Public Investment, the Rate of Return, and Optimal Fiscal Policy in a Stochastically Growing Economy

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Abstract

This study examines the optimal fiscal policy in a stochastic endogenous growth model with private and public capital. The government is willing to actualize a socially optimal equilibrium using a lump-sum tax and government debt linked to public investments, subject to the budget constraint under the golden rule of public finance. A socially optimal fiscal policy states that a deterministic rate of return on government bonds sets the marginal product of public capital. Moreover, public investments optimally adjust the ratio of private capital to public capital to equate the rates of return on such capital. The presence of stochastic disturbances results in a disparity between the optimal marginal products of the two types of capital, as reported in previous empirical studies. This disparity significantly affects the socially optimal growth rate in response to investment risk.

Keywords: Public capital; Optimal fiscal policy; Economic growth
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1. Introduction

The effectiveness of government investments and their optimization has been a long-lasting subject of both theoretical and empirical analysis in public economics. The outstanding study by Arrow and Kurz (1970) derived an optimal fiscal policy in a general equilibrium model with private and public capital and provided an application of the Pontryagin maximum principle to policy analysis. Subsequently, many theoretical studies have analyzed the optimal rule for public investments (e.g., Pestieau, 1974; Ogura and Yohe, 1977; Okuno and Yakita, 1981).[1]

In the literature on endogenous growth, Futagami et al. (1993) developed an endogenous growth model through the accumulation of private and public capital.[2] They introduced public capital as public input, enabling a linearly homogenous production technology with respect to reproducible capital inputs. Within this deterministic endogenous growth model with private and public capital, some studies have investigated the optimal fiscal policy (Turnovsky, 1997; Gómez, 2004; Tamai, 2008; Agénor, 2009).[3] Although with minor differences, these and prior studies have derived the equalization of the marginal products of private and public capital.

The marginal product of public capital that indicates the rate of return on public investments is a major empirical concern. Mera (1973) used Japanese regional data to present a pioneering study on the productivity of public infrastructure. Thereafter, Aschauer (1989) employed the data of the U.S. to estimate the output elasticity of public capital. These two leading studies showed that public capital is positively associated with output. Most subsequent studies also have provided empirical evidence for the positive growth effect of public capital.[4]

An estimation of the output elasticity of public capital enables us to calculate the marginal product of public capital, which is related to the optimal rule for public investments. Regarding this point, recent empirical studies reported that the marginal products of two types of capital are not equalized (e.g., Lighthart and Suárez, 2011; Bom and Ligthart, 2014; Gupta et al., 2014). Specifically, Lighthart and Suárez (2011) implied a marginal product of public capital of 27.5%, which is substantially higher than the marginal product of private capital that is reflected in the long-term real rate of interest.[5]

More recently, Bom and Ligthart (2014) suggested that the marginal product of public capital of 16% is substantially above the marginal user cost of public capital of 14% (depreciation rate 10% plus long-term real interest rate 4%). Gupta et al. (2014) also showed that the marginal productivity of public capital is larger than that of private capital using data of developing countries. The estimated values of the rates of return differ among empirical studies because they use different data sources and empirical methods. However, all of these studies showed that the

[1] More recently, Coto-Martínez (2006) examines the macroeconomic effects of fiscal policy under imperfect competition.

[2] Futagami et al. (1993) extends the model of Barro (1990) by substituting the public capital stock for the productive government expenditure as the public input. Greiner (1998) examine the optimal fiscal policy in the Barro (1990) model. See Irmen and Kuehnel (2009) for the survey of extended models of Barro (1990), and Kneller et al. (1999) for the empirical investigation of Barro model.

[3] Agénor (2009) considers the optimal fiscal policy including optimal allocation rules of investment and maintenance expenditure in the extended model presented by Rioja (2003) and Kalaitzidakis and Kalyvitis (2004).

[4] See Lighthart and Suárez (2011), Pereira and Andraz (2013) and Bom and Ligthart (2014) for recent surveys of this literature.

[5] They assume the public capital-to-GDP ratio of 50% (the US in the early 2000s). In OECD countries, long-term real interest rates of sovereign debt have been less than 5% (Afonso and Rault 2010).

marginal product of public capital is larger than that of private capital.

The actual disparity between the marginal products of private and public capital suggests that a shortage of public capital or some disturbance factors may exist. Some prior studies have contributed to this issue. Ogura and Yohe (1977) derived the optimal conditions for government investments under the general setting of capital market imperfection. Alternatively, Okuno and Yakita (1981) also derived such optimal conditions by incorporating a different income distribution.[6] In major countries, public investments in infrastructure remain at a certain level even today (Figure 1) and, as previously stated, positively impact economic growth.

[Figure 1]

After the global financial crisis, the macroeconomic effects of public investment gathered attention again, and the importance of fiscal institutions and debt-financing of public investment has been recognized as it has an influence to the macroeconomic effects (IMF, 2014, Ch. 3). Within an endogenous growth framework and a plausible fiscal rule, this issue will indeed be more deeply discussed from theoretical viewpoints and through an analysis of optimal fiscal policy.

Uncertainty is most appropriate for the realistic modeling in this literature because future expectations are important in investment decisions.[7] Indeed, numerous preceding empirical studies implied that risk has a significant relation with the rate of return and economic growth (e.g., Kormendi and Meguire, 1985; Grier and Tullock, 1989; Ramey and Ramey, 1995; Barro, 1998; Bredin and Fountas, 2005; Imbs, 2007). In particular, Imbs (2007) found a positive relation between the investment rate and the volatility of the economic growth rate. Therefore, this study focuses on the presence of a stochastic disturbance, which is inherent in capital investments.

In connection with our focus, some studies examined the optimal fiscal policy and investigated the growth and welfare effects of fiscal policy within a balanced government budget (Turnovsky, 1999; Ott and Soretz, 2004; Wang and Hu, 2007; Tamai, 2013). These studies are based on the endogenous growth model of Barro (1990) that introduces productive government expenditures. Although Tamai (2014) developed a stochastic growth model with private and public capital, he focused on the relation between debt financing of public investments and economic growth. In his model, public investment is financed by distortionary tax and debt, and so optimal fiscal policy is out of the scope of his analysis.

The analysis of optimal fiscal policy and rates of return on the two types of capital under uncertainty are not yet underway. Therefore, this study clarifies the optimal rule for public investment under uncertainty and characterizes the relation among the rate of return, the growth rate, and risk. We extend the deterministic growth model of Futagami et al. (1993) into a stochastic growth model by incorporating the dynamic equations of private and public capital accumulation as stochastic differential equations.

This study elucidates the relation between the rate of return, economic growth, and risk under

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[6] Arrow and Lind (1970) is the prominent study of public investment, which is perfectly substitutable for private investment, under uncertainty. Fisher (1973) suggests that the Arrow and Lind theorem does not precisely apply to investments that produce the output of pure public goods. Ogura and Yohe (1977) investigates the socially optimal rate of return of government investment by incorporating the complementarity between private and public capital.

[7] Recently, Haddow et al (2013) and Bloom (2014) report that an increase in uncertainty was observed during the recent crises and the crisis raised a dispersion of future expectation. Therefore, an analysis on optimal government investment under uncertainty is needed.

an optimal fiscal policy. Regarding an optimal fiscal policy, we show that a plausible fiscal rule, *the golden rule of public finance*, enables actualizing the socially optimal equilibrium. Given the presence of a stochastic disturbance, the economic agent for deciding capital investments requires a risk premium to invest in riskier capital. Under an optimal fiscal policy, the risk premium equalizes the rates of return on private and public capital that are composed of the marginal products of two capitals and the risk premium. Then, the term of the risk premium reflects the disparity between the marginal products of private and public capital. An increase in risk expands or shrinks the gap between the marginal products according to the portfolio share.

In addition, we show that this gap affects the optimal growth rate in response to a risk increase. A risk increase not only increases the mean and variance of the optimal growth rate but also reduces them in proportion to the portfolio share and the gap between the marginal products. The mixed empirical evidence of the relation between risk and growth can be explained as an optimal policy response to an increase in risk. This study provides a natural extension of the deterministic endogenous growth model with private and public capital and generalizes the optimal fiscal policy and its results.

This study is organized as follows. Section 2 explains the basic setting of our mathematical model and derives the stochastic balanced growth equilibrium in a social optimum and a decentralized economy through the optimal fiscal policy. Section 3 provides an equilibrium analysis of the interaction among deep parameters and economic growth. Further, we characterize the socially optimal rates of return on private and public capital. Section 4 presents a discussion on the extensions of the basic model. Finally, Section 5 concludes this study.

2. The model

2.1. Social optimum

Consider a closed economy with single final good and two capital inputs. Time is continuous and is represented by t.[8] The population is normalized to unity and is constant over time. The final good is producible using private and public capital. Let dY denote the flow of output over instance $(t, t+dt)$, and the production function of the final good is $F(K,G)$, where F is twice continuously differentiable function satisfying constant returns to scale, $F_x = \partial F/\partial x > 0$, $F_{xx} = \partial^2 F/\partial x^2 < 0$ $(x = K, G)$, K denotes the stock of private capital and G represents the stock of public capital; $dY = F(K,G)dt$.

Following Arrow and Kurz (1970, Ch.4), we impose a further assumption on the properties of the production function:

$\lim_{x\to 0} F_x = +\infty$, $\lim_{x\to\infty} F_x = 0$, $\lim_{y\to 0} F_x \in \mathbb{R}_+$, and $\lim_{y\to\infty} F_x \in \mathbb{R}_+$ $(x = K, G; y = K, G; x \neq y)$.

Note that we have $F_{KK} - F_{KG} < 0$ and $F_{GG} - F_{GK} < 0$.[9]

Private and public capital are accumulated by making investments in each type of capital with stochastic disturbances. Formally, the dynamic capital accumulation equations are assumed to be

[8] All economic variables are the function with respect to t. For a simplification, we omit (t) from the notation.

[9] Since the production function is constant returns to scale, F_K and F_G are homogenous of degree zero. Therefore, $F_{KK}K + F_{KG}G = 0$ and $F_{GG}G + F_{GK}K = 0$. In the other words, $F_{KK}K = -F_{KG}G < 0$ and $F_{GG}G = -F_{GK}K < 0$; $F_{KG} > 0$ and $F_{GK} > 0$ (note $F_{KG} = F_{GK}$).

