda(u-1)))\\
&= \frac{-2}{-\sqrt{ab\sqrt{\lambda}}}\int_{0}^{1}\frac{du}{\sqrt{u(u-1)(u-2)}}(1+\mathrm{O}\left(\lambda(u-1)\right))\\
&\sim \frac{-2}{-\sqrt{ab\sqrt{\lambda}}}\int_{0}^{1}\frac{du}{\sqrt{u(u-1)(u-2)}} := \frac{C_{(1,2)}}{-\sqrt{ab\sqrt{\lambda}}}.
\end{aligned}
$$

Secondly, we look at $a_{21} = \int_{\delta_1}\omega_2$ and similarly it holds that

$$
\begin{aligned}
a_{21} &= \frac{-2}{-\sqrt{ab}}\int_{-\sqrt{\lambda}}^{0}\frac{xdx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}}(1+\mathrm{O}(x))\\
&\overset{s=x+\sqrt{\lambda}}{=\!=\!=} \frac{-2}{-\sqrt{ab}}\int_{0}^{\sqrt{\lambda}}\frac{(s-\sqrt{\lambda})\cdot ds}{\sqrt{s(s-\sqrt{\lambda})(s-2\sqrt{\lambda})}}\left(1+\mathrm{O}(s-\sqrt{\lambda})\right)\\
&\overset{s=u\cdot\sqrt{\lambda}}{=\!=\!=} \frac{-2}{-\sqrt{ab}}\int_{0}^{1}\frac{(u-1)\sqrt{\lambda}\cdot\sqrt{\lambda}du\cdot\left(1+\mathrm{O}((u-1)\cdot\sqrt{\lambda})\right)}{\sqrt{u\cdot\sqrt{\lambda}\cdot(u\cdot\sqrt{\lambda}-\sqrt{\lambda})\cdot(u\cdot\sqrt{\lambda}-2\sqrt{\lambda})}}\\
&= \frac{-2\sqrt{\sqrt{\lambda}}}{-\sqrt{ab}}\int_{0}^{1}\frac{(u-1)du}{\sqrt{u(u-1)(u-2)}}\left(1+\mathrm{O}((u-1)\cdot\sqrt{\lambda})\right)\\
&\sim \frac{-2\sqrt{\sqrt{\lambda}}}{-\sqrt{ab}}\int_{0}^{1}\frac{(u-1)du}{\sqrt{u(u-1)(u-2)}} := \frac{C'_{(1,2)}\sqrt{\sqrt{\lambda}}}{-\sqrt{ab}}.
\end{aligned}
$$

Thirdly, let δ_2 contain only $-\sqrt{\lambda}, 0, \sqrt{\lambda}$ and a. Then, it follows that

$$\begin{aligned}
a_{12} &= \int_{\delta_2} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}} \\
&= \int_{\delta_2} \frac{dx}{\sqrt{x(x-b)}} \frac{1}{\sqrt{x}} \left(1 + \frac{\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \frac{1}{\sqrt{x}} \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \frac{1}{\sqrt{x}} \left(1 + \mathrm{O}(\frac{1}{x})\right) \\
&= \int_{\delta_2} \frac{dx}{x^2\sqrt{(x-b)}} \left(1 + \frac{\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \left(1 + \mathrm{O}(\frac{1}{x})\right),
\end{aligned}$$

for $|a| < |x| < |b|$. Since δ_2 doesn't contain b, $\frac{1}{\sqrt{(x-b)}}$ is holomorphic and therefore bounded on δ_2 by $C \in \mathbb{C}$. Then, a_{12} is bounded by

$$C \int_{\delta_2} \frac{dx}{x^2} \left(1 + \frac{\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \left(1 + \mathrm{O}(\frac{1}{x})\right) = 0.$$

Lastly, we deal with $a_{22} = \int_{\delta_2} \omega_2$. Similarly, it holds that

$$\begin{aligned}
a_{22} &= \int_{\delta_2} \frac{xdx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}} \\
&= \int_{\delta_2} \frac{xdx}{\sqrt{x(x-b)}} \frac{1}{\sqrt{x}} \left(1 + \frac{\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \frac{1}{\sqrt{x}} \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \frac{1}{\sqrt{x}} \left(1 + \mathrm{O}(\frac{1}{x})\right) \\
&= \int_{\delta_2} \frac{dx}{x\sqrt{(x-b)}} \left(1 + \frac{\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \left(1 + \mathrm{O}(\frac{1}{x})\right) \\
&= C \int_{\delta_2} \frac{dx}{x} \left(1 + \frac{\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \left(1 + \mathrm{O}(\frac{1}{x})\right) = 2\pi\sqrt{-1}C.
\end{aligned}$$

Also, we could have

$$\begin{aligned}
a_{22} &= \int_{\delta_2} \frac{xdx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}} \\
&= \int_{\delta_2} \frac{xdx}{\sqrt{x(x-a)(x-b)}} \frac{1}{\sqrt{x}} \left(1 + \frac{\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \frac{1}{\sqrt{x}} \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \\
&= \int_{\delta_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} \left(1 + \mathrm{O}(\frac{\sqrt{\lambda}}{x})\right) \sim \int_{\delta_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} := C_{a,b},
\end{aligned}$$

where δ_2 contains 0 and a. Therefore, we could get the asymptotics of the matrix $A_\lambda^{(8)}$ and finish the proof of Lemma 5.3.1. $\square$

Lemma 5.3.2. *Under the same assumptions as in Lemma 5.3.1, as $\lambda \to 0$, it holds that*

$$B_\lambda^{(8)} \sim \frac{1}{\sqrt{-ab}} \begin{pmatrix} \frac{C_{(1,2)}}{\sqrt{\sqrt{\lambda}}}, & C''_{a,b}\sqrt{-ab} \\ -C'_{(1,2)} \cdot \sqrt{\sqrt{\lambda}} & C'_{a,b}\sqrt{-ab} \end{pmatrix},$$

where $C''_{a,b} := -2\int_a^b \frac{dx}{x\sqrt{x(x-a)(x-b)}}$ *and* $C'_{a,b} := -2\int_a^b \frac{dx}{\sqrt{x(x-a)(x-b)}}$.

Proof of Lemma 5.3.2. Again, we estimate all the four elements one by one. Firstly, let γ_1 contain only 0 and $\sqrt{\lambda}$. By Cauchy Integral Theorem, we can get that

$$\begin{aligned}
b_{11} &= -2\int_0^{\sqrt{\lambda}} \omega_1 = -2\int_0^{\sqrt{\lambda}} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}} \\
&= -2\int_0^{\sqrt{\lambda}} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}} \frac{1}{\sqrt{-a}} \frac{1}{\sqrt{-b}} \left(1 + \frac{x}{2a} + \mathrm{O}(x^2)\right)\left(1 - \frac{x}{2b} + \mathrm{O}(x^2)\right) \\
&= \frac{-2}{-\sqrt{ab}} \int_0^{\sqrt{\lambda}} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}} (1 + \mathrm{O}(x)) \\
&\xlongequal{s=x-\sqrt{\lambda}} \frac{-2}{-\sqrt{ab}} \int_{-\sqrt{\lambda}}^0 \frac{ds}{\sqrt{s(s+\sqrt{\lambda})(s+2\sqrt{\lambda})}} \left(1 + \mathrm{O}(s+\sqrt{\lambda})\right) \\
&\xlongequal{s=-u\cdot\sqrt{\lambda}} \frac{-2}{-\sqrt{ab}} \int_1^0 \frac{-\sqrt{\lambda}\cdot du \cdot \left(1 + \mathrm{O}((-u+1)\sqrt{\lambda})\right)}{\sqrt{-u\cdot\sqrt{\lambda}\cdot(-u\cdot\sqrt{\lambda}+\sqrt{\lambda})\cdot(-u\cdot\sqrt{\lambda}+2\sqrt{\lambda})}} \\
&= \frac{-2}{-\sqrt{ab\sqrt{\lambda}}} \int_1^0 \frac{-du}{-\sqrt{-u(u-1)(u-2)}} \left(1 + \mathrm{O}((-u+1)\cdot\sqrt{\lambda})\right) \\
&\sim \frac{-2}{-\sqrt{ab\sqrt{\lambda}}} \int_1^0 \frac{du}{\sqrt{-u(u-1)(u-2)}} = \frac{-2}{-\sqrt{ab\sqrt{\lambda}}} \int_0^1 \frac{\sqrt{-1}du}{\sqrt{u(u-1)(u-2)}} := \frac{C_{(1,2)}}{\sqrt{-ab\sqrt{\lambda}}}.
\end{aligned}$$

