ON FROBENIUS SPLIT ABELIAN FIBER SPACES OVER CURVES
（曲線上の FROBENIUS 分裂するアーベルファイバー空間について）

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Abstract. In this thesis, we study the Frobenius split property of Abelian fiber spaces over curves in positive characteristic. A variety is called Frobenius split if its absolute Frobenius morphism splits. The Frobenius split property is a global and strong condition for a projective variety over a field of positive characteristic. In particular, if a variety X has a fibration structure $\pi : X \to Y$, then the Frobenius split property of X imposes strong conditions on Y and fibers of π. We are interested in the Frobenius split property of varieties X if $\pi : X \to C$ is an elliptic fibration, and more generally, Abelian fiber spaces over curves. To be more precise, we explore Frobenius split elliptic surfaces and Abelian fiber spaces over curves.

As a consequence, we can classify Frobenius split Abelian fiber spaces over curves and elliptic fibrations with only tame fibers, in terms of the degree of the top higher direct image, non-ordinary points, the multiplicities of multiple fibers, and the characteristic of the base field.

Contents

Notations

- k is an algebraically closed field of characteristic $p > 0$.
- A variety is a reduced, irreducible and separated scheme of finite type over k.
- A curve or surface are a one and a two dimensional variety, respectively.
- ω_X is the canonical sheaf of a smooth variety X.
- K_X is the canonical divisor of a smooth variety X.
- $h^i(X, \mathcal{F})$ is the dimension of $H^i(X, \mathcal{F})$ as a k-vector space.
- $\lceil \alpha \rceil$ is the integer defined by $\alpha \leq \lceil \alpha \rceil < \alpha + 1$ for every number $\alpha \in \mathbb{R}$.
- $\lfloor \alpha \rfloor$ is the integer defined by $\alpha - 1 < \lfloor \alpha \rfloor \leq \alpha$ for every number $\alpha \in \mathbb{R}$.
- For a $\mathbb{Q}$-divisor $D = \sum_i^r d_i D_i$ such that D_i is a prime divisor and $d_i \in \mathbb{Q}$, we define the round up $\lceil D \rceil$ (resp. round down $\lfloor D \rfloor$) as $\sum_i^r \lceil d_i \rceil D_i$ (resp. $\sum_i^r \lfloor d_i \rfloor D_i$).
- For divisors D_1 and D_2, $D_1 \equiv D_2$ means that D_1 is numerically equivalent to D_2 (see Theorem 2.1.21).
- Given a vector bundle $\mathcal{E}$, $\mathrm{Sym}^m \mathcal{E}$ denotes the m-th symmetric power of $\mathcal{E}$ (see Proposition 2.1.20).
- If $\mathcal{E}$ is a vector bundle on a variety X, $\mathbb{P}(\mathcal{E})$ denotes the projective bundle which is defined by $\mathbf{Proj}_X \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathrm{Sym}^m \mathcal{E}$ (Fact 1.0.2).
- P_i is the divisor such that $F_{b_i} = m_i P_i$, where F_{b_i} is the multiple fiber (see Theorem 2.2.6).
- m_i is the multiplicities of a multiple fiber of an Abelian fiber space $\pi : X \to C$ (see Theorem 2.2.6).
- a_i is the coefficient of the divisor P_i which appears in the canonical bundle formula of Abelian fiber spaces (see Theorem 2.2.6).

1. Introduction

In this thesis, we work on a variety defined over a field of characteristic $p > 0$. We discuss a Frobenius split variety X over an algebraically closed field k of characteristic $p > 0$, i.e., a variety whose absolute Frobenius morphism splits. We study this Frobenius split property when a variety has fibration structures over curves, and especially we are interested in the Frobenius split property of elliptic fibrations, more generally Abelian fiber spaces. As a main result, we classify Frobenius split Abelian fiber spaces over curves and elliptic fibrations with only tame fibers in terms of the degree of the top higher direct image, non-ordinary points, the multiplicities of multiple fibers and the characteristic of the base field. Further, we explore the existence of multiple fibers of elliptic surfaces. We give some examples of multiple fibers of Frobenius split elliptic surfaces with only tame fibers.

The absolute Frobenius morphism

$$F_X : X \to X$$

is just an identity map on the topological space of X, but as the morphism between coordinate rings, it is the p-th power map

$$\begin{array}{rcl} F_X^\sharp : \mathcal{O}_X & \to & (F_X)_*\mathcal{O}_X. \\ f & \mapsto & f^p \end{array}$$

Here, $(F_X)_*\mathcal{O}_X$ is just $\mathcal{O}_X$ as a sheaf. However, we can regard that $(F_X)_*\mathcal{O}_X$ as an $\mathcal{O}_X$-module via this $F_X^\sharp$. If this p-th power map $F_X^\sharp : \mathcal{O}_X \to (F_X)_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-module, we call X Frobenius split. To be more precise, a Frobenius split variety is defined as follows.

Definition 1.0.1. [MR, Definition 2] *Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$. We say that X is Frobenius split (for short, F-split) if the map $F_X^\sharp : \mathcal{O}_X \to (F_X)_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-module. Namely, there is a splitting morphism $\phi : (F_X)_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\phi \circ F_X^\sharp = \mathrm{id}_{\mathcal{O}_X}$.*

The notion of F-split varieties was introduced by Mehta and Ramanathan [MR] in the 1980's to investigate the cohomology of Schubert varieties in positive characteristic. The F-split property is a global and strong condition on projective varieties in positive characteristic, and F-split varieties satisfy some nice properties although the definition of F-split varieties is simple. For example, any F-split varieties satisfy the Kodaira vanishing theorem in positive characteristic (see Theorem 2.1.14). When a morphism $\pi : X \to Y$ satisfies $\pi_*\mathcal{O}_X = \mathcal{O}_Y$, the image of an F-split variety is also F-split (see Lemma 2.1.12). F-split varieties are also considered as a generalization of the ordinarity of Abelian varieties. Let A be an Abelian variety over an algebraically closed field k of characteristic $p > 0$. We can define the p-rank of A by the structure of p-torsion points of A. If the p-rank is equal to the dimension of A, it is said to be ordinary. It is known that A is ordinary if and only if A is F-split (see Remark 2.1.9). Moreover, it was shown by Mehta and Srinivas [MS], that smooth projective varieties with the trivial cotangent bundle are ordinary in the sense of Bloch-Kato [BlKa] if and only if they are F-split.

Since the introduction of F-split varieties in [MR], they have appeared appeared in many areas of algebraic geometry. In particular, F-split varieties are useful for studying birational geometry and representation theory in positive characteristic. In birational geometry, we have to consider the Minimal Model Program (for short, MMP). In characteristic 0, the Kodaira vanishing theorem or the Kawamata-Viehweg vanishing theorem are crucial theorems when we need to run the MMP. On the other hand, in positive characteristic, we have counter examples of these vanishing theorems. So it was expected to be difficult to run the MMP in positive characteristic. However, in recent years, the notion of F-split varieties are getting to be in an important position in birational geometry in positive characteristic. Hacon and Xu [HX] showed that the existence of minimal models under some assumption. In their proof, the notion of F-split varieties are important.

By the way, when we have a morphism $\pi : X \to Y$ between smooth projective varieties, it is a natural question of whether the F-split property of X (resp. Y) is inherited by that of Y (resp. X). For this question, we have several answers as follows.

- If π is a finite morphism and Y is F-split, then the condition of X to be F-split is given by a splitting section of Y (see Theorem 2.1.8) and the ramification divisor of π ([ST, Theorem 5.7]).
- If π satisfies $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and X is F-split, then Y is also F-split (see Lemma 2.1.12).

In general, the F-split property of X often implies strong conditions for the families of varieties. To be more precise, when a morphism $\pi : X \to Y$ satisfies $\pi_*\mathcal{O}_X = \mathcal{O}_Y$, the condition of X to be F-split implies some strong conditions on not only Y but also fibers of π. In fact, if X is F-split, then the general fibers of π are also F-split by the similar argument in [GLSTZP, Theorem 2.1]. The simplest nontrivial families of varieties are ruled surfaces. For a ruled surface $\pi : X \to C$, the condition of X to be F-split can be characterized as the follows:

Fact 1.0.2. *(c.f., Theorem 2.1.21)* [GT, Proposition 3.1] [MS1, Remark 1] *Let $\pi : X \to C$ be a relatively minimal ruled surface over an algebraically closed field k of characteristic $p > 0$. Suppose that X is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E})$ for a rank 2 vector bundle $\mathcal{E}$ on C. Then X is F-split if and only if $\mathcal{E}$ is decomposable and C is F-split.*

1.1. **Main Results.** Main results are based on the paper [Shira] which states the classification of F-split elliptic surfaces and Abelian fiber spaces over curves with only tame fibers. As we see before, we have a characterization of F-split ruled surfaces in Fact 1.0.2. Thus we are interested in the case where the genus of general fibers are one. To be more precise, we have the following question.

Question 1.1.1. *Which relatively minimal elliptic fibrations $\pi : X \to C$ are F-split? More generally, which Abelian fiber spaces over curves are F-split?*

Here, an Abelian fiber space is a generalization of an elliptic fibration. The general fibers of elliptic fibrations are elliptic curves, and the general fibers of an Abelian fiber space are Abelian varieties. Abelian fiber spaces over curves have some properties which are similar to elliptic fibrations. In particular, we have the Kodaira's canonical bundle formula in Abelian fiber space over curves, which is due to Yasuda [Ya] (see Theorem 2.2.6).

In this thesis, we classify n-dimensional F-split Abelian fiber spaces over curves and elliptic fibrations with only tame fibers in terms of the degree of $R^{n-1}\pi_*\mathcal{O}_X$, the multiplicities of multiple fibers, non-ordinary points, and the characteristic of the base field p. We denote $(b_1, \cdots, b_r; m_1, \cdots, m_r)$ as the multiple fibers $\pi^{-1}(b_i)$ with multiplicities m_i. After changing coordinates appropriately, we have the following theorem:

Theorem 1.1.2. *(c.f., Theorem 3.3.6) Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space over an algebraically closed field of characteristic $p > 0$ and H a hyperplane section of X. Assume $(K_X^2 \cdot H^{n-2}) = 0$ and set $\mathcal{L}_{n-1} := R^{n-1}\pi_*\mathcal{O}_X$. Further assume that $\mathcal{L}_{n-1}$ is a locally free sheaf on C and all jumping numbers are 1. The only possibilities for C are either $\mathbb{P}^1$ or an ordinary elliptic curve.*

If C is an ordinary elliptic curve, then X is F-split if and only if $\mathrm{ord}(K_X) \mid (p-1)$ and the general fibers of π are ordinary Abelian varieties.

Conditions in the case $C = \mathbb{P}^1$ are dependent on $\deg(\mathcal{L}_{n-1}^{-1})$. *Set $D = \sum_i \frac{m_i - 1}{m_i} b_i$ the divisor on $\mathbb{P}^1$ where $\pi^{-1}(b_i)$ is a multiple fiber and m_i is the multiplicity of the fiber at b_i, $(\beta_1 : \alpha_1), \cdots, (\beta_r : \alpha_r) \in \mathbb{P}^1$ non-ordinary points, and $n_1, \cdots, n_r$ the order of non-ordinary points. We have the following possible configurations for multiplicities of multiple fibers, depending on* $\deg(\mathcal{L}_{n-1}^{-1})$.

(1) *If* $\deg(\mathcal{L}_{n-1}^{-1}) = 0$ *then X is F-split if and only if the general fibers of π are ordinary Abelian varieties, and $(\mathbb{P}^1, D)$ is F-split.*
(2) *If* $\deg(\mathcal{L}_{n-1}^{-1}) = 1$ *then X is F-split if and only if the general fibers of π are ordinary Abelian varieties, and one of the following conditions occur*
  (a) *There are no multiple fibers,*
  (b) *There is one multiple fiber $(0; m)$ and global section s of $\mathcal{L}_{n-1}^{\otimes(1-p)}$ with degree $(p-1)$ such that $s = (\alpha_1 x - \beta_1 y)^{n_1}(\alpha_2 x - \beta_2 y)^{n_2} \cdots (\alpha_r x - \beta_r y)^{n_r}$ on $\mathbb{P}^1$ where at least any one of the coefficients of $x^i y^{p-1-i}$, $0 \leq i < \lceil \frac{p}{m} \rceil$ are nonzero,*
  (c) *There are two multiple fibers $(0, \infty; 2, 2)$, $p > 2$ and global sections s of $\mathcal{L}_{n-1}^{\otimes(1-p)}$ with degree $(p-1)$ such that $s = (\alpha_1 x - \beta_1 y)^{n_1}(\alpha_2 x - \beta_2 y)^{n_2} \cdots (\alpha_r x - \beta_r y)^{n_r}$ on $\mathbb{P}^1$ where the coefficient of $(xy)^{\frac{p-1}{2}}$ is nonzero,*
(3) *If* $\deg(\mathcal{L}_{n-1}^{-1}) = 2$ *then X is F-split if and only if the general fibers of π are ordinary Abelian varieties, there are no multiple fibers and a global section s of $\mathcal{L}_{n-1}^{\otimes(1-p)}$ of degree $2(p-1)$ such that $s = (\alpha_1 x - \beta_1 y)^{n_1}(\alpha_2 x - \beta_2 y)^{n_2} \cdots (\alpha_r x - \beta_r y)^{n_r}$ on $\mathbb{P}^1$ where the coefficient of $(xy)^{p-1}$ is nonzero.*

We emphasize that the words "possible" in Theorem 1.1.2 mean that the existence of such multiple fibers in the above list as an algebraic variety is not known. However, in Section 3.4, we discuss the existence of multiple fibers in the above list in the 2-dimensional case. Namely, if such multiple fibers in the above list exist as algebraic varieties, we give an example; otherwise, we prove the nonexistence of multiple fibers as an algebraic variety by using a necessary condition for the algebraicity of the elliptic surface due to Katsura and Ueno [KU]. Eventually, we have the following F-split elliptic surfaces with only tame fibers, including the existence of multiple fibers:

Theorem 1.1.3. *(c.f., Corollary 3.4.13) For relatively minimal elliptic fibrations $\pi : X \to C$ over an algebraically closed fields of positive characteristic $p > 0$, every case in Theorem 1.1.2 arises except where $\chi(\mathcal{O}_X) = 0$. In this case there are no elliptic fibrations $\pi : X \to \mathbb{P}^1$ where π has only one multiple fiber, two multiple fibers with different multiplicities, and three multiple fibers with multiplicities $(2, 2, d)$, $(d \geq 3)$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$.*

In Theorem 1.1.3, our assumption is more natural than an Abelian fiber space since we can remove the assumption of jumping numbers, and the degree of $R^1\pi_*\mathcal{O}_X$ is equal to the anti-Euler characteristic $-\chi(\mathcal{O}_X)$.