$dK = dI_K + \sigma_K K dz_K$ and $dG = dI_G + \sigma_G G dz_G$ where dI_x is the flow of investments in capital x; dz_x represents the Wiener process with a mean of zero and standard deviation of one; and coefficient $\sigma_x > 0$ denotes the standard deviation of the growth rate of private capital ($x = K, G$). Let λ be the correlation coefficient of dz_K and dz_G. Note that the coefficient λ satisfies $|\lambda| \leq 1$.[10]

The social planner allocates the output of the economy to consumption and investments in private and public capital. We assume that the investments are reversible. For the planner, the total wealth of the economy is the sum of private and public capital. Let $W \equiv K + G$, $s_K \equiv K/W$, and $s_G \equiv G/W$. The dynamic equation of wealth accumulation becomes

$$dW = [F(s_K, s_G)W - C]dt + [s_K \sigma_K dz_K + s_G \sigma_G dz_G]W. \tag{1}$$

Then, the optimization problem of the planner is

$$\max_{C, s_K, s_G} E \int_0^\infty \frac{C^{1-\theta}}{1-\theta} e^{-\rho t} dt \qquad \text{s.t. } s_K + s_G = 1 \text{ and (1).}$$

Solving the problem obtains the optimal ratio of consumption to wealth and the ratio of capital to wealth (See Appendix A):

$$C = \alpha W, \tag{2}$$

$$F_K - F_G = [(\sigma_K - \lambda \sigma_G) s_K \sigma_K - (\sigma_G - \lambda \sigma_K) s_G \sigma_G]\theta, \tag{3}$$

where

$$\alpha \equiv \frac{\rho}{\theta} + (\theta - 1)\left[\frac{F(s_K, s_G)}{\theta} - \frac{(s_K \sigma_K)^2 + 2\lambda s_K s_G \sigma_K \sigma_G + (s_G \sigma_G)^2}{2}\right].$$

These optimal conditions and the dynamic equation of wealth generate the socially optimal equilibrium.

Equation (2) represents an optimal consumption-wealth ratio. When $\theta = 1$, the second term of α vanishes; $\alpha = \rho$. Then, stochastic disturbances do not affect the optimal consumption-wealth ratio because the income and substitution effects of a change in the variances of the two types of capital are exactly offsetting. However, the net effect of a change in the variances of the two typed of capital is generally non-zero when $\theta \neq 1$.

Equation (3) shows the equalization of the risk-adjusted marginal product of private and public capital.[11] Indeed, equation (3) without stochastic fluctuations implies that the marginal products of the two capitals are equalized at the optimum (e.g., Arrow and Kurz, 1970; Turnovsky, 1997; Gómez, 2004; Tamai, 2008). Therefore, the magnitude relation between the marginal product of private capital and of public capital depends on the magnitude relation between the risk spread of private capital and of public capital. The properties of the production function and the deep parameters σ_K, σ_G, and λ are important to determining the sign of the right hand side of equation (3). We provide further analysis in the next section.

The optimal ratio of private capital to total wealth s_K^* or the optimal ratio of private capital to total wealth s_G^* is uniquely determined by equation (3) and $s_K + s_G = 1$. Hereafter, the variable with an asterisk represents the socially optimal balanced growth equilibrium value of the endogenous variables. Geometrical analysis easily indicates the existence of a unique optimal capital portfolio. Let $\Delta r(s_K)$ be the left hand side of equation (3), i.e., the difference between

[10] See Chang (2004, pp.90-91) for the detail.

[11] On condition that the marginal product of private capital equals arbitrary constant cut-off point, Ogura and Yohe (1977) derives a similar result that implies the market distortion, although our result is not derived from capital market distortion but the gap between the marginal products of capital in the first-best.

the deterministic parts of the rate of return.[12] Further, let $\psi(s_K)$ be the right hand side of equation (3), i.e., the difference between the volatility parts.[13] Depending on the sizes of the deep parameters, these two functions are illustrated in Figure 2a-2d. In each case, a unique intersection point exists.

[Figure 2a-2d]

Note that the socially planned economy is on the balanced growth path of socially optimal equilibrium. By the reversibility of investment, private and public capital jump to their optimal values at initial time (Turnovsky, 1997; Gómez, 2004): If the private (public) capital share at initial time is larger than its optimal level, an infinite rate of negative investment in private (public) capital can be executed. Therefore, the socially planned economy does not exhibit transitional dynamics. We will discuss the case of irreversible investments in Section 4.

Summarizing the above analysis, we have the following proposition (See Appendix A):

Proposition 1. *If θ is sufficiently large to assure a positive consumption-wealth ratio, then the socially optimal equilibrium is uniquely determined and is governed by equations (1)-(3).*

In the socially optimal equilibrium, equation (1) with the optimal capital portfolio leads to the following equilibrium growth rate:

$$\frac{dW}{W} = [F(s_K^*, s_G^*) - \alpha^*]dt + s_K^* \sigma_K dz_K + s_G^* \sigma_G dz_G. \tag{4}$$

Then, the expected value and variance of equation (4) are

$$\frac{E\left(\frac{dW}{W}\right)}{dt} = \frac{F(s_K^*, s_G^*) - \rho}{\theta} + (\theta - 1)\left[\frac{(s_K^* \sigma_K)^2 + 2\lambda s_K^* s_G^* \sigma_K \sigma_G + (s_G^* \sigma_G)^2}{2}\right] \equiv \gamma^*, \tag{5}$$

$$\frac{\operatorname{var}\left(\frac{dW}{W}\right)}{dt} = (s_K^* \sigma_K)^2 + 2\lambda s_K^* s_G^* \sigma_K \sigma_G + (s_G^* \sigma_G)^2 \equiv \omega^*. \tag{6}$$

Equation (4) provides the equilibrium growth path of this economy.[14] Equation (5) shows that the expected growth rate is affected by risk only through portfolio share changes if $\theta = 1$, and that the expected growth rate is positively (negatively) associated with the variances of growth in wealth if $\theta > 1$ ($\theta < 1$). This result is based on the consumption/saving choice under uncertainty; recalling equation (2). Equation (6) implies that the variance in the growth in wealth depends on not only the variances of the two capitals but also on the portfolio shares because the capital stock size is important for the investment risk that increases proportionally with capital stock.

2.2. The decentralized economy

To solve the dynamic system in a decentralized economy, some specifications of the fiscal rules

[12] The Δr function has the properties: $\Delta r(0) = +\infty$, $\Delta r(1) = -\infty$ and $\Delta r'(s_K) = F_{KK} + F_{GG} - 2F_{KG} < 0$.
[13] The ψ function has the properties: $\psi(0) = -(\sigma_G - \lambda\sigma_K)\sigma_G\theta$, $\psi(1) = (\sigma_K - \lambda\sigma_G)\sigma_K\theta$ and $\psi'(s_K) = [(\sigma_K - \sigma_G)^2 + 2(1-\lambda)\sigma_K\sigma_G]\theta > 0$. Furthermore, $\psi(1) - \psi(0) = [(\sigma_K - \sigma_G)^2 + 2(1-\lambda)\sigma_K\sigma_G]\theta = \psi'(s_K) > 0$.
[14] Indeed, we can solve equation (4) as $W(t) = W(0)\exp[(\gamma^* - \omega^*/2)t + s_K^*\sigma_K z_K(t) + s_G^*\sigma_G z_G(t)]$.

are required for the issuance of public debt and taxation to finance public investment. In reality, many developed countries adopt explicit or implicit fiscal rules. The golden rule of public finance proposed by Musgrave (1939) is well-known. The golden rule is that, "*over the economic cycle, the Government will borrow only to invest and not to fund current spending*" (HM Treasury, 2008). In the United Kingdom and Japan, a deficit in public finance is restricted by the golden rule, and it was at one time, effective in Germany.[15]

The golden rule has some merits: IMF (2014, Ch.3) summarized them as: "*The golden rule takes into account that borrowing to finance productive public investment could pay for itself over the long-term, both through user fees and through higher tax revenues resulting from higher output" and "if public investment is productive, a balanced current budget is consistent with a positive steady-state ratio of public debt to GDP and with optimal fiscal policy."* Thus, the golden rule is a plausible and considerable fiscal rule to actualize the optimal equilibrium.

Incorporating the golden rule of public finance into the model, public investment is financed using public infrastructure bonds. Therefore, we have

$$B = G. \tag{7a}$$

Then, B denotes public infrastructure bonds that finance public infrastructure projects.

The government imposes a lump-sum tax and issues public bonds to finance investments in public capital and interest payment of its bond. The lump-sum tax is equal to the reward for public capital;

$$dT = F_G G dt, \tag{7b}$$

which can be interpreted as the user fee of public capital.[16] Then, we obtain the budget constraint of the government as

$$dB = B dR_B + dH - dT = B dR_B + dH - F_G G dt, \tag{8}$$

where dH denotes government investment expenditures in public capital and dR_B stands for the rate of return on the government bond. Furthermore, the dynamic equation of public capital is

$$dG = dH + \sigma_G G dz_G. \tag{9}$$

Using equations (7a)-(9), the golden rule requires that the tax revenue except for diffusion term of public capital accumulation must allot interest payment:

$$B dR_B = dT + \sigma_G G dz_G. \tag{10}$$

In the decentralized equilibrium, the representative household chooses consumption size and decides on the asset allocation that maximizes his or her lifetime utility. As in Turnovsky (2000, Ch.15), the government issues public bonds $B = pb$ where p is the price of the bonds and b is the outstanding public debts.[17] The rate of return on the government bond follows $dR_B = r_B dt + \sigma_B dz_B$. Therefore, the household can allocate its wealth between private capital and public bonds. The total wealth of the household is defined as $A \equiv K + B$.

The after-tax budget constraint of the household is

$$dA = [\{n_K r_K + n_B r_B\}A - C]dt + [n_K \sigma_K dz_K + n_B \sigma_B dz_B]A, \tag{11}$$

where r_K is the expected rate of return on private capital, $n_K \equiv K/A$ and $n_B \equiv B/A$. Note that

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15 In 2011 after the global financial crisis, The Budget Responsibility and National Audit Act was established in UK. By the Acts, the UK government budget is assessed by the Office for Budget Responsibility. In Japan, it is specified in the Article 4 of Public Finance Law. In Germany, the constitution are amended for strengthening fiscal discipline in 2009. Before the amendment to the constitution, there was express provisions of golden rule in the present Germany constitution. Further issues are discussed in Section 5.

