Topologically stable gapless phases in nonsymmorphic superconductors

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We study the topological stability of nodes in nonsymmorphic superconductors (SCs). In particular, we demonstrate that line nodes in nonsymmorphic odd-parity SCs are protected by the interplay between topology and nonsymmorphic symmetry. As an example, it is shown that the E_{2u}-superconducting state of UPt$_3$ hosts the topologically stable line node at the Brillouin zone face. Our theory indicates that the existence of spin-orbit coupling is essential for protecting such a line node, complementing Norman's group theory argument. Developing the topological arguments, we also propose a generalization to point nodes and to other symmetry cases beyond the group theory arguments.

I. INTRODUCTION

Nontrivial node structures are a salient feature in unconventional superconductors (SCs), offering valuable clues to aid in the understanding of the symmetry of Cooper pairs. The node structures are detected by the power-law behaviors of temperature dependence, such as the specific heat and NMR relaxation rates [1,2]. Since the 1980s, superconductivity in heavy fermion materials has attracted much attention, as these materials may be candidates for unconventional SCs [3]. At the early stage, the group theory is applied to the superconducting states in order to limit possible Cooper pairs [4] by use of spin-orbit coupling (SOC) and crystal structure in heavy fermion materials. From a group-theoretical basis, Blount showed the impossibility of line nodes in odd-parity SCs in the presence of SOC [5]; this is called Blount's theorem. In contrast, real candidate materials of heavy fermion odd-parity SCs such as UPt$_3$ have often suggested the existence of line nodes [6]. To resolve this contradiction, Norman pointed out the possibility of a stable line node on the Brillouin zone (BZ) face in odd-parity SCs due to nonsymmorphic symmetry [7], which is a counter example of Blount's theorem. Furthermore, Mickliz and Norman proved that twofold screw symmetry generally forces an odd representation of a Cooper pair to vanish on the BZ face perpendicular to the screw axis within the group theory [8].

Recently, unconventional SCs have been seen in a new light, i.e., from the viewpoint of topology [9–18]. The stability of nodes is given by topological numbers. The topological approach enables us to classify symmetric and accidental nodes in a unified way, and it may connect the topological structures of bulk nodes to surface zero-energy states via the bulk-edge correspondence [19–28]. Using this method, two of the present authors proved the topological version of Blount's theorem [15] and updated the conventional Blount's theorem by connecting a bulk line node with a surface zero-energy flat band instability. At the same time, the reinterpretation may also raise new theoretical questions regarding the connection between Norman's argument and the topological approach, i.e., the topological stability of line nodes in nonsymmorphic odd-parity SCs.

In addition to unconventional SCs, similar gapless phases have been suggested in the context of topological semimetals, such as Dirac/Weyl semimetals [29–44] and line nodal semimetals [45–54]. Among them, tight-binding model studies in orthorhombic perovskite SrIrO$_3$ showed stable line nodes at the BZ face [55–57]. Because of nonsymmorphic symmetry and strong SOC in SrIrO$_3$, the line node is topologically protected [58]. Recently, versatile topological semimetals [59–65] and insulators [66–78] with a nontrivial influence of nonsymmorphic symmetry have been anticipated theoretically.

In this paper, we establish a general theory to treat the topological stability of nodes in nonsymmorphic SCs. Our theory enables us to take into account nonsymmorphic crystal in a topological manner, and it is a natural extension of the previous work [15]. The obtained results include the topological Blount's theorem. In a generalized framework, we will find that the line node proposed by Norman is exactly protected by the interplay between topology and nonsymmorphic symmetry, the stability of which is characterized by a mirror topological number. In addition to the topological number, we also reveal that SOC plays a central role in protecting the line node. In the absence of SOC, the Fermi surface acquires a fourfold degeneracy at the BZ face, and the line node disappears. We apply our theory to the E_{2u} superconducting state of UPt$_3$, and we show the existence of nonsymmorphic-symmetry-protected nodal rings at the BZ face by taking into account an antisymmetric SOC. In addition, the topological approach predicts nontrivial nonsymmorphic-symmetry-protected nodes beyond Norman's argument. Thus, our results not only connect the group theory studies with the topological classification, but they also provide a guiding principle in the search for nonsymmorphic-symmetry-protected nodes.

The paper is organized as follows. In Sec. II, we construct the Bogoliubov–de Gennes (BdG) Hamiltonian, taking into account nonsymmorphic crystals. This part is at the heart of the mechanism of nonsymmorphic-symmetry-protected nodes. In Sec. III, the stability of line nodes in nonsymmorphic odd-parity SCs is discussed in two different ways: group-theoretical classification of possible Cooper pairs in Sec. III A, and topological classification of BdG Hamiltonians in Sec. III B. We apply the topological argument to the E_{2u}-representation

superconducting state of UPt$_3$ in Sec. III C. In Sec. IV, we mention a possible generalization of nonsymmorphic-symmetry-protected nodes. Finally, we summarize this paper in Sec. V.

II. FORMULATION

First, we generalize the basis function of the underlying Hamiltonian in order to take into account nonsymmorphic crystals [79]. A nonsymmorphic crystal has at least two atoms in the unit cell, and these atoms are separated by a nonprimitive lattice vector. To involve the nonprimitive lattice vector in a tight-binding Hamiltonian, we use Löwdin orbitals $\varphi_\alpha(\boldsymbol{r} - \boldsymbol{R} - \boldsymbol{r}_\alpha)$ [80], where $\boldsymbol{R}$ is a Bravais lattice (BL) vector and $\boldsymbol{r}_\alpha$ denotes the position of an atom α. Here, α ($\alpha = 1, \ldots, m$) describes spin, sublattice indices, and orbital degrees of freedom. The wave functions centered at different sites (or with different indices α) are orthogonal to each other. The basis function, which has a discrete translational invariance in terms of BL vectors, is given by the linear combination of Löwdin orbitals:

$$\phi_{\boldsymbol{k},\alpha}(\boldsymbol{r}) = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{R}} e^{i\boldsymbol{k}\cdot(\boldsymbol{R}+\boldsymbol{r}_\alpha)} \varphi_\alpha(\boldsymbol{r} - \boldsymbol{R} - \boldsymbol{r}_\alpha), \quad (1)$$

where N is the number of primitive unit cells in the crystal. The function $\phi_{\boldsymbol{k},\alpha}$ obeys the Bloch condition: $\phi_{\boldsymbol{k},\alpha}(\boldsymbol{r} + \boldsymbol{R}) = e^{i\boldsymbol{k}\cdot\boldsymbol{R}} \phi_{\boldsymbol{k},\alpha}(\boldsymbol{r})$, and, due to a nonprimitive lattice vector $\boldsymbol{r}_\alpha$, it satisfies the additional condition $\phi_{\boldsymbol{k}+\boldsymbol{G},\alpha}(\boldsymbol{r}) = e^{i\boldsymbol{G}\cdot\boldsymbol{r}_\alpha} \phi_{\boldsymbol{k},\alpha}(\boldsymbol{r})$, where $\boldsymbol{G}$ is a reciprocal-lattice (RL) vector. If $\boldsymbol{r}_\alpha = \boldsymbol{0}$, the Löwdin orbital reduces to the Wanner function. Using the Löwdin orbitals, the tight-binding Hamiltonian is given by

$$H_{\alpha\beta}(\boldsymbol{k}) = \int d\boldsymbol{r} \, \phi^*_{\boldsymbol{k},\alpha}(\boldsymbol{r}) \mathcal{H} \phi_{\boldsymbol{k},\beta}(\boldsymbol{r}), \quad (2)$$

where $\mathcal{H}$ is the single-particle Hamiltonian. The tight-binding Hamiltonian satisfies

$$H_{\alpha\beta}(\boldsymbol{k} + \boldsymbol{G}) = e^{-i\boldsymbol{G}\cdot\boldsymbol{r}_\alpha} H_{\alpha\beta}(\boldsymbol{k}) e^{i\boldsymbol{G}\cdot\boldsymbol{r}_\beta}. \quad (3)$$

We model nonsymmorphic superconductors (SCs) phenomenologically using the Löwdin orbitals. We introduce a creation operator of the wave function $\phi_{\boldsymbol{k},\alpha}$,

$$c^\dagger_{\boldsymbol{k},\alpha} = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{R}} e^{i\boldsymbol{k}\cdot(\boldsymbol{R}+\boldsymbol{r}_\alpha)} c_\alpha(\boldsymbol{R} + \boldsymbol{r}_\alpha)^\dagger, \quad (4)$$

where $c_\alpha(\boldsymbol{R} + \boldsymbol{r}_\alpha)^\dagger$ is a creation operator of an electron with index α located at $\boldsymbol{R} + \boldsymbol{r}_\alpha$. Equation (4) satisfies $c^\dagger_{\boldsymbol{k}+\boldsymbol{G},\alpha} = e^{i\boldsymbol{G}\cdot\boldsymbol{r}_\alpha} c^\dagger_{\boldsymbol{k},\alpha}$. The Bogoliubov–de Gennes (BdG) Hamiltonian is given by

$$H_{\text{BdG}} = \frac{1}{2} \sum_{\boldsymbol{k},\alpha,\beta} \left(c^\dagger_{\boldsymbol{k},\alpha}, c_{-\boldsymbol{k},\alpha} \right) \tilde{H}(\boldsymbol{k}) \begin{pmatrix} c_{\boldsymbol{k},\beta} \\ c^\dagger_{-\boldsymbol{k},\beta} \end{pmatrix}, \quad (5)$$

with

$$\tilde{H}(\boldsymbol{k}) = \begin{pmatrix} \mathcal{E}_{\alpha\beta}(\boldsymbol{k}) & \Delta_{\alpha\beta}(\boldsymbol{k}) \\ \Delta_{\alpha\beta}(\boldsymbol{k})^\dagger & -\mathcal{E}_{\alpha\beta}(-\boldsymbol{k})^T \end{pmatrix}, \quad (6)$$

where $\mathcal{E}_{\alpha\beta}(\boldsymbol{k}) = H_{\alpha\beta}(\boldsymbol{k}) - \mu\delta_{\alpha\beta}$ is the normal Hamiltonian, the gap function $\Delta_{\alpha\beta}(\boldsymbol{k})$ satisfies $\Delta_{\alpha\beta}(-\boldsymbol{k}) = -\Delta_{\beta\alpha}(\boldsymbol{k})$ due to the