Secondly, substitute ω_1 with ω_2 and similarly it holds that

$$\begin{aligned}
b_{21} &= \frac{-2}{-\sqrt{ab}} \int_0^{\sqrt{\lambda}} \frac{xdx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})}} (1 + \mathrm{O}(x)) \\
&\xlongequal{s=x-\sqrt{\lambda}} \frac{-2}{-\sqrt{ab}} \int_{-\sqrt{\lambda}}^0 \frac{(s+\sqrt{\lambda})ds}{\sqrt{s(s+\sqrt{\lambda})(s+2\sqrt{\lambda})}} \left(1 + \mathrm{O}(s+\sqrt{\lambda})\right) \\
&\xlongequal{s=-u\cdot\sqrt{\lambda}} \frac{-2}{-\sqrt{ab}} \int_1^0 \frac{(-u+1)\sqrt{\lambda}\cdot(-\sqrt{\lambda})du\cdot\left(1 + \mathrm{O}((-u+1)\cdot\sqrt{\lambda})\right)}{\sqrt{-u\cdot\sqrt{\lambda}\cdot(-u\cdot\sqrt{\lambda}+\sqrt{\lambda})\cdot(-u\cdot\sqrt{\lambda}+2\sqrt{\lambda})}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-2\sqrt{\sqrt{\lambda}}}{-\sqrt{ab}} \int_1^0 \frac{(u-1)du}{-\sqrt{-u(u-1)(u-2)}} \left(1 + \mathrm{O}((-u+1)\cdot\sqrt{\lambda})\right) \\
&\sim \frac{-2\sqrt{\sqrt{\lambda}}}{-\sqrt{ab}} \int_0^1 \frac{(u-1)du}{\sqrt{-u(u-1)(u-2)}} \\
&= \frac{-2\sqrt{\sqrt{\lambda}}}{-\sqrt{ab}} \int_0^1 \frac{-\sqrt{-1}(u-1)du}{\sqrt{u(u-1)(u-2)}} := \frac{\sqrt{-\sqrt{\lambda}}C'_{(1,2)}}{\sqrt{ab}}.
\end{aligned}
$$

Thirdly, let γ_2 contain only a and b. Then, it follows that

$$
\begin{aligned}
b_{12} &= \int_{\gamma_2} \frac{dx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}} \\
&= \int_{\gamma_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} \frac{1}{\sqrt{x}} \left(1 + \frac{\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \frac{1}{\sqrt{x}} \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \\
&= \int_{\gamma_2} \frac{dx}{x\sqrt{x(x-a)(x-b)}} \left(1 + \mathrm{O}(\frac{\sqrt{\lambda}}{x})\right) \sim \int_{\gamma_2} \frac{dx}{x\sqrt{x(x-a)(x-b)}} := C''_{a,b}.
\end{aligned}
$$

Lastly, it holds that

$$
\begin{aligned}
b_{22} &= \int_{\gamma_2} \frac{xdx}{\sqrt{x(x-\sqrt{\lambda})(x+\sqrt{\lambda})(x-a)(x-b)}} \\
&= \int_{\gamma_2} \frac{xdx}{\sqrt{x(x-a)(x-b)}} \frac{1}{\sqrt{x}} \left(1 + \frac{\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \frac{1}{\sqrt{x}} \left(1 + \frac{-\sqrt{\lambda}}{2x} + \mathrm{O}(\frac{\lambda}{x^2})\right) \\
&= \int_{\gamma_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} \left(1 + \mathrm{O}(\frac{\sqrt{\lambda}}{x})\right) \sim \int_{\gamma_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} := C'_{a,b}.
\end{aligned}
$$

and this finishes the proof of Lemma 5.3.1. $\square$

The asymptotic results in Theorem 5.3.1 for the Bergman kernels are obtained by combining (2.9) and the following lemma.

Lemma 5.3.3. *Let $Z^{(8)}_\lambda$ denote the period matrix of $X^{(8)}_\lambda$. Then, as $\lambda \to 0$, it holds that*

$$
(\operatorname{Im} Z^{(8)}_\lambda)^{-1} \sim \frac{1}{c - \operatorname{Im}^2\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}} \begin{pmatrix} c & -\operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\} \\ -\operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\} & 1 \end{pmatrix}.
$$

Proof of Lemma 5.3.3. By Lemma 5.3.1, we know as $\lambda \to 0$ that

$$A^{-1} \sim \begin{pmatrix} \frac{1}{-\sqrt{ab\sqrt{\lambda}}}C_{(1,2)}, & 0 \\ \frac{C'_{(1,2)}\sqrt{\sqrt{\lambda}}}{-\sqrt{ab}} & C_{a,b} \end{pmatrix}^{-1} = \frac{-\sqrt{ab}\sqrt{\sqrt{\lambda}}}{C_{a,b}C_{(1,2)}} \begin{pmatrix} C_{a,b} & 0 \\ \frac{C'_{(1,2)}\sqrt{\sqrt{\lambda}}}{\sqrt{ab}} & \frac{1}{-\sqrt{ab\sqrt{\lambda}}}C_{(1,2)} \end{pmatrix}.$$

Therefore, it follows that

$$\begin{aligned} Z =& A^{-1}B \\ \sim& \frac{-\sqrt{ab}\sqrt{\sqrt{\lambda}}}{C_{a,b}C_{(1,2)}} \begin{pmatrix} C_{a,b} & 0 \\ \frac{C'_{(1,2)}\sqrt{\sqrt{\lambda}}}{\sqrt{ab}} & \frac{1}{-\sqrt{ab\sqrt{\lambda}}}C_{(1,2)} \end{pmatrix} \frac{1}{\sqrt{-ab}} \begin{pmatrix} \frac{C_{(1,2)}}{-\sqrt{\sqrt{\lambda}}} & C''_{a,b}\sqrt{-ab} \\ -C'_{(1,2)}\sqrt{\sqrt{\lambda}} & C'_{a,b}\sqrt{-ab} \end{pmatrix} \\ =& \frac{\sqrt{-\sqrt{\lambda}}}{C_{a,b}C_{(1,2)}} \begin{pmatrix} \frac{C_{a,b}C_{(1,2)}}{\sqrt{\sqrt{\lambda}}}, & C_{a,b}C''_{a,b}\sqrt{-ab} \\ \frac{2C'_{(1,2)}C_{(1,2)}}{\sqrt{ab}} & C'_{(1,2)}C''_{a,b}\sqrt{-\sqrt{\lambda}} + \frac{C_{(1,2)}C'_{a,b}}{\sqrt{-\sqrt{\lambda}}} \end{pmatrix} \\ \sim& \frac{\sqrt{-\sqrt{\lambda}}}{C_{a,b}C_{(1,2)}} \begin{pmatrix} \frac{C_{a,b}C_{(1,2)}}{\sqrt{\sqrt{\lambda}}}, & C_{a,b}C''_{a,b}\sqrt{-ab} \\ \frac{2C'_{(1,2)}C_{(1,2)}}{\sqrt{ab}} & \frac{C_{(1,2)}C'_{a,b}}{\sqrt{-\sqrt{\lambda}}} \end{pmatrix} = \begin{pmatrix} \sqrt{-1} & \frac{-C''_{a,b}\sqrt{ab\sqrt{\lambda}}}{C_{(1,2)}} \\ \frac{2C'_{(1,2)}\sqrt{-\sqrt{\lambda}}}{C_{a,b}\sqrt{ab}} & \frac{C'_{a,b}}{C_{a,b}} \end{pmatrix}. \end{aligned}$$