1.2. **The idea for the proof.** The idea for the proof of Theorem 1.1.2 is to characterize the F-split property of the total space X by the relative Frobenius morphism $F_{X/C}$ as follows:

Proposition 1.2.1. *(c.f., Proposition 3.1.1) Let $\pi : X \to C$ be a surjective morphism with $\pi_*\mathcal{O}_X = \mathcal{O}_C$ from an n-dimensional smooth projective variety to a smooth projective curve C over an algebraically closed field k of characteristic $p > 0$. Then the absolute Frobenius action of X*

$$F_X^* : H^n(X, \omega_X) \to H^n(X, \omega_X^{\otimes p})$$

is nonzero if and only if the composition of the following morphisms

$$H^1(C, R^{n-1}\pi_*\omega_X) \xrightarrow{F_C^*} H^1\left(C, F_C^*(R^{n-1}\pi_*\omega_X)\right) \xrightarrow{F_{X/C}^*} H^1(C, R^{n-1}\pi_*\omega_X^{\otimes p})$$

is nonzero.

An F-split variety is defined by the absolute Frobenius morphism. The absolute Frobenius morphism is defined by just mapping the p-th powers. Then the absolute Frobenius morphism is not a k-morphism, except for $k = \mathbb{F}_p$. This results in many problems when we consider families of varieties $\pi : X \to Y$ since the absolute Frobenius morphism is not compatible with the restriction to fibers. To be more precise, when we restrict the p-th power map $F_X^{\#} : \mathcal{O}_X \to (F_X)_*\mathcal{O}_X$ to the fiber E, the induced map $(F_X^{\#})|_E : \mathcal{O}_E \to ((F_X)_*\mathcal{O}_X)|_E$ is not the p-th power map on E. However, we have the relative Frobenius morphism $F_{X/Y}$ for a morphism $\pi : X \to Y$. The relative Frobenius morphism $F_{X/Y}$ is defined as the universal map of the fiber product of F_Y and π. The relative Frobenius morphism is compatible with the restriction to the fibers. Thus, it is useful to study the F-split property for families of varieties. In particular, if Y is a curve C, then we have the characterization of the total space X to be F-split via the relative Frobenius morphism $F_{X/C}$ in Proposition 1.2.1.

In Proposition 1.2.1, the absolute Frobenius action on the base curve F_C^* is just the p-th power map. In contrast, the relative Frobenius action $F_{X/C}^*$ is complicated in general. However, the relative Frobenius action $F_{X/C}^*$ in Proposition 1.2.1 has remarkable properties in the case of Abelian fiber spaces over curves $\pi : X \to C$. We can interpret the relative Frobenius action $F_{X/C}^*$ as the ordinarities of the general fibers as follows;

Proposition 1.2.2. *(c.f., Proposition 3.2.3) Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space, and we assume that $\mathcal{L}_{n-1} := R^{n-1}\pi_*\mathcal{O}_X$ is a locally free sheaf on C. We denote a fiber product of F_C and π as $X^{(p)}$ and the projection $X^{(p)} \to C$ as $\pi^{(p)}$. If the general fibers of π are ordinary Abelian varieties, then the relative Frobenius action*

$$F_{X/C}^* : R^{n-1}\pi_*^{(p)}\mathcal{O}_{X^{(p)}} \to R^{n-1}\pi_*^{(p)}\left((F_{X/C})_*\mathcal{O}_X\right)$$

is isomorphic to the following morphism up to rational functions:

$$\times s : F_C^* R^{n-1}\pi_*\mathcal{O}_X \to R^{n-1}\pi_*\mathcal{O}_X,$$

where s in $H^0(C, \mathcal{L}_{n-1}^{\otimes(1-p)})$ is a non-zero section; otherwise, $F_{X/C}^*$ is zero map.

One has the relative Frobenius morphism $F_{X/C}^{\sharp} : \mathcal{O}_{X^{(p)}} \to (F_{X/C})_*\mathcal{O}_X$ and we denote the cokernel of $F_{X/C}^{\sharp}$ as $\mathcal{B}_{X/C}$. Taking the higher direct image of $\pi^{(p)}$, we have the exact sequence $R^{n-1}\pi_*^{(p)}\mathcal{O}_{X^{(p)}} \to R^{n-1}\pi_*^{(p)}((F_{X/C})_*\mathcal{O}_X) \to R^{n-1}\pi_*^{(p)}\mathcal{B}_{X/C} \to 0$. The first morphism of this sequence can be considered as the element of $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{L}_{n-1}^{\otimes p}, \mathcal{L}_{n-1})$ since we assume the local freeness of $R^{n-1}\pi_*\mathcal{O}_X$. Since $\mathcal{L}_{n-1}$ is a line bundle, this vector space is isomorphic to $H^0(C, \mathcal{L}_{n-1}^{\otimes(1-p)})$. So this morphism is the element of $H^0(C, \mathcal{L}_{n-1}^{\otimes(1-p)})$ up to rational functions. By this identification, this morphism has only zeros in the support of $R^{n-1}\pi_*^{(p)}\mathcal{B}_{X/C}$ and fibers at these points are non-ordinary Abelian varieties. So we call these points non-ordinary points (see Definition 3.2.1). If every point is non-ordinary point, then this element is trivial in $H^0(C, \mathcal{L}_{n-1}^{\otimes(1-p)})$. So the relative Frobenius action is zero map. Otherwise we have a non-zero element s in $H^0(C, \mathcal{L}_{n-1}^{\otimes(1-p)})$ and this s is determined up to scalars by non-ordinary points and their orders (see Lemma 3.2.4).

By Propositions 1.2.1 and 1.2.2, we consider the composition map in Proposition 1.2.1 which consists of the absolute Frobenius action F_C^* and the global section of $\mathcal{L}_{n-1}^{\otimes(1-p)}$ which has zeros at non-ordinary points. The first morphism F_C^* induces the F-split pair (for the definition of F-split pair, see Definition 3.3.3) $\left(C, \sum_i \frac{a_i}{m_i} \cdot b_i\right)$, where a_i is the coefficient which appears in the canonical bundle formula, m_i is the multiplicity of a multiple fiber and $\pi^{-1}(b_i)$

is the multiple fiber. In Theorem 1.1.2, since we assume that $R^{n-1}\pi_*\mathcal{O}_X$ is locally free and all jumping numbers are 1, this a_i is always $m_i - 1$. Then we can use Watanabe's classification [Wa, Theorem 4.2] of the F-split pair $\left(C, \sum_i \frac{m_i-1}{m_i} \cdot b_i\right)$ (see Theorem 3.3.4). We can calculate the composition map in Proposition 1.2.1 on case-by-case by Watanabe's results. After this calculation, we obtain the statement of Theorem 1.1.2.

Moreover, we explore the existence of multiple fibers of elliptic surfaces which are obtained in Theorem 1.1.2. When we have relatively minimal elliptic fibrations $\pi : X \to C$ with multiple fibers, some of them does not exist as algebraic varieties. Therefore we are interested in the existence of elliptic fibration and multiple fibers in Theorem 1.1.2. In Section 3.4, we will give some examples of multiple fibers of elliptic surfaces if they exist. However how should we prove the nonexistence of elliptic surfaces and multiple fibers if they does not exist ? The result of Katsura and Ueno provides an answer for this question (see Theorem 3.4.11). To be more precise, Theorem 3.4.11 gives a necessary condition for algebraicities of elliptic surfaces. By Theorem 3.4.11 and examples which we will give in Section 3.4, we have F-split elliptic surfaces with only tame fibers including the existence of multiple fibers, i.e., we obtained Theorem 1.1.3.

In Section 3.4, we will give some examples of F-split elliptic surfaces. In Example 3.4.1 in char $k = 2$, where it is called Shioda Kummer surfaces, this is a new example of F-split elliptic surfaces. It has an elliptic fibration structure, then we show that such surfaces satisfy F-split property as a corollary of Theorem 1.1.2. The other examples are known as F-split surfaces by other method. We will show that these examples satisfy F-split property by using elliptic fibration structures.

The organization of this paper is as follows. In Section 2, we will present some basic notions of F-split varieties and Abelian fiber spaces. Section 3 will be based on the paper [Shira] and we will discuss the Frobenius action of the total spaces of fibrations over the curves $\pi : X \to C$ in Section 3.1. We will characterize X to be F-split via the relative Frobenius morphism $F_{X/C}$. In Section 3.2, we will treat Abelian fiber spaces, where we can interpret the relative Frobenius action $F^*_{X/C}$ just as the multiplication map under the assumption that the general fibers are ordinary Abelian varieties. In Section 3.3, we will state Theorem 1.1.2. In Section 3.4, we will consider the existence of multiple fibers in the list of Theorem 1.1.2 and show examples of F-split elliptic surfaces.

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2. Basic facts about Frobenius split varieties and Abelian fiber spaces over curves

In this section, we will survey basic facts about Frobenius split varieties and Abelian fiber spaces over curves. In Section 2.1, we will study some properties about Frobenius split varieties. In Section 2.2, we will consider Abelian fiber spaces over curves and their properties. Especially, we recall the canonical bundle formula of Abelian fiber spaces over curves, due to [Ya]. In section 2.3, we treat the canonical bundle formula in elliptic fibration and jumping number with tame fibers.

2.1. **Frobenius split varieties.**

We review the definition of the Frobenius morphism and Frobenius split varieties that was introduced by Mehta and Ramanathan [MR]. We treat the absolute Frobenius morphism and some properties that Frobenius split varieties satisfy in this section.

Definition 2.1.1. *Let X be a smooth projective variety. Then the absolute Frobenius morphism*

$$F_X : X \to X$$

is the identity map on the topological space of X, and the p-th power map on the structure sheaf $\mathcal{O}_X$

$$\begin{array}{rcl} F_X^\sharp : \mathcal{O}_X & \to & (F_X)_*\mathcal{O}_X, \\ f & \mapsto & f^p. \end{array}$$

Note that the action of $\mathcal{O}_X$ to $(F_X)_*\mathcal{O}_X$ is defined by p-th power. To be more precise, the action is defined by $a * f := a^p f$ for $a, f \in \mathcal{O}_X$.

For any morphism $\pi : X \to Y$ of smooth projective varieties, the following diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "F_X"'] & Y \arrow[d, "F_Y"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

commutes. If we take a fiber product of (π, F_Y), then we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dd, "F_X"'] \arrow[dr, "F_{X/Y}"] & & Y \arrow[dd, "F_Y"] \\ & X \times_Y Y \arrow[ur] \arrow[dl] & \\ X \arrow[rr, "\pi"] & & Y. \end{tikzcd}$$

Definition 2.1.2. *Let $\pi : X \to Y$ be a morphism between smooth projective varieties. The relative Frobenius morphism is the universal map of the above diagram*

$$F_{X/Y} : X \to X^{(p)} := X \times_Y Y.$$

We denote the first projection by $W : X^{(p)} \to X$ and the second projection by $\pi^{(p)} : X^{(p)} \to Y$.

Definition 2.1.3. [MR, Definition 2] *Let X be a smooth projective variety. We say that X is Frobenius split (for short, F-split) if the map $F_X^\sharp : \mathcal{O}_X \to (F_X)_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-module. Namely, there is a morphism $\varphi : (F_X)_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\varphi \circ F_X^\sharp = \mathrm{id}_{\mathcal{O}_X}$. We call this φ a splitting morphism of X.*

We give some examples of F-split varieties. Firstly we see that $\mathbb{P}^1$ is F-split just by the definition of F-split as follows.