Fermi statistics, and μ is the chemical potential. Since the gap function should be consistent with the structure of nonsymmorphic crystals, we require $\Delta_{\alpha\beta}(\boldsymbol{k} + \boldsymbol{G}) = e^{-i\boldsymbol{G}\cdot\boldsymbol{r}_\alpha} \Delta_{\alpha\beta}(\boldsymbol{k}) e^{i\boldsymbol{G}\cdot\boldsymbol{r}_\beta}$. Combining it with Eq. (3), the BdG Hamiltonian has the constraint under an RL vector $\boldsymbol{G}$:

$$\tilde{H}(\boldsymbol{k} + \boldsymbol{G}) = \tilde{V}_{\boldsymbol{G}} \tilde{H}(\boldsymbol{k}) \tilde{V}^\dagger_{\boldsymbol{G}}, \quad \tilde{V}_{\boldsymbol{G}} = \begin{pmatrix} V_{\boldsymbol{G}} & 0 \\ 0 & V_{\boldsymbol{G}} \end{pmatrix}, \quad (7)$$

with $V_{\boldsymbol{G}} = \text{diag}\,[e^{-i\boldsymbol{G}\cdot\boldsymbol{r}_1}, \ldots, e^{-i\boldsymbol{G}\cdot\boldsymbol{r}_m}]$ ($\alpha = 1, \ldots, m$).

In the following, we summarize discrete symmetries that are relevant to the stability of nodes. To start with, we introduce particle-hole symmetry (PHS), time-reversal symmetry (TRS), and spatial-inversion symmetry (IS) as follows:

$$\text{PHS}: C\tilde{H}(\boldsymbol{k})C^\dagger = -\tilde{H}(-\boldsymbol{k}), \quad C = \begin{pmatrix} 0 & \delta_{\alpha\beta} \\ \delta_{\alpha\beta} & 0 \end{pmatrix} K, \quad (8)$$

$$\text{TRS}: \mathcal{T}\tilde{H}(\boldsymbol{k})\mathcal{T}^\dagger = \tilde{H}(-\boldsymbol{k}), \quad \mathcal{T} = \begin{pmatrix} T_{\alpha\beta} & 0 \\ 0 & T^*_{\alpha\beta} \end{pmatrix}, \quad (9)$$

$$\text{IS}: \mathcal{P}\tilde{H}(\boldsymbol{k})\mathcal{P}^\dagger = \tilde{H}(-\boldsymbol{k}), \quad \mathcal{P} = \begin{pmatrix} P_{\alpha\beta} & 0 \\ 0 & \eta_P P^*_{\alpha\beta} \end{pmatrix}. \quad (10)$$

Here, $T \equiv U_t K$, and U_t and P are $m \times m$ unitary matrices satisfying $U_t = -U_t^T$ and $P^2 = \mathbf{1}_m$. K is the complex conjugation operator, and $\mathbf{1}_m$ is the identity matrix with rank m. From Eqs. (9) and (10), TRS and IS, respectively, require $U_t H(\boldsymbol{k}) U_t^\dagger = H^*(-\boldsymbol{k})$ and $U_t \Delta(\boldsymbol{k}) U_t^T = \Delta^*(-\boldsymbol{k})$, and $P H(\boldsymbol{k}) P^\dagger = H(-\boldsymbol{k})$ and $P \Delta(\boldsymbol{k}) P^T = \eta_P \Delta(-\boldsymbol{k})$, where η_P describes the parity of the gap function, i.e., $\eta_P = +1$ for even parity and $\eta_P = -1$ for odd parity. For even-parity gap functions, we have $[C, \mathcal{P}] = 0$, while for odd-parity gap functions, $\{C, \mathcal{P}\} = 0$. Hereafter, we assume $[C, \mathcal{T}] = [\mathcal{P}, \mathcal{T}] = 0$ unless otherwise specified.

In addition to the nonspatial symmetries, crystal symmetry may stabilize nodal structure. An element of a space group G is given as $\{g|\boldsymbol{\tau}\}$ with a point group element g and a translation $\boldsymbol{\tau}$. Under $\{g|\boldsymbol{\tau}\}$, $\boldsymbol{x}$ transforms as $\boldsymbol{x} \to D(g)\boldsymbol{x} + \boldsymbol{\tau}$. For $\{g|\boldsymbol{\tau}\}$, $c^\dagger_{\boldsymbol{k}\alpha}$ transforms as (see Appendix A)

$$\{g|\boldsymbol{\tau}\} c^\dagger_{\boldsymbol{k},\alpha} \{g|\boldsymbol{\tau}\}^{-1} = e^{-iD(g)\boldsymbol{k}\cdot\boldsymbol{\tau}} c^\dagger_{D(g)\boldsymbol{k},\beta} U_{\beta\alpha}(g)$$

$$\equiv c^\dagger_{D(g)\boldsymbol{k},\beta} D_{\boldsymbol{k}}(\{g|\boldsymbol{\tau}\})_{\beta\alpha}, \quad (11)$$

where $D(g)$ and $U(g)$ are matrix representations of g in real space and the space of α, respectively. When $\{g_1|\boldsymbol{\tau}_1\}$ and $\{g_2|\boldsymbol{\tau}_2\}$ are elements of the little group leaving $\boldsymbol{k}$ invariant, the associative property of $D_{\boldsymbol{k}}$, $D_{\boldsymbol{k}}(\{g_1|\boldsymbol{\tau}_1\})D_{\boldsymbol{k}}(\{g_2|\boldsymbol{\tau}_2\}) = D_{\boldsymbol{k}}(\{g_1 g_2 | D(g_1)\boldsymbol{\tau}_2 + \boldsymbol{\tau}_1\})$, leads to

$$U(g_1)U(g_2) = \omega^{\boldsymbol{k}}_{g_1,g_2} U(g_1 g_2), \quad (12)$$

where $\omega^{\boldsymbol{k}}_{g_1,g_2} \equiv e^{i\boldsymbol{k}\cdot[D(g_1)^{-1}\boldsymbol{\tau}_1 - D(g_2)^{-1}D(g_1)^{-1}\boldsymbol{\tau}_1]}$ is a factor system in the group theory [82]. Here, the factor system $\omega^{\boldsymbol{k}}_{g_1,g_2}$ is nontrivial only if $\boldsymbol{k}$ is in a high-symmetric subspace on the BZ face. Furthermore, if $U(g_1 g_2) = t_{g_1,g_2} U(g_2 g_1)$ with $t_{g_1,g_2} = \pm 1$, the commutation relation between $D_{\boldsymbol{k}}(\{g_1|\boldsymbol{\tau}_1\})$ and $D_{\boldsymbol{k}}(\{g_2|\boldsymbol{\tau}_2\})$ becomes

$$U(g_1)U(g_2) = t_{g_1,g_2} \alpha^{\boldsymbol{k}}_{g_1,g_2} U(g_2)U(g_1), \quad (13)$$

where $\alpha^{\boldsymbol{k}}_{g_1,g_2} \equiv \omega^{\boldsymbol{k}}_{g_1,g_2} / \omega^{\boldsymbol{k}}_{g_2,g_1}$.

From Eq. (11), H_{BdG} in Eq. (5) transforms as

$$H_{\text{BdG}} \rightarrow \sum c^\dagger_{D(g)k} D_k(\{g|\tau\}) \mathcal{E}(k) D_k(\{g|\tau\})^\dagger c_{D(g)k}$$
$$+ \sum c^\dagger_{D(g)k} D_k(\{g|\tau\}) \Delta(k) D_{-k}(\{g|\tau\})^T c_{-D(g)k}$$
$$+ \cdots \tag{14}$$

under $\{g|\tau\}$. Since the normal Hamiltonian is invariant under G, we have

$$D_k(\{g|\tau\}) \mathcal{E}(k) D_k(\{g|\tau\})^\dagger = \mathcal{E}(D(g)k). \tag{15}$$

Moreover, in order for $\{g|\tau\}$ to be the symmetry of the superconducting state, the gap function should obey

$$D_k(\{g|\tau\}) \Delta(k) D_{-k}(\{g|\tau\})^T = \eta_{C,g} \Delta(D(g)k), \tag{16}$$

with $\eta_{C,g} = \pm 1$. For $\eta_{C,g} = 1(-1)$, the right-hand side of Eq. (14) coincides with H_{BdG} trivially (by performing the π-gauge rotation of $c^\dagger_{k\alpha}$). The phase factors $e^{-iD(g)k\cdot\tau}$ are canceled in Eqs. (15) and (16), so we have

$$U(g) \mathcal{E}(k) U(g)^\dagger = \mathcal{E}(D(g)k), \tag{17}$$

$$U(g) \Delta(k) U(g)^T = \eta_{C,g} \Delta(D(g)k). \tag{18}$$

In the matrix form of the BdG Hamiltonian, Eqs. (17) and (18) are summarized as

$$\tilde{U}(g) \tilde{H}(k) \tilde{U}(g)^\dagger = \tilde{H}(D(g)k), \tag{19}$$

with $\tilde{U}(g) = \text{diag}[U(g), \eta_{C,g} U(g)^*]$. We also have $C\tilde{U}(g) = \eta_{C,g} \tilde{U}(g) C$.