Since Z is symmetric, this implies that

$$ab\sqrt{-1}C''_{a,b}C_{a,b} = 2C_{(1,2)}C'_{(1,2)},$$

namely

$$2ab\sqrt{-1}\int_0^a \frac{dx}{\sqrt{x(x-a)(x-b)}} \int_a^b \frac{dx}{x\sqrt{x(x-a)(x-b)}} = C_{(1,2)}C'_{(1,2)}.$$

Moreover, as $\lambda \to 0$ we know that,

$$\operatorname{Im} Z \sim \begin{pmatrix} 1 & \operatorname{Im}\left\{\frac{-C''_{a,b}\sqrt{ab\sqrt{\lambda}}}{C_{(1,2)}}\right\} \\ \operatorname{Im}\left\{\frac{2C'_{(1,2)}\sqrt{-\sqrt{\lambda}}}{C_{a,b}\sqrt{ab}}\right\} & \operatorname{Im}\left\{\frac{C'_{a,b}}{C_{a,b}}\right\} \end{pmatrix} =: \begin{pmatrix} 1 & \operatorname{Im}\left\{c'\lambda^{\frac{1}{4}}\right\} \\ \operatorname{Im}\left\{c'\lambda^{\frac{1}{4}}\right\} & c \end{pmatrix},$$

where

$$c := \operatorname{Im}\left\{\frac{C'_{a,b}}{C_{a,b}}\right\} = \operatorname{Im}\left\{\frac{\int_a^b \frac{dx}{\sqrt{x(x-a)(x-b)}}}{\int_0^a \frac{dx}{\sqrt{x(x-a)(x-b)}}}\right\} = \operatorname{Im}\left\{\tau\left(\frac{b}{a}\right)\right\} > 0.$$

Since is $\operatorname{Im} Z$ positive definite, we get that $c > \operatorname{Im}^2\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}$. $\square$

Proof of Theorem 5.3.1. By (2.9), we know that the coefficient of the Bergman kernel in local coordinate $z = \sqrt{x}$ is given by

$$\begin{aligned}
k_\lambda(z) =& \sum_{i,j=1}^{2} (\operatorname{Im}^{-1} Z)_{ij} \frac{4z^{2(2-i)} \cdot \overline{z}^{2(2-j)}}{|(z^4-\lambda)(z^2-a)(z^2-b)|} \\
=& 4 \cdot \frac{(\operatorname{Im}^{-1} Z)_{11} + (\operatorname{Im}^{-1} Z)_{12}\overline{z}^2 + (\operatorname{Im}^{-1} Z)_{21} \cdot z^2 + (\operatorname{Im}^{-1} Z)_{22}|z|^4}{|(z^4-\lambda)(z^2-a)(z^2-b)|} \\
\sim& 4 \cdot \frac{c - \operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}(\overline{z}^2 + z^2) + |z|^4}{|(z^4-\lambda)(z^2-a)(z^2-b)|} \cdot \frac{1}{c - \operatorname{Im}^2\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}},
\end{aligned}$$

as $\lambda \to 0$. We can see that the leading term asymptotic expansion of $k_\lambda(z)$ is

$$\frac{4}{c} \cdot \frac{c + |z|^4}{|z^4(z^2-a)(z^2-b)|}.$$

Subtracting the leading term from $k_\lambda(z)$, we determine the two-term asymptotic expansion as follows. As $\lambda \to 0$, it holds that

$$\begin{aligned}
& k_\lambda(z) - \frac{4}{c} \cdot \frac{c + |z|^4}{|z^4(z^2-a)(z^2-b)|} \\
\sim& \frac{4}{|z^4(z^2-a)(z^2-b)|} \left\{ \frac{c - \operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}(z^2 + \overline{z}^2) + |z|^4}{c - \operatorname{Im}^2\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}} - \frac{c + |z|^4}{c} \right\} \\
\sim& \frac{4}{|z^4(z^2-a)(z^2-b)|} \left\{ \frac{-\operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}(z^2 + \overline{z}^2)c + \{c + |z|^4\} \cdot \operatorname{Im}^2\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}}{\left\{c - \operatorname{Im}^2\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}\right\} c} \right\} \\
\sim& \frac{4\operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}}{|z^4(z^2-a)(z^2-b)|} \left\{ \frac{-(z^2 + \overline{z}^2)c + \{c + |z|^4\} \cdot \operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}}{c^2} \right\} \\
\sim& \frac{4\operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\}}{|z^4(z^2-a)(z^2-b)|} \left\{ \frac{-(z^2 + \overline{z}^2)}{c} \right\},
\end{aligned}$$

which implies that

$$k_\lambda(z) \sim \frac{4}{c|z^4(z^2-a)(z^2-b)|} \left\{ c + |z|^4 - (z^2 + \overline{z}^2)\operatorname{Im}\left\{c' \cdot \lambda^{\frac{1}{4}}\right\} \right\}.$$

Taking the logarithm, as $\lambda \to 0$, we will get the following two-term asymptotic formula

$$\begin{aligned}
\log k_\lambda(z) &\sim \log \frac{4}{c|z^4(z^2-a)(z^2-b)|} + \log\left\{\left(c+|z|^4\right) - (z^2+\overline{z}^2)\,\mathrm{Im}\left\{c'\cdot\lambda^{\frac{1}{4}}\right\}\right\} \\
&= \log \frac{4\left(c+|z|^4\right)}{c|z^4(z^2-a)(z^2-b)|} + \log\left\{1 - \frac{(z^2+\overline{z}^2)\,\mathrm{Im}\left\{c'\cdot\lambda^{\frac{1}{4}}\right\}}{c+|z|^4}\right\} \\
&\sim \log \frac{4\left(c+|z|^4\right)}{c|z^4(z^2-a)(z^2-b)|} - \frac{(z^2+\overline{z}^2)\,\mathrm{Im}\left\{c'\cdot\lambda^{\frac{1}{4}}\right\}}{c+|z|^4}. \qquad \square
\end{aligned}$$

5.4 Cusp I: hyperelliptic curves and Jacobians

For $X_\lambda^{(6)}$, we will analyze the asymptotics of the matrices $A_\lambda^{(6)}$ and $B_\lambda^{(6)}$ respectively. To make the statements precise, we assume that $p(x)$ is a polynomial of degree at least 2 with distinct roots a_j such that $|a_1| < |a_2| < \dots$ and $\lambda \in \mathbb{C} \setminus \{0, \cup_j a_j\}$.