16 Similar assumptions have used in Arrow and Kurz (1970) and Okuno and Yakita (1981).

17 This expression provides the relations such as $A(0) = \bar{K}/n_K$ and $p(0) = n_G\bar{K}/n_K\bar{b}$; $G(0) = p(0)\bar{b} = n_G\bar{K}/n_K$.

we have $r_K = F_K$. Then, the optimization problem of the household is

$$\max_{C, n_K, n_B} E \int_0^\infty \frac{C^{1-\theta}}{1-\theta} e^{-\rho t} dt \qquad \text{s.t. } n_K + n_B = 1 \text{ and (11)}.$$

Solving the problem, we have the optimal consumption and ratio for capital that satisfy the following equations (See Appendix B):

$$C = \beta A, \tag{12}$$

$$r_K - r_B = [(\sigma_K - \mu\sigma_B) n_K \sigma_K + (\mu\sigma_K - \sigma_B) n_B \sigma_B]\theta, \tag{13}$$

where

$$\beta \equiv \frac{\rho}{\theta} + (\theta - 1)\left[\frac{n_K r_K + n_B r_B}{\theta} - \frac{n_K^2 \sigma_K^2 + 2\mu n_K n_B \sigma_K \sigma_B + n_B^2 \sigma_B^2}{2}\right].$$

Note that r_K, r_B and $\sigma_B dz_B$ will be endogenously determined, and both r_K and r_B will be constant over time in equilibrium, as explained later.

Equation (12) corresponds to equation (2) in the previous section, and the same explanation can be applied. Equation (13) shows the equalization of the rate of return on private capital and public bonds. Two assets have different risks. Therefore, depending on the relative degree of investment risk, the households require a risk premium if they are going to hold risky assets.

Using $A \equiv K + B$, $r_K = F_K$, $n_K \equiv K/A$, $n_B \equiv B/A$, equations (8), (9) and (10), the resource constraint becomes

$$dK + dG = [F(K, G) - C]dt + \sigma_K K dz_K + \sigma_G G dz_G. \tag{14}$$

2.3. The balanced growth decentralized equilibrium and optimal fiscal policy

Arrow and Kurz (1970, Ch. 8) showed that the publicly optimal policy is controllable (without taxes) if and only if the debt is equal to the public capital stock at the initial time. In our stochastic growth model, the socially optimal equilibrium with constant portfolio share attains a balanced growth equilibrium. Not only is the equalization of initial debt and public capital required but the sequential equalization of their growth rates is required as well. The golden rule of public finance introduced in Subsection 2.2 assures equalization of initial debt, public capital, and balanced growth, as shown below.

In equilibrium, the share of assets must be constant over time. The rates of return on assets, dR_K and dR_B, are determined as the level that n_K and n_B are constant. If the rates are determined, the equilibrium ratio of private consumption to wealth is also determined by equation (12). Then, the government's budget constraint determines the equilibrium growth rate of public capital subject to equilibrium conditions. We now consider the determination of equilibrium values and the attainability of the socially optimal equilibrium.

First, the expected rate of return on private capital in equilibrium becomes

$$r_K = F_K(n_K, n_B). \tag{15}$$

Note that $n_B = n_G \equiv G/A$ holds under equation (7a). Furthermore, recall that equations (7a)-(9) lead to $dR_B = F_G dt + \sigma_G dz_G$;

$$r_B = F_G, \tag{16}$$

$$\sigma_B dz_B = \sigma_G dz_G. \tag{17}$$

Note that equation (17) leads to $\mu = \lambda$. Equations (16) and (17) imply that the rate of return on public bonds is equal to the rate of return on public capital; the government must pay the interest of public bonds that equals the risk adjusted rate of return on public capital.

These rates of return must assure that n_K and n_B are constant over time. Substituting s_K and s_G for n_K and n_B respectively, equations (13), (15), (16) and (17) provide equation (3). By Proposition 1, we obtain a uniquely determined constant share of assets (i.e., $s_K^* = n_K^*$ and $s_G^* = n_G^* = n_B^*$). A constant share of assets implies the balanced growth equilibrium, which is

$$\frac{dA}{A} = \frac{dK}{K} = \frac{dB}{B} = \frac{dG}{G}. \tag{18}$$

To attain equation (18), the drift term of each growth rate must coincide with the drift term of the other growth rates, and the diffusion term must also coincide with the diffusion term of the other growth rates. Furthermore, under the golden rule, the investment in public capital is financed by only infrastructure bonds. The level of public investment is endogenously determined according to equilibrium asset allocation that satisfies (15)-(18). Therefore, the government investment in public capital follows

$$dH = [F(n_K, n_B) - \beta]Gdt - n_K G[\sigma_G dz_G - \sigma_K dz_K], \tag{19}$$

where the first term of the right hand side denotes a positive constant expected growth rate of public capital and the second term represents the stochastic part of government investment.

Equation (19) indicates that the expected growth rate of public capital is equal to the expected growth rate of private capital; the government must invest in public capital and maintains the average growth rate such that it equals the average growth rate of private capital. Furthermore, equation (19) shows that the stochastic part of public investments adjusts the growth rate of public capital to the balanced growth rate; the government must manage total investments to equalize the growth rates of the two types of capital.

Then, the following proposition is established (See Appendix C).

Proposition 2. *The optimal fiscal policy consists of (7a) and (7b), so that the golden rule of public finance applies. In the decentralized economy under the golden rule of public finance, a unique balanced growth equilibrium exists that satisfies equations (7)-(19). The balanced growth equilibrium in the decentralized economy is identical to the socially optimal equilibrium;* $s_K^* = n_K^*$, $s_G^* = n_G^* = n_B^*$ *and* $\alpha^* = \beta^*$.

Using Proposition 2, the decentralized equilibrium under the optimal fiscal policy is characterized by the equations in the first subsection of Section 2. Recall Proposition 1 and equations (4)-(6). These equations depend on production technology and some deep parameters. Therefore, we need to investigate the equilibrium properties of balanced growth equilibrium to clarify the optimal fiscal policy. In the next section, we provide a comparative statics analysis of balanced growth equilibrium.

3. Equilibrium analysis

3.1. Preliminary results

In this subsection, we characterize the relationship among equilibrium economic variables and deep parameters that relates to the variance and covariance of stochastic fluctuations. Total differentiation of equation (3) provides the effects of a change in $\sigma_K, \sigma_G, \lambda$ on the portfolio share (See Appendix D):

$$\frac{\partial s_K^*}{\partial \sigma_K} = -\frac{\partial s_G^*}{\partial \sigma_K} = \frac{[2s_K^*\sigma_K - \lambda s_K^*\sigma_G + \lambda s_G^*\sigma_G]\theta}{a_{11} - a_{12}}, \tag{20a}$$

$$\frac{\partial s_K^*}{\partial \sigma_G} = -\frac{\partial s_G^*}{\partial \sigma_G} = -\frac{[2s_G^*\sigma_G + \lambda s_K^*\sigma_K - \lambda s_G^*\sigma_K]\theta}{a_{11} - a_{12}}, \tag{20b}$$

$$\frac{\partial\, s_K^*}{\partial \lambda} = -\frac{\partial s_G^*}{\partial \lambda} = \frac{(s_G^* - s_K^*)\sigma_K\sigma_G}{a_{11} - a_{12}} \gtreqless 0 \Leftrightarrow s_K^* \gtreqless s_G^*, \tag{20c}$$

where $a_{11} - a_{12} = F_{KK}^* - F_{GK}^* - F_{KG}^* + F_{GG}^* - [(\sigma_K - \sigma_G)^2 + 2(1-\lambda)\sigma_K\sigma_G]\theta < 0$.

[Table 1]

The signs of these effects are ambiguous in a general case. Using actual data of G5 countries, we can focus on the plausible case. Table 1 shows observed and estimated values of of σ_K, σ_G, λ, s_K, and s_G. The data shows that the private capital share is larger than public capital share. By this observation, we focus on the case where $s_K^* > s_G^*$. Furthermore, calculations based on the data lead to $2s_K^*\sigma_K - \lambda s_K^*\sigma_G + \lambda s_G^*\sigma_G > 0$ and $2s_G^*\sigma_G + \lambda s_K^*\sigma_K - \lambda s_G^*\sigma_K > 0$ for all countries of Table 1. Then, the sign of equations (20a)-(20c) are determined as follows (See Appendix D):

$$\frac{\partial s_K^*}{\partial \sigma_K} < 0, \frac{\partial s_K^*}{\partial \sigma_G} > 0, \frac{\partial s_K^*}{\partial \lambda} > 0, \frac{\partial s_G^*}{\partial \sigma_K} > 0, \frac{\partial s_G^*}{\partial \sigma_G} < 0, \frac{\partial s_G^*}{\partial \lambda} < 0. \tag{21}$$

We discuss these results of the comparative statics analysis using Figure 2b. A change in σ_K, σ_G and λ does not affect the locus of the Δr curve. In contrast, a change in σ_K, σ_G and λ influences the locus of the ψ curve.

If $2s_K^*\sigma_K - \lambda s_K^*\sigma_G + \lambda s_G^*\sigma_G > 0$, an increase in σ_K moves the ψ curve upward. Then, s_K^* decreases. The mechanism behind this result is explained as follows: for large asset share of private capital, an increase in the variance of private investment shocks raises the investment risk of private capital. It brings about the portfolio switch toward public bonds and an increase in the risk premium on private capital. The effects of a rise in σ_G are diametrically opposite to the effects of an increase in σ_K.

Regarding the effect of a rise in λ on the portfolio shares, we explain it in a similar manner. An increase in λ moves the ψ curve downward (upward) if $s_K^* > s_G^*$ ($s_K^* < s_G^*$). Then, an increase in λ increases (decreases) s_K^*. In response to an increase in λ, uniform portfolio shares are not influenced. However, under non-uninform portfolio shares, households must change the portfolio share to manage an increase in the covariance of the stochastic growth rate of two types of capital, including a change in the risk premium. In particular, for $s_K^* > s_G^*$, an increase in λ relatively increases the investment risk of public bonds and brings about the portfolio switch toward private capital.

Using the partial differentiation of the production function F, we derive the effects of a change in σ_K, σ_G, and λ on the marginal products of private capital and public capital (See Appendix D):

$$\frac{\partial F_K^*}{\partial \sigma_K} > 0, \frac{\partial F_K^*}{\partial \sigma_G} < 0, \frac{\partial F_K^*}{\partial \lambda} < 0, \frac{\partial F_G^*}{\partial \sigma_K} < 0, \frac{\partial F_G^*}{\partial \sigma_G} > 0, \frac{\partial F_G^*}{\partial \lambda} > 0. \tag{22}$$

A change in the marginal products of private and public capital depends on a change in s_K^* or s_G^*. A decrease in s_K^* increases the marginal product of private capital by the diminishing marginal productivity of capital and the Edgeworth-Pareto complementarity of two capital inputs. The effect on the marginal product of public capital can be explained in an analogous manner. These results show that an increase in σ_K affects Δr according to the relative size of investment risks, and σ_G decreases Δr. Consequently, high σ_G tends to produce $\Delta r < 0$.