Since we are interested in the influence of the crystal symmetry on the nodes, we focus on the behavior of the BdG Hamiltonian near a node at k_0, where the position of a node is defined by $\det[\tilde{H}(k_0)] = 0$. We assume that k_0 lies in a high-symmetry subspace of the BZ, and $\{g|\tau\}$ belongs to the little group of k_0, i.e., $D(g)k_0 - k_0$ is an RL vector. With the condition (7), the space-group operation $\{g|\tau\}$ on the BdG Hamiltonian at $k + k_0$ yields

$$\tilde{U}_{k_0}(g) \tilde{H}(k + k_0) \tilde{U}_{k_0}(g)^\dagger = \tilde{H}(D(g)k + k_0), \tag{20}$$

where $\tilde{U}_{k_0}(g) \equiv \tilde{V}^\dagger_{D(g)k_0-k_0} \tilde{U}(g)$. Hence, nodes at k_0 obey the symmetry operation $\tilde{U}_{k_0}(g)$ rather than $\tilde{U}(g)$. Consider the commutation relation between $\{g_1|\tau_1\}$ and $\{g_2|\tau_2\}$, which belong to the little group of k_0. The product of $\tilde{U}_{k_0}(g_1)$ and $\tilde{U}_{k_0}(g_2)$ is calculated as (see Appendix B)

$$\tilde{U}_{k_0}(g_1) \tilde{U}_{k_0}(g_2) = \omega^{k_0}_{g_1,g_2} \tilde{U}_{k_0}(g_1 g_2). \tag{21}$$

Therefore, $\tilde{U}_{k_0}$ satisfies the same relationship as Eq. (12), implying that V_G gives the correct factor system. In addition, if $\tilde{U}_{k_0}(g_1 g_2) = t_{g_1,g_2} \tilde{U}_{k_0}(g_2 g_1)$, we obtain

$$\tilde{U}_{k_0}(g_1) \tilde{U}_{k_0}(g_2) = t_{g_1,g_2} \alpha^{k_0}_{g_1,g_2} \tilde{U}_{k_0}(g_2) \tilde{U}_{k_0}(g_1), \tag{22}$$

which coincides with Eq. (13).

In closing this section, we remark on a few properties of the factor system $\omega^k_{g_1,g_2}$. In the case in which $\{g_1|\tau_1\}$ and $\{g_2|\tau_2\}$ are an order-2 operator, i.e., twofold screw (rotation), glide (reflection), and spatial-inversion symmetries, $\alpha^k_{g_1,g_2}$ of the factor system is simplified as

$$\alpha^k_{g_1,g_2} = e^{iD(g_1)D(g_2)k\cdot\{[D(g_2)\tau_1 - \tau_1] - [D(g_1)\tau_2 - \tau_2]\}}. \tag{23}$$

In addition, PHS and TRS act trivially in real space, i.e., $D(C) = D(T) = \mathbf{1}_3$, so $\omega^k_{g,C} = \omega^k_{g,T} = 1$ for any $\{g|\tau\} \in G$ and k. In the following sections, Eqs. (22) and (23) are essential for nodes protected by nonsymmorphic symmetry.

III. LINE NODE IN NONSYMMORPHIC ODD-PARITY SUPERCONDUCTORS

In this section, we revisit a line node in odd-parity SCs predicted by Micklitz and Norman [7,8]. As the minimal condition, consider a time-reversal invariant odd-parity SC with twofold screw symmetry. The superconducting state possesses PHS C ($C^2 = 1$), TRS $\mathcal{T}$ ($\mathcal{T}^2 = -1$), IS $\mathcal{P}$ ($\mathcal{P}^2 = 1$ and $\{C, \mathcal{P}\} = 0$), and the twofold screw symmetry whose axis is perpendicular to a line node. When the screw axis is chosen to be the z axis, the twofold screw operator is described algebraically by $\{C_{2z}|\frac{1}{2}\hat{z}\}$, where $\hat{z}$ is a unit lattice vector along the z axis, and C_{2z} is a twofold rotation operator around the z axis. The matrix representation of $\{C_{2z}|\frac{1}{2}\hat{z}\}$ in Eq. (11) is $D_k(\{C_{2z}|\frac{1}{2}\hat{z}\}) = e^{-i\frac{k_z}{2}} U(C_{2z})$. Combining the twofold screw with P, a mirror-reflection operator is also defined as

$$D_k\left(\left\{PC_{2z}\Big|\frac{1}{2}\hat{z}\right\}\right) = D_k\left(\left\{M_{xy}\Big|\frac{1}{2}\hat{z}\right\}\right)$$
$$= e^{-i\frac{k_z}{2}} U(M_{xy}), \tag{24}$$

where M_{xy} is the mirror-reflection operator with respect to the xy plane. Due to the spinor representation of rotation, we have $U(C_{2z})^2 = U(M_{xy})^2 = -1$. By calculating the factor system in Eq. (23), the commutation relation between $D_k(\{M_{xy}|\frac{1}{2}\hat{z}\})$ and P yields

$$U(M_{xy})P = e^{ik_z} PU(M_{xy}). \tag{25}$$

Here, we implicitly assume that P commutes with $U(M_{xy})$ as usual. More general cases are discussed in Sec. IV. In what follows, we elucidate the existence of stable line nodes in odd-parity SCs in two different ways. In Sec. III A, we rely on the group-theoretical method by focusing on the symmetry of Cooper pairs. Then, in Sec. III B, we develop a topological approach.

A. Group-theoretical approach

In this subsection, we prove the existence of a stable line node at the BZ face based on the group-theoretical method. The following argument essentially follows Norman's argument [7] but with some simplification. When TRS and IS are present in a normal metal, Kramer's doublet exists at an arbitrary k, labeled by $|k\rangle$ and $PT|k\rangle$. Here, $|k\rangle$ represents an electronic state with momentum k and a pseudospin β (under PT: $\beta \rightarrow -\beta$). $|k\rangle$ and $PT|k\rangle$ correspond to spin-up and spin-down electronic states in the SOC free limit. Also, $P|k\rangle$ and $T|k\rangle$ describe Kramer's doublet at $-k$. When electrons at k and $-k$ form a Cooper pair, we have a single even-parity pairing $(k, Tk) - (PTk, Pk)$ and three odd-parity pairings (k, Pk), (PTk, Tk), and $(k, Tk) + (PTk, Pk)$, where $(.,.)$ represents the electron pairs forming the Cooper pair. Introducing the d-vector representation, each spin-triplet pairing is described by $-d_x + id_y$, $d_x + id_y$, and d_z, respectively. Here, we assume time-reversal invariant spin-triplet SCs that require $d \in \mathbb{R}^3$. To

obtain a line node in three-dimensional momentum space, it is necessary to fulfill $\boldsymbol{d} = \boldsymbol{0}$ along a curve on the Fermi surface. However, it is vanishingly improbable to satisfy the three conditions on the Fermi surface at the same time. Thus, we need crystal symmetry. A line node may appear on a cross line between the Fermi surface and a higher symmetric plane where some of $\boldsymbol{d}$ vanishes, so here we consider mirror-reflection symmetry. On the mirror-invariant plane, an electronic state $|k\rangle$ is an eigenstate of the mirror-reflection operator. Without loss of generality, we assume that $|k\rangle$ has the mirror-reflection eigenvalue $+i$. Then, the mirror-reflection eigenvalue of other electrons is systematically determined by the commutation relation between T, P, and $U(M_{xy})$. Likewise, the mirror-reflection eigenvalue of Cooper pairs is given by the product of that of two electrons, and it takes ± 1. We have a mirror-reflection symmetry protected line node if mirror-reflection symmetry forces all components of the $\boldsymbol{d}$ vector to vanish simultaneously on the mirror-invariant plane.

First, consider the mirror-reflection symmetry in Eq. (24) and the mirror-invariant plane at $k_z = 0$. From Eq. (25), $[P, U(M_{xy})] = 0$. We also have $[T, U(M_{xy})] = 0$. With the antiunitarity of T in mind, $|k\rangle$ and $P|k\rangle$ take the mirror-reflection eigenvalue $+i$, and $T|k\rangle$ and $PTk\rangle$ take $-i$. Thus, the Cooper pairs (k, Tk) and (PTk, Pk) have the mirror-reflection eigenvalue $+1$, whereas (k, Pk) and (PTk, Tk) have -1. [See Fig. 1(a).] Hence, when the Cooper pair takes the mirror-reflection eigenvalue $+1$, only the d_z component consisting of (k, Tk) and (PTk, Pk) survives on the mirror-invariant plane, while when the Cooper pair takes -1, the other d_x and d_y components are nonvanishing. That is, whichever mirror-reflection eigenvalue you take, d_x, d_y, and d_z cannot vanish simultaneously, which means that mirror-reflection symmetry does not allow a line node in spin-triplet SCs. Accordingly, a line node at $k_z = 0$ is unstable in time-reversal invariant spin-triplet SCs with and without mirror-reflection symmetry. This result is known as Blount's theorem [5].

Next, consider the mirror-invariant plane at $k_z = \pi$. From Eq. (25), we obtain $\{P, U(M_{xy})\} = 0$ in addition to $[T, U(M_{xy})] = 0$, leading to $+i$ for $|k\rangle$ and $PT|k\rangle$, and $-i$ for $P|k\rangle$ and $T|k\rangle$. In contrast to the mirror-invariant plane at $k_z = 0$, all of the Cooper pairs (k, Tk), (PTk, Pk), (k, Pk), and (PTk, Tk) have the same mirror-reflection eigenvalue $+1$. [See Fig. 1 (b).] That is, all components of the $\boldsymbol{d}$ vector vanish simultaneously when the Cooper pair takes -1, leading to a stable line node at the BZ face [7,8]. This result does not contradict Blount's theorem since the commutation relation between P and $U(M_{xy})$ changes at the BZ face. To sum up, the mirror-reflection symmetry allows a symmetry-protected line node only when $\{P, U(M_{xy})\} = [T, U(M_{xy})] = 0$ and the Cooper pair is odd under the mirror-reflection operation.