Lemma 5.4.1. *Under the same assumptions as in Theorem 1.2.6, as $\lambda \to 0$, it holds that*

$$A_\lambda^{(6)} \sim \begin{pmatrix}
C_{(1,2)}^{(1)}\cdot\lambda^{-\frac{1}{4}} & 0 & \cdots & \cdots & \cdots & 0 \\
C_{(1,2)}^{(2)}\cdot\lambda^{\frac{1}{4}} & \alpha_{22} & 0 & \cdots & 0 & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots \\
\vdots & \alpha_{g-1,2} & \cdots & \cdots & \alpha_{g-1,g-1} & 0 \\
C_{(1,2)}^{(g)}\cdot\sqrt{\lambda}^{\frac{1}{2}+g-2} & \alpha_{g,2} & \cdots & \cdots & \cdots & \alpha_{g,g}
\end{pmatrix}$$

where $C_{(1,2)}^{(k)} := -2\int_0^1 \frac{(u-1)^{k-1}du}{\sqrt{u(u-1)(u-2)}} \cdot \frac{1}{\sqrt{-a_1(-a_2)(-a_3)\dots}}$, *and* $\alpha_{ij}(i \geq j \geq 2)$ *are non-zero constants depending on* $p(x)$.

Proof. We will estimate all the $g \times g$ elements one by one. Firstly, as $\lambda \to 0$, it holds that

$$\begin{aligned}
a_{11} &= \int_{\delta_1} \omega_1 = -2\int_{-\sqrt{\lambda}}^{0} \frac{dx}{y} = -2\int_{-\sqrt{\lambda}}^{0} \frac{dx}{\sqrt{x(x^2-\lambda)(x-a_1)(x-a_2)(x-a_3)...}} \\
&\overset{Taylor}{=\!=\!=} -2\int_{-\sqrt{\lambda}}^{0} \frac{dx}{\sqrt{x(x^2-\lambda)}} \frac{1}{\sqrt{-a_1(-a_2)(-a_3)...}}(1+\mathrm{O}(x)) \\
&\overset{s=x+\sqrt{\lambda}}{=\!=\!=} -2\int_{0}^{\sqrt{\lambda}} \frac{ds}{\sqrt{s(s-\sqrt{\lambda})(s-2\sqrt{\lambda})}} \frac{1}{\sqrt{-a_1(-a_2)(-a_3)...}}(1+\mathrm{O}(s+\sqrt{\lambda})) \\
&\overset{s=u\cdot\sqrt{\lambda}}{=\!=\!=} -2\int_{0}^{1} \frac{\sqrt{\lambda}du}{\sqrt{\sqrt{\lambda}u(\sqrt{\lambda}u-\sqrt{\lambda})(\sqrt{\lambda}u-2\sqrt{\lambda})}} \frac{(1+\mathrm{O}((u-1)\sqrt{\lambda}))}{\sqrt{-a_1(-a_2)(-a_3)...}} \\
&\sim -2\int_{0}^{1} \frac{\sqrt{\lambda}du}{\sqrt{\sqrt{\lambda}u(\sqrt{\lambda}u-\sqrt{\lambda})(\sqrt{\lambda}u-2\sqrt{\lambda})}} \frac{1}{\sqrt{-a_1(-a_2)(-a_3)...}} \\
&= -2\int_{0}^{1} \frac{du}{\sqrt{u(u-1)(u-2)}} \frac{1}{\sqrt{-a_1(-a_2)(-a_3)...}} \cdot \frac{1}{\lambda^{\frac{1}{4}}} =: C^{(1)}_{(1,2)} \cdot \lambda^{-\frac{1}{4}}.
\end{aligned}$$

For a_{21}, there is an extra x in the original integrand and thus an extra $\sqrt{\lambda}(u-1)$ in the final numerator. Then, it follows that

$$a_{21} \sim -2\int_{0}^{1} \frac{(u-1)du}{\sqrt{u(u-1)(u-2)}} \frac{1}{\sqrt{-a_1(-a_2)(-a_3)...}} \cdot \lambda^{\frac{1}{4}} =: C^{(2)}_{(1,2)} \cdot \lambda^{\frac{1}{4}}.$$

And similarly, as $\lambda \to 0$, it holds that

$$a_{g1} \sim -2\int_{0}^{1} \frac{(u-1)^{g-1}du}{\sqrt{u(u-1)(u-2)}} \frac{\sqrt{\lambda}^{\frac{1}{2}+g-2}}{\sqrt{-a_1(-a_2)(-a_3)...}} =: C^{(g)}_{(1,2)} \cdot \sqrt{\lambda}^{\frac{1}{2}+g-2}.$$

Secondly, δ_2 contains only $-\sqrt{\lambda}, 0, \sqrt{\lambda}$ and a_1, but not $a_2, a_3, ...$ Then, it follows that

$$\begin{aligned}
a_{12} &= \int_{\delta_2} \frac{dx}{\sqrt{x(x^2-\lambda)(x-a_1)(x-a_2)...}} \\
&= \int_{\delta_2} \frac{dx}{x^2\sqrt{(x-a_2)(x-a_3)...}} \left(1+\frac{\lambda}{2x^2}+\mathrm{O}(\frac{\lambda^2}{x^4})\right)\left(1+\mathrm{O}(\frac{1}{x})\right),
\end{aligned}$$

for $|a_1| < |x| < |a_2| < |a_3| < ....$ Since δ_2 doesn't contain $a_2, a_3, ...,$ $\frac{1}{\sqrt{(x-a_2)(x-a_3)...}}$ is holomorphic and therefore bounded on δ_2 by $C \in \mathbb{C}$. Then, it holds that

$$a_{12} = C\int_{\delta_2} \frac{dx}{x^2}\left(1+\frac{\lambda}{2x^2}+\mathrm{O}(\frac{\lambda^2}{x^4})\right)\left(1+\mathrm{O}(\frac{1}{x})\right) = 0.$$

However, such a phenomenon will not happen for $a_{j2}(j \geq 2)$, since there is an extra x^{j-1} in the numerator, which makes $a_{j2}(j \geq 2)$ asymptotic as $\lambda \to 0$ to non-zero constants denoted by α_{j2}. Thirdly, δ_3 contains only $-\sqrt{\lambda}, 0, \sqrt{\lambda}, a_1, a_2$ and a_3, but not $a_4, a_5, ...$ Then, it follows that

$$\begin{aligned} a_{13} &= \int_{\delta_3} \frac{dx}{\sqrt{x(x^2-\lambda)(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)...}} \\ &= \int_{\delta_3} \frac{dx}{x^3\sqrt{(x-a_4)(x-a_5)...}} \left(1 + \frac{\lambda}{2x^2} + \mathrm{O}(\frac{\lambda^2}{x^4})\right)\left(1 + \mathrm{O}(\frac{1}{x})\right), \end{aligned}$$

for $|a_3| < |x| < |a_4| < |a_5| < ....$ Since δ_3 doesn't contain $a_4, a_5, ...$, $\frac{1}{\sqrt{(x-a_4)(x-a_5)...}}$ is holomorphic and therefore bounded on δ_3 by $C' \in \mathbb{C}$. Then, it holds that

$$a_{13} = C' \int_{\delta_3} \frac{dx}{x^3} \left(1 + \frac{\lambda}{2x^2} + \mathrm{O}(\frac{\lambda^2}{x^4})\right)\left(1 + \mathrm{O}(\frac{1}{x})\right) = 0.$$

For a_{23}, same argument also gives that $a_{23} = 0$. However, for $a_{j3} (j \geq 3)$ this is not the case, since there is an extra x^{j-1} in the numerator, which makes $a_{j3} (j \geq 3)$ asymptotic as $\lambda \to 0$ to non-zero constants denoted by α_{j3}. Lastly, repeating the above process, we will conclude that all elements above the diagonal in the matrix A vanishes. □

Lemma 5.4.2. *Under the same assumptions as Theorem 1.2.6, as $\lambda \to 0$, it holds that*

$$B_\lambda^{(6)} \sim \begin{pmatrix} \sqrt{-1} \cdot C_{(1,2)}^{(1)} \cdot \lambda^{-\frac{1}{4}} & \beta_{12} & \dots & \beta_{1,g} \\ -\sqrt{-1} \cdot C_{(1,2)}^{(2)} \cdot \lambda^{\frac{1}{4}} & \beta_{22} & \dots & \beta_{2,g} \\ \vdots & \vdots & & \vdots \\ (-1)^{g-1}\sqrt{-1} \cdot C_{(1,2)}^{(g)} \cdot \sqrt{\lambda}^{\frac{1}{2}+g-2} & \beta_{g,2} & \dots & \beta_{g,g} \end{pmatrix},$$

where β_{ij} are constants depending on $p(x)$.