We now examine the effects of a change in $\sigma_K, \sigma_G, \lambda$ on the debt-to-GDP ratio. Using the definition of portfolio shares, the debt-to-GDP ratio becomes

$$\phi \equiv \frac{B}{F(K,G)} = \frac{s_G}{F(s_K, s_G)}. \tag{23}$$

The partial differentiation of equation (23) with respect to s_K^* and s_G^* leads to

$$\frac{s_K^*}{\phi^*}\frac{\partial \phi^*}{\partial s_K^*} = -\frac{\epsilon_K}{s_G^*} \text{ and } \frac{s_G^*}{\phi^*}\frac{\partial \phi^*}{\partial s_G^*} = \frac{\epsilon_K}{s_K^*}, \tag{24}$$

where $\epsilon_K \equiv \frac{F_K s_K}{F}$, $\epsilon_G \equiv \frac{F_G s_G}{F}$ and $\epsilon_K + \epsilon_G = 1$. Equation (24) implies that the equilibrium debt-to-GDP ratio ϕ^* is positively (negatively) associated with s_G^* (s_K^*). Combining the results of the effects of a change in $\sigma_K, \sigma_G,$ and λ on the portfolio share in the benchmark case with (24), we obtain the following proposition:

Proposition 3. *Suppose that* $s_K^* > s_G^*$, $2s_K^*\sigma_K - \lambda s_K^*\sigma_G + \lambda s_G^*\sigma_G > 0$, *and* $2s_G^*\sigma_G + \lambda s_K^*\sigma_K - \lambda s_G^*\sigma_K > 0$ *is initially satisfied to assure relations such as (21). Then, (a) an increase in* σ_K *increases the equilibrium debt-to-GDP ratio. (b) In contrast, an increase in* σ_G *reduces the equilibrium debt-to-GDP ratio. (c) An increase in* λ *reduces the equilibrium debt-to-GDP ratio.*

Proposition 3-(a) is interpreted as follows. Through an increase in σ_K, the household switches its private capital to public bonds. In contrast, the change in asset portfolios affects the equilibrium output depending on the relative risk spread. Because the former effect on asset portfolios dominates the latter effect on GDP, an increase in σ_K increases ϕ^*. Tamai (2014) also derived Proposition 3-(a). He assumes that $B < G$, $\lambda = 1$, and $\sigma_G = 0$ hold under distortionary taxation. Although his model setting is not the same as ours, our situation includes the case studied by him. In this paper, we impose two different stochastic disturbances and there is no distortionary taxation. The relative size of σ_K to σ_G is important for this result; the result of Tamai (2014) does not hold without any condition.

We explain Proposition 3-(b) using a similar (but opposite) method, as Proposition 3-(c) by taking notice of the interpretation of $\partial s_x^*/\partial \lambda$ $(x = K, G)$. A noteworthy point from Proposition 3-(a), 3-(b), and 3-(c) is that mixed shocks bring about structural change of equilibrium properties. The comparative statics without any condition shows that the responses of s_K^* and s_G^* to an increase in λ depends on the size of s_K^* and s_G^*. Although $2s_K^*\sigma_K - \lambda s_K^*\sigma_G + \lambda s_G^*\sigma_G > 0$ and $2s_G^*\sigma_G + \lambda s_K^*\sigma_K - \lambda s_G^*\sigma_K > 0$ are initially assumed, a large increase in σ_K changes asset portfolio. Therefore, after large shocks, the responses of economic variables to additional shocks are also changed; in particular, once the economy has a high debt-to-GDP ratio by large shocks, the economy is confronted with further increase in the debt-GDP ratio in response to strengthening correlation between two stochastic disturbances.

3.2. Optimal rate of return

The results discussed in the previous subsection enable us to characterize the socially optimal condition for capital allocation. We investigate the relationship between the optimal rates of return on two types of capital, which has long been an important issue for empirical studies because pioneering theoretical studies, such as Arrow and Kurz (1970), presented the socially optimal conditions for public investment in the deterministic dynamic general equilibrium model. Our stochastic endogenous growth model appends a significant result explained hereafter in this study.

When $\lambda > 0$, four cases exist including the extreme case such as $\lambda = 1$; Figure 2a-2d. First, we consider the case in which $0 < \sigma_G/\sigma_K < \lambda$; Figure 2a. In this case, $\sigma_K > \sigma_G$ and $\psi > 0$ hold. Then, $\Delta r^* > 0$, i.e. $F_K^* > F_G^*$:

$$0 < \frac{\sigma_G}{\sigma_K} < \lambda \Rightarrow 0 < \sigma_G < \lambda\sigma_K \Rightarrow \sigma_G < \sigma_K \Rightarrow \lambda\sigma_G < \sigma_K \Rightarrow \psi > 0 \Rightarrow F_K^* > F_G^*.$$

The risk premium on private capital is generated. To equalize the risk adjusted rate of return on two assets, the household selects the portfolio shares that satisfy $F_K^* > F_G^*$. The case in which $0 < 1/\lambda < \sigma_G/\sigma_K$ in Figure 2c is just the opposite of the first case and, therefore, is explained using a method similar (but opposite) to that of the first case:

$$0 < \frac{1}{\lambda} < \frac{\sigma_G}{\sigma_K} \Rightarrow 0 < \sigma_K < \lambda\sigma_G \Rightarrow \sigma_K < \sigma_G \Rightarrow \lambda\sigma_K < \sigma_G \Rightarrow \psi < 0 \Rightarrow F_K^* < F_G^*.$$

After that, we consider the case in which $0 < \lambda < \sigma_G/\sigma_K < 1/\lambda$; Figure 2b. There exist critical portfolio shares that switch the risk premium on public bonds to the risk premium on private capital, and vice versa. Defined κ as its critical value satisfies $\psi(\kappa) = 0$. Solving $\psi(\kappa) = 0$ with respect to κ, we obtain

$$\kappa = \frac{(\sigma_G - \lambda\sigma_K)\sigma_G}{(\sigma_K - \lambda\sigma_G)\sigma_K + (\sigma_G - \lambda\sigma_K)\sigma_G} > 0.$$

Note that a smaller s_K^* provides a larger F_K^* and a smaller F_G^*. Therefore, for small s_K^*, we have $F_K^* > F_G^*$. In contrast, $F_K^* < F_G^*$ for large s_K^*. When $-1 \le \lambda \le 0$, the corresponding figure is Figure 2b. This situation can be explained in an analogous manner in Figure 2b for the case in which $\lambda > 0$. Formally, we have

$$s_K^* \gtreqless \kappa \Leftrightarrow \psi \gtreqless 0 \Leftrightarrow F_K^* \gtreqless F_G^*.$$

Finally, we consider the case in which $\lambda = 1$. When $\lambda = 1$, the accumulation of private and public capital is affected by the same stochastic process with different diffusion coefficients. Therefore, the diffusion coefficients of private capital growth and public capital growth are significant to determining Δr^*. Indeed, we have $F_K^* - F_G^* = (\sigma_K - \sigma_G)(s_K^*\sigma_K + s_G^*\sigma_G)\theta$:

$$\sigma_K \gtreqless \sigma_G \Leftrightarrow \psi \gtreqless 0 \Leftrightarrow F_K^* \gtreqless F_G^*.$$

The sign of Δr^* depends on the sign of $(\sigma_K - \sigma_G)$. Two possible intersection points A and B exist in Figure 2d.

These results are summarized as follows**.**

Proposition 4. *(a)* $0 < \sigma_G/\sigma_K < \lambda$. *The marginal product of private capital is larger than the marginal product of public capital;* $F_K^* > F_G^*$. *(b)* $0 < \lambda < \sigma_G/\sigma_K < 1/\lambda$ *or* $-1 \le \lambda \le 0$. *If* $s_K^* > \kappa$ $(s_K^* < \kappa)$, *the marginal product of private capital is larger (smaller) than the marginal product of public capital;* $F_K^* > F_G^*$ $(F_K^* < F_G^*)$. *(c)* $0 < 1/\lambda < \sigma_G/\sigma_K$. *The marginal product of private capital is smaller than the marginal product of public capital;* $F_K^* < F_G^*$. *(d)* $\lambda = 1$. *If* σ_K *is larger (smaller) than* σ_G, *the marginal product of private capital is larger (smaller) than the marginal product of public capital;* $F_K^* > F_G^*$ $(F_K^* < F_G^*)$.

In a deterministic economy without stochastic disturbances, the optimal allocation of capital is represented by the equalization of the marginal product of private capital and public capital (e.g., Arrow and Kurz, 1970; Turnovsky, 1997; Gómez, 2004; Tamai, 2008). However, Proposition 4 indicates that the equalization of the marginal product of private capital and public capital is an exceptional case; it holds if and only if $\psi = 0$. Some studies reported empirical evidence on $F_K^* < F_G^*$ (e.g., Lighthart and Suárez, 2011; Bom and Lighthart, 2014; Gupta et al., 2014). From Proposition 4, $F_K^* < F_G^*$ holds if σ_G/σ_K is sufficiently large. In other words, larger risk of public capital investment provides $F_K^* < F_G^*$.

We should check the result to fit to actual data as discussed in Turnovsky (2000, Ch. 15). Parameters in Table 1 imply that Germany, Japan, the United Kingdom, and the United States are in the situation of Proposition 4 (b), and France is in the situation of Proposition 4 (c).[18] We should investigate Proposition 4 using by the data. By the definition of ϵ_G, we have $F_G = \epsilon_G F/G$. Using this equation and values of ϵ_G and the average products of private and public capital, we can calculate the marginal product of private capital and marginal product of public capital. Table 2 shows the average products calculated from estimated value and plausible value of the elasticity of output with respect to public capital. The elasticity of output with respect to public capital is estimated in the range from 0.1 to 0.4 by various empirical studies. We set the values of France, Germany, Japan, the UK, and the US to 0.15, 0.15, 0.27, 0.2, and 0.27, respectively.[19]

[Table 2]

Using equation (3) and the data of Table 1 and Table 2, we derive the results of Table 3a for whole period of the data. Estimated values of the difference between two marginal products are negative except for France and Japan. On the other hand, theoretical values of the difference between two marginal products are positive in Germany, Japan and the United States, and negative in France and the United Kingdom. Regarding the United Kingdom and Japan, estimated and theoretical values are the same sign, although the results show that estimated and theoretical values are different signs in France, Germany, and the United States. The differences between estimated and theoretical values are 0.009% point in the UK; 0.3% point in Germany; 1.7% point in Japan, 5.3% point in France; 6.7% point in the US.