Although it is not clear in Norman's original argument, it is important for the SOC to have a stable line node in odd-parity SCs. Without the SOC, there is fourfold degeneracy on the Fermi surface at $k_z = \pi$: As mentioned above, $|k\rangle$ and $PT|k\rangle$ have the same eigenvalue of $U(M_{xy})$ at $k_z = \pi$. Since $|k\rangle$ and $PT|k\rangle$ have the same momentum $\boldsymbol{k}$, there is twofold degeneracy at each $\boldsymbol{k}$ in the $U(M_{xy}) = i$ subsector. In the absence of SOC, on the other hand, spin is a good quantum number, so $|k\rangle$ and $PT|k\rangle$ can be written as $|k\uparrow\rangle$ and $PT|k\uparrow\rangle$, respectively. In this case, we also have full spin-rotation
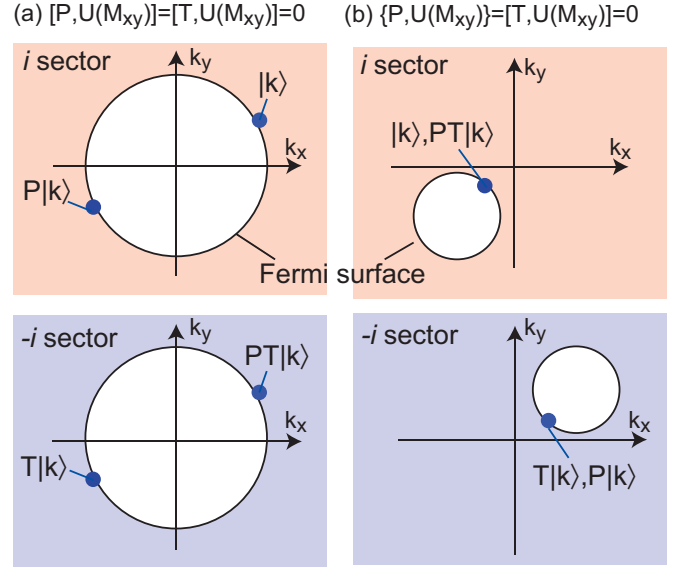

FIG. 1. Schematic illustration of the mirror-reflection eigenvalue of electron states, assuming that $U(M_{xy})|k\rangle = +i|k\rangle$. In the presence of TRS and IS, Kramer's degeneracy occurs at any $\boldsymbol{k}$. The blue points indicate the electron states on the Fermi surface. $|k\rangle$ and $PT|k\rangle$ are the Kramer's pair located at $\boldsymbol{k}$ and $P|k\rangle$ and $T|k\rangle$ at $-\boldsymbol{k}$. If mirror-reflection symmetry is present, each electronic state has the mirror-reflection eigenvalue $\pm i$ on the mirror-invariant plane, which is determined systematically according to the commutation relation between P, T, and $U(M_{xy})$. Parts (a) and (b) describe the case of $[P, U(M_{xy})] = [T, U(M_{xy})] = 0$ and of $\{P, U(M_{xy})\} = [T, U(M_{xy})] = 0$, respectively. The upper (lower) figure represent the mirror-reflection eigenspace with $+i$ ($-i$). The Cooper pair has the mirror-reflection eigenvalue $+1$ (-1) if two electrons that form a Cooper pair lie in the different (same) eigenspace. Note that, in case (b), the Fermi surface needs a noncentrosymmetric shape in each mirror eigenspace in order to avoid a fully gapped SC.

symmetry, which can flip the spin and the eigenvalue of $U(M_{xy})$ at the same time. Thus, using spin-rotation symmetry, we obtain $|k\downarrow\rangle$ and $PT|k\downarrow\rangle$, which have the same energy and momentum as $|k\uparrow\rangle$ and $PT|k\uparrow\rangle$, but they have the different eigenvalue $-i$ of $U(M_{xy})$. In total, we have fourfold degeneracy on the Fermi surface at $k_z = \pi$.

In this situation, we cannot have a stable line node in general. Because of the additional degeneracy, there are additional possible Cooper pairs $(k\uparrow, Tk\downarrow)$, $(PTk\uparrow, Pk\downarrow)$, $(k\uparrow, Pk\downarrow)$, and $(PTk\uparrow, Tk\downarrow)$, which take the mirror-reflection eigenvalue -1. Thus, even when the Cooper pair is odd under the mirror reflection, the $\boldsymbol{d}$ vector of the additional Cooper pair survives at $k_z = \pi$. Consequently, no stable line node can be obtained.

B. Topological approach

Here we prove the stability of the line node from a topological point of view. We assume that line nodes exist at $k_z = 0$ and $k_z = \pi$. Let $\tilde{H}(\boldsymbol{k})$ be the BdG Hamiltonian defined by Eq. (6). From Eq. (19), the action of $\{M_{xy}|\frac{1}{2}\hat{z}\}$ on the BdG Hamiltonian is

$$\tilde{U}(M_{xy})\tilde{H}(k_x, k_y, k_z)\tilde{U}(M_{xy})^\dagger = \tilde{H}(k_x, k_y, -k_z). \quad (26)$$

For mirror-reflection symmetry, Eq. (18) becomes $U(M_{xy})\Delta(k_x,k_y,k_z)U(M_{xy})^\dagger = \eta_{C,M}\Delta(k_x,k_y,-k_z)$, under which $\tilde{U}(M_{xy})$ = diag$[U(M_{xy}),\eta_{C,M}U(M_{xy})^*]$ obeys $C\tilde{U}(M_{xy}) = \eta_{C,M}\tilde{U}(M_{xy})C$. We label the position of a line node as $\mathbf{k}_M$ for $k_z = 0$ and $\mathbf{k}'_M$ for $k_z = \pi$, which are invariant under the mirror-reflection operation up to an RL vector. From Eq. (20), we have

$$\tilde{U}(M_{xy})\tilde{H}(\mathbf{k}+\mathbf{k}_M)\tilde{U}(M_{xy})^\dagger = \tilde{H}(D(M_{xy})\mathbf{k}+\mathbf{k}_M), \quad (27)$$

$$\tilde{U}_{\mathbf{k}'_M}(M_{xy})\tilde{H}(\mathbf{k}+\mathbf{k}'_M)\tilde{U}_{\mathbf{k}'_M}(M_{xy})^\dagger = \tilde{H}(D(M_{xy})\mathbf{k}+\mathbf{k}'_M), \quad (28)$$

where $\tilde{U}_{\mathbf{k}'_M}(M_{xy}) = V^\dagger_{-2\pi\hat{z}}\tilde{U}(M_{xy}) = V_{2\pi\hat{z}}\tilde{U}(M_{xy})$. It is found from Eqs. (21) and (25) that $V_{2\pi\hat{z}}$ gives rise to the commutation relation $[\mathcal{P},\tilde{U}(M_{xy})] = \{\mathcal{P},\tilde{U}_{\mathbf{k}'_M}(M_{xy})\} = 0$. Hence, the commutation relation between mirror reflection and inversion differs between the mirror-invariant planes at $\mathbf{k}_M$ and $\mathbf{k}'_M$. As shown below, this difference is crucial for protecting a line node at the BZ face.

To argue the topological stability of the line node, we employ the Clifford algebra extension method [14,15,83,84], which leads to the correct topological classification of the gapped systems. For gapless nodes, one can consider a sphere or a circle enclosing the gapless nodes in the momentum space, instead of the whole BZ. Since the Hamiltonian of the nodes is fully gapped on the sphere or the circle, the topological classification of gapped systems is also applicable to the gapless nodes. Following Ref. [15], we expand the BdG Hamiltonian around a line node, yielding a massless Dirac Hamiltonian,

$$\tilde{H}(\mathbf{k}+\mathbf{k}_M) \simeq v_1 k_z \gamma_1 + v_2 k_\parallel \gamma_2, \quad (29)$$

where $k_\parallel$ is momentum parallel to the mirror-invariant plane, and we neglect terms of order k^n ($n > 2$). The γ matrices satisfy $\gamma_1^2 = \gamma_2^2 = \mathbf{1}_m$ and $\{\gamma_1,\gamma_2\} = 1$. The same expansion is applied to the BdG Hamiltonian at $\mathbf{k}+\mathbf{k}'_M$ as well. For the Hamiltonian expanded around the line node, a symmetry operation is relevant only when it does not change the position of the line node. PHS, TRS, and IS transfer $\mathbf{k}_M$ to $-\mathbf{k}_M$ ($\mathbf{k}'_M$ to $-\mathbf{k}'_M$) in the BZ, only their combinations are meaningful. For this reason, we consider the combined symmetry operators $C\mathcal{P}$ and $C\mathcal{T}$, where $\mathcal{PT}$ is constructed from the combination of $C\mathcal{P}$ and $C\mathcal{T}$. Furthermore, the mirror-reflection operation (27) or (28) also does not change the position of the line node and is also relevant to the line node stability.