Proof. Firstly, for the first column of B we can make the change of coordinates (similar to the proof of Lemma 5.2.1) by setting $x = (-u+1) \cdot \sqrt{\lambda}$, which yields that $b_{i1} \sim \sqrt{-1} \cdot a_{i1} \cdot (-1)^{i-1}$ for $1 \leq i \leq g$. Secondly, γ_2 contains only a_1 and a_2, and we can get that $|x^2| > |\lambda|$ for small λ. The Taylor expansion for $\frac{1}{\sqrt{x^2-\lambda}}$ will then guarantee that b_{i2} is asymptotic to a non-zero constants which depends on $a_1, a_2, ...$, denoted by β_{i2}. Lastly, similar arguments work for other columns and the proof could be completed. □

Now, we state the results on the asymptotic behaviors of the period matrix of $X_\lambda^{(6)}$.

Lemma 5.4.3. *Let $Z_\lambda^{(6)}$ denote the period matrix of $X_\lambda^{(6)}$. Then, as $\lambda \to 0$, it holds that*

$$(\operatorname{Im} Z_\lambda^{(6)})^{-1} \sim \begin{pmatrix} 1 & \mathrm{O}(\lambda^{\frac{1}{4}}) & \dots & \mathrm{O}(\lambda^{\frac{1}{4}}) \\ \mathrm{O}(\lambda^{\frac{1}{4}}) & \mathrm{O}(1) & \dots & \mathrm{O}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{O}(\lambda^{\frac{1}{4}}) & \mathrm{O}(1) & \dots & \mathrm{O}(1) \end{pmatrix},$$

where the involved constants depend on $p(x)$.

Proof of Lemma 5.4.3. We know that

$$A^{-1} \sim \frac{\lambda^{\frac{1}{4}}}{C^{(1)}_{(1,2)}\alpha_{22}\cdot\ldots\cdot\alpha_{gg}} \begin{pmatrix} \alpha_{22}\cdot\ldots\cdot\alpha_{gg} & 0 & \ldots & \ldots & \ldots & 0 \\ \mathrm{O}(\lambda^{\frac{1}{4}}) & \mathrm{O}(\lambda^{-\frac{1}{4}}) & 0 & \ldots & 0 & \vdots \\ \mathrm{O}(\lambda^{\frac{1}{4}}) & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 0 & \vdots \\ \vdots & \mathrm{O}(\lambda^{-\frac{1}{4}}) & \ldots & \ldots & \mathrm{O}(\lambda^{-\frac{1}{4}}) & 0 \\ \mathrm{O}(\lambda^{\frac{1}{4}}) & \mathrm{O}(\lambda^{-\frac{1}{4}}) & \ldots & \ldots & \ldots & \mathrm{O}(\lambda^{-\frac{1}{4}}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\lambda^{\frac{1}{4}}}{C^{(1)}_{(1,2)}} & 0 & \ldots & \ldots & \ldots & 0 \\ \mathrm{O}(\lambda^{\frac{1}{2}}) & \mathrm{O}(1) & 0 & \ldots & 0 & \vdots \\ \mathrm{O}(\lambda^{\frac{1}{2}}) & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 0 & \vdots \\ \vdots & \mathrm{O}(1) & \ldots & \ldots & \mathrm{O}(1) & 0 \\ \mathrm{O}(\lambda^{\frac{1}{2}}) & \mathrm{O}(1) & \ldots & \ldots & \ldots & \mathrm{O}(1) \end{pmatrix}$$

as $\lambda \to 0$. Therefore, it follows that

$$Z = A^{-1}B \sim \begin{pmatrix} \sqrt{-1} & C^{(2)}\lambda^{\frac{1}{4}} & \ldots & C^{(g)}\lambda^{\frac{1}{4}} \\ C^{(2)}\lambda^{\frac{1}{4}} & \mathrm{O}(1) & \ldots & \mathrm{O}(1) \\ \vdots & \vdots & \ddots & \vdots \\ C^{(g)}\lambda^{\frac{1}{4}} & \mathrm{O}(1) & \ldots & \mathrm{O}(1) \end{pmatrix},$$

where $C^{(j)}$ are constants depending on $a_j (2 \leq j \leq g)$. Moreover, as $\lambda \to 0$, we know that

$$\operatorname{Im} Z \sim \begin{pmatrix} 1 & \operatorname{Im}\left\{C^{(2)}\lambda^{\frac{1}{4}}\right\} & \ldots & \operatorname{Im}\left\{C^{(g)}\lambda^{\frac{1}{4}}\right\} \\ \operatorname{Im}\left\{C^{(2)}\lambda^{\frac{1}{4}}\right\} & r_{2,2} & \ldots & r_{2,g} \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Im}\left\{C^{(g)}\lambda^{\frac{1}{4}}\right\} & r_{g,2} & \ldots & r_{g,g} \end{pmatrix}, \tag{5.5}$$

which yields that

$$(\operatorname{Im} Z)^{-1} = \begin{pmatrix} 1 & \mathrm{O}(\lambda^{\frac{1}{4}}) & \ldots & \mathrm{O}(\lambda^{\frac{1}{4}}) \\ \mathrm{O}(\lambda^{\frac{1}{4}}) & \mathrm{O}(1) + \mathrm{O}(\lambda^{\frac{1}{2}}) & \ldots & \mathrm{O}(1) + \mathrm{O}(\lambda^{\frac{1}{2}}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{O}(\lambda^{\frac{1}{4}}) & \mathrm{O}(1) + \mathrm{O}(\lambda^{\frac{1}{2}}) & \ldots & \mathrm{O}(1) + \mathrm{O}(\lambda^{\frac{1}{2}}) \end{pmatrix} := C_{ij}. \qquad \square$$

Finally, we will prove Theorem 1.2.6 by Lemma 5.4.3.