Example 2.1.4. [Sch, Example 3.4] *Let X be projective line $\mathbb{P}^1$. Since $(F_X)_*\mathcal{O}_X$ is a rank p vector bundle on $\mathbb{P}^1$, we can describe $(F_X)_*\mathcal{O}_X$ as follows:*

$$(F_X)_*\mathcal{O}_X \cong \mathcal{O}_X(a_1) \oplus \cdots \oplus \mathcal{O}_X(a_p),$$

where all of the values of a_i's are integers such that $a_1 \leq \cdots \leq a_p$. We can determine these values of a_i's. Indeed, taking global sections $H^0(X, -)$, a_p is 0. Tensoring with $\mathcal{O}_X(1)$ to the above isomorphism, we have a canonical isomorphism $(F_X)_*\mathcal{O}_X \otimes \mathcal{O}_X(1) \cong (F_X)_*\mathcal{O}_X(p)$ by the projection formula. We have the following equation:

$$h^0(X, \mathcal{O}_X(p)) = h^0(X, \mathcal{O}_X(a_1+1)) + \cdots + h^0(X, \mathcal{O}_X(a_p+1)).$$

Since $a_1 \leq \cdots \leq a_{p-1}$, we see that $a_1 = \cdots = a_{p-1} = -1$. Then we have the following isomorphism:

$$(F_X)_*\mathcal{O}_X \cong \mathcal{O}_X(-1) \oplus \cdots \oplus \mathcal{O}_X(-1) \oplus \mathcal{O}_X.$$

Obviously, if we choose the p-th projection as the splitting morphism of $F_X^\sharp$, $\mathbb{P}^1$ is F-split.

It is difficult to check whether a variety X is F-split or not just by the definition even for projective space $\mathbb{P}^N$. However it is known that $\mathbb{P}^N$ are F-split because of the following theorem.

Theorem 2.1.5. [Hart1, Corollary 6.4] *Let X be projective space $\mathbb{P}^N$. Then for any integer $m \in \mathbb{Z}$, the Frobenius push-forward $(F_X)_*\mathcal{O}_X(m)$ is isomorphic to $\bigoplus_{i=1}^{p^N} \mathcal{O}_X(m_i)$ for some integer $m_i \in \mathbb{Z}$.*

If we admit Theorem 2.1.5, we can see that $\mathbb{P}^N$ is F-split similar to $\mathbb{P}^1$. We want to have other examples of F-split varieties. Now we observe splitting morphisms of the absolute Frobenius morphism. A F-split variety has splitting morphisms of the absolute Frobenius morphism. The existence of splitting morphisms is equivalent to the surjectivity of the trace map. Let X be an n-dimensional smooth projective variety. For a close point $x \in X$, a system of local coordinates $t_1, \cdots, t_n$ of X at x is a minimal system of generators of the maximal ideal of the local ring $\mathcal{O}_{X,x}$. The monomial $t_1^{i_1} \cdots t_n^{i_n}$ will be denoted by $t^{\mathbf{i}}$, where $\mathbf{i} := (i_1, \cdots, i_n) \in \mathbb{Z}^n_{\geq 0}$.

Definition 2.1.6. [BK, Lemma 1.3.6] *Let X be an n-dimensional smooth projective variety and $t_1, \cdots, t_n$ a system of local coordinates at $x \in X$. We define the additive map Tr for any elements of $f := \sum_{\mathbf{i}} c_{\mathbf{i}} t^{\mathbf{i}} \in \mathcal{O}_{X,x} \subset k[[t_1, \cdots, t_n]]$ by*

$$\mathrm{Tr}\left(\sum_{\mathbf{i}} c_{\mathbf{i}} t^{\mathbf{i}}\right) := \sum c_{\mathbf{i}}^{1/p} t^{\mathbf{j}},$$

where $c_{\mathbf{i}} \in k$ and the summation on the right side is taken over those $\mathbf{i}$ such that $\mathbf{i} = \mathbf{p} - \mathbf{1} + p\mathbf{j}$ for some $\mathbf{j} \in \mathbb{Z}^n_{\geq 0}$. In particular $\mathrm{Tr}(f) \in \mathcal{O}_{X,x}$.

Observation 2.1.7. *Let $\mathcal{B}$ the cokernel of $\mathcal{O}_X \to (F_X)_*\mathcal{O}_X$ i.e., we have the short exact sequence*

$$0 \to \mathcal{O}_X \to (F_X)_*\mathcal{O}_X \to \mathcal{B} \to 0.$$

If we apply $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ to the above sequence, we have the long exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{B}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}((F_X)_*\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\epsilon} \mathcal{O}_X \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{B}, \mathcal{O}_X) \to \cdots.$$

By the Grothendieck duality [Hart, Exercise 6.10, Chapter III]*, $\mathcal{H}om_{\mathcal{O}_X}((F_X)_*\mathcal{O}_X, \mathcal{O}_X)$ is isomorphic to $(F_X)_*\omega_X^{\otimes(1-p)}$. Then we have the following map which is called the trace map*

$$\widehat{\tau} : (F_X)_*\omega_X^{\otimes(1-p)} \quad \to \quad \mathcal{O}_X.$$

Theorem 2.1.8. [BK, Theorem 1.3.8] *Let X be an n-dimensional smooth projective variety. The above map $\epsilon : \mathcal{H}om_{\mathcal{O}_X}((F_X)_*\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{O}_X$ is identified with the trace map*

$$\widehat{\tau} : (F_X)_*\omega_X^{\otimes(1-p)} \quad \to \quad \mathcal{O}_X$$

which is given by at any closed point $x \in X$ by

$$\widehat{\tau}(f(dt_1 \wedge \cdots \wedge dt_n)^{1-p}) = \mathrm{Tr}(f),$$

for all $f \in \mathcal{O}_{X,x}$. Here $t_1, \cdots, t_n$ is a system of local coordinates at $x \in X$. Then X is F-split if and only if there exists a nonzero section $s \in H^0(X, \omega_X^{\otimes(1-p)})$ such that $\mathrm{Tr}(s) = 1$. We call such a nonzero section s splitting section of X.

Equivalently, by the Serre duality of the trace map $\widehat{\tau}$, we have the following the absolute Frobenius action.

Remark 2.1.9. [BK, Remark 1.3.9] *Let X be an n-dimensional smooth projective variety. Then X is F-split if and only if the absolute Frobenius action*

$$F_X^* : H^n(X, \omega_X) \to H^n(X, \omega_X^{\otimes p})$$

is nonzero. For example, an elliptic curve X is F-split if and only if X is not supersingular. Moreover, a variety of general type is never F-split by Theorem 2.1.8. Then smooth projective F-split curves are only $\mathbb{P}^1$ or ordinary elliptic curves.

Definition 2.1.10. *Let X be an n-dimensional Abelian variety. We say X is ordinary if the map*

$$\begin{array}{rcl} F_X^* : H^n(X, \mathcal{O}_X) & \to & H^n(X, \mathcal{O}_X) \\ \xi & \mapsto & \xi^p \end{array}$$

is nonzero.

Remark 2.1.11. *An ordinary Abelian variety X is originally defined by its p-rank. It is known that ordinarity of X by p-rank is equivalent to Definition 2.1.10 (see* [MS2, Example 5.4]*).*

The following lemma states that an image of F-split variety is also F-split when the fibers of a morphism are connected as a geometric interpretation.

Lemma 2.1.12. [BK, Lemma 1.1.8] *Let $f : X \to Y$ be a morphism of smooth projective varieties such that $\mathcal{O}_Y$ is isomorphic to $f_*\mathcal{O}_X$. If X is F-split, then Y is also F-split.*

Proof. Let φ be a splitting morphism of $F_X^\sharp : \mathcal{O}_X \to (F_X)_*\mathcal{O}_X$. Taking push-forward f_* to this φ, we have the following,

$$f_*\varphi : f_*(F_X)_*\mathcal{O}_X \to f_*\mathcal{O}_X.$$

Since $f_*((F_X)_*\mathcal{O}_X) = (F_Y)_*(f_*\mathcal{O}_X) \cong (F_Y)_*\mathcal{O}_Y$, this $f_*\varphi$ maps $(F_Y)_*\mathcal{O}_Y$ to $\mathcal{O}_Y$. Note that this $f_*\varphi$ maps 1 to 1. Then this $f_*\varphi$ is a splitting morphism of $F_Y^\sharp : \mathcal{O}_Y \to (F_Y)_*\mathcal{O}_Y$, since $f_*\varphi \circ F_Y^\sharp(a) = f_*\varphi(a^p) = f_*\varphi(a * 1) = a \cdot f_*\varphi(1) = a$. $\square$

It is famous that in positive characteristic, we have counter-examples of the Kodaira vanishing theorem which is due to Raynaud [Ray1]. But if X is F-split, X satisfies the Kodaira vanishing theorem as follows.

Lemma 2.1.13. [MR, Proposition 1] *Let X be an F-split variety and $\mathcal{L}$ a line bundle. For all large $m \gg 0$ and fixed i, if $H^i(X, \mathcal{L}^{\otimes m}) = 0$, then $H^i(X, \mathcal{L}) = 0$.*

Proof. Since X is F-split, then the following p-th power map splits,

$$F_X^\sharp : \mathcal{O}_X \to (F_X)_* \mathcal{O}_X.$$

Tensoring with $\mathcal{L}$ to this $F_X^\sharp$, the following morphism also splits

$$\mathcal{L} \to (F_X)_* \mathcal{L}^{\otimes p}.$$

Taking i-th cohomology, we have the following

$$H^i(X, \mathcal{L}) \to H^i(X, \mathcal{L}^{\otimes p}).$$

Note that this morphism between i-th cohomology also splits, so this is injective especially. If we repeat same operation, we have the following injective sequence,

$$H^i(X, \mathcal{L}) \hookrightarrow H^i(X, \mathcal{L}^{\otimes p}) \hookrightarrow \cdots \hookrightarrow H^i(X, \mathcal{L}^{\otimes p^e}) \hookrightarrow \cdots$$

Thus $H^i(X, \mathcal{L}^{\otimes m}) = 0$ for all large $m \gg 0$ implies $H^i(X, \mathcal{L}) = 0$. $\square$

Theorem 2.1.14. [MR, Proposition 2] *Let X be an F-split variety. X satisfies the Kodaira vanishing theorem. That is if $\mathcal{L}$ is an ample line bundle, then $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \dim X$.*

Proof. By the Serre vanishing theorem, for large $m \gg 0$, $H^i(X, \omega_X \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$. By the Serre duality, we have $H^i(X, \mathcal{L}^{\otimes(-m)}) = 0$ for $i < \dim X$. By Lemma 2.1.13, this implies $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \dim X$. $\square$

If a variety is F-split, we can define special closed subschemes of F-split varieties which are called "compatibly split subschemes". This special subschemes are also F-split. Let Y be a closed subscheme of X and $\mathcal{I}_Y$ the ideal sheaf defining Y. Then we have the commutative diagram as follows;

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_Y \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_*\mathcal{I}_Y \arrow[r] & F_*\mathcal{O}_X \arrow[r] & F_*\mathcal{O}_Y \arrow[r] & 0
\end{tikzcd}$$

Definition 2.1.15. [MR, Definition 3] [BK, Definition 1.1.3] *Let X be an F-split variety and Y a closed subscheme of X. Y is called compatibly split if there exists a splitting morphism φ of X such that*

$$\varphi((F_X)_* \mathcal{I}_Y) \subset \mathcal{I}_Y.$$

Especially if Y is compatibly split, then Y is F-split, since the following diagram commutes,

$$\begin{tikzcd}
0 \arrow[r] & F_*\mathcal{I}_Y \arrow[r] \arrow[d] & F_*\mathcal{O}_X \arrow[r] \arrow[d, "\varphi"'] & F_*\mathcal{O}_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_Y \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_Y \arrow[r] & 0.
\end{tikzcd}$$

Similar to the F-split property of an ambient variety X, it is difficult to check a compatible split property of closed subschemes Y just by definition. However, we can find compatibly split subschemes of X by Theorem 2.1.8 if Y is a divisor.

Let X be an n-dimensional smooth projective variety and φ a splitting morphism of X. By Theorem 2.1.8, we have the splitting section s in $H^0(X, \omega_X^{\otimes(1-p)})$. If Y is a divisor of X, then the canonical inclusion $\mathcal{O}_X((1-p)Y) \hookrightarrow \mathcal{O}_X$ induces the following exact sequence;

$$0 \to H^0(X, \omega_X^{\otimes(1-p)} \otimes \mathcal{O}_X((1-p)Y)) \xrightarrow{\iota} H^0(X, \omega_X^{\otimes(1-p)}).$$

Proposition 2.1.16. *(c.f.,* [BK, Exercise 1.3.E, (3)]*) Let X be an n-dimensional F-split smooth projective variety and Y a divisor of X. We denote φ by a splitting morphism of X. Then Y is compatibly split with φ if and only if there exists a global section t in $H^0\left(X, \omega_X^{\otimes(1-p)} \otimes \mathcal{O}_X\left((1-p)Y\right)\right)$ such that $\iota(t) = s$.*

Proof. We can regard φ as a global section s in $H^0(X, \omega_X^{\otimes(1-p)})$ by Theorem 2.1.8. If Y is compatibly split, $\varphi|_{F_*\mathcal{O}_X(-Y)}$ is an element of $\mathrm{Hom}_{\mathcal{O}_X}\left((F_X)_*\mathcal{O}_X(-Y), \mathcal{O}_X(-Y)\right)$. This vector space is isomorphic to $H^0(X, \omega_X^{\otimes(1-p)} \otimes \mathcal{O}_X((1-p)Y))$, since we have the following isomorphism between sheaves;

$$\mathcal{H}om_{\mathcal{O}_X}\left((F_X)_*\mathcal{O}_X(-Y), \mathcal{O}_X(-Y)\right) \cong (F_X)_*\left(\omega_X^{\otimes(1-p)} \otimes \mathcal{O}_X((1-p)Y)\right) \tag{1}$$

by [Hart, Exercise 6.10]. Therefore we have a section t in $H^0(X, \omega_X^{\otimes(1-p)} \otimes \mathcal{O}_X((1-p)Y))$ which corresponds to $\varphi|_{F_*\mathcal{O}_X(-Y)}$ and t goes to s via ι.