For the massless Dirac Hamiltonian, the gap opening at nodes is equivalent to the existence of a mass term. Hence, the line node can be stable if Eq. (29) does not have any mass term under the symmetry constraint. The Clifford algebra extension method allows us to count all of the possible mass terms in Dirac Hamiltonians and clarify relevant topological numbers. Mathematically, the space of mass terms is described by the classifying space, C_i ($i = 0,1$) and R_j ($j = 0,\ldots,7$), and the topological number is defined by the zeroth homotopy group of the classifying space, $\pi_0(C_i)$ and $\pi_0(R_j)$ (cf. Refs. [15,83] for more information). If the topological number is zero, a mass term creates a gap, i.e., the line node is unstable, whereas if the topological number is nonzero, such a mass term is forbidden by topology and symmetry. In preparation

for the calculation, we define a set of Clifford algebras. We have the complex Clifford algebra $Cl_n = \{l_1,\ldots,l_n\}$ with $\{l_i,l_j\} = \delta_{ij}$ when the Hamiltonian does not have any antiunitary symmetry, whereas we adopt the real Clifford algebra $Cl_{p,q} = \{e_1,\ldots,e_p;e_{p+1},\ldots,e_{p+q}\}$ when the Hamiltonian has antiunitary symmetry, where e_i is a Clifford algebra satisfying $\{e_i,e_j\} = 0$ ($i \neq j$), and $e_i^2 = -1$ ($1 \leqslant i \leqslant p$) and $e_i^2 = +1$ ($p+1 \leqslant i \leqslant q$). l_i and e_i are constructed from γ_i's and symmetry operators for the underlying Hamiltonian. In addition, in order to input the imaginary number i in the real Clifford algebra, we introduce a generator J ($J^2 = -1$), which anticommutes only with antiunitary operators.

For illustration purposes, we first examine the line node stability in odd-parity SCs without assuming mirror-reflection symmetry. Odd-parity superconductivity implies the anticommutation relation $\{C,\mathcal{P}\} = 0$ with $\mathcal{P}^2 = 1$. In the case of TRS breaking odd-parity SCs, we have γ_1, γ_2, J, and $C\mathcal{P}$. Adjusting the anticommutation relation between them, the set of Clifford algebra is constructed as $Cl_{2,2} = \{C\mathcal{P},JC\mathcal{P};\gamma_1,\gamma_2\}$, where $(C\mathcal{P})^2 = (JC\mathcal{P})^2 = -1$. According to Ref. [15], we calculate the Clifford algebra extension problem in terms of γ_2, leading to $Cl_{2,1} \to Cl_{2,2}$ and the classifying space R_7. Since $\pi(R_7) = 0$ [83], a line node is topologically unstable. On the other hand, in the case of time-reversal invariant odd-parity SCs, we need to add $C\mathcal{T}$ in the above set. Hence, the set of Clifford algebra becomes $Cl_{3,2} = \{C\mathcal{P},JC\mathcal{P},C\mathcal{T};\gamma_1,\gamma_2\}$, where $(C\mathcal{T})^2 = -1$. The Clifford algebra extension is $Cl_{3,1} \to Cl_{3,2}$, resulting in R_6 and $\pi_0(R_6) = 0$. Thus, a line node is topologically unstable as well. These results imply that an additional symmetry is necessary to stabilize a line node in odd-parity SCs with and without TRS. In what follows, we calculate the Clifford algebra extension problem in time-reversal invariant odd-parity SCs with mirror-reflection symmetry, and we compare it with the group-theoretical results.

First, consider a line node in the mirror-invariant plane at $k_z = 0$, in which the mirror-reflection operator satisfies $[\mathcal{P},\tilde{U}(M_{xy})] = [\mathcal{T},\tilde{U}(M_{xy})] = 0$. The commutation relation between $U(M_{xy})$ and the combined operators becomes

$$C\mathcal{P}\tilde{U}(M_{xy}) = \eta_{C,M}\tilde{U}(M_{xy})C\mathcal{P}, \quad (30a)$$

$$C\mathcal{T}\tilde{U}(M_{xy}) = \eta_{C,M}\tilde{U}(M_{xy})C\mathcal{T}. \quad (30b)$$

In the presence of mirror-reflection symmetry, we have γ_1, γ_2, J, $C\mathcal{P}$, $C\mathcal{T}$, and $\tilde{U}(M_{xy})$ as candidates of the Clifford algebra. Taking into account the sign of $\eta_{C,M}$, these algebras are packed in the set of Clifford algebras as $Cl_{3,3} = \{C\mathcal{P},JC\mathcal{P},C\mathcal{T};\gamma_1,\gamma_2,\gamma_1\tilde{U}(M_{xy})\}$ for $\eta_{C,M} = +1$ and $Cl_{3,2} \otimes Cl_{0,1} = \{C\mathcal{P},JC\mathcal{P},C\mathcal{T};\gamma_1,\gamma_2\} \otimes \{;JC\mathcal{T}\tilde{U}(M_{xy})\}$ for $\eta_{C,M} = -1$, where $JC\mathcal{T}\tilde{U}(M_{xy})$ with $[JC\mathcal{T}\tilde{U}(M_{xy})]^2 = +1$ commutes with the other Clifford algebras and does not affect the extension problem. Calculating the Clifford algebra extension problem in terms of γ_2, we obtain $\pi_0(R_7) = 0$ for $\eta_{M,C} = +1$ and $\pi_0(R_6) = 0$ for $\eta_{C,M} = -1$. Therefore, a line node is topologically unstable in both cases. As a result, mirror-reflection symmetry cannot stabilize a line node in time-reversal invariant odd-parity SCs. This result, together with the result without mirror-reflection symmetry, is the topological version of Blount's theorem [15].

Next, consider a line node at $k_z = \pi$. Taking into account the effect of $V_{2\pi\hat{z}}$, we replace $\tilde{U}(M_{xy})$ with $\tilde{U}_{\mathbf{k}'_M}(M_{xy})$. Then,

the nontrivial factor system changes the commutation relation between IS and the mirror-reflection symmetry operator, so we obtain

$$CP\tilde{U}_{k'_M}(M_{xy}) = -\eta_{C,M}\tilde{U}_{k'_M}(M_{xy})CP, \quad (31a)$$

$$CT\tilde{U}_{k'_M}(M_{xy}) = \eta_{C,M}\tilde{U}_{k'_M}(M_{xy})CT. \quad (31b)$$

Hence, the commutation relation with CP changes. For $\eta_{C,M} = +1$, the set of Clifford algebras is given by $Cl_{4,2} = \{CP, JCP, CT, J\gamma_1\tilde{U}_{k'_M}(M_{xy}); \gamma_1, \gamma_2\}$ and the Clifford algebra extension becomes $Cl_{4,1} \to Cl_{4,2}$, leading to R_5 and $\pi_0(R_5) = 0$. Thus, a line node is topologically unstable. On the other hand, for $\eta_{C,M} = -1$, the set of Clifford algebras is constructed as $Cl_{3,2} \otimes Cl_{1,0} = \{CP, JCP, CT; \gamma_1, \gamma_2\} \otimes \{CT\tilde{U}_{k'_M}(M_{xy}); \}$. Here, $CT\tilde{U}_{k'_M}(M_{xy})$ with $[CT\tilde{U}_{k'_M}(M_{xy})]^2 = -1$ commutes with the other Clifford algebras and thus plays a role of the complex factor. Then, the set of Clifford algebras changes to the complex case [83], $Cl_{3,2} \otimes Cl_{1,0} \simeq Cl_5$. The Clifford algebra extension becomes $Cl_4 \to Cl_5$, leading to C_0 and $\pi_0(C_0) = \mathbb{Z}$. As a result, a line node at the BZ face can be topologically stable for mirror-odd (and odd-parity) pairing states. This result reproduces Norman's result, in spite of the fact that the argument is completely different. In Sec. III C, based on a recently proposed model of UPt₃, we show that the line node at the BZ face actually has a nontrivial topological number.

In the topological approach, we can generalize the above result to TRS breaking odd-parity SCs. The Clifford algebra extension is given by removing CT from the set of Clifford algebras. On the mirror-invariant plane at $k_z = 0$, the set of Clifford algebras is constructed as $Cl_{2,3} = \{CP, JCP; \gamma_1, \gamma_2, \gamma_1\tilde{U}(M_{xy})\}$ for $\eta_{C,M} = +1$ and $Cl_{3,2} = \{CP, JCP, J\gamma_1\tilde{U}(M_{xy}); \gamma_1, \gamma_2\}$ for $\eta_{C,M} = -1$. From the Clifford algebra extension problem in terms of γ_2, we obtain $\pi_0(R_0) = \mathbb{Z}$ for $\eta_{C,M} = +1$ and $\pi_0(R_6) = 0$ for $\eta_{C,M} = -1$. Thus, a line node at $k_z = 0$ is topologically stable when $\eta_{C,M} = +1$. On the other hand, on the mirror-invariant plane at $k_z = \pi$, the mirror-reflection operator is $\tilde{U}_{k'_M}(M_{xy})$, which obeys $CP\tilde{U}_{k'_M}(M_{xy}) = -\eta_{C,M}\tilde{U}_{k'_M}(M_{xy})CP$. Since the only difference between the above commutation relation and Eq. (30a) is a minus sign in $\eta_{C,M}$, we can obtain the topological structure at $k_z = \pi$ from that at $k_z = 0$ by changing the sign of $\eta_{C,M}$. Therefore, a line node at $k_z = \pi$ can be topologically stable when $\eta_{C,M} = -1$. In conclusion, there exists a topologically stable line node at $k_z = \pi$ regardless of TRS when the Cooper pair is odd under the mirror-reflection operation. It should be noted here that the possibility of stable line nodes at $k_z = 0$ in the above is overlooked in the group-theoretical method: Although only the original mirror-reflection symmetry exists at $k_z = 0$, there may be a stable line node. An example of the stable line node at $k_z = 0$ has been given in Appendix 3 in Ref. [15]. This result suggests that the topological approach is more powerful than the group-theoretical method.