[Table 3a and 3b]

These findings also provide interesting policy implications. First, the result for the UK shows that theoretical value is nearly equal to actual value, and suggests that optimal equilibrium is attained in the United Kingdom under the golden rule of public finance. The result for Germany

[18] For UK, we have $\lambda < 0$ (See Table 1). By calculations, we have $\sigma_G/\sigma_K = 1.176$ and $1/\lambda = 1.06$ for FR; $\sigma_G/\sigma_K = 1$ and and $1/\lambda = 1.17$ for GR; $\sigma_G/\sigma_K = 1.05$ and $1/\lambda = 1.21$ for Japan; $\sigma_G/\sigma_K = 1.28$ and $1/\lambda = 1.77$ for Untied States.

[19] The values from 0.2 to 0.3 are well used as ϵ_G in endogenous growth model. However, empirical studies reported values lower than 0.2 in European countries, and values higher than 0.2 in Japan and US. For example, 0.083 for France (Cadot et al., 2006); 0.169 for Germany (Kemmerling and Stephan, 2002); 0.2, 0.278 for Japan (Mera, 1973; Hayashi, 2009); 0.2 for UK (Lynde and Richmond 1993); 0.39, 0.27 for US (Aschauer, 1989; Duggal et al., 1999). See also Lighthart and Suárez (2011), Pereira and Andraz (2013) and Bom and Ligthart (2014). Taking into account these findings, we set the values.

also shows 0.3% point of difference between estimated and theoretical value, although the sign of the disparity of the two marginal productivity is different from theoretical prediction. However, for the period 1995-2010, the sign of theoretical value is consistent with the sign of estimated value and the difference is smaller than that for whole period (Table 3b). Therefore, these results imply that Germany actualizes optimal equilibrium within a plausible error range for the period that the golden rule was explicitly provided. The result for Japan shows 1.7 percentage point of difference between estimated and theoretical value, although the theoretical prediction is correct in the sign of disparity of the two rates of return. For the period 1994-2000 before the ratio of government investment to GDP is less than 5%, the difference is 0.39 % point. Therefore, Japan was in the neighborhood of optimal equilibrium for past. However, in present, Japan is far from optimal equilibrium and have over supply of public capital.

The result for France implies that public capital is over-supplied compared with the optimal level. The result for the US shows that about 6.7% points of the difference between the theoretical and actual value, and implies that fiscal policy fails to achieve optimal equilibrium in the United States. In particular, the result shows a shortage of public capital in United States. Taking into account theoretical values, US has the shortage of public capital as reported by empirical studies (e.g. Lighthart and Suárez 2011; Bom and Lighthart 2014).

These results imply that our model elucidates the disparity of the two marginal products in the economy under the golden rule for at least the UK, and might also apply to Germany and Japan within acceptable error range for fixed periods when the golden rule was binding. Regarding France and the US, it provides the basis of an evaluation of the dynamic efficiency of public investment for France and the US.

3.3. Optimal growth rate

The rate of return is a key determinant of economic growth rate in an endogenous growth model. To end this section, we examine the effects of a change in σ_K, σ_G, and λ on the expected equilibrium growth rate and the variance of the equilibrium growth rate. The partial differentiation of equation (5) yields

$$\frac{\partial \gamma^*}{\partial \sigma_K} = (F_K^* - F_G^*)\frac{\partial s_K^*}{\partial \sigma_K} + (\theta - 1)(s_K^*\sigma_K + \lambda s_G^*\sigma_G)s_K^*, \tag{25a}$$

$$\frac{\partial \gamma^*}{\partial \sigma_G} = -(F_K^* - F_G^*)\frac{\partial s_G^*}{\partial \sigma_G} + (\theta - 1)(s_G^*\sigma_G + \lambda s_K^*\sigma_K)s_G^*, \tag{25b}$$

$$\frac{\partial \gamma^*}{\partial \lambda} = (F_K^* - F_G^*)\frac{\partial s_K^*}{\partial \lambda} + (\theta - 1)s_K^* s_G^*\sigma_K\sigma_G. \tag{25c}$$

In each equation, the first term reflects the productivity effect through a change in the portfolio shares in response to increases in the variance or the covariance of two stochastic capital growth rates. The second term denotes the net effect on the consumption-to-wealth ratio, reflecting both income and substitute effects. Note that the middle parenthesis of the second terms in (25a) and (25b) are positive from the data of Table 1. Therefore, we focus on the case where $s_K^*\sigma_K + \lambda s_G^*\sigma_G > 0$ and $s_G^*\sigma_G + \lambda s_K^*\sigma_K > 0$.

The second term is positive (negative) when $\theta > 1$ $(\theta < 1)$. Risk-averse households decrease the consumption-wealth ratio because higher variances are the same as a decrease in income. In contrast, the first term is ambiguous because the portfolio shares depend on the relative degree of two different investment risks. We deduce its sign using the results of comparative statics and Proposition 3.

The partial differentiation of equation (6) leads to

$$\frac{\partial \omega^*}{\partial \sigma_K} = 2\left[\frac{F_K^* - F_G^*}{\theta}\frac{\partial s_K^*}{\partial \sigma_K} + (s_K^*\sigma_K + \lambda s_G^*\sigma_G)s_K^*\right], \tag{26a}$$

$$\frac{\partial \omega^*}{\partial \sigma_G} = 2\left[-\frac{F_K^* - F_G^*}{\theta}\frac{\partial s_G^*}{\partial \sigma_G} + (s_G^*\sigma_G + \lambda s_K^*\sigma_K)s_G^*\right], \tag{26b}$$

$$\frac{\partial \omega^*}{\partial \lambda} = 2\left[\frac{F_K^* - F_G^*}{\theta}\frac{\partial s_K^*}{\partial \lambda} + s_K^* s_G^*\sigma_K\sigma_G\right]. \tag{26c}$$

Similar to the effects on the expected growth rate, each equation can be decomposed into two analogous effects. However, the variance of the equilibrium growth rate is not directly affected by the consumption-to-wealth ratio. Therefore, the sign of the first term depends on the disparity between the marginal products and the effect of a change in the deep parameter on the portfolio shares, and the second term of equations (26a)-(26c) are always positive regardless of the value of θ.

The results provided in the previous instance are summarized as the following proposition:

Proposition 5. *Suppose that* $\theta \geq 1$, $s_K^*\sigma_K + \lambda s_G^*\sigma_G > 0$, *and* $s_G^*\sigma_G + \lambda s_K^*\sigma_K > 0$ *holds. (a) An increase in* σ_K *increases (has the possibility of decreasing) the mean and variance of the equilibrium growth rate if* $F_K^* < F_G^*$ $(F_K^* > F_G^*)$. *(b) An increase in* σ_G *has the possibility to decrease (increases) the mean and variance of the equilibrium growth rate if* $F_K^* < F_G^*$ $(F_K^* > F_G^*)$. *(c) The possibility exists that an increase in* λ *has the possibility to decrease the mean and variance of the equilibrium growth rate if* $F_K^* < F_G^*$, *whereas an increase in* λ *increases the mean and variance of the equilibrium growth rate if* $F_K^* > F_G^*$.

Using Proposition 4, in our model, a sufficiently large σ_G/σ_K leads to $F_K^* < F_G^*$. For Proposition 5 when σ_G/σ_K is sufficiently large, an increase in risk of private capital increases the mean growth rate but spreads the realized growth rate; an increase in risk of public capital reduces the mean growth rate and the variance of the growth rate. Using a stochastic growth model with productive public expenditures, Turnovsky (1999) clarified that the coefficient of relative risk aversion plays a key role in determining the effect of an increase in risk on economic growth. This mechanism is common to stochastic growth models with productive public expenditures or public capital (e.g., Ott and Soretz, 2004; Wang and Hu, 2007; Tamai, 2013, 2014).

In particular, Tamai (2014) also examined the effects of an increase in risk on the mean and volatility of the economic growth rate in a stochastic endogenous growth model with private and public capital by incorporating $B < G$, $\lambda = 1$, and $\sigma_G = 0$. In his model, the key determinant of growth effect is the size of the portfolio change through an increase in risk of private investment. Since our model has two different investment risks, the disparity between marginal products is also important for the determination of the growth effect under optimal fiscal policy.

4. Discussion

In this section, we discuss the assumptions and some extensions to the basic model. First, we address the irreversibility of investment and the next pick up the issue of government budgetary system. Thereafter, the optimal investment condition is reexamined by incorporating the difference in the depreciation rates of private and public capital. Finally, we consider the necessity of income taxation if a distortion of factor income distribution exists.

Irreversibility of investment. The irreversibility of investment is important in investment decision making. Gomez (2004) analyzed the optimal fiscal policy in a deterministic model with private and public capital under irreversibility of investments. The irreversibility does not allow negative investment. Then, the initial capital values cannot immediately jump to the optimal level. Therefore, it shows that the economy exhibits transitional dynamics as far as arriving at the optimal equilibrium. The same results are derived in our stochastic model. When the economy reaches the optimal equilibrium, our basic results hold. As shown in the previous section, results derived from our model seems fit for replicating observed values.

Rule for public finance. In an endogenous growth model with private and public capital, some studies have investigated the growth and welfare effects of fiscal policy under a fiscal rule, including the golden rule of public finance (e.g., Greiner and Semmler, 2000; Ghosh and Mourmouras, 2004; Greiner, 2007, 2010; Minea and Villieu, 2009; Groneck, 2011).[20] According to the IMF (2014), many advanced countries other than G5 countries also have improved the design of their fiscal rules. Outside of our purpose, investigating the attainability of the first-best equilibrium within the various fiscal rules is important. In this study, we focus on the only stochastic balanced growth path under an optimal fiscal policy within the golden rule; this study presents a basis of the future extended model.

The difference in depreciation rates of private and public capital. In previous sections, we assume that the depreciation rate of private capital and that of public capital are equal to zero. Two different depreciation rates results in a disparity between the marginal products of two types of capital at the optimum. Equation (3) becomes

$$F_K - F_G = \delta_K - \delta_G + [(\sigma_K - \lambda\sigma_G)s_K\sigma_K - (\sigma_G - \lambda\sigma_K)s_G\sigma_G]\theta,$$

The difference between the depreciation rates affects a critical value of the sign of Δr. However, we derive the same result by substituting the *net* marginal products for all equations in the previous sections.