Conversely if there is a section t in $H^0\left(X, \omega_X^{\otimes(1-p)} \otimes \mathcal{O}_X\left((1-p)Y\right)\right)$ such that $\iota(t) = s$, then we have a morphism $\psi : F_*\mathcal{O}_X(-Y) \to \mathcal{O}_X(-Y)$ which corresponds to t via the isomorphism (1) such that the diagram

$$\begin{tikzcd} F_*\mathcal{O}_X(-Y) \arrow[r, "\psi"] \arrow[d] & \mathcal{O}_X(-Y) \arrow[d] \\ F_*\mathcal{O}_X \arrow[r, "\varphi"'] & \mathcal{O}_X \end{tikzcd}$$

commutes. Therefore this ψ is obtained by restricting φ to $F_*\mathcal{O}_X(-Y)$. $\square$

The following proposition is useful to study an F-split property of a blown-up F-split variety.

Proposition 2.1.17. [LMP, Proposition 2.1] *Let X be a smooth projective F-split variety and Y a closed subscheme of codimension $d \geq 2$. Let $B_Y(X)$ be the blow-up of X along Y. Suppose that section s of $H^0(X, \omega_X^{\otimes(1-p)})$ gives a splitting section of X. Then s vanishes to order at least $(d-1)(p-1)$ generically along Y if and only if s extends to a splitting section $\widetilde{s}$ of $B_Y(X)$.*

We give an example that an F-split variety which is obtained by blown-up of $\mathbb{P}^2$ at general nine points by Proposition 2.1.17.

Example 2.1.18. *Let C_1 and C_2 be two distinct smooth cubic curves that are ordinary in $\mathbb{P}^2$. Note that if the smooth cubic curve $C = \{g(x,y,z) = 0\}$ in $\mathbb{P}^2$ is ordinary if and only if the coefficient of $(xyz)^{p-1}$ in g^{p-1} is nonzero. We assume that the two curves C_1 and C_2 intersect at nine distinct points, which are in the general position. The pencil generated by C_1 and C_2 gives a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. If we blow-up $\mathbb{P}^2$ at these nine points, which is denoted by X, we obtain a morphism $\pi : X \to \mathbb{P}^1$.*

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^1 \\ X \arrow[u, "f"] \arrow[ur, "\pi"'] & \end{tikzcd}$$

By Proposition 2.1.17, this X is F-split.

Remark 2.1.19. *In Example 2.2.5, we will show that this X has an elliptic fibration structure. Therefore this is an example of F-split elliptic surface.*

Finally, we characterize an F-split relatively minimal ruled surface in Theorem 2.1.21.

Proposition 2.1.20. [GT, Proposition 3.1] *Let X be an F-split variety and $\mathcal{E}$ the rank r vector bundle which is isomorphic to $\bigoplus_{i=1}^{r}\mathcal{L}_i$, where $\mathcal{L}_i$'s are line bundles on X. Then the projective bundle $\mathbb{P}(\mathcal{E})$ is also F-split.*

Proof. Since X is F-split, then we have the splitting map as follows,

$$(F_X)_*\mathcal{O}_X \to \mathcal{O}_X, \tag{2}$$

which sends 1 to 1. Consider r-tuple positive integers $(m_1, \cdots, m_r) \in \mathbb{Z}_{\geq 0}^{\oplus r}$. If all m_i's are divisible by p, then we have a map as follows by tensoring with $\mathcal{L}_1^{m_1/p} \otimes \cdots \otimes \mathcal{L}_r^{m_r/p}$ to the morphism (2),

$$(F_X)_*\left(\mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r}\right) \to \mathcal{L}_1^{m_1/p} \otimes \cdots \otimes \mathcal{L}_r^{m_r/p} \hookrightarrow \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathrm{Sym}^m\mathcal{E}.$$

Otherwise (if one m_i is not divisible by p at least), we define the map

$$(F_X)_*\left(\mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r}\right) \to \mathrm{Sym}^m\mathcal{E}.$$

to be zero map. Combine two cases, we have the morphism

$$(F_X)_* \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathrm{Sym}^m\mathcal{E} \to \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathrm{Sym}^m\mathcal{E},$$

which sends 1 to 1. This morphism induces a morphism between projective bundles

$$(F_{\mathbb{P}_X(\mathcal{E})})_*\mathcal{O}_{\mathbb{P}_X(\mathcal{E})} \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}.$$

This is a splitting map of $F^\sharp_{\mathbb{P}_X(\mathcal{E})}$. ☐

The following Theorem 2.1.21 is a characterization of F-split ruled surface.

Theorem 2.1.21. [GT, Proposition 3.1] [MS, Remark 1] *Let $\pi : X \to C$ be a relatively minimal ruled surface over C. Suppose that X is isomorphic to $\mathbb{P}(\mathcal{E})$ for a rank 2 vector bundle $\mathcal{E}$. Then X is F-split if and only if C is F-split and $\mathcal{E}$ is decomposable.*

Proof. When C is F-split and $\mathcal{E}$ is decomposable, the assertion was already proved in Proposition 2.1.20. If C is not F-split, then X is not F-split by Lemma 2.1.12. So we will show that if C is F-split and $\mathcal{E}$ is not decomposable, then X is not F-split.

Since $\mathcal{E}$ is indecomposable, we have $e \leq 0$ by [Hart, Theorem 2.12, Chapter V]. Then C is an ordinary elliptic curve. By [Hart, Corollary 2.11, Chapter V], we have the following;

$$(1-p)K_X \equiv 2(p-1)S + e(p-1)F,$$

where S is the section, F is the fiber and $e := -\deg\mathcal{E}$. When $e < 0$, $(1-p)K_X$ is not linearly equivalent to any effective divisors by [La, p.71, Case II] since $\mathcal{E}$ is indecomposable. This implies X is not F-split by Theorem 2.1.8. In the case of $e = 0$, X is not F-split by the following Lemma 2.1.22. ☐

Lemma 2.1.22. [MS1, Remark 1] *Let $\pi : X \to C$ be a relatively minimal ruled surface on C which is an ordinary elliptic curve. Suppose that X is isomorphic to $\mathbb{P}(\mathcal{E})$ for the rank 2 vector bundle $\mathcal{E}$ with $e := -\deg\mathcal{E} = 0$ and the extension class $\xi \in \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C)$ of $\mathcal{E}$ is nonzero. Then this X is not F-split.*

2.2. **Abelian fiber spaces and elliptic fibrations.** In this subsection, we briefly review definitions, examples, and especially, the canonical bundle formula of an Abelian fiber space. Abelian fiber spaces are considered as a generalization of elliptic fibrations toward higher dimensions.

Definition 2.2.1. [Ya, p.55] *Let X be an n-dimensional smooth projective variety and C a smooth projective curve. A surjective morphism $\pi : X \to C$ is said to be an $(n-1)$-Abelian fiber space if $\pi_*\mathcal{O}_X = \mathcal{O}_C$ and almost all fibers are $(n-1)$-dimensional Abelian varieties. Especially, we call an 1-Abelian fiber space an elliptic fibration.*

In general, when we have a surjective morphism to an integral curve $f : X \to C$, f is the flat morphism by [Hart, Proposition 9.7]. Then every Abelian fiber space over curve $\pi : X \to C$ is a flat morphism but not smooth. Then we have a fiber $F_b := \pi^{-1}(b)$, $b \in C$ such that F_b is not smooth. This fiber F_b is called a *singular fiber*. Some Abelian fiber spaces have singular fibers, but we have fibers not even reduced as follows;

Definition 2.2.2. [Ya, p.57] *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space and $F_b := \pi^{-1}(b)$, $b \in C$ a fiber of π at $b \in C$. A fiber F_b is said to be a multiple fiber of π with multiplicities m if $m \geq 2$ and $F_b = mP$ with $\sum_{i=1}^{r} n_i E_i$ such that $(n_1, \cdots, n_r) = 1$, where E_i's are prime divisors on X.*

We will define a *wild* fiber which appears only in positive characteristic for Abelian fiber spaces. This fiber often appears in low characteristic, i.e., char$k = 2$ or 3. By the flat base change theorem [Hart, Proposition 9.3], we get $R^{n-1}\pi_*\mathcal{O}_X \otimes k(b) \cong H^{n-1}(F_b, \mathcal{O}_{F_b})$ for all $b \in C$. Since the general fibers of π are Abelian varieties, $\dim H^{n-1}(F_b, \mathcal{O}_{F_b}) = 1$ for an open set $b \in U \subset C$. Therefore $R^{n-1}\pi_*\mathcal{O}_X$ is a line bundle over U. On the other hand, since C is a smooth curve, we have $R^{n-1}\pi_*\mathcal{O}_X = \mathcal{L} \oplus \mathcal{T}$, where $\mathcal{L}$ is a locally free sheaf and $\mathcal{T}$ is the torsion part which is contained in $C \setminus U$.

Definition 2.2.3. [Ya, Definition, p.58] *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space and $R^i\pi_*\mathcal{O}_X := \mathcal{L}_i \oplus \mathcal{T}_i$, where $\mathcal{L}_i$ is a locally free sheaf, and $\mathcal{T}_i$ is a torsion part. Let b be a closed point of C. The fiber $F_b := \pi^{-1}(b)$ is said to be wild if $b \in \mathrm{Supp}(\mathcal{T}_{n-1})$. If a fiber is not wild, it is called tame.*

Remark 2.2.4. *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space. There are no wild fibers in characteristic 0.*

We defined the several types of fibers, i.e., singular, multiple, tame and wild fibers. We show some examples of Abelian fiber spaces, elliptic fibrations and their fibers.

Example 2.2.5. (1) an example of an elliptic fibration and singular fiber

We recall a pencil of plane cubics in $\mathbb{P}^2$ in Example 2.1.18. Let C_1 be a smooth cubic curve and C_2 any other cubics (we do not assume the smoothness of C_2) in $\mathbb{P}^2$. We assume that C_1 and C_2 intersect at general nine points. Taking blow-up of $\mathbb{P}^2$ at these nine points, we have an elliptic fibration structure $f : X \to \mathbb{P}^1$. Indeed, If we choose a general member C of this pencil, then the genus of the strict transform $g(\widetilde{C}) = 1$, where $\widetilde{C}$ is the strict transform of C by f. Indeed, we have the following equation:

$$\begin{aligned}
2g(\widetilde{C})-2 &= \widetilde{C}\cdot(K_X+\widetilde{C}) \\
&= \left(f^*C-\sum_{i=1}^{9}E_i\right)\cdot\left(f^*K_{\mathbb{P}^2}+\sum_{i=1}^{9}E_i+f^*C-\sum_{i=1}^{9}E_i\right) \\
&= \left(f^*C-\sum_{i=1}^{9}E_i\right)\cdot(f^*K_{\mathbb{P}^2}+f^*C) \\
&= 0.
\end{aligned}$$

Therefore, the genus of $\widetilde{C}$ is 1. We have some types of singular fibers which depends on C_2. If we take C_2 as a nodal curve, then the strict transform of C_2 gives a singular fiber of f which is denoted by I_1 in Kodaira's table. Moreover if we take C_2 as a cusp curve, then the strict transform gives a singular fiber II in Kodaira's table.

(2) an example of an elliptic fibration and a multiple fiber

We recall the construction of the Halphen pencil by [CD, p.347]. This construction is similar to the case (1). Let C_1 be a smooth cubic curve in $\mathbb{P}^2$ such that it passes through the nine distinct points $p_1,\cdots,p_9$ that are in the general position and C_2 be a curve of degree $3m$ that passes through $p_1,\cdots,p_9$ with multiplicity m. A pencil generated by mC_1 and C_2 gives a rational map from $\mathbb{P}^2$ to $\mathbb{P}^1$. If we blow-up $\mathbb{P}^2$ at these nine points (denoted by X), then we have the morphism $\pi : X\to\mathbb{P}^1$. We show that π is an elliptic fibration with one multiple fiber of multiplicity m. If we choose a general member C of this pencil, then the genus of the strict transform $g(\widetilde{C})=1$, where $\widetilde{C}$ is a strict transform by this blow-up. Indeed, we have the following equation by intersection theory:

$$\begin{aligned}
2g(\widetilde{C})-2 &= \widetilde{C}\cdot(K_X+\widetilde{C}) \\
&= \left(f^*C-\sum_{i=1}^{9}mE_i\right)\cdot\left(f^*K_{\mathbb{P}^2}+\sum_{i=1}^{9}E_i+f^*C-\sum_{i=1}^{9}mE_i\right) \\
&= \left(f^*C-\sum_{i=1}^{9}mE_i\right)\cdot\left(f^*K_{\mathbb{P}^2}+f^*C-\sum_{i=1}^{9}(m-1)E_i\right) \\
&= 3m(3m-3)+\left(\sum_{i=1}^{9}mE_i\right)\cdot\left(\sum_{i=1}^{9}(m-1)E_i\right) \\
&= 9m(m-1)-9m(m-1) \\
&= 0.
\end{aligned}$$

Therefore, $g(\widetilde{C})=1$. In particular, the strict transform of mC_1 gives a multiple fiber of π with multiplicity m.