Finally, we present the topological number of nodal rings in the mirror-invariant plane. (Generally, a line node on a plane forms a nodal ring.) As shown in the above, a nodal ring is characterized by an integer. The topological number on the

mirror-invariant plane is defined by

$$Q_\lambda \equiv n^>_{\text{occ},\lambda} - n^<_{\text{occ},\lambda} \in \mathbb{Z}, \quad (32)$$

where $n^>_{\text{occ},\lambda}$ ($n^<_{\text{occ},\lambda}$) is the number of occupied states with mirror-reflection eigenvalue λ outside (inside) a nodal ring. We readily verify that Q_λ is nontrivial only if $[CP, \tilde{U}(M_{xy})] = 0$ in TRS breaking odd-parity SCs or $[CP, \tilde{U}(M_{xy})] = \{CT, \tilde{U}(M_{xy})\} = 0$ in time-reversal invariant odd-parity SCs. This is because CP symmetry leads to $n^{>(<)}_{\text{occ},\lambda} = N_\lambda - n^{>(<)}_{\text{occ},\lambda}$ when $\{CP, \tilde{U}(M_{xy})\} = 0$. ($N_\lambda$ is the total number of eigenstates with λ and does not depend on k.) This means that $n^>_{\text{occ},\lambda} = n^<_{\text{occ},\lambda} = \frac{N_\lambda}{2}$, leading to $Q_\lambda = 0$. In the same way, CT symmetry leads to $Q_\lambda = 0$ when $[CT, \tilde{U}(M_{xy})] = 0$. Applying Q_λ to time-reversal invariant odd-parity SCs, the commutation relations at $k_z = 0$ are given by Eqs. (30a) and (30b), so Q_λ is always trivial. On the other hand, the commutation relations at $k_z = \pi$ are given by Eqs. (31a) and (31b). That is, Q_λ is nontrivial only when $\eta_{C,M} = -1$. Accordingly, Q_λ coincides with the above argument. Note that the absence of SOC leads to $Q_\lambda = 0$ even when $\eta_{C,M} = -1$. (See Appendix C.)

C. Application to superconducting state in UPt₃

We demonstrate the nonsymmorphic-symmetry-protected line node concretely in the tight-binding model for the E_{2u}-superconducting state of UPt₃ B-phase [85]. The BdG Hamiltonian is given by

$$\mathcal{E}_{mm'ss'}(k) = \xi(k)\delta_{m,m'}\delta_{s,s'} + a_{mm'}(k)\delta_{s,s'} + (-1)^{3-m}\alpha g(k) \cdot s_{ss'}\delta_{m,m'}, \quad (33a)$$

$$\Delta_{mm'ss'}(k) = \frac{\Delta}{\sqrt{2}}[\Gamma^a_{mm'ss'}(k) + i\Gamma^b_{mm'ss'}(k)], \quad (33b)$$

where $m = 1,2$ and $s = \uparrow, \downarrow$ are indexes of sublattice and spin, respectively. $\Gamma^a(k)$ and $\Gamma^b(k)$ represent the order parameter in the superconducting state of a two-dimensional irreducible representation E_{2u}. Taking into account the local violation of inversion symmetry, which gives rise to the sublattice-dependent Zeeman-type SOC [86,87], each term in the normal Hamiltonian is described as

$$\xi(k) = 2t\sum_{i=1}^{3}\cos k_\parallel \cdot e_i + 2t_z\cos k_z - \mu, \quad (34a)$$

$$a_{11} = a_{22} = 0, \quad (34b)$$

$$a_{12}(k) = a_{21}(k)^* = 2t'\cos\frac{k_z}{2}\sum_{i=1}^{3}e^{ik_\parallel \cdot r_i}, \quad (34c)$$

$$g(k) = \hat{z}\sum_{i=1}^{3}\sin k_\parallel \cdot e_i, \quad (34d)$$

with $k_\parallel = (k_x, k_y, 0)$. As in Fig. 2(a), $e_1 = (1,0,0)$, $e_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, and $e_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$ are unit vectors in the two-dimensional triangular lattice, and $r_1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2})$, $r_2 = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2})$, and $r_3 = (0, -\frac{1}{\sqrt{3}}, \frac{1}{2})$ are nonprimitive lattice vectors in two dimensions. The symmetry-allowed gap
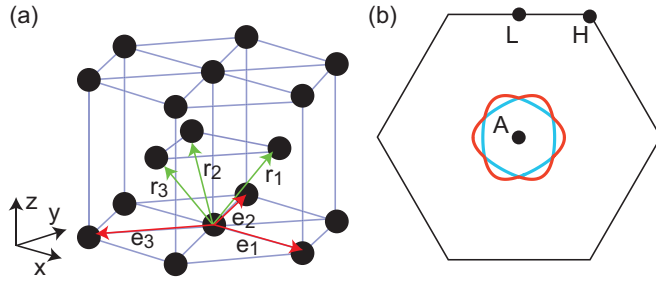
FIG. 2. (a) Crystal structure of UPt$_3$, where black points indicate the position of U ions [85]. (b) Fermi surfaces at $k_z = \pi$ from the tight-binding Hamiltonian (33a) with the parameters $(t, t_z, t', \alpha, \mu) = (1, -4, 1, 2, 12)$. The red and blue lines represent doubly degenerate Fermi surfaces around the A point. If $\alpha = 0$, these Fermi surfaces are overlapped and cause fourfold degeneracy.

function is constructed from

$$\Gamma^a(\boldsymbol{k}) = [\delta\{p_x(\boldsymbol{k})s_x - p_y(\boldsymbol{k})s_y\}\sigma_0$$
$$+ f_{(x^2-y^2)z}(\boldsymbol{k})s_z\sigma_x - d_{yz}(\boldsymbol{k})s_z\sigma_y]is_y, \quad (35a)$$

$$\Gamma^b(\boldsymbol{k}) = [\delta\{p_x(\boldsymbol{k})s_x + p_y(\boldsymbol{k})s_y\}\sigma_0$$
$$+ f_{xyz}(\boldsymbol{k})s_z\sigma_x - d_{xz}(\boldsymbol{k})s_z\sigma_y]is_y, \quad (35b)$$

where $s_\alpha = (1_2, s)$ and $\sigma_\alpha = (1_2, \sigma)$ are the identity and Pauli matrices in the spin and sublattice spaces. The p-wave, f-wave, and d-wave components of the basis function are

$$p_x(\boldsymbol{k}) = \sum_{i=1}^{3} e_i^x \sin \boldsymbol{k}_\parallel \cdot \boldsymbol{e}_i, \quad (36a)$$

$$p_y(\boldsymbol{k}) = \sum_{i=1}^{3} e_i^y \sin \boldsymbol{k}_\parallel \cdot \boldsymbol{e}_i, \quad (36b)$$

$$f_{(x^2-y^2)z}(\boldsymbol{k}) = -\sin\frac{k_z}{2}\text{Re}\left[\frac{1}{2}e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_1} + \frac{1}{2}e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_2} - e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_3}\right], \quad (36c)$$

$$f_{xyz}(\boldsymbol{k}) = -\frac{\sqrt{3}}{2}\sin\frac{k_z}{2}\text{Re}[e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_1} - e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_2}], \quad (36d)$$

$$d_{yz}(\boldsymbol{k}) = -\sin\frac{k_z}{2}\text{Im}\left[\frac{1}{2}e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_1} + \frac{1}{2}e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_2} - e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_3}\right], \quad (36e)$$

$$d_{xz}(\boldsymbol{k}) = -\frac{\sqrt{3}}{2}\sin\frac{k_z}{2}\text{Im}[e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_1} - e^{i\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_2}]. \quad (36f)$$

As shown in Ref. [85], there exist six stable nodal rings at the BZ face when parameters are set in such a way that the Fermi surfaces appear around the A point, and the effect of the Zeeman-type SOC is included. [See Fig. 2(b).] We point out in the following that the obtained nodal rings have nontrivial topological numbers due to nonsymmorphic symmetry.

The symmetries relevant to the stability of the nodal rings are $C = \sigma_0 s_0 \tau_x K$, $\mathcal{P} = \sigma_x s_0 \tau_z$, and $\tilde{U}(M_{xy}) = i\sigma_0 s_z \tau_0$, where $\tau_\alpha = (1_2, \tau)$ describes the identity and Pauli matrices in the Nambu space. At this point, the symmetry operators satisfy $\{C\mathcal{P}, \tilde{U}(M_{xy})\} = 0$, namely, there is no symmetry-protected line node since $\mathcal{Q}_\lambda = 0$. Following UPt$_3$ having nonsymmor-
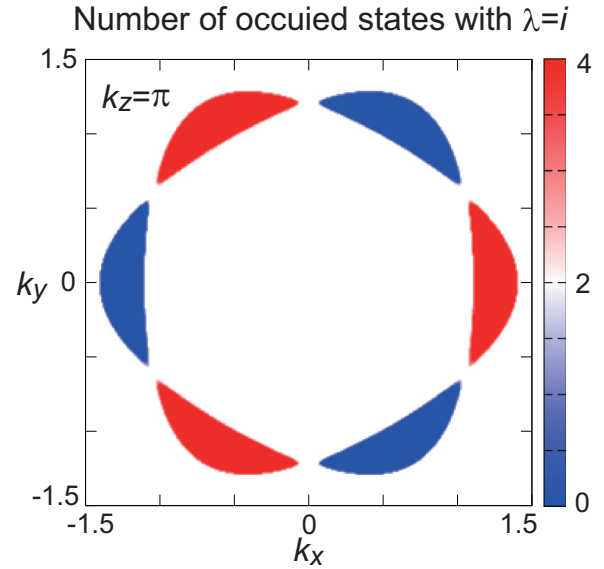

FIG. 3. Number of occupied states with mirror-reflection eigen-value $+i$ as a function of (k_x, k_y), which is calculated by numerically diagonalizing the BdG Hamiltonian (33a) and (33b) with the parameters $(t, t_z, t', \alpha, \mu, \Delta, \delta) = (1, -4, 1, 2, 12, 0.1, 0.04)$. The red and blue regions indicate the inside of nodal rings, indicating that they are protected by the mirror topological number $\mathcal{Q}_\lambda = \pm 2$.