Proof of Theorem 1.2.6. The Bergman kernel of $X_\lambda^{(6)}$ in the local coordinate $z = \sqrt{x}$ is

$$\begin{aligned}
k_\lambda^{(6)}(z) &= \sum_{i,j=1}^{g} (\operatorname{Im}^{-1} Z)_{ij} \frac{4z^{2(i-1)} \cdot \overline{z}^{2(j-1)}}{|(z^4-\lambda)(z^2-a_1)(z^2-a_2)...|} \\
&\sim \frac{4}{|(z^4-\lambda)(z^2-a_1)(z^2-a_2)...|} \left\{ 1 + \sum_{i,j=2}^{g} C_{ij} z^{2(i-1)}\overline{z}^{2(j-1)} + 2\sum_{j=2}^{g} C_{jg} \cdot \operatorname{Re}\left(z^{2(j-1)}\right) \right\} \\
&= \frac{4}{|z^4(z^2-a_1)(z^2-a_2)...|} \left\{ 1 + \mathrm{O}(z^4) + \mathrm{O}(\lambda^{\frac{1}{4}}) \cdot \operatorname{Re}\left(\sum_{j=2}^{g} z^{2(j-1)} \right) \right\} + \mathrm{O}(\lambda^{\frac{1}{2}}),
\end{aligned}$$

as $\lambda \to 0$. We further obtain that

$$\log k_\lambda^{(6)}(z) = \log \frac{4\left(1+\mathrm{O}(z^4)\right)}{|z^4(z^2-a_1)(z^2-a_2)...|} + \frac{\mathrm{O}(\lambda^{\frac{1}{4}}) \cdot \operatorname{Re}\left(\sum_{j=2}^{g} z^{2(j-1)}\right)}{1+\mathrm{O}(z^4)},$$

as $\lambda \to 0$ for small $|z| \neq 0$. Note that both the leading and the subleading terms above is harmonic with respect to λ, which vanishes under the $\partial_\lambda \bar{\partial}_\lambda$ operator. $\square$

Remark For the Jacobians of $X_\lambda^{(6)}$ (of genus g), the Bergman kernel can be written as

$$\frac{1}{\det \operatorname{Im} Z_\lambda^{(6)}} dW \wedge d\overline{W} =: K_\lambda(W) dW \wedge d\overline{W},$$

for $W \in \mathbb{C}^g$. Then, as $\lambda \to 0$ for $|W| \neq 0$ sufficiently small, it holds by (5.5) that

$$\log K_\lambda(W) = -\log\det \operatorname{Im} Z_\lambda^{(6)} \sim -\log\left(C + \left(\mathrm{O}(\lambda^{\frac{1}{4}})\right)^2 \right) \sim C' + \mathrm{O}(\lambda^{\frac{1}{2}}),$$

where C and C' depend on p. In particular, if $p(x) = (x-a)(x-b)$, then as $\lambda \to 0$ it follows that

$$\log K_\lambda(W) \sim -\log c_1 + \frac{1}{c_1} \cdot \operatorname{Im}^2 \left\{ c_2 \cdot \lambda^{\frac{1}{4}} \right\},$$

which is the precise result for the genus-two case. So, it seems that the Jacobian of $X_\lambda^{(6)}$ remains non-degenerate, since $\det \operatorname{Im} Z(\lambda) \sim \exp(-C') < +\infty$.

5.5 Cusp II: genus-two curves with precise coefficients

We first prove the following Lemma 5.5.1 and Lemma 5.5.2, by analyzing asymptotics of two matrices $A_\lambda^{(7)} := \left(\int_{\delta_j} \omega_i\right)_{ij}$ and $B_\lambda^{(7)} := \left(\int_{\gamma_j} \omega_i\right)_{ij}$, respectively.

Lemma 5.5.1. *Under the same assumptions as Theorem 1.2.7, as $\lambda \to 0$, it holds that*

$$A_\lambda^{(7)} \sim \begin{pmatrix} \frac{2\pi}{\sqrt{ab}\sqrt{\lambda}} & 0 \\ C\lambda\sqrt{\lambda} & C_{ab} \end{pmatrix},$$

where $C_{ab} := -2\int_0^a \frac{dx}{\sqrt{x(x-a)(x-b)}}$ *and C is a constant depending on a and b.*

Proof of Lemma 5.5.1. We estimate all the four elements one by one. Firstly, $a_{11} = \int_{\delta_1} \omega_1$, where δ_1 only contains 0 and λ^2. As $\lambda \to 0$, it follows that

$$\begin{aligned}
a_{11} &= -2\int_0^{\lambda^2} \omega_1 = -2\int_0^{\lambda^2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}} \\
&= -2\int_0^{\lambda^2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \frac{1}{\sqrt{-a}} \frac{1}{\sqrt{-b}} \left(1 + \frac{x}{2a} + \mathrm{O}(x^2)\right) \left(1 - \frac{x}{2b} + \mathrm{O}(x^2)\right) \\
&= \frac{-2}{-\sqrt{ab}} \int_0^{\lambda^2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} (1 + \mathrm{O}(x)) \\
&\sim \frac{-2}{-\sqrt{ab}} \int_0^{\lambda^2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \overset{q=x-\lambda^2}{=\!=\!=} \frac{1}{-\sqrt{ab}} \int_{-\lambda^2}^0 \frac{-2dq}{\sqrt{q(q+\lambda^2)(q+\lambda^2-\lambda)}} \\
&\overset{q=-\lambda^2\cdot v}{=\!=\!=} \frac{-2}{-\sqrt{ab}} \int_1^0 \frac{-\lambda^2 dv}{\sqrt{-\lambda^2 v(-\lambda^2 v + \lambda^2)(-\lambda^2 v + \lambda^2 - \lambda)}} \\
&= \frac{1}{-\sqrt{ab}} \frac{-2}{-\lambda\sqrt{-1}} \int_0^1 \frac{dv}{\sqrt{v(v-1)(v-1+\frac{1}{\lambda})}} \sim \frac{1}{-\sqrt{ab}} \frac{1}{-\lambda\sqrt{-1}} \int_{\tilde{\gamma}} \frac{dv}{v\sqrt{-1+\frac{1}{\lambda}}} \\
&= \frac{1}{-\sqrt{ab}} \frac{1}{-\lambda\sqrt{-1}} \frac{2\pi\sqrt{-1}}{\sqrt{-1+\frac{1}{\lambda}}} \sim \frac{2\pi}{\sqrt{ab}\sqrt{\lambda}}.
\end{aligned}$$

Secondly, it holds that

$$\begin{aligned}
a_{21} &= -2\int_0^{\lambda^2} \omega_1 = -2\int_0^{\lambda^2} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}} \\
&= -2\int_0^{\lambda^2} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \frac{1}{\sqrt{-a}} \frac{1}{\sqrt{-b}} \left(1 + \frac{x}{2a} + \mathrm{O}(x^2)\right) \left(1 - \frac{x}{2b} + \mathrm{O}(x^2)\right) \\
&= \frac{-2}{-\sqrt{ab}} \int_0^{\lambda^2} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} (1 + \mathrm{O}(x)) \sim \frac{-2}{-\sqrt{ab}} \int_0^{\lambda^2} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \\
&\underset{q=x-\lambda^2}{\overset{q=-\lambda^2\cdot v}{=\!=\!=}} \frac{-2}{-\sqrt{ab}} \int_1^0 \frac{-\lambda^2(\lambda^2(1-v))dv}{\sqrt{-\lambda^2 v(-\lambda^2 v + \lambda^2)(-\lambda^2 v + \lambda^2 - \lambda)}} \\
&= \frac{\lambda}{-\sqrt{ab}} \frac{-2}{\sqrt{-1}} \int_0^1 \frac{(v-1)dv}{\sqrt{v(v-1)(v-1+\frac{1}{\lambda})}} \sim \frac{\lambda\sqrt{\lambda}}{\sqrt{ab}} \frac{2}{\sqrt{-1}} \int_0^1 \sqrt{\frac{v-1}{v}} dv := C\lambda\sqrt{\lambda}.
\end{aligned}$$

Thirdly, let δ_2 contain only $0, \lambda, \lambda^2$ and a. Then, it follows that

$$\begin{aligned} a_{12} &= \int_{\delta_2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}} \\ &= \int_{\delta_2} \frac{dx}{x^2\sqrt{x-b}} \left(1+\frac{\lambda}{2x}+\mathrm{O}(\frac{\lambda^2}{x^2})\right)\left(1+\frac{\lambda^2}{2x}+\mathrm{O}(\frac{\lambda^4}{x^2})\right)\left(1+\mathrm{O}(\frac{1}{x})\right), \end{aligned}$$

for $|a| < |x| < |b|$. Since δ_2 doesn't contain b, $\frac{1}{\sqrt{(x-b)}}$ is holomorphic and therefore bounded on δ_2 by $C \in \mathbb{C}$. Then, a_{12} is asymptotically bounded by

$$C\int_{\delta_2} \frac{dx}{x^2} \left(1+\frac{\lambda}{2x}+\mathrm{O}(\frac{\lambda^2}{x^2})\right)\left(1+\frac{\lambda^2}{2x}+\mathrm{O}(\frac{\lambda^4}{x^2})\right)\left(1+\mathrm{O}(\frac{1}{x})\right) = 0.$$