(3) an example of 2-Abelian fiber space and a wild fiber

We recall the construction of 2-Abelian fiber space with wild fiber by [Ya, p.65, Example]. Let E be an ordinary elliptic curve and $a\in E$ a point of order p. Then the group $\mathbb{Z}/p\mathbb{Z}=\langle\sigma\rangle$ acts on E by

$$\begin{aligned}
\sigma : E &\to E, \\
x &\mapsto x+a.
\end{aligned}$$

The group $\mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{P}^1$ by

$$\sigma : t\mapsto t+1,$$

where t is a coordinate of the affine line $\mathbb{A}^1$ in $\mathbb{P}^1$. Therefore, the group $\mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{P}^1 \times E \times E$. We have a 2-Abelian fiber space

$$f : X := (\mathbb{P}^1 \times E \times E)/\langle \sigma \rangle \to \mathbb{P}^1/\langle \sigma \rangle \cong \mathbb{P}^1.$$

(4) an example of 2-Abelian fiber space

We recall the construction of 2-Abelian fiber space by [Ya, Example, p.66]. Assume that the characteristic of the base field k is $p \equiv 1 \pmod 6$. Let C be a smooth projective curve defined by the equation

$$t^2 = x^p - x.$$

The genus of C is given by $g(C) = \frac{1}{2}(p-1)$. Let E and E' be ordinary elliptic curves, $a \in E$ be a point of order p, and $a' \in E'$ a point of order 6. The groups $\langle \sigma \rangle = \mathbb{Z}/p\mathbb{Z}$ and $\langle \tau \rangle = \mathbb{Z}/6\mathbb{Z}$ act on C, E, and E' by

$$\begin{aligned} \sigma : (x,t) \mapsto (x+1,t) \quad &, \quad \tau : (x,t) \mapsto (\omega x, -\omega t) \text{ on } C, \\ \sigma : z \mapsto z + a \quad &, \quad \tau : z \mapsto z \text{ on } E, \\ \sigma : z' \mapsto z' \quad &, \quad \tau : z' \mapsto z' + a' \text{ on } E', \end{aligned}$$

where ω is a primitive cube root of unity. Since $C \times E \to (C \times E)/\langle \sigma \rangle$ is an étale morphism, we have an elliptic fibration

$$f_0 : X_0 := (C \times E)/\langle \sigma \rangle \to C/\langle \sigma \rangle \cong \mathbb{P}^1.$$

We set $X := (X_0 \times E')/\langle \tau \rangle$ and $f : X \to \mathbb{P}^1/\langle \tau \rangle \cong \mathbb{P}^1$. This f is a 2-Abelian fiber space.

In the case of elliptic surfaces, we have the canonical bundle formula, i.e., a formula about the relative canonical bundle $\omega_{X/C}$. This formula was firstly proved by Kodaira [Kod] in characteristic 0 and Bombieri-Mumford [BM] in arbitrary characteristic. In the case of Abelian fiber spaces over curves with some good conditions, we have a similar formula which is due to Yasuda as follows;

Theorem 2.2.6. [Ya, Theorem 0.1] *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space with $(K_X^2 \cdot H^{n-2}) = 0$, where H is a hyperplane section of X. Let $R^i\pi_*\mathcal{O}_X = \mathcal{L}_i \oplus \mathcal{T}_i$, where $\mathcal{L}_i$ is a locally free sheaf, and $\mathcal{T}_i$ is a torsion sheaf of $R^i\pi_*\mathcal{O}_X$. Let $l(\mathcal{T}_i)$ be the length of $\mathcal{T}_i$. Then we have*

$$\omega_X \cong \pi^*(\mathcal{L}_{n-1}^{-1} \otimes \omega_C) \otimes \mathcal{O}_X\left(\sum_i a_i P_i\right),$$

where

(1) $m_iP_i = F_{b_i}$ *are the multiple fibers of* π,

(2) $0 \leq a_i \leq m_i - 1$,

(3) $a_i = m_i - n_i$ *if F_{b_i} is a tame fiber, where the jumping number* $n_i := \min\{n \in \mathbb{Z} \mid \dim H^0(\omega_{nP_i}) > 0\}$,

(4) $\chi(\mathcal{O}_X) = \sum_{i=1}^{n}(-1)^i(\deg\mathcal{L}_i + l(\mathcal{T}_i))$.

Remark 2.2.7. *We emphasize that, even for the tame fiber, it is possible that the jumping numbers are greater than one if $n \geq 3$. This is different from the case of elliptic surfaces.*

2.3. **The canonical bundle formula of elliptic fibrations.** In this subsection, we consider Theorem 2.2.6 in elliptic fibration. Especially in the case of elliptic fibrations with only tame fibers, the canonical bundle formula in Theorem 2.2.6 is simpler than the other cases; see Corollary 2.3.5. First of all, we consider the condition $(K_X^2 \cdot H^{n-2}) = 0$ in Theorem 2.2.6. In elliptic fibration, this condition is equivalent to π is relatively minimal, i.e., there are no (-1)-curves in all of fibers of π as follows;

Lemma 2.3.1. *(c.f.,* [Katsu, Theorem 3.2.4]*) Let $\pi : X \to C$ be an elliptic fibration. The following conditions are equivalent;*

(A) $K_X^2 = 0$,

(B) *K_X is π-nef, i.e., $(K_X \cdot D) \geq 0$ for all curve D such that $\pi(D)$ is a point,*

(C) *π is relatively minimal.*

Proof. (A) $\Rightarrow$ (B) Since the intersection number does not change under linearly equivalence, we may fix a certain divisor L such that $\mathcal{L} \cong \mathcal{O}_X(L)$. By Theorem 2.2.6, we have the canonical bundle formula. For all curve D such that $\pi(D)$ is a point, we have the following equation by the intersection theory;

$$\begin{aligned}(K_X \cdot D) &= \left(\pi^*(L^{-1} + K_C) + \sum_i a_i P_i\right) \cdot D \\ &= (L^{-1} + K_C) \cdot \pi_* D + \sum_i a_i (P_i \cdot D) \\ &= 0.\end{aligned}$$

The second equation is followed by the projection formula for intersection numbers and third one is followed since $\sum_i a_i P_i$ is contained in some unions of fibers.

(B) $\Rightarrow$ (C) This implication is followed by a general argument of algebra surfaces. Indeed, If we have a rational curve E in a fiber of π such that $E^2 = -1$, then we have the following by adjunction formula,

$$\begin{aligned}2g(E) - 2 &= E \cdot (E + K_X) \\ &\Leftrightarrow (E \cdot K_X) = -1\end{aligned}$$

Then K_X is not π-nef.

(C) $\Rightarrow$ (A) This implication is followed by the argument in [Katsu, Theorem 3.2.4] as follows. Let F_b be a general smooth fiber of π. Since F_b is an elliptic curve, then $\omega_{F_b} \cong \mathcal{O}_{F_b}$. We take r general points in C and consider the r general fibers $F_{b_1}, \cdots, F_{b_r}$. We have the exact sequence as follows;

$$0 \to \mathcal{O}_X\left(-\sum_{i=1}^r F_{b_i}\right) \to \mathcal{O}_X \to \bigoplus_{i=1}^r \mathcal{O}_{F_{b_i}} \to 0. \tag{3}$$

Tensoring with $\omega_X \otimes \mathcal{O}_X(\sum_{i=1}^r F_{b_i})$ to (3), we have

$$0 \to \omega_X \to \omega_X \otimes \mathcal{O}_X\left(\sum_{i=1}^r F_{b_i}\right) \to \bigoplus_{i=1}^r \mathcal{O}_{F_{b_i}} \to 0.$$

Therefore we have the following inequality;

$$\begin{aligned}h^0\left(X, \omega_X \otimes \mathcal{O}_X\left(\sum_{i=1}^r F_{b_i}\right)\right) &\geq h^0(X, \omega_X) + \sum_{i=1}^r h^0(F_{b_i}, \mathcal{O}_{F_{b_i}}) - h^1(X, \omega_X) \\ &= h^0(X, \omega_X) + r - h^1(X, \omega_X)\end{aligned}$$

Since we take r general points, the linear system $|K_X + \sum_{i=1}^r F_{b_i}|$ has nontrivial members if r is taken largely enough. Let D be a member of the linear system $|K_X + \sum_{i=1}^r F_{b_i}|$ and K_X is linearly equivalent to

$$\sum_{i=1}^r -F_{b_i} + D.$$

For any fiber F_b, we have $(D \cdot F_b) = 0$. Indeed

$$\begin{aligned}(D \cdot F_b) &= \left(K_X + \sum_{i=1}^{r} F_{b_i}\right) \cdot F_b \\ &= (K_X \cdot F_b),\end{aligned}$$

and by the adjunction formula, we have $2g(F_b) - 2 = F_b \cdot (F_b + K_X)$. Since π is elliptic fibration, $g(F_b) = 1$ and $F_b^2 = 0$, so $(K_X \cdot F_b) = 0$. Taking r appropriately, we may assume that $K_X \sim \sum_{i=1}^{r'} b_i F_{b_i} + D$, where $b_i \in \mathbb{Z}$ and D is contained in a union of fibers but not containing any fibers. Let D_i be a connected component of D. Since D_i is contained in a fiber, $D_i^2 \leq 0$. If $D_i^2 < 0$, then we have at least one prime component E of D_i such that $(D_i \cdot E) < 0$ and this E satisfies $E^2 < 0$. This prime divisor E satisfies the following;

$$\begin{aligned}(K_X \cdot E) &= \sum_{i=1}^{r'} b_i (F_{b_i} \cdot E) + (D \cdot E) \\ &= (D_i \cdot E) < 0\end{aligned}$$

So the prime divisor satisfies the condition $(K_X \cdot E) < 0$ and $E^2 < 0$. This implies E is a (-1)-curve. This contradicts the condition of minimality of π. Therefore $D_i^2 = 0$. Eventually we have

$$\begin{aligned}K_X^2 &= \left(\sum_{i=1}^{r'} b_i F_{b_i} + D\right)^2 \\ &= \sum_j D_j^2 \quad (j \text{ is a number of connected components of } D) \\ &= 0\end{aligned}$$

□

By Lemma 2.3.1, we may replace the condition $K_X^2 = 0$ with "relatively minimal" in Theorem 2.2.6.

Next, we consider some numerical invariants associated with multiple fibers of an elliptic fibration in Theorem 2.2.6. Let $\pi : X \to C$ be a relatively minimal elliptic fibration and $F_b = mP$ a multiple fiber of multiplicities m. We denote the order of $\mathcal{O}_X(P) \otimes \mathcal{O}_P$ in $\mathrm{Pic}(\mathcal{O}_P)$ by ν. We have the following Lemma;

Lemma 2.3.2. [Katsu, Lemma 3.2.11] *With the same notation as above, we have the following;*

(A) $h^0(P, \mathcal{O}_{nP})$ *is a nondecreasing function of* n,
(B) $h^0(P, \mathcal{O}_{(\nu+1)P}) = 2$, *and*
(C) ν *divides* m *and* $a + 1$.

Proof. Let n_1 and n_2 be positive integers such that $n_1 > n_2 \geq 1$. We have a exact sequence;

$$0 \to \mathrm{Ker}(r) \to \mathcal{O}_{n_1 P} \xrightarrow{r} \mathcal{O}_{n_2 P} \to 0.$$

Them we have a surjective map

$$H^1(P, \mathcal{O}_{n_1 P}) \to H^1(P, \mathcal{O}_{n_2 P}) \to H^2(P, \mathrm{Ker}(r)) = 0$$

and the function $h^1(P, \mathcal{O}_{nP})$ is a nondecreasing function of n. On the other hand, we have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(-nP)) + \chi(\mathcal{O}_{nP})$ by the following exact sequence;

$$0 \to \mathcal{O}_X(-nP) \to \mathcal{O}_X \to \mathcal{O}_{nP} \to 0.$$

By the Riemann-Roch theorem, $\chi(\mathcal{O}_X(-nP)) = \frac{(-nP)(-nP-K_X)}{2} + \chi(\mathcal{O}_X)$. Since P is contained in the fiber of π, $P^2 = P{\cdot}K_X = 0$, so $\chi(\mathcal{O}_X(-nP)) = \chi(\mathcal{O}_X)$. This implies $\chi(\mathcal{O}_{nP}) = 0$. Therefore $h^0(P, \mathcal{O}_{nP}) = h^1(P, \mathcal{O}_{nP})$ and (A) is proved.