phic space group $P6_3/mmc$, the BdG Hamiltonian satisfies Eq. (7) with the nonprimitive lattice vector $\boldsymbol{\tau} = (0, -\frac{1}{\sqrt{3}}, \frac{1}{2})$. It follows that

$$\mathcal{E}(\boldsymbol{k} + \boldsymbol{G}) = V_{\boldsymbol{G}}\mathcal{E}(\boldsymbol{k})V_{\boldsymbol{G}}^\dagger, \quad (37)$$

$$\Delta(\boldsymbol{k} + \boldsymbol{G}) = V_{\boldsymbol{G}}\Delta(\boldsymbol{k})V_{\boldsymbol{G}}^\dagger, \quad (38)$$

with $\boldsymbol{G} = m_1\boldsymbol{b}_1 + m_2\boldsymbol{b}_2 + m_3\boldsymbol{b}_3$ ($m_1, m_2, m_3 \in \mathbb{Z}$) and

$$V_{\boldsymbol{G}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\boldsymbol{G}\cdot\boldsymbol{\tau}} \end{pmatrix}_\sigma \otimes s_0. \quad (39)$$

Here, $\boldsymbol{b}_1 = \frac{4\pi}{\sqrt{3}}(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$, $\boldsymbol{b}_2 = \frac{4\pi}{\sqrt{3}}(0, 1, 0)$, and $\boldsymbol{b}_3 = 2\pi(0, 0, 1)$ are the RL vectors. Therefore, using Eq. (28), the mirror-reflection operator at $k_z = \pi$ is

$$\tilde{U}_{\boldsymbol{k}_M'}(M_{xy}) = \tilde{V}_{2\pi\hat{z}}\tilde{U}(M_{xy}) = i\sigma_z s_z \tau_0. \quad (40)$$

Thus, the mirror-reflection operator satisfies $[C\mathcal{P}, \tilde{U}_{\boldsymbol{k}_M'}(M_{xy})] = 0$ at $k_z = \pi$, which is attributed to the nontrivial factor system between P and $\{U(M_{xy})|\frac{1}{2}\hat{z}\}$. To verify that the nodal rings are protected by the topological number, we evaluate $\mathcal{Q}_{+i}$ in the mirror-invariant plane at $k_z = \pi$. In Fig. 3, we show the number of occupied states with the mirror-reflection eigenvalue $+i$ by numerically diagonalizing the BdG Hamiltonian. The red and blue regions indicate the inside of the nodal rings. By calculating $\mathcal{Q}_{+i}$ according to the definition (32), we find that the red and blue nodal rings have $\mathcal{Q}_{+i} = -2$ and $+2$, respectively. That is, the six nodal rings are topologically protected with help from nonsymmorphic symmetry. It must be noted that the nodal rings disappear when $\alpha = 0$, implying that the Zeeman-type SOC plays an important role in protecting the nodal rings.

More generally, the heavy fermion superconductor UPt$_3$ exhibits multiple superconducting phases in the field-temperature phase [6]. The order parameter that covers the enter range of superconducting phases is described by $\Delta(\boldsymbol{k}) = \Delta[\eta_a \Gamma^a(\boldsymbol{k}) + \eta_b \Gamma^b(\boldsymbol{k})]$ with $(\eta_a, \eta_b) = (1, i\eta)/\sqrt{1 + \eta^2}$ ($\eta \in \mathbb{R}$). By tuning a real parameter η, we obtain the A phase ($\eta = \infty$), the B phase ($0 < \eta < \infty$), and the C phase ($\eta = 0$). In contrast to the B phase, the A and C phases recover TRS. Nevertheless, as discussed in Sec. III B, a nonsymmorphic-symmetry-protected line node is stable independent of TRS. Thus, the nodal rings are robust for any η, as long as the mirror reflection symmetry (40) and the Zeeman-type SOC exist.

IV. GENERALIZATION TO OTHER SYSTEMS

Up to this point, we have discussed nonsymmorphic-symmetry-protected line nodes bearing Norman's discussion in mind. An advantage of the topological approach is that one can generalize the argument to other nodes and other symmetry classes systematically.

For this purpose, we consider a generic node at $\boldsymbol{k}_0$ described by the massless Dirac Hamiltonian,

$$\tilde{H}(\boldsymbol{k} + \boldsymbol{k}_0) \simeq \sum_{i=1}^{p+1} v_i k_i \gamma_i, \tag{41}$$

where k_i's are momentum on a p-dimensional sphere enclosing the node, the γ matrices satisfy $\{\gamma_i, \gamma_j\} = 0$ ($i \neq j$), and p specifies the transverse dimension of nodes, which we call the codimension of nodes (Codim. for short). For example, $p = 0$, $p = 1$, and $p = 2$ represent a gapless superconductor (a surface node), a line node, and a point node in three-dimensional momentum space.

For symmetries protecting the node, we consider PHS, TRS, IS, and the mirror reflection symmetry M. (Without loss of generality, we assume $M^2 = -1$.) In a manner similar to Sec. III B, the topological stability of the node depends on the commutation relation between $C\mathcal{P}$, $C\mathcal{T}$, and $\tilde{M}$, where $\tilde{M} = \text{diag}[M, \eta_{C,M} M^*]$ is the mirror-reflection operator in the Nambu space. Introducing $\eta_{S,M}$ to specify the commutation relation between $S(= C, \mathcal{P}, \mathcal{T})$ and $\tilde{M}$ as $S\tilde{M} = \eta_{S,M} \tilde{M} S$, the commutation relations between $C\mathcal{P}$, $C\mathcal{T}$, and $\tilde{M}$ are given by $\eta_{CP,M} = \eta_{C,M} \eta_{P,M}$ and $\eta_{CT,M} = \eta_{C,M} \eta_{T,M}$. We label $\tilde{M}$ with these commutation relations as $\tilde{M}^{\eta_{CP,M}}$ ($\tilde{M}^{\eta_{CP,M}, \eta_{CT,M}}$) for the time-reversal breaking (invariant) case. In addition, the parity of the gap function takes either even-parity ($[C, \mathcal{P}] = 0$) or odd-parity ($\{C, \mathcal{P}\} = 0$) for each mirror symmetry class. Solving the Clifford algebra extension problem in terms of γ_{p+1} systematically [15], we obtain the corresponding classifying space and the topological number for each Codim., as shown in Table I.

Finally, we take into account the influence of the factor system on the topological classification. Toward that end, we define a nonsymmorphic mirror operator as the combination of spatial inversion P and twofold screw symmetries $\{C_{2x_\perp} | \boldsymbol{\tau}_\perp\}$, $\{P C_{2x_\perp} | \boldsymbol{\tau}_\perp\} \equiv \{M | \boldsymbol{\tau}_\perp\}$, where $C_{2x_\perp}$ is a twofold rotation operator in terms of the $x_\perp$ axis, and $\boldsymbol{\tau}_\perp$ is a nonprimitive lattice vector along the $x_\perp$ axis ($2\boldsymbol{\tau}_\perp$ is a primitive lattice vector). As discussed in the previous section, only the nontrivial factor system changes the commutation relation between $\tilde{M}$ and

TABLE I. Classification of nodes under PHS, TRS, IS, and mirror-reflection symmetry [15]. The first and second columns represent the symmetry operations and the parity of the gap function. The third column lists the commutation relation between $C\mathcal{P}$, $C\mathcal{T}$, and $\tilde{M}$, where $\tilde{M}^{\eta_{CP,M}}$ ($\tilde{M}^{\eta_{CP,M}, \eta_{CT,M}}$) represents the time-reversal breaking (invariant) case. The fourth column shows the corresponding classifying spaces. The following columns represent the topological numbers for each codimension.

Symmetry	Parity	Mirror	Classifying space	$p=0$	$p=1$	$p=2$
$\{C\mathcal{P}, \tilde{M}\}$	Even	$\tilde{M}^+$	R_{p+3}	0	$2\mathbb{Z}$	0
		$\tilde{M}^-$	R_{p+1}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
	Odd	$\tilde{M}^+$	R_{p-1}	0	$\mathbb{Z}$	$\mathbb{Z}_2$
		$\tilde{M}^-$	R_{p-3}	0	0	0
$\{C\mathcal{P}, C\mathcal{T}, \tilde{M}\}$	Even	$\tilde{M}^{++}$	R_{p+4}	$2\mathbb{Z}$	0	0
		$\tilde{M}^{-+}$	R_{p+2}	$\mathbb{Z}_2$	0	$2\mathbb{Z}$
		$\tilde{M}^{+-}$	R_{p+3}	0	$2\mathbb{Z}$	0
		$\tilde{M}^{--}$	C_{p+1}	0	$\mathbb{Z}$	0
	Odd	$\tilde{M}^{++}$	R_{p-2}	0	0	$\mathbb{Z}$
		$\tilde{M}^{-+}$	R_{p-4}	$2\mathbb{Z}$	0	0
		$\tilde{M}^{+-}$	C_{p+3}	0	$\mathbb{Z}$	0
		$\tilde{M}^{--}$	R_{p-3}	0	0	0

$C\mathcal{P}$, which causes the change of mirror-reflection symmetry between the mirror-invariant plane at $k_\perp = 0$ and at $k_\perp = \pi$:

Symmetry	$k_\perp = 0$		$k_\perp = \pi$	
$\{C\mathcal{P}, \tilde{M}\}$	$\tilde{M}^{\eta_{CP,M}}$	$\Longrightarrow$	$\tilde{M}^{-\eta_{CP,M}}$	(42)
$\{C\mathcal{P}, C\mathcal{T}, \tilde{M}\}$	$\tilde{M}^{\eta_{CP,M}, \eta_{CT,M}}$	$\Longrightarrow$	$\tilde{M}^{-\eta_{CP,M}, \eta_{CT,M}}$	