Lastly,

$$\begin{aligned} a_{22} &= \int_{\delta_2} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}} \\ &= \int_{\delta_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} \frac{1}{\sqrt{x}} \left(1+\frac{\lambda}{2x}+\mathrm{O}(\frac{\lambda^2}{x^2})\right)\left(1+\frac{\lambda^2}{2x}+\mathrm{O}(\frac{\lambda^4}{x^2})\right) \\ &= \sim \int_{\delta_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} := C_{a,b}, \end{aligned}$$

where δ_2 contains 0 and a. Therefore, we could get the asymptotics of the matrix A and finish the proof of Lemma 5.5.1. $\square$

Lemma 5.5.2. *Under the same assumptions as Theorem 1.2.7, as $\lambda \to 0$, it holds that*

$$B_\lambda^{(7)} \sim \begin{pmatrix} \frac{2\sqrt{-1}\log\lambda}{-\sqrt{ab}\sqrt{\lambda}} & C''_{a,b} \\ \frac{2}{\sqrt{ab}}\sqrt{-\lambda} & C'_{ab} \end{pmatrix},$$

where $C'_{ab} := -2\int_a^b \frac{dx}{\sqrt{x(x-a)(x-b)}}$ *and* $C''_{ab} := -2\int_a^b \frac{dx}{x\sqrt{x(x-a)(x-b)}}$.

Proof of Lemma 5.5.2. Again, all the four elements are estimated one by one. Firstly, let γ_1 contain only λ and λ^2, and we get that

$$\begin{aligned} -2\int_0^\lambda \omega_1 &= -2\int_0^\lambda \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}} \\ &= -2\int_0^\lambda \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \frac{1}{\sqrt{-a}} \frac{1}{\sqrt{-b}} \left(1+\frac{x}{2a}+\mathrm{O}(x^2)\right)\left(1-\frac{x}{2b}+\mathrm{O}(x^2)\right) \\ &= \frac{-2}{-\sqrt{ab}} \int_0^\lambda \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} (1+\mathrm{O}(x)) \sim \frac{-2}{-\sqrt{ab}} \int_0^\lambda \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \\ &\sim \frac{1}{-\sqrt{ab}} \frac{2\sqrt{-1}\log\lambda}{\sqrt{\lambda}}, \end{aligned}$$

as $\lambda \to 0$. Thus, by Cauchy Integral Theorem we know that

$$b_{11} = -2\int_0^{\lambda} \omega_1 + 2\int_0^{\lambda^2} \omega_1 \sim \frac{1}{-\sqrt{ab}} \frac{2\sqrt{-1}\log\lambda}{\sqrt{\lambda}} - \frac{2\pi}{\sqrt{ab}\sqrt{\lambda}} \sim \frac{2\sqrt{-1}\log\lambda}{-\sqrt{ab}\sqrt{\lambda}}.$$

Secondly, substitute ω_1 with ω_2 and similarly it holds that

$$\begin{aligned}
b_{21} &= -2\int_{\lambda^2}^{\lambda} \omega_2 = -2\int_{\lambda^2}^{\lambda} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}} \\
&= -2\int_{\lambda^2}^{\lambda} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \frac{1}{\sqrt{-a}} \frac{1}{\sqrt{-b}} \left(1+\frac{x}{2a}+\mathrm{O}(x^2)\right)\left(1-\frac{x}{2b}+\mathrm{O}(x^2)\right) \\
&= \frac{-2}{-\sqrt{ab}} \int_{\lambda^2}^{\lambda} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} (1+\mathrm{O}(x)) \sim \frac{-2}{-\sqrt{ab}} \int_{\lambda^2}^{\lambda} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)}} \\
&\overset{x=\frac{\lambda}{s}}{\underset{t=\frac{1}{\lambda}}{=\!=}} \frac{1}{-\sqrt{ab}} \frac{2}{\lambda} \int_1^t \frac{\lambda ds}{s\sqrt{s(s-1)(s-t)}} \sim \frac{2}{-\sqrt{ab}} \int_1^t \frac{ds}{s^2\sqrt{s-t}} \\
&\sim \frac{2}{-\sqrt{ab}} \left(-\frac{\sqrt{-1}}{\sqrt{t}}\right) = \frac{2}{\sqrt{ab}}\sqrt{-\lambda},
\end{aligned}$$

as $\lambda \to 0$. Thirdly, let γ_2 contain only a and b. Then, it follows that

$$\begin{aligned}
b_{12} &= \int_{\gamma_2} \frac{dx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}} \\
&= \int_{\gamma_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} \frac{1}{\sqrt{x}} \left(1+\frac{\lambda}{2x}+\mathrm{O}(\frac{\lambda^2}{x^2})\right) \frac{1}{\sqrt{x}} \left(1+\frac{\lambda^2}{2x}+\mathrm{O}(\frac{\lambda^4}{x^2})\right) \\
&= \int_{\gamma_2} \frac{dx}{x\sqrt{x(x-a)(x-b)}} \left(1+\mathrm{O}(\frac{\lambda}{x})\right) \sim \int_{\gamma_2} \frac{dx}{x\sqrt{x(x-a)(x-b)}} := C''_{a,b}.
\end{aligned}$$

Lastly, it holds that

$$\begin{aligned}
b_{22} &= \int_{\gamma_2} \frac{xdx}{\sqrt{x(x-\lambda)(x-\lambda^2)(x-a)(x-b)}} \\
&= \int_{\gamma_2} \frac{xdx}{\sqrt{x(x-a)(x-b)}} \frac{1}{\sqrt{x}} \left(1+\frac{\lambda}{2x}+\mathrm{O}(\frac{\lambda^2}{x^2})\right) \frac{1}{\sqrt{x}} \left(1+\frac{\lambda^2}{2x}+\mathrm{O}(\frac{\lambda^4}{x^2})\right) \\
&= \int_{\gamma_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} \left(1+\mathrm{O}(\frac{\lambda}{x})\right) \sim \int_{\gamma_2} \frac{dx}{\sqrt{x(x-a)(x-b)}} := C'_{a,b}.
\end{aligned}$$

and this finishes the proof of Lemma 5.5.2. $\square$

The asymptotic results for the Bergman kernels are obtained by combining (2.10) and the following lemma.

Lemma 5.5.3. *Under the assumptions as in Theorem 1.2.7, let $Z_\lambda^{(7)}$ denote the period matrix of $X_\lambda^{(7)}$. Then, as $\lambda \to 0$, it holds that*

$$(\operatorname{Im} Z_\lambda^{(7)})^{-1} \sim \frac{\pi}{-c\log|\lambda| - \operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\}^2} \begin{pmatrix} c & -\operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\} \\ -\operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\} & -\log|\lambda| \end{pmatrix},$$

where $c'' = -\frac{\sqrt{ab}}{\pi}\int_a^b \frac{dx}{x\sqrt{x(x-a)(x-b)}}$.