Next, we consider the following exact sequence;

(4) $0 \to \mathcal{O}_X(-\nu P)/\mathcal{O}_X(-(\nu+1)P) \to \mathcal{O}_X/\mathcal{O}_X(-(\nu+1)P) \to \mathcal{O}_X/\mathcal{O}_X(-\nu P) \to 0.$

Since $\mathcal{O}_X(-\nu P)/\mathcal{O}_X(-(\nu+1)P) \cong \mathcal{O}_X/\mathcal{O}_X(-P) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\nu P)$ and the definition of ν, this sheaf is isomorphic to $\mathcal{O}_P$. Therefore the exact sequence (4) is the following;

$$0 \to \mathcal{O}_P \to \mathcal{O}_{(\nu+1)P} \to \mathcal{O}_{\nu P} \to 0,$$

and we have $0 \to H^0(P, \mathcal{O}_P) \to H^0(P, \mathcal{O}_{(\nu+1)P}) \to H^0(P, \mathcal{O}_{\nu P})$. There are constants in $H^0(P, \mathcal{O}_P)$ and $H^0(P, \mathcal{O}_{\nu P})$ respectively, and constants in $H^0(P, \mathcal{O}_{(\nu+1)P})$ are mapped into constants in $H^0(P, \mathcal{O}_{\nu P})$, so $h^0(P, \mathcal{O}_{(\nu+1)P}) = 2$ and (B) is proved.

Finally, the following sheaf

$$(\mathcal{O}_X(P) \otimes \mathcal{O}_P)^{\otimes m}$$

is isomorphic to $\mathcal{O}_P$ since $\mathcal{O}_X(mP)$ is $\pi^*\mathcal{O}_C(b)$. Then ν divides m. Since the dualizing sheaf of P is isomorphic to $\mathcal{O}_P$, adjunction formula implies

$$\begin{aligned} \mathcal{O}_P \cong \omega_P &\cong \omega_X \otimes \mathcal{O}_X(P) \otimes \mathcal{O}_P \\ &\cong \mathcal{O}_X((a+1)P) \otimes \mathcal{O}_P \\ &\cong (\mathcal{O}_X(P) \otimes \mathcal{O}_P)^{\otimes(a+1)}. \end{aligned}$$

Therefore ν divides $a+1$. □

By Lemma 2.3.2, we have a numerical relation between ν and m for tame fibers as follows;

Lemma 2.3.3. [Katsu, Theorem 3.2.13] *Let $\pi : X \to C$ be a relatively minimal elliptic fibration with only tame fibers and $F_b = mP$ is a multiple fiber of π with multiplicities m. Then ν is equal to m.*

Proof. Since F_b is a tame fiber, $h^0(P, \mathcal{O}_{mP}) = 1$. Since $h^0(P, \mathcal{O}_{nP})$ is a nondecreasing function, so $\nu + 1 \geq m$ by (A) and (B) in Lemma 2.3.2. By (C) in Lemma 2.3.2, $\nu = m$. □

Remark 2.3.4. *In the case of wild fibers for elliptic fibrations, multiplicities m and the integer a appearing in Theorem 2.2.6 are closely related to the order of normal bundle $\mathcal{O}_X(P)|_P$ and the characteristic p which is due to Bombieri-Mumford* [BM]. *However we do not know whether such a relation exists in higher dimensional Abelian fiber spaces or not.*

For a multiple fiber $F_b = mP$, it is known that the dualizing sheaf ω_P is trivial by Theorem [Ba, Corollary 7.9]. Therefore if a relatively minimal elliptic fibration $\pi : X \to C$ has only tame fibers i.e., $R^1\pi_*\mathcal{O}_X$ is locally free, then the canonical bundle formula in Theorem 2.2.6 is written simply as follows;

Corollary 2.3.5. *Let $\pi : X \to C$ be relatively minimal elliptic fibration such that $\mathcal{L} := R^1\pi_*\mathcal{O}_X$ is locally free. Then we have the canonical bundle formula*

$$\omega_X \cong \pi^*(\mathcal{L}^{-1} \otimes \omega_C) \otimes \mathcal{O}_X\left(\sum_i (m_i - 1)P_i\right),$$

where m_i is the multiplicities of a multiple fiber $F_{b_i} = m_iP_i$ and $\chi(\mathcal{O}_X) = -\deg\mathcal{L}$.

3. On Frobenius split Abelian fiber spaces over curves

This section is based on the work on the paper [Shira] which states the classification of F-split Abelian fiber spaces over curves with only tame fibers.

3.1. **Frobenius split varieties via the relative Frobenius morphism.** In this subsection, we treat families of varieties over a smooth curve. For a morphism $\pi : X \to C$ such that $\mathcal{O}_C$ is isomorphic to $\pi_*\mathcal{O}_X$, we investigate the Frobenius action of X via the relative Frobenius morphism $F_{X/C}$.

Firstly, we recall Remark 2.1.9. Let $\mathcal{B}$ be a cokernel of $\mathcal{O}_X \to (F_X)_*\mathcal{O}_X$. That is, we have the short exact sequence

$$0 \to \mathcal{O}_X \to (F_X)_*\mathcal{O}_X \to \mathcal{B} \to 0.$$

If we apply $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ to the above sequence, we have the long exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{B}, \mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}((F_X)_*\mathcal{O}_X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{B}, \mathcal{O}_X) \to \cdots.$$

By the Grothendieck duality, $\mathrm{Hom}_{\mathcal{O}_X}((F_X)_*\mathcal{O}_X, \mathcal{O}_X)$ is isomorphic to $H^0(X, \omega_X^{\otimes(1-p)})$. Then X is F-split if and only if the map

$$\begin{array}{rcl} \widehat{\tau} : H^0(X, \omega_X^{\otimes(1-p)}) & \to & H^0(X, \mathcal{O}_X) \\ s & \longmapsto & \mathrm{Tr}(s) \end{array}$$

is surjective by Theorem 2.1.8. By the Serre duality, the surjectivity of $\widehat{\tau}$ is equivalent to the non-vanishing of the dual map

$$F_X^* := \widehat{\tau}^\vee : H^n(X, \omega_X) \to H^n(X, \omega_X^{\otimes p}).$$

Note that this F_X^* is induced by just tensoring ω_X to $\mathcal{O}_X \to (F_X)_*\mathcal{O}_X$.

In the previous section, we recalled the relative Frobenius morphism; in particular, when we have a map $\pi : X \to Y$, we see that $F_X = W \circ F_{X/Y}$, where the projection $W : X^{(p)} \to X$; see Definition 2.1.2. By using this, if we apply the same operation $H^n(X, - \otimes \omega_X)$, then we have the following proposition.

Proposition 3.1.1. *Let $\pi : X \to C$ be a surjective morphism with $\pi_*\mathcal{O}_X = \mathcal{O}_C$ from an n-dimensional smooth projective variety to a smooth projective curve C. Then the absolute Frobenius action of X*

$$F_X^* : H^n(X, \omega_X) \to H^n(X, \omega_X^{\otimes p})$$

is nonzero if and only if a composition of the following morphisms

$$H^1(C, R^{n-1}\pi_*\omega_X) \xrightarrow{F_C^*} H^1\left(C, F_C^*(R^{n-1}\pi_*\omega_X)\right) \xrightarrow{F_{X/C}^*} H^1(C, R^{n-1}\pi_*\omega_X^{\otimes p})$$

is nonzero.

Proof. The p-th power map

$$\mathcal{O}_X \to (F_X)_*\mathcal{O}_X$$

consists of two morphisms as the composition

$$\mathcal{O}_X \to W_*\mathcal{O}_{X^{(p)}} \to W_*\left((F_{X/C})_*\mathcal{O}_X\right), \tag{5}$$

owing to $F_X = W \circ F_{X/C}$. Tensoring with ω_X to (5), we have the composition map

$$\omega_X \to W_*\mathcal{O}_{X^{(p)}} \otimes \omega_X \to W_*\left((F_{X/C})_*\mathcal{O}_X\right) \otimes \omega_X. \tag{6}$$

By the projection formula,

$$W_*\mathcal{O}_{X^{(p)}} \otimes \omega_X \cong W_*W^*\omega_X.$$

Similarly,

$$\begin{aligned} W_*\left((F_{X/C})_*\mathcal{O}_X\right) \otimes \omega_X &\cong W_*\left((F_{X/C})_*\mathcal{O}_X \otimes W^*\omega_X\right) \\ &\cong W_*\left((F_{X/C})_*(\mathcal{O}_X \otimes F_{X/C}^* W^*\omega_X)\right) \\ &\cong W_*\left((F_{X/C})_*\omega_X^{\otimes p}\right). \end{aligned}$$

Taking $H^n(X, -)$ of (6), we have

$$H^n(X, \omega_X) \to H^n(X, W_*W^*\omega_X) \to H^n\left(X, W_*\left((F_{X/C})_*\omega_X^{\otimes p}\right)\right).$$

Since W and $F_{X/C}$ are affine morphisms, we have

$$H^n(X, W_*W^*\omega_X) \cong H^n(X^{(p)}, W^*\omega_X),$$

$$H^n\left(X, W_*\left((F_{X/C})_*\omega_X^{\otimes p}\right)\right) \cong H^n(X, \omega_X^{\otimes p}).$$

The absolute Frobenius action F_X^* consists of the following two morphisms:

$$H^n(X, \omega_X) \xrightarrow{W^*} H^n(X^{(p)}, W^*\omega_X) \xrightarrow{F_{X/C}^*} H^n(X, \omega_X^{\otimes p}).$$

By degeneration of the Leray spectral sequence, for any coherent sheaf $\mathcal{F}$ on X (resp. $\mathcal{F}'$ on $X^{(p)}$), we have the following isomorphisms:

$$H^n(X, \mathcal{F}) \cong H^1(C, R^{n-1}\pi_*\mathcal{F}), \quad \left(\text{resp. } H^n(X^{(p)}, \mathcal{F}') \cong H^1(C, R^{n-1}\pi_*^{(p)}\mathcal{F}')\right).$$

Then we have the following composition:

$$H^1(C, R^{n-1}\pi_*\omega_X) \xrightarrow{W^*} H^1\left(C, R^{n-1}\pi_*^{(p)}(W^*\omega_X)\right) \xrightarrow{F_{X/C}^*} H^1(C, R^{n-1}\pi_*\omega_X^{\otimes p}).$$

Since F_C is flat, we can apply the flat base change theorem [Hart, Proposition 9.3] to $R^{n-1}\pi_*^{(p)}(W^*\omega_X)$. Then, $R^{n-1}\pi_*^{(p)}(W^*\omega_X)$ is isomorphic to $F_C^*R^{n-1}\pi_*\omega_X$. Therefore, F_X^* is nonzero if and only if the composition of the following two morphisms

$$H^1(C, R^{n-1}\pi_*\omega_X) \xrightarrow{F_C^*} H^1(C, F_C^*R^{n-1}\pi_*\omega_X) \xrightarrow{F_{X/C}^*} H^1(C, R^{n-1}\pi_*\omega_X^{\otimes p})$$

is nonzero. □

3.2. **Pullback of the relative Frobenius morphism on a curve.** In this section, we consider Abelian fiber spaces. In Proposition 3.1.1, we have two morphisms: the absolute Frobenius action F_C^* and the relative Frobenius action $F_{X/C}^*$. The former is simply defined by all cohomology classes on C mapped to its p-th powers, but how should we interpret the relative Frobenius action on C? In Abelian fiber spaces, this action can be interpreted as the ordinarity of the general fiber and a multiplication map by a certain global section on C.

First, we consider the cokernel of the relative Frobenius morphism. Let $\pi : X \to C$ be a morphism from n-dimensional smooth projective variety to a smooth projective curve and $\mathcal{B}_{X/C}$ the cokernel of $\mathcal{O}_{X^{(p)}} \to (F_{X/C})_*\mathcal{O}_X$. Applying the higher direct image functor $R^{n-1}\pi_*^{(p)}-$ to the short exact sequence, where the projection $\pi^{(p)} : X^{(p)} \to C$; see Definition 2.1.2,

$$0 \to \mathcal{O}_{X^{(p)}} \to (F_{X/C})_*\mathcal{O}_X \to \mathcal{B}_{X/C} \to 0,$$

we have the following exact sequence:

$$R^{n-1}\pi_*^{(p)}\mathcal{O}_{X^{(p)}} \to R^{n-1}\pi_*^{(p)}\left((F_{X/C})_*\mathcal{O}_X\right) \to R^{n-1}\pi_*^{(p)}\mathcal{B}_{X/C} \to 0.$$

In particular, we denoted the following map by $F^*_{X/C}$:

$$R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}} \to R^{n-1}\pi^{(p)}_*\left((F_{X/C})_*\mathcal{O}_X\right).$$

Definition 3.2.1. *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space and we assume that $R^{n-1}\pi_*\mathcal{O}_X$ is a locally free sheaf on C. We call points in the support of $R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C}$ non-ordinary points and the length of $R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C}$ at a non-ordinary point b in C is called the order of b.*

Lemma 3.2.2. *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space, and we assume that $R^{n-1}\pi_*\mathcal{O}_X$ is a locally free sheaf on C. If the general fibers of π are ordinary Abelian varieties, then the relative Frobenius action*

$$F^*_{X/C} : R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}} \to R^{n-1}\pi^{(p)}_*\left((F_{X/C})_*\mathcal{O}_X\right)$$

is injective; otherwise, $F^*_{X/C}$ is a zero map.