Unpaid factor and income tax. In the previous section, we assume that the government can impose a lump-sum tax (or charge the user a public capital fee). However, some types of public capital may be treated as common wealth. Then, public capital is an unpaid factor for economic agents. Therefore, the factor income distribution in such a case satisfies $r_K K = F(K,G)$. Because total output is distributed as a reward for providing private capital, the rate of return on private capital in a decentralized economy differs from its socially optimal rate.

An income tax is necessary to fill the gap and the tax rate τ should be set to $\tau = 1 - \epsilon_K = \epsilon_G$. In other words, the income tax rate is equal to the output elasticity with respect to public capital.[21] Hence, tax revenue is equal to the government's revenue in Section 3. If no other distortion exists, Proposition 2 is still alive. Without income taxes, the socially optimal

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[20] Greiner (2010) points out that Minea and Villieu (2009) has reached some misleading conclusions.

[21] This optimal tax rule is a similar to Barro (1990) tax rule of growth-maximizing and welfare-maximizing under the Cobb-Douglas production function. Note that our tax rule is the first best to correct distortionary allocation and that Barro rule is the second best under distortionary taxation. Regarding the relation between growth-maximizing and welfare-maximizing, Misch et al. (2013) investigate two endogenous growth models of Barro (1990) and Futagami et al. (1993).

equilibrium is not attainable and the decentralized equilibrium results in over-accumulation of private capital and under-accumulation of public capital.

5. Conclusion

This study developed a stochastic endogenous growth model with private and public capital by incorporating capital accumulation with two different stochastic disturbances. In this study, we derived the optimal fiscal policy that the first-best equilibrium is attainable in a decentralized economy. We also provided a general characterization of the optimal rate of return on public capital investment and the optimal growth rate of a stochastic balanced growth equilibrium.

We showed that the marginal product of public (private) capital exceeds that of private (public) capital if the ratio of diffusion coefficient of public investment to that of private investment is sufficiently large. The disparity between the marginal products of private and public capital is important to determine the socially optimal growth rate by standing face to face with two different stochastic disturbances inherent in investments in two different-types of capital. Numerical analysis based on estimated values shows that our model can gives a good value for the difference between two marginal products in the UK economy, and might be applicable to Germany and Japan when the golden rule was active.

These theoretical and numerical findings are consistent with some empirical findings in the literature on the relations between public capital and economic growth and on the relation between risk and economic growth. This study provides one theoretical background to explain the gap between the marginal products of two capitals and the basis of criteria to evaluate the dynamic efficiency of public investment. Furthermore, in this study, we made some conceivable extensions of the basic model. In particular, in addition to the conditions in this study, an income tax is required to attain the first-best equilibrium in a decentralized economy under a distortion of factor income.

Finally, we indicate future directions for this study. As discussed in Section 4, we should investigate the attainability of the first-best equilibrium under the other feasible fiscal policy rules. Then, it is necessary to incorporate an endogenous labor supply and taxation on consumption and factor income into our basic model. Furthermore, the irreversibility of investment is also important for extending our basic model. Although same results will hold in the long-run, the optimal fiscal policy under transitional dynamics is also valuable. However, it is difficult to derive analytical solutions in the case of irreversible investments. Numerical analysis will be helpful to characterize this case. These issues present significant avenues for future investigation and this study provides a good analytical basis.

Appendix (Some appendices can be moved to online materials)

A: Derivation of equations (2) and (3) and Proof of Proposition 1

The Bellman equation that corresponds to the optimization problem of social planner is

$$0 = \max_{C,s_K}\left[\frac{C^{1-\theta}}{1-\theta} - \rho J + \{F(s_K, 1-s_K)W - C\}J_W \right.$$
$$\left. + \{s_K^2\sigma_K^2 + 2\lambda s_K(1-s_K)\sigma_K\sigma_G + (1-s_K)^2\sigma_G^2\}\frac{W^2 J_{WW}}{2}\right].$$

The optimal consumption and portfolio rules are governed by

$$C^{-\theta} = J_W, \tag{A1}$$

$$(F_K - F_G) + [s_K\sigma_K^2 + \lambda(1-2s_K)\sigma_K\sigma_G - (1-s_K)\sigma_G^2]\frac{WJ_{WW}}{J_W} = 0, \tag{A2}$$

where $F_K = F_K(K,G) = F_K(s_K, 1-s_K)$ and $F_G = F_G(K,G) = F_G(s_K, 1-s_K)$.[22] To solve (A1) and (A2) with respect to C and s_K, we guess that the value function is

$$J = \frac{\alpha^{-\theta}W^{1-\theta}}{1-\theta}. \tag{A3}$$

Using equations (A1), (A2) and (A3), we obtain (2) and (3).

We now consider the existence and uniqueness of the optimal share of private capital to wealth or, equivalently, the share of public capital to wealth. Let the function P be

$$P(s_K) \equiv F_K - F_G - [(\sigma_K - \lambda\sigma_G)s_K\sigma_K + (\lambda\sigma_K - \sigma_G)(1-s_K)\sigma_G]\theta.$$

Note that total differentiation of (3) and $s_K + s_G = 1$ gives $a_{11}ds_K + a_{12}ds_G = 0$ and $ds_K + ds_G = 1$, where $a_{11} \equiv F_{KK} - F_{GK} - \{(\sigma_K - \lambda\sigma_G)\sigma_K\}\theta$ and $a_{12} \equiv F_{KG} - F_{GG} - \{(\lambda\sigma_K - \sigma_G)\sigma_G\}\theta$. Using these equations and the Inada conditions, we have

$$P'(s_K) \equiv \mathrm{a}_{11} - a_{12} = F_{KK} - F_{GK} - F_{KG} + F_{GG} - [(\sigma_K - \sigma_G)^2 + 2(1-\lambda)\sigma_K\sigma_G]\theta < 0,$$
$$P(0) = +\infty, P(1) = -\infty.$$

Then, using the intermediate value theorem, there exists the value s_K^* that satisfies $P(s_K^*) = 0$.

Finally, we consider the determination of the undetermined coefficient α that must satisfy

$$0 = \frac{C^{1-\theta}}{1-\theta} - \rho J + [F(s_K, s_G)W - C]J_W + [s_K^2\sigma_K^2 + 2\lambda s_K s_G\sigma_K\sigma_G + s_G^2\sigma_G^2]\frac{W^2 J_{WW}}{2}.$$

Using equations (2) and (A3), and after some manipulations, we have

$$\alpha - \rho + (1-\theta)[F(s_K, s_G) - \alpha] - \frac{[s_K^2\sigma_K^2 + 2\lambda s_K s_G\sigma_K\sigma_G + s_G^2\sigma_G^2](1-\theta)\theta}{2} = 0.$$

Solving this equation with respect to α, we obtain α in the form of Proposition 1.

B: Derivation of equations (11) and (12)

Equations (11) and (12) with β is derived in the same way as in Appendix A. The Bellman equation that corresponds to the consumer's problem is

[22] By the assumption, F_x is the homogenous of degree zero ($x = K, G$). Therefore, we have

$$F_x(K,G) = F_x\left(\frac{K}{W}, \frac{G}{W}\right) = F_x(s_K, s_G) = F_x(s_K, 1-s_K).$$

$$0 = \max_{C, n_K} \left[\frac{C^{1-\theta}}{1-\theta} - \rho V + \{(n_K r_K + (1-n_K) r_B) A - C\} V_A \right.$$
$$\left. + \{n_K^2 \sigma_K^2 + 2\mu n_K (1-n_K) \sigma_K \sigma_B + (1-n_K)^2 \sigma_B^2\} \frac{A^2 V_{AA}}{2} \right].$$

The optimality conditions for consumption and portfolio are

$$C^{-\theta} = V_A, \tag{B1}$$

$$(r_K - r_B) + [n_K \sigma_K^2 + \mu(1-2n_K)\sigma_K \sigma_B - (1-n_K)\sigma_B^2] \frac{A V_{AA}}{V_A} = 0. \tag{B2}$$

We deduce that the value function is

$$V = \frac{\beta^{-\theta} A^{1-\theta}}{1-\theta}. \tag{B3}$$

Using equations (B1)-(B3), we arrive at

$$C = \beta W,$$
$$r_K - r_B = [(\sigma_K - \mu \sigma_B) n_K \sigma_K + (\mu \sigma_K - \sigma_B) n_B \sigma_B] \theta.$$

The undetermined coefficient β must satisfy

$$0 = \frac{C^{1-\theta}}{1-\theta} - \rho V + [\{n_K r_K + n_B r_B\} A - C] V_A + [n_K^2 \sigma_K^2 + 2\lambda n_K n_B \sigma_K \sigma_B + n_B^2 \sigma_B^2] \frac{A^2 V_{AA}}{2}.$$

In the same manner as in Appendix A, using this equation, equations (11) and (B3), we obtain β in the main text.

C: Proof of Proposition 2

We begin our analysis to derive each growth rate. Using (10), we obtain the growth rate of total wealth:

$$\frac{dA}{A} = [n_K r_K + n_B r_B - c] dt + n_K \sigma_K dz_K + n_B \sigma_B dz_B,$$

where $c \equiv C/A$ is the ratio of consumption to wealth.

Suppose that the investment in public capital follows

$$dH = \chi G dt - \sigma_X G dz_X. \tag{C1}$$

Equation (8), (9) and (18) lead to

$$\frac{dB}{B} = (r_B - F_G + \chi) dt + \sigma_B dz_B - \sigma_X dz_X,$$
$$\frac{dG}{G} = \chi dt - \sigma_X dz_X + \sigma_G dz_G.$$

Comparing dB/B with dG/G gives $(r_B - F_G) dt = \sigma_G dz_G - \sigma_B dz_B$. In other words, $r_B = F_G$ and $\sigma_G dz_G = \sigma_B dz_B$ are required. Then, $\mu = \lambda$ holds.