Proof of Lemma 5.5.3. By Lemma 5.1.1, we know that

$$A^{-1} \sim \begin{pmatrix} \frac{2\pi}{\sqrt{ab}\sqrt{\lambda}} & 0 \\ C\lambda\sqrt{\lambda} & C_{ab} \end{pmatrix}^{-1} = \frac{1}{\frac{2\pi}{\sqrt{ab}\sqrt{\lambda}}C_{ab}} \begin{pmatrix} C_{ab} & -C\lambda\sqrt{\lambda} \\ 0 & \frac{2\pi}{\sqrt{ab}\sqrt{\lambda}} \end{pmatrix},$$

as $\lambda \to 0$. Therefore, it follows that

$$Z = A^{-1}B \sim \frac{1}{\frac{2\pi}{\sqrt{ab}\sqrt{\lambda}}C_{ab}} \begin{pmatrix} C_{ab} & -C\lambda\sqrt{\lambda} \\ 0 & \frac{2\pi}{\sqrt{ab}\sqrt{\lambda}} \end{pmatrix} \cdot \begin{pmatrix} \frac{2\sqrt{-1}\log\lambda}{-\sqrt{ab}\sqrt{\lambda}} & C''_{a,b} \\ \frac{2}{\sqrt{ab}}\sqrt{-\lambda} & C'_{ab} \end{pmatrix}$$

$$= \frac{\sqrt{ab}\sqrt{\lambda}}{2\pi C_{ab}} \begin{pmatrix} \frac{2\sqrt{-1}C_{ab}\log\lambda}{-\sqrt{ab}\sqrt{\lambda}} - \frac{2\sqrt{-1}C\lambda^2}{\sqrt{ab}}, & C_{ab}C''_{a,b} - C\lambda\sqrt{\lambda}C'_{ab} \\ \frac{4\pi\sqrt{-1}}{ab} & \frac{2\pi C'_{ab}}{\sqrt{ab}\sqrt{\lambda}} \end{pmatrix} \sim \begin{pmatrix} \frac{\sqrt{-1}\log\lambda}{-\pi} & \frac{C''_{a,b}\sqrt{ab}\sqrt{\lambda}}{2\pi} \\ \frac{\sqrt{\lambda}}{C_{ab}}\frac{2\sqrt{-1}}{\sqrt{ab}} & \frac{C'_{ab}}{C_{ab}} \end{pmatrix}$$

Since Z is symmetric, this implies that

$$C_{ab}C''_{a,b}ab = 4\pi\sqrt{-1},$$

namely

$$ab\int_0^a \frac{dx}{\sqrt{x(x-a)(x-b)}}\int_a^b \frac{dx}{x\sqrt{x(x-a)(x-b)}} = \pi\sqrt{-1}.$$

Moreover, as $\lambda \to 0$ we know that,

$$\operatorname{Im} Z \sim \begin{pmatrix} -\frac{\log|\lambda|}{\pi} & \frac{\operatorname{Im}\{C''_{ab}\sqrt{ab\lambda}\}}{2\pi} \\ 2\operatorname{Re}\left\{\frac{\sqrt{\lambda}}{C_{ab}\sqrt{ab}}\right\} & \operatorname{Im}\left\{\frac{C'_{ab}}{C_{ab}}\right\} \end{pmatrix} =: \frac{1}{\pi}\begin{pmatrix} -\log|\lambda| & \operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\} \\ \operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\} & c \end{pmatrix}. \tag{5.6}$$

We could also derive that $c > 0$, due to the fact that $\operatorname{Im} Z$ positive definite. Also, it has $-c\log|\lambda| > \operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\}^2$. Moreover, we have

$$(\operatorname{Im} Z)^{-1} \sim \pi\begin{pmatrix} -\log|\lambda| & \operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\} \\ \operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\} & c \end{pmatrix}^{-1}$$

$$= \frac{\pi}{-c\log|\lambda| - \operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\}^2}\begin{pmatrix} c & -\operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\} \\ -\operatorname{Im}\left\{c''\lambda^{\frac{1}{2}}\right\} & -\log|\lambda| \end{pmatrix},$$

which proves Lemma 5.1.3. □

Proof of Theorem 1.2.7. By (2.10), we know that near the cusp $(0,0)$, the coefficient of the Bergman kernel in local coordinate $z = \sqrt{x}$ is given by

$$\begin{aligned}
k_\lambda^{(7)}(z) =& \sum_{i,j=1}^{2} (\mathrm{Im}^{-1} Z)_{ij} \frac{4z^{2(2-i)} \cdot \overline{z}^{2(2-j)}}{|(z^2-\lambda)(z^2-\lambda^2)(z^2-a)(z^2-b)|} \\
=& 4 \cdot \frac{(\mathrm{Im}^{-1} Z)_{11} + (\mathrm{Im}^{-1} Z)_{12}\overline{z}^2 + (\mathrm{Im}^{-1} Z)_{21} \cdot z^2 + (\mathrm{Im}^{-1} Z)_{22}|z|^4}{|(z^2-\lambda)(z^2-\lambda^2)(z^2-a)(z^2-b)|} \\
\sim& 4 \cdot \frac{c - \mathrm{Im}\left\{c''\lambda^{\frac{1}{2}}\right\}\overline{z}^2 - \mathrm{Im}\left\{c''\lambda^{\frac{1}{2}}\right\} \cdot z^2 - \log|\lambda| \cdot |z|^4}{|(z^2-\lambda)(z^2-\lambda^2)(z^2-a)(z^2-b)|} \cdot \frac{\pi}{-c\log|\lambda| - \mathrm{Im}\left\{c''\lambda^{\frac{1}{2}}\right\}^2},
\end{aligned}$$

as $\lambda \to 0$. We can see that the leading term asymptotic expansion of $k_\lambda^{(7)}(z)$ is

$$\frac{4\pi}{c\,|(z^2-a)(z^2-b)|}.$$

Subtracting the leading term from $k_\lambda^{(7)}(z)$, we determine the two-term asymptotic expansion as follows. As $\lambda \to 0$, it has

$$k_\lambda^{(7)}(z) \sim \left\{\frac{1}{c} + \frac{1}{-\log|\lambda| \cdot |z|^4}\right\} \cdot \frac{4\pi}{|(z^2-a)(z^2-b)|}.$$

Taking the logarithm, we will know that, as $\lambda \to 0$,

$$\log k_\lambda^{(7)}(z) = \log \frac{4\pi}{c\,|(z^2-a)(z^2-b)|} + \frac{c}{-\log|\lambda| \cdot |z|^4} + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right). \qquad \square$$

We remark that for the Jacobian varieties of $X_\lambda^{(7)}$ (more generally $X_\lambda^{(9)}$), as $\lambda \to 0$, hyperbolic growth appears again by (5.6) in the proof of Lemma 5.5.3.

Theorem 5.5.1. *Under the same assumptions as Theorem 1.2.7, as $\lambda \to 0$, it holds that*

$$\partial\bar{\partial} \log k_\lambda^{(7)}(z) \sim \frac{c}{|z|^4} \left\{ \frac{d\lambda \wedge d\bar{\lambda}}{2|\lambda|^2(-\log|\lambda|)^3} - \frac{d\lambda \wedge d\bar{z}}{\bar{z}\lambda(-\log|\lambda|)^2} - \frac{dz \wedge d\bar{\lambda}}{z\bar{\lambda}(-\log|\lambda|)^2} + \frac{4dz \wedge d\bar{z}}{|z|^2(-\log|\lambda|)} \right\}.$$

Concluding remarks Although Bergman kernels near different types of singularities behave differently, it might be interesting if there exists some special coordinate working for the cusp case, in comparison to the pinching coordinate for the node case, so that results for general curves can be further obtained. Geometric interpretations of the coefficients are also appreciated.

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