Proof. We have the following exact sequence:

$$R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}} \xrightarrow{F^*_{X/C}} R^{n-1}\pi^{(p)}_*\left((F_{X/C})_*\mathcal{O}_X\right) \to R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C} \to 0.$$

Let y be a general closed point of C and $k(y)$ the residue field at y. Tensoring with $k(y)$ to the above exact sequence, we have the following exact sequence:

$$(7)\quad \begin{aligned} R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}} \otimes k(y) \to R^{n-1}\pi^{(p)}_*\left((F_{X/C})_*\mathcal{O}_X\right) \otimes k(y) &\to R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C} \otimes k(y) \\ &\to 0. \end{aligned}$$

Let E and $E^{(p)}$ be the closed fiber of π and $\pi^{(p)}$. The relative Frobenius morphism $F_{X/C}$ induces the relative Frobenius morphism $F_{E/\mathrm{Spec} k(y)} : E \to E^{(p)}$. By the flat base change theorem, the first map of the above sequence is the following:

$$H^{n-1}(E^{(p)}, \mathcal{O}_{E^{(p)}}) \to H^{n-1}(E, \mathcal{O}_E).$$

Therefore the exact sequence (7) is equivalent to the following exact sequence:

$$(8)\qquad \begin{array}{ccc} H^{n-1}(E, \mathcal{O}_E) & \to & H^{n-1}(E, \mathcal{O}_E) \to R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C} \otimes k(y) \to 0. \\ \xi & \mapsto & \xi^p \end{array}$$

Since $F_{X/C}$ is affine, $R^i(F_{X/C})_*\mathcal{O}_X = 0$ for $i > 0$. By degeneration of the Leray spectral sequence $E_2^{i,j} = R^i\pi^{(p)}_*(R^j(F_{X/C})_*\mathcal{O}_X) \Rightarrow E^{i+j} = R^{i+j}\pi_*\mathcal{O}_X$, we have the following isomorphism:

$$\begin{aligned} R^{n-1}\pi^{(p)}_*\left((F_{X/C})_*\mathcal{O}_X\right) &\cong R^{n-1}(\pi^{(p)} \circ F_{X/C})_*\mathcal{O}_X \\ &= R^{n-1}\pi_*\mathcal{O}_X. \end{aligned}$$

By the flat base change theorem, we also have the following isomorphism:

$$\begin{aligned} R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}} &\cong R^{n-1}\pi^{(p)}_*(W^*\mathcal{O}_X) \\ &\cong F_C^*R^{n-1}\pi_*\mathcal{O}_X. \end{aligned}$$

We assume that $R^{n-1}\pi_*\mathcal{O}_X$ is locally free. Thus $R^{n-1}\pi^{(p)}_*\left((F_{X/C})_*\mathcal{O}_X\right)$ and $R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}}$ are line bundles since the general fibers of π are Abelian varieties.

If E is an ordinary Abelian variety, the first map of (8) is isomorphism. Therefore, $R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C} \otimes k(y) = 0$ for almost all closed points $y \in C$. Nakayama's lemma implies $R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C}$ is torsion sheaf. Since the image of $F^*_{X/C}$ is a subsheaf of locally free sheaf $R^{n-1}\pi^{(p)}_*\left((F_{X/C})_*\mathcal{O}_X\right)$, we have the surjective map between line bundles from $R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}}$

to the image of $F^*_{X/C}$, so $R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}}$ is isomorphic to the image of $F^*_{X/C}$. Then $F^*_{X/C}$ is injective.

Similarly, if all of the fibers of π are not ordinary, then $R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C} \otimes k(y)$ is a one-dimensional vector space. This implies $R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C}$ is a rank 1 locally free sheaf. Therefore, $R^{n-1}\pi^{(p)}_*((F_{X/C})_*\mathcal{O}_X)$ is isomorphic to $R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C}$. So $F^*_{X/C}$ is zero map. □

The following proposition claims that this $F^*_{X/C}$ is just multiplication by a global section on C if the general fibers of π are ordinary.

Proposition 3.2.3. *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space, and we assume that $\mathcal{L}_{n-1} := R^{n-1}\pi_*\mathcal{O}_X$ is a locally free sheaf on C. If the general fibers of π are ordinary Abelian varieties, then the relative Frobenius action*

$$F^*_{X/C} : R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}} \to R^{n-1}\pi^{(p)}_*\left((F_{X/C})_*\mathcal{O}_X\right)$$

is isomorphic to the following morphism up to rational functions:

$$\times s : F^*_C R^{n-1}\pi_*\mathcal{O}_X \to R^{n-1}\pi_*\mathcal{O}_X,$$

where s in $H^0(C, \mathcal{L}^{\otimes(1-p)}_{n-1})$ is a non-zero section; otherwise, $F^*_{X/C}$ is zero map.

Proof. If the general fibers of π are ordinary (resp. not ordinary), then $F^*_{X/C}$ is injective (resp. a zero map) by Lemma 3.2.2. If the general fibers of π are ordinary, then we have the following short exact sequence:

$$0 \to F^*_C R^{n-1}\pi_*\mathcal{O}_X \xrightarrow{F^*_{X/C}} R^{n-1}\pi_*\mathcal{O}_X \to R^{n-1}\pi^{(p)}_*\mathcal{B}_{X/C} \to 0.$$

Since the degree of $F^*_C R^{n-1}\pi_*\mathcal{O}_X$ is $p \cdot \deg R^{n-1}\pi_*\mathcal{O}_X$, the relative Frobenius action

$$F^*_{X/C} : F^*_C R^{n-1}\pi_*\mathcal{O}_X \to R^{n-1}\pi_*\mathcal{O}_X \tag{9}$$

is an element of $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{L}^{\otimes p}_{n-1}, \mathcal{L}_{n-1}) \cong H^0(C, \mathcal{L}^{\otimes(1-p)}_{n-1})$, where $\mathcal{L}_{n-1} := R^{n-1}\pi_*\mathcal{O}_X$. Note that the locally free sheaf $\mathcal{L}_{n-1}$ is a line bundle since the general fibers of π are Abelian varieties. Therefore, the relative Frobenius action $F^*_{X/C}$in (9) is isomorphic to the morphism up to rational function

$$\times s : F^*_C R^{n-1}\pi_*\mathcal{O}_X \to R^{n-1}\pi_*\mathcal{O}_X,$$

for a nonzero section s in $H^0(C, \mathcal{L}^{\otimes(1-p)}_{n-1})$. □

In Definition 3.2.1, we defined non-ordinary points. So the relative Frobenius action $F^*_{X/C}$ in Proposition 3.2.3 is uniquely determined up to scalar as follows.

Lemma 3.2.4. *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space, and we assume that $\mathcal{L}_{n-1} := R^{n-1}\pi_*\mathcal{O}_X$ is a locally free sheaf on C. Let $b_1, \cdots, b_r$ be non-ordinary points and $n_1, \cdots, n_r$ the order of b_i's. If the general fibers of π are ordinary Abelian varieties, then the relative Frobenius action*

$$F^*_{X/C} : R^{n-1}\pi^{(p)}_*\mathcal{O}_{X^{(p)}} \to R^{n-1}\pi^{(p)}_*\left((F_{X/C})_*\mathcal{O}_X\right)$$

is isomorphic to the following morphism up to scalars:

$$\times s : F^*_C R^{n-1}\pi_*\mathcal{O}_X \to R^{n-1}\pi_*\mathcal{O}_X$$

where s in $H^0(C, \mathcal{L}^{\otimes(1-p)}_{n-1})$ is a nonzero section which has zeros only at the non-ordinary points and the order of the zero at a point b_i is n_i. Otherwise, $F^*_{X/C}$ is a zero map.

Proof. By Proposition 3.2.3, $F^*_{X/C}$ is isomorphic to nonzero section $s \in H^0(C, \mathcal{L}_{n-1}^{\otimes(1-p)})$ up to rational functions. Since we assume that $R^{n-1}\pi_*\mathcal{O}_X$ is locally free, this $\times s$ is the map as follows,

$$0 \to \mathcal{L}_{n-1}^{\otimes p} \xrightarrow{\times s} \mathcal{L}_{n-1} \to R^{n-1}\pi_*^{(p)}\mathcal{B}_{X/C} \to 0.$$

$R^{n-1}\pi_*^{(p)}\mathcal{B}_{X/C}$ has support at non-ordinary points b_i's with the order n_i's. Then s has zero points at b_i with the order n_i. □

3.3. **The absolute Frobenius action on the base curve.** Before we prove Theorem 1.1.2, we will prove some technical lemmas to simplify the arguments and refer to a theorem due to Watanabe [Wa, Theorem 4.2], which is important in the proof of Theorem 1.1.2.

Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space and H a hyperplane section of X with $(K_X^2 \cdot H^{n-2}) = 0$. By Theorem 2.2.6, we have the following formula

$$\omega_X \cong \pi^*(\mathcal{L}_{n-1}^{-1} \otimes \omega_C) \otimes \mathcal{O}_X\left(\sum_i a_i \cdot P_i\right),$$

where $\mathcal{L}_{n-1}$ is a locally free part of $R^{n-1}\pi_*\mathcal{O}_X$, $\pi^{-1}(b_i)$ is a multiple fiber, and the values of m_i are the multiplicities of the multiple fibers.

Lemma 3.3.1. *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space with $(K_X^2 \cdot H^{n-2}) = 0$. Then for any integer $l \in \mathbb{Z}$,*

$$\pi_*\mathcal{O}_X\left(\sum_i la_i \cdot P_i\right) \cong \mathcal{O}_C\left(\sum_i \left\lfloor \frac{la_i}{m_i} \right\rfloor \cdot b_i\right).$$

Proof. For simplicity, we show the case of $l \geq 0$. Since the problem is local, we will show that $\pi_*\mathcal{O}_X(la_i \cdot P_i) \cong \mathcal{O}_C\left(\left\lfloor \frac{la_i}{m_i}\right\rfloor \cdot b_i\right)$. This was shown by similar way in [Ba, Claim 7.5.13] for the case of $0 \leq la_i < m_i$. Suppose that $la_i := m_i l_i + \alpha_i$, where $l_i \in \mathbb{Z}_{\geq 0}$ and $\alpha_i \in \mathbb{Z}_{\geq 0}$, such that $0 \leq \alpha_i < m_i$. By the projection formula, we have the following isomorphism:

$$\begin{aligned} \pi_*\mathcal{O}_X(la_i \cdot P_i) &\cong \pi_*\mathcal{O}_X\left((m_i l_i + \alpha_i) \cdot P_i\right) \\ &\cong \mathcal{O}_C(l_i \cdot b_i) \otimes \pi_*\mathcal{O}_X(\alpha_i \cdot P_i). \end{aligned}$$

Since $\pi_*\mathcal{O}_X(\alpha_i \cdot P_i) \cong \mathcal{O}_C$, $\pi_*\mathcal{O}_X(la_i \cdot P_i)$ is isomorphic to $\mathcal{O}_C(l_i \cdot b_i) \cong \mathcal{O}_C\left(\left\lfloor \frac{la_i}{m_i}\right\rfloor \cdot b_i\right)$. □

Lemma 3.3.2. *Let a and m be positive integers such that $m \geq 2$ and $0 \leq a < m$, and p a prime number. Then, $\lfloor p \cdot \frac{a}{m} \rfloor = \lceil (p-1) \cdot \frac{a}{m} \rceil$ if $a = m-1$.*

Proof. We may assume that $p = m\alpha + \beta$ such that $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ and $0 \leq \beta \leq m-1$. By an easy calculation, we have the following:

$$\begin{aligned} \left\lfloor p \cdot \tfrac{m-1}{m} \right\rfloor &= p - \left\lceil \tfrac{p}{m} \right\rceil \\ &= p - \alpha - \left\lceil \tfrac{\beta}{m} \right\rceil. \end{aligned} \tag{10}$$

On the other hand, we have the following:

$$\begin{aligned} \left\lceil (p-1) \cdot \tfrac{m-1}{m} \right\rceil &= p - 1 - \left\lfloor \tfrac{p-1}{m} \right\rfloor \\ &= p - 1 - \left\lfloor \tfrac{m\alpha+\beta-1}{m} \right\rfloor \\ &= p - 1 - \alpha - \left\lfloor \tfrac{\beta-1}{m} \right\rfloor. \end{aligned} \tag{11}$$

When $\beta = 0$, $\lfloor p \cdot \frac{m-1}{m} \rfloor = p - \alpha$, and $\lceil (p-1) \cdot \frac{m-1}{m} \rceil = p - 1 - \alpha + 1 = p - \alpha$. When $\beta \neq 0$, $\lfloor p \cdot \frac{m-1}{m} \rfloor = p - \alpha - 1$ and $\lceil (p-1) \cdot \frac{m-1}{m} \rceil = p - 1 - \alpha$. Therefore, we have assertion. □

Definition 3.3.3. [Hara, Definition 2.4] *Let X be a d-dimensional smooth projective variety and D an effective $\mathbb{Q}$-Weil divisor on X such that the coefficient of D in every irreducible component is less than* 1.

(1) *We say that the pair (X, D) is F-split if the Frobenius morphism*

$$F : H^d(X, \mathcal{O}_X(K_X)) = H^d(X, \mathcal{O}_X(K_X + D)) \to H^d(X, \mathcal{O}_X(p(K_X + D)))$$

is injective.

(2) *We say that (X, D) is strongly F-split if for every $n > 0$ and for every nonzero $f \in H^0(X, \mathcal{O}_X(nH))$, there exists $e > 0$ such that the map*

$$F : H^d(X, \mathcal{O}_X(K_X)) \xrightarrow{F^e} H^d(X, \mathcal{O}_X(q(K_X + D))) \xrightarrow{\times f} H^d(X, \mathcal{O}_X(q(K_X + D)) + nH)$$

is injective, where H is an ample Cartier divisor on X, and $q := p^e$.

The following theorem is a key of our proof of Theorem 3.3.6 which was firstly proved in [Wa, Theorem 4.2] and stated as the notion of pairs in [Hara, Proposition 2.6].

Theorem 3.3.4. [Wa, Theorem 4.2] [Hara, Proposition 2.6] *Let D be a $\mathbb{Q}$-divisor on $\mathbb{P}^1$ of the form*

$$D = \sum_{i=1}^{r} \frac{d_i - 1}{d_i} P_i.$$

We will denote D by $(P_1, \cdots, P_r; d_1, \cdots, d_r)$, and if $r \leq 3$, we will denote D simply by $(d_1, \cdots, d_r)$ which does not depend on the order of P_i's.

(1) *$(\mathbb{P}^1, D)$ is strongly F-split if and only if $r \leq 2$ or one of the following holds:*
 (a) *$D = (2, 2, d), d \geq 2$, and $p \neq 2$,*
 (b) *$D = (2, 3, 3)$ or $(2, 3, 4)$ and $p > 3$,*
 (c) *$D = (2, 3, 5)$ and $p > 5$,*

(2) *Strongly F-split is always F-split in the sense of Definition 3.3.3. If $(\mathbb{P}^1, D)$ is F-split but not strongly F-split, then $\deg D = 2$. More explicitly D is one of the following:*
 (a) *$D = (3, 3, 3)$ or $(2, 3, 6)$ and $p \equiv 1 \pmod 3$,*
 (b) *$D = (2, 4, 4)$ and $p \equiv 1 \pmod 4$,*
 (c) *If $D = (\infty, 0, -1, \lambda; 2, 2, 2, 2)$ with $\lambda \in k$, $\lambda \neq 0, -1$, then $(\mathbb{P}^1, D)$ is F-split if and only if $p = 2n+1$ such that the coefficient of x^n in the expansion of $(x+1)^n(x-\lambda)^n$ is not zero.*

Remark 3.3.5. *In the last case (c) of (2) in Theorem 3.3.4, we need to take an isomorphism $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ such that distinct three points map to the distinct points in the set $\{0, -1, \infty\}$. We fix an isomorphism σ_1 and denote another point $\lambda \in k$. For this σ_1, F-splitness of $(\mathbb{P}^1, D)$ depends on the coefficient of x^n in the expansion of $s := (x + 1)^n(x - \lambda)^n$. Since we have six choices as σ, we have the following six points depending on σ;*

$$\lambda, \frac{1}{\lambda}, -\lambda - 1, \frac{-1}{1 + \lambda}, \frac{-\lambda}{\lambda + 1}, \frac{\lambda + 1}{-\lambda}.$$

So we have six divisors as follows;

$$D_1 := (\infty, 0, -1, \lambda; 2, 2, 2, 2) \quad , \quad D_2 := (\infty, 0, -1, \frac{1}{\lambda}; 2, 2, 2, 2),$$
$$D_3 := (\infty, 0, -1, -\lambda - 1; 2, 2, 2, 2) \quad , \quad D_4 := (\infty, 0, -1, \frac{-1}{1 + \lambda}; 2, 2, 2, 2),$$
$$D_5 := (\infty, 0, -1, \frac{-\lambda}{\lambda + 1}; 2, 2, 2, 2) \quad , \quad D_6 := (\infty, 0, -1, \frac{\lambda + 1}{-\lambda}; 2, 2, 2, 2).$$

For each D_i's, F-splitness of $(\mathbb{P}^1, D_i)$ are characterized as follows;

(1) *If* $D_1 = (\infty, 0, -1, \lambda; 2, 2, 2, 2)$ *with* $\lambda \in k$, $\lambda \neq 0, -1$, *then* $(\mathbb{P}^1, D_1)$ *is* F*-split if and only if* $p = 2n + 1$ *such that the coefficient of* x^n *in the expansion of* $s_1 := (x+1)^n(x-\lambda)^n$ *is not zero.*
(2) *If* $D_2 = (\infty, 0, -1, \frac{1}{\lambda}; 2, 2, 2, 2)$ *with* $\lambda \in k$, $\lambda \neq 0, -1$, *then* $(\mathbb{P}^1, D_2)$ *is* F*-split if and only if* $p = 2n + 1$ *such that the coefficient of* x^n *in the expansion of* $s_2 := (x+1)^n(\lambda x-1)^n$ *is not zero.*
(3) *If* $D_3 = (\infty, 0, -1, -1-\lambda; 2, 2, 2, 2)$ *with* $\lambda \in k$, $\lambda \neq 0, -1$, *then* $(\mathbb{P}^1, D_3)$ *is* F*-split if and only if* $p = 2n + 1$ *such that the coefficient of* x^n *in the expansion of* $s_3 := (x+1)^n(x+\lambda+1)^n$ *is not zero.*
(4) *If* $D_4 = (\infty, 0, -1, \frac{-1}{\lambda+1}; 2, 2, 2, 2)$ *with* $\lambda \in k$, $\lambda \neq 0, -1$, *then* $(\mathbb{P}^1, D_4)$ *is* F*-split if and only if* $p = 2n + 1$ *such that the coefficient of* x^n *in the expansion of* $s_4 := (x+1)^n\{(\lambda+1)x+1\}^n$ *is not zero.*
(5) *If* $D_5 = (\infty, 0, -1, \frac{-\lambda}{\lambda+1}; 2, 2, 2, 2)$ *with* $\lambda \in k$, $\lambda \neq 0, -1$, *then* $(\mathbb{P}^1, D_5)$ *is* F*-split if and only if* $p = 2n + 1$ *such that the coefficient of* x^n *in the expansion of* $s_5 := (x+1)^n\{(\lambda+1)x+\lambda\}^n$ *is not zero.*
(6) *If* $D_6 = (\infty, 0, -1, \frac{\lambda+1}{-\lambda}; 2, 2, 2, 2)$ *with* $\lambda \in k$, $\lambda \neq 0, -1$, *then* $(\mathbb{P}^1, D_6)$ *is* F*-split if and only if* $p = 2n + 1$ *such that the coefficient of* x^n *in the expansion of* $s_6 := (x+1)^n(\lambda x+\lambda+1)^n$ *is not zero.*

Note that these conditions (1) $\sim$ *(6) are equivalent to each other. Indeed, by* [Hart, Corollary 4.22], *the coefficient of* x^n *in* s_1 *is*

$$(h_1)_p(\lambda) := \sum_{i=0}^{n} \binom{n}{i}^2 \lambda^i (-1)^i \in k[\lambda],$$

where $n := \frac{p-1}{2}$. *Then the coefficients* $(h_p)_i$ *of* x^n *for the others* s_i *are the following;*

$$(h_p)_2 := \lambda^n (h_p)_1(\frac{1}{\lambda}) = \lambda^n \sum_{i=0}^{n} \binom{n}{i}^2 \frac{1}{\lambda^i}(-1)^i,$$

$$(h_p)_3 := (h_p)_1(-1-\lambda) = \sum_{i=0}^{n} \binom{n}{i}^2 (-\lambda-1)^i(-1)^i,$$

$$(h_p)_4 := (1+\lambda)^n (h_p)_1(\frac{-1}{1+\lambda}) = (1+\lambda)^n \sum_{i=0}^{n} \binom{n}{i}^2 (\frac{-1}{\lambda+1})^i(-1)^i,$$

$$(h_p)_5 := (1+\lambda)^n (h_p)_1(\frac{-\lambda}{\lambda+1}) = (1+\lambda)^n \sum_{i=0}^{n} \binom{n}{i}^2 (\frac{-\lambda}{\lambda+1})^i(-1)^i,$$

$$(h_p)_6 := \lambda^n (h_p)_1(\frac{-\lambda-1}{\lambda}) = \lambda^n \sum_{i=0}^{n} \binom{n}{i}^2 (\frac{-\lambda-1}{\lambda})^i(-1)^i.$$

Since $(h_p)_1, \cdots, (h_p)_6$ *are the same as* $\pm(h_1)_p(\lambda)$ *as polynomials, the conditions i.e., whether the coefficient of* x^n *of each polynomials are nonzero or not, are equivalent to each other. Since* $\lambda \neq 0, -1$ *and each* $(h_p)_i$*'s are the same as* $\pm(h_p)_1(\lambda)$*, the roots* λ*'s of each equations coincide. Therefore* F*-splitness of* $(\mathbb{P}^1, D_i)$ *are equivalent to each other.*

Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space. We denote $(b_1, \cdots, b_r; m_1, \cdots, m_r)$ as the multiple fibers $\pi^{-1}(b_i)$ with multiplicities m_i. In the following Theorem 3.3.6, when C is $\mathbb{P}^1$ and $r \geq 2$, one may apply an action of $\mathrm{PGL}(\mathbb{P}^1)$ to the base curve $\mathbb{P}^1_{\{X:Y\}}$. More precisely, we choose three points from b_i's and take an isomorphism of $\mathbb{P}^1$ such that these points map to the distinct points in the set $\{0, -1, \infty\}$. We have six ways to choose this isomorphism, but we fix one isomorphism $\sigma : \mathbb{P}^1_{\{x:y\}} \to \mathbb{P}^1_{\{X:Y\}}$. For the original fibration $\pi : X \to C$, we will have another fibration $\pi' : X' \to C$ by taking a fiber product of π and the isomorphism σ of $\mathbb{P}^1$ as follows;

$$\begin{tikzcd} X' := X \times_{\mathbb{P}^1} \mathbb{P}^1 \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^1_{\{x:y\}} \arrow[r, "\cong", "\sigma"'] & \mathbb{P}^1_{\{X:Y\}}. \end{tikzcd}$$

We put $(\beta_i : \alpha_i)$, $\alpha_i, \beta_i \in k$ the coordinates of non-ordinary points in $\mathbb{P}^1_{\{x:y\}}$ but not $\mathbb{P}^1_{\{X:Y\}}$.

Since X and X' are isomorphic, F-split property of X does not change by taking such a fiber product. The following Theorem 3.3.6 is obtained by taking a fiber product as above if $r \geq 2$.

In this situation, we classify F-split Abelian fiber spaces over curves with only tame fibers as follows;

Theorem 3.3.6. *Let $\pi : X \to C$ be an $(n-1)$-Abelian fiber space over an algebraically closed field of characteristic $p > 0$ and H a hyperplane section of X. Assume $(K_X^2 \cdot H^{n-2}) = 0$ and set $\mathcal{L}_{n-1} := R^{n-1}\pi_*\mathcal{O}_X$. Further assume that $\mathcal{L}_{n-1}$ is a locally free sheaf on C and all jumping numbers are 1. The only possibilities for C are either $\mathbb{P}^1$ or an ordinary elliptic curve.*

If C is an ordinary elliptic curve, then X is F-split if and only if $\mathrm{ord}(K_X) \mid (p-1)$ and the general fibers of π are ordinary Abelian varieties.

Conditions in the case $C = \mathbb{P}^1$ are dependent on $\deg(\mathcal{L}_{n-1}^{-1})$. Set $D = \sum_i \frac{m_i - 1}{m_i} b_i$ the divisor on $\mathbb{P}^1$ where $\pi^{-1}(b_i)$ is a multiple fiber and m_i is the multiplicity of the fiber at b_i, $(\beta_1 : \alpha_1), \cdots, (\beta_r : \alpha_r) \in \mathbb{P}^1$ non-ordinary points, and $n_1, \cdots, n_r$ the order of non-ordinary points. We have the following possible configurations for multiplicities of multiple fibers, depending on $\deg(\mathcal{L}_{n-1}^{-1})$

1. *If $\deg(\mathcal{L}_{n-1}^{-1}) = 0$ then X is F-split if and only if the general fibers of π are ordinary Abelian varieties, and $(\mathbb{P}^1, D)$ is F-split.*
2. *If $\deg(\mathcal{L}_{n-1}^{-1}) = 1$ then X is F-split if and only if the general fibers of π are ordinary Abelian varieties, and one of the following conditions occur*
   - (a) *There are no multiple fibers,*
   - (b) *There is one multiple fiber $(0; m)$ and global section s of $\mathcal{L}_{n-1}^{\otimes(1-p)}$ with degree $(p-1)$ such that $s = (\alpha_1 x - \beta_1 y)^{n_1}(\alpha_2 x - \beta_2 y)^{n_2} \cdots (\alpha_r x - \beta_r y)^{n_r}$ on $\