Equations (7a) and (13) lead to

$$\frac{dK}{K} = \frac{[F(n_K, n_B) - c - n_B \chi] dt}{n_K} + \frac{n_B \sigma_X dz_X}{n_K} + \sigma_K dz_K.$$

A comparison among the diffusion terms of equations dB/B, dG/G and dK/K gives

$$n_K \sigma_K dz_K + n_B \sigma_B dz_B = -\sigma_X dz_X + \sigma_G dz_G = \frac{n_B \sigma_X dz_X}{n_K} + \sigma_K dz_K. \tag{C2}$$

The right and middle terms of equation (C2) imply

$$\sigma_X dz_X = n_K [\sigma_G dz_G - \sigma_K dz_K].$$

In contrast, the left and middle terms of equation (C2) with this equation show

$$n_K\sigma_K dz_K + n_B\sigma_B dz_B = -n_K[\sigma_G dz_G - \sigma_K dz_K] + \sigma_G dz_G.$$

Therefore, we arrive at

$$\sigma_B dz_B = \sigma_G dz_G. \tag{16}$$

Equation (16) does not contradict with previous conditions.

We notice $c = \beta$ and the comparison among the drift terms of equations dA/A, dB/B, dG/G and dK/K to obtain

$$n_K r_K + n_B r_B - \beta = \frac{F(n_K, n_B) - \beta - n_B\chi}{n_K} = \chi = r_B - F_G + \chi. \tag{C3}$$

The two middle terms of equation (C3) lead to

$$\frac{F(n_K, n_B) - \beta - n_B\chi}{n_K} = \chi \Rightarrow F(n_K, n_B) - \beta = \chi. \tag{19}$$

The left and middle left terms of equation (C3) with equation (16) provide

$$n_K r_K + n_B r_B - \beta = \frac{F(n_K, n_B) - \beta - n_B\chi}{n_K} \Rightarrow n_K r_K + n_B r_B = F(n_K, n_B). \tag{C4}$$

The right and middle right terms of equation (D3) provide

$$\chi = r_B - F_G + \chi \Rightarrow r_B = F_G. \tag{15}$$

Equation (C4) with (14) and (15) is consistent with a linear homogenous production function. Furthermore, equation (15) is also consistent with previous necessary conditions.

Taking into account equations (15), (16), (19), and (20), we recall

$$0 = \frac{C^{1-\theta}}{1-\theta} - \rho V + \{F(n_K, n_G)A - C\}V_A + \{n_K^2\sigma_K^2 + 2\lambda n_K n_G \sigma_K \sigma_G + n_G^2\sigma_G^2\}\frac{A^2 V_{AA}}{2}, \tag{C5}$$

$$C = \beta W,$$

$$F_K - F_G = [n_K\sigma_K^2 + \lambda(n_G - n_K)\sigma_K\sigma_G - n_G\sigma_G^2]\theta, \tag{C6}$$

$$0 = \frac{C^{1-\theta}}{1-\theta} - \rho J + \{F(s_K, s_G)W - C\}J_W + \{s_K^2\sigma_K^2 + 2\lambda s_K s_G \sigma_K \sigma_G + s_G^2\sigma_G^2\}\frac{W^2 J_{WW}}{2}, \tag{C7}$$

$$C = \alpha W,$$

$$F_K - F_G = [s_K\sigma_K^2 + \lambda(s_G - s_K)\sigma_K\sigma_G - n_G\sigma_G^2]\theta. \tag{C8}$$

Equations (C6) and (C8) show $n_x = s_x$ $(x = K, G)$. Given the assumptions in this study, $A = W$ holds. Then, equations (A3), (B3), (C5), and (C7) lead to $\alpha = \beta$. Therefore, all key equations in the decentralized equilibrium are identical to those of the socially optimal equilibrium.

D: Comparative statics

Total differentiations of equation (3) and $s_K + s_G = 1$ lead to

$$\begin{pmatrix} a_{11} & a_{12} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} ds_K^* \\ ds_G^* \end{pmatrix} = \begin{pmatrix} a_3 \\ 0 \end{pmatrix} d\sigma_K + \begin{pmatrix} a_4 \\ 0 \end{pmatrix} d\sigma_G + \begin{pmatrix} s_G - s_K \\ 0 \end{pmatrix}\sigma_K\sigma_G d\lambda, \tag{D1}$$

where $a_3 \equiv [2s_K\sigma_K - \lambda s_K\sigma_G + \lambda s_G\sigma_G]\theta$ and $a_4 \equiv -[2s_G\sigma_G + \lambda s_K\sigma_K - \lambda s_G\sigma_K]\theta$. Recall

$$a_{11} - a_{12} = F_{KK} - F_{GK} - F_{KG} + F_{GG} - [(\sigma_K - \sigma_G)^2 + 2(1-\lambda)\sigma_K\sigma_G]\theta < 0.$$

Applying Cramer's rule to the system (D1), we obtain

$$\frac{\partial s_K^*}{\partial \sigma_K} = -\frac{\partial s_G^*}{\partial \sigma_K} = \frac{a_3}{a_{11} - a_{12}},$$

$$\frac{\partial s_K^*}{\partial \sigma_G} = -\frac{\partial s_G^*}{\partial \sigma_G} = \frac{a_4}{a_{11} - a_{12}},$$

$$\frac{\partial s_K^*}{\partial \lambda} = -\frac{\partial s_G^*}{\partial \lambda} = \frac{(s_G - s_K)\sigma_K \sigma_G}{a_{11} - a_{12}} \gtreqless 0 \Leftrightarrow s_K^* \gtreqless s_G^*.$$

Calculations lead to $a_3/\theta = 2s_K\sigma_K - \lambda s_K\sigma_G + \lambda s_G\sigma_G > 0$ (0.0177 for FR; 0.01 for GR; 0.0201 for JP; 0.0210 for UK; 0.0344 for US) and $-a_4/\theta = s_G\sigma_G + \lambda s_K\sigma_K - \lambda s_G\sigma_K > 0$ (0.0171 for FR; 0.0116 for GR; 0.0181 or JP; 0.0127 for UK; 0.0180 for US). Further, the data shows $s_K^* > s_G^*$. Then, we have

$$\frac{\partial s_K^*}{\partial \sigma_K} = -\frac{\partial s_G^*}{\partial \sigma_K} = \frac{a_3}{a_{11} - a_{12}} < 0,$$

$$\frac{\partial s_K^*}{\partial \sigma_G} = -\frac{\partial s_G^*}{\partial \sigma_G} = \frac{a_4}{a_{11} - a_{12}} > 0,$$

$$\frac{\partial s_K^*}{\partial \lambda} = -\frac{\partial s_G^*}{\partial \lambda} = \frac{(s_G - s_K)\sigma_K \sigma_G}{a_{11} - a_{12}} > 0.$$

The partial differentiation of the production function F leads to

$$\frac{\partial F_K^*}{\partial \sigma_K} = (F_{KK}^* - F_{KG}^*)\frac{\partial s_K^*}{\partial \sigma_K}, \frac{\partial F_K^*}{\partial \sigma_G} = (F_{KK}^* - F_{KG}^*)\frac{\partial s_K^*}{\partial \sigma_G}, \text{and } \frac{\partial F_K^*}{\partial \lambda} = (F_{KK}^* - F_{KG}^*)\frac{\partial s_K^*}{\partial \lambda}.$$

$$\frac{\partial F_G^*}{\partial \sigma_K} = (F_{GK}^* - F_{GG}^*)\frac{\partial s_K^*}{\partial \sigma_K}, \frac{\partial F_G^*}{\partial \sigma_G} = (F_{GK}^* - F_{GG}^*)\frac{\partial s_K^*}{\partial \sigma_G}, \text{and } \frac{\partial F_G^*}{\partial \lambda} = (F_{GK}^* - F_{GG}^*)\frac{\partial s_K^*}{\partial \lambda}.$$

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Tables

Table 1: Estimated values of σ_K, σ_G, λ, s_K, and s_G

	σ_K	σ_G	λ	s_K	s_G
FR	0.017	0.020	0.944	0.83	0.17
GR	0.013	0.013	0.853	0.85	0.15
JP	0.019	0.020	0.823	0.72	0.28
UK	0.009	0.036	-0.310	0.80	0.20
US	0.027	0.021	0.563	0.76	0.24

Data: FR (1994-2014; Eurostat); GR (1995-2014; Eurostat); JP (1994-2014; Annual Report on National Accounts of 2015); UK (1997-2014; Capital Stocks, Consumption of Fixed Capital 2015); US (1997-2014; Annual National Data from BEA database).
Note: The values of standard deviation are calculated using by Mathematica. The correlation coefficients are calculated based on the residual values. Private and public capital share are average for data periods.

Table 2. Estimated values of the elasticity of output with respect to public capital and average products

	F/K	F/G	ϵ_G
FR	0.408	1.961	0.15
GR	0.366	2.096	0.15
JP	0.482	1.240	0.27
UK	0.530	2.120	0.2
US	0.449	1.460	0.27

Data: FR (1994-2014; Eurostat); GR (1995-2014; Eurostat); JP (1994-2014; Annual Report on National Accounts of 2015); UK (1997-2014; Private and public capital from same data of Table 1, GDP from "United Kingdom National Accounts, The Blue Book, 2015 Edition"); US (1997-2014; All variables from Annual National Data, BEA database).

Table 3a. Estimated and theoretical values of marginal products (%) for whole period

	F_K *(estimated)*	F_G *(estimated)*	$F_K - F_G$ *(estimated)*	$F_K - F_G$ *(theoretical)*
FR	34.67	29.41	5.277	-0.012
GR	31.13	31.43	-0.304	0.002
JP	35.15	33.47	1.682	0.002
UK	42.36	42.41	-0.042	-0.033
US	32.76	39.42	-6.663	0.067

Data: See data sources of Table 1 and 2.
Note: Following Groneck (2011), we set $\theta = 2.5$ for all countries.

Table 3b. Estimated and theoretical values of marginal products (%) for fixed period

	F_K *(estimated)*	F_G *(estimated)*	$F_K - F_G$ *(estimated)*	$F_K - F_G$ *(theoretical)*
GR	31.32	31.13	0.189	0.002
JP	36.65	36.26	0.039	0.003

Data: See data sources of Table 1 and 2.
Note: Calculation method is same as Table 3a. The values are based on the data during period 1995-2010 for Germany, and period 1994-2000 for Japan.

Figures

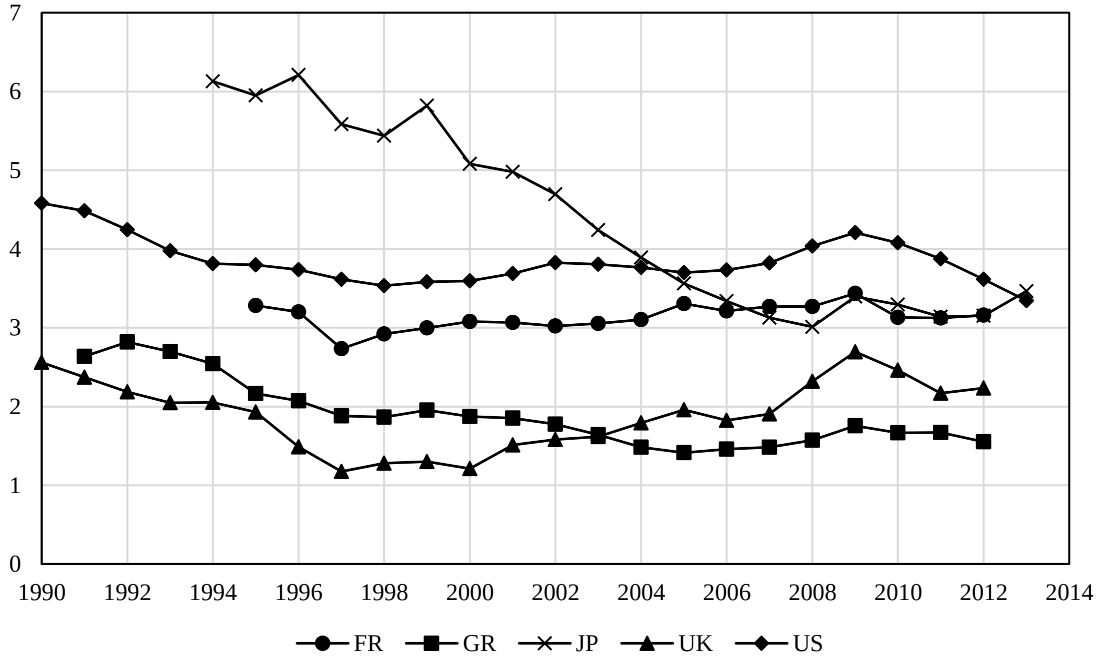

Figure 1: The ratio of government investment to GDP in G5 countries (%)

Data: Gross capital formation (nominal, national currency, general government), GDP (nominal, national currency) from OECD stat.

Note: In April 2005, British Nuclear Fuels Limited (BNFL) transferred nuclear installations to the Nuclear Decommissioning Authority (NDA). The transferred value is 15.6 billion pound sterling. In this figure, the UK value in 2005 includes this value. National Accounts of OECD Countries 2008, Volume II.

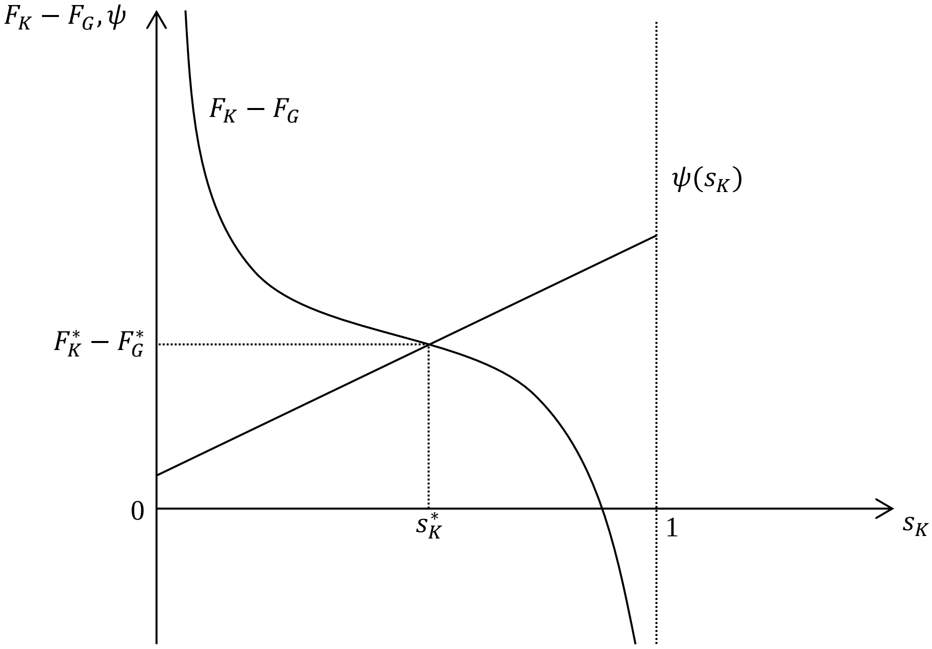

Figure 2a: $0 < \sigma_G/\sigma_K < \lambda$

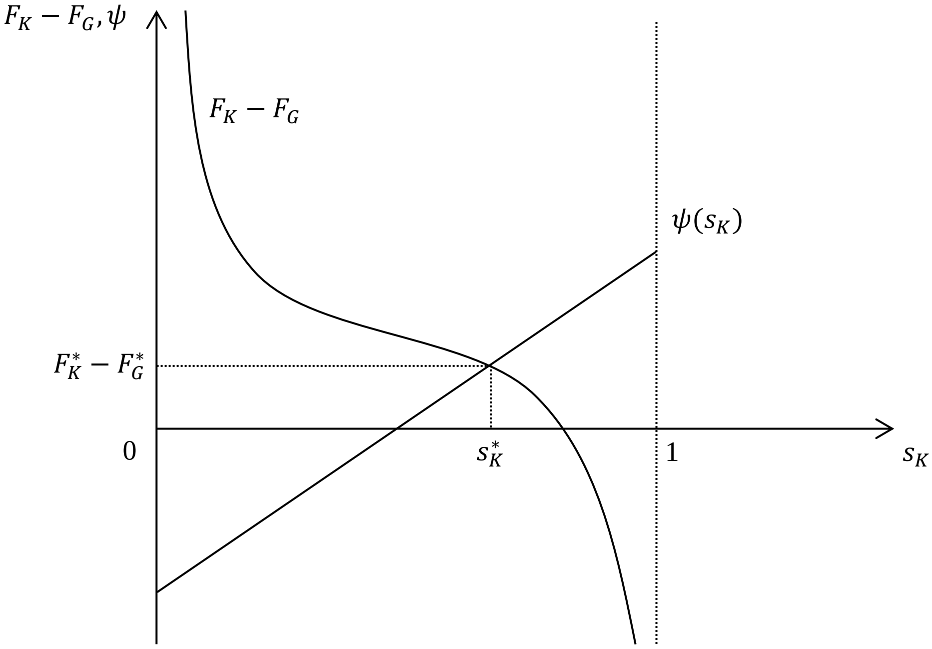

Figure 2b: $0 < \lambda < \sigma_G/\sigma_K < 1/\lambda$ or $-1 \leq \lambda \leq 0$

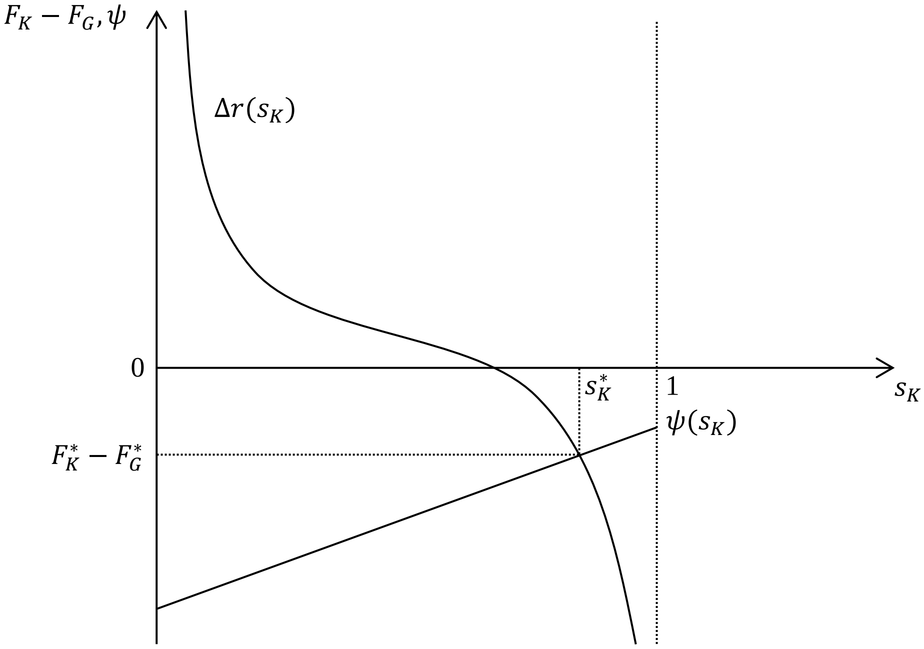

Figure 2c: $0 < 1/\lambda < \sigma_G/\sigma_K$

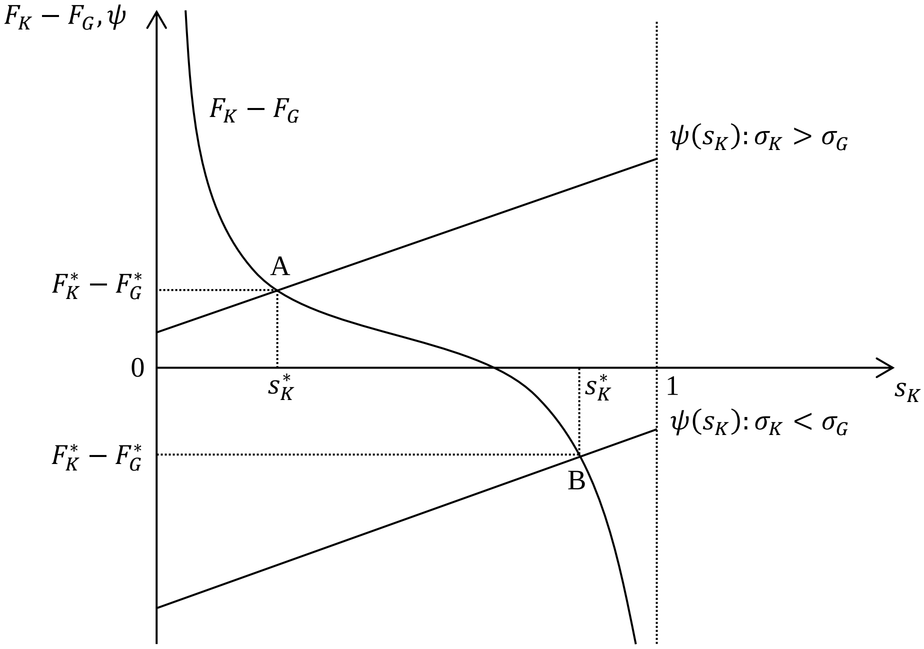

Figure 2d: $\lambda = 1$