Comparing Table I with Eq. (42), a nonsymmorphic-symmetry-protected line node in odd-parity SCs with and without TRS is classified by

Parity	Codim.	$k_\perp = 0$		$k_\perp = \pi$	
Odd	$p = 1$	$\tilde{M}^-$	$\Longrightarrow$	$\tilde{M}^+$	(43)
Odd	$p = 1$	$\tilde{M}^{--}$	$\Longrightarrow$	$\tilde{M}^{+-}$	

A line node in both cases is protected by the $\mathbb{Z}$ topological number on the BZ face and is characterized by $\mathcal{Q}_\lambda$. The E_{2u} superconducting state of UPt$_3$ B-phase belongs to the first line in Eq. (43). Furthermore, provided that $[M, T] = [M, P] = 0$ with spin-singlet or spin-triplet SCs in mind, we find two types of nonsymmorphic-symmetry-protected point nodes in Table I as follows:

Parity	Codim.	$k_\perp = 0$		$k_\perp = \pi$	
Odd	$p = 2$	$\tilde{M}^-$	$\Longrightarrow$	$\tilde{M}^+$	(44)
Even	$p = 2$	$\tilde{M}^{++}$	$\Longrightarrow$	$\tilde{M}^{-+}$	

On the BZ face, a point node in TRS breaking odd-parity SCs is protected by the $\mathbb{Z}_2$ topological number, whereas one in time-reversal invariant even-parity SCs is protected by the $2\mathbb{Z}$ topological number. Similarly to the line node, the nonsymmorphic symmetry plays a crucial role in protecting these point nodes, because a point node is topologically unstable in the mirror-invariant plane at $k_\perp = 0$.

V. SUMMARY

We argued the topological stability of nodes in nonsymmorphic SCs, taking into account the influence of the factor system

on the topological classification. The important point is that nonsymmorphic symmetry leads to a nontrivial factor system at BZ faces, which is reflected as the change of the commutation relation between spatial-inversion and mirror-reflection operators in some cases. Adding a nontrivial factor system in the topological classification allows us to deal with a node in nonsymmorphic SCs in the same manner as symmorphic SCs. Although we focused on the order-2 symmetries in this paper, Eqs. (13) and (22) are generally satisfied for all space-group operations, but we need a topological method beyond the Clifford algebra extension method, which is outside the scope of this paper.

In the topological approach, we found a nonsymmorphic-symmetry-protected line (43) and point nodes (44), which can be considered to be gapless superconducting states analogous to nonsymmorphic-symmetry-protected topological semimetals. Therefore, our findings will enlarge the category of topological gapless phases and facilitate our understanding of gapless superconductors with a nonsymmorphic crystal structure such as UPt$_3$.

We comment briefly on the bulk-boundary correspondence for non-symmetry-protected line nodes. Usually, a line node induced surface zero-energy flat band is robust as long as the line node is protected by nonspatial symmetry CT [20,27]. In contrast, a crystal symmetry supported surface zero-energy flat band accidentally occurs, so one vanishes by adding crystal symmetry-breaking perturbations such as the surface Rashba SOC [28]. In addition, making a surface parallel to a line node may break nonsymmorphic symmetry. For this reason, we expect that the surface flat band induced by a non-symmetry-protected line node may be unstable unless other mechanisms protect it.

APPENDIX A: GROUP OPERATION ON $c^\dagger_{k,\alpha}$

Here, we define the group operation on $c^\dagger_{k,\alpha}$. For an element $\{g|\tau\}$ of a space group G, the group operation is defined by

$$\{g|\tau\}c^\dagger_{k,\alpha}\{g|\tau\}^{-1} = \frac{1}{\sqrt{N}}\sum_R e^{ik\cdot(R+r_\alpha)}c^\dagger_\beta$$
$$\times [D(g)R + \Delta_{\beta\alpha} + r_\beta]U_{\beta\alpha}(g), \quad \text{(A1)}$$

with $\Delta_{\beta\alpha} = D(g)r_\alpha - r_\beta + \tau$. If the system is invariant under G, there exists a BL vector R' such that $R' = D(g)R + \Delta_{\beta\alpha}$ for an arbitrary $\{g|\tau\} \in G$. Then Eq. (A1) reduces to

$$\{g|\tau\}c^\dagger_{k,\alpha}\{g|\tau\}^{-1} = e^{-iD(g)k\cdot\tau}c^\dagger_{D(g)k,\beta}U_{\beta\alpha}(g), \quad \text{(A2)}$$

where we use $k \cdot r = D(g)k \cdot D(g)r$. A similar definition of the group operation is given in Ref. [75].

APPENDIX B: DERIVATION OF EQ. (21)

The matrix element of the combination of $\tilde{U}_{k_0}(g_1)$ and $\tilde{U}_{k_0}(g_2)$ is described by

$$[\tilde{U}_{k_0}(g_1)\tilde{U}_{k_0}(g_2)]_{\alpha\gamma} = \sum_\beta e^{i\theta(g_1,g_2)}\tilde{U}(g_1)_{\alpha\beta}\tilde{U}(g_2)_{\beta\gamma}, \quad \text{(B1)}$$

with $\theta(g_1,g_2) = [D(g_1)k_0 - k_0] \cdot r_\alpha + [D(g_2)k_0 - k_0] \cdot r_\beta$. By the symmetry of the Löwdin orbitals, if $U(g_1)_{\alpha\beta} \neq 0$, there exists a BL vector R' for the inverse element of $\{g_1|\tau_1\}$ such that $D(g_1)^{-1}r_\alpha - r_\beta = D(g_1)^{-1}\tau_1 + R_{g_1}$, where $R_{g_1} := R' - D(g_1)^{-1}R$ is the BL vector. Using this property, $\theta(g_1,g_2)$ is rewritten as

$$\theta(g_1,g_2) = k_0 \cdot \{D(g_1)^{-1}\tau_1 - D(g_2)^{-1}D(g_1)^{-1}\tau_1$$
$$+ D(g_2)^{-1}D(g_1)^{-1}r_\alpha - r_\alpha + R_{g_1} - D(g_2)^{-1}R_{g_1}\}. \quad \text{(B2)}$$

Substituting Eq. (B2) into Eq. (B1), we obtain

$$\tilde{U}_{k_0}(g_1)\tilde{U}_{k_0}(g_2) = \omega^{k_0}_{g_1,g_2}\tilde{U}_{k_0}(g_1g_2), \quad \text{(B3)}$$

where the third and fourth terms in Eq. (B2) become

$$e^{ik_0\cdot\{D(g_2)^{-1}D(g_1)^{-1}r_\alpha - r_\alpha\}}[\tilde{U}(g_1)\tilde{U}(g_2)]_{\alpha\gamma}$$
$$= e^{i\{D(g_1g_2)k_0 - k_0\}\cdot r_\alpha}\tilde{U}(g_1g_2)_{\alpha\gamma}$$
$$= \tilde{U}_{k_0}(g_1g_2)_{\alpha\gamma}, \quad \text{(B4)}$$

and the fifth and sixth terms in Eq. (B2) vanish such that

$$e^{ik_0[R_{g_1} - D(g_2)^{-1}R_{g_1}]} = e^{-i[D(g_2)k_0 - k_0]\cdot R_{g_1}} = 1. \quad \text{(B5)}$$

APPENDIX C: VANISHING OF THE MIRROR TOPOLOGICAL NUMBER IN THE ABSENCE OF SOC

As shown in Sec. III A, a line node can be unstable in the absence of SOC even when mirror-odd Cooper pairs are at the BZ face. Here, we prove this statement from a topological point of view and show that the instability of a line node occurs irrespective of TRS. We start from the condition that $\{P, U(M_{xy})\} = \{C, \tilde{U}(M_{xy})\} = 0$ and SOC is absent. The absence of SOC in the normal Hamiltonian allows spin-rotational symmetry, $[e^{i\theta n\cdot S}, H(k)] = 0$, where $S = \frac{1}{2}(s_x, s_y, s_z)$ are the generators of spin rotation, and $e^{i\theta n\cdot S}$ represents the spin rotation about an n axis within $0 \leqslant \theta < 2\pi$. Without loss of generality, we can choose $\theta = \pi$ and $n \parallel \hat{x}$. The spin-rotation operator anticommutes with the mirror-reflection operator, $\{e^{i\pi S_x}, U(M_{xy})\} = 0$. Then, the combination of P and $e^{i\pi S_x}$ satisfies $[Pe^{i\pi S_x}, U(M_{xy})] = 0$, leading to $n_{\text{occ},\lambda}(k_\parallel) = n_{\text{occ},\lambda}(-k_\parallel)$ for any $k_\parallel$ and λ, where $k_\parallel$ is momentum on the mirror-invariant plane. [$n_{\text{occ},\lambda}(k_\parallel)$ is the number of occupied states with λ at $k_\parallel$]. Since $\{C, \tilde{U}(M_{xy})\} = 0$, PHS leads to $n_{\text{occ},\lambda}(k_\parallel) = N_\lambda - n_{\text{occ},\lambda}(-k_\parallel) = N_\lambda - n_{\text{occ},\lambda}(k_\parallel)$, resulting in $n_{\text{occ},\lambda}(k_\parallel) = \frac{N_\lambda}{2}$ for any $k_\parallel$. That is, $\mathcal{Q}_\lambda = 0$. The same argument is applicable to time-reversal invariant odd-parity SCs. In this case, an inversion-symmetric Fermi surface with fourfold degeneracy occurs on the mirror-invariant plane. As a result, the presence of SOC is of significant importance in stabilizing a nodal ring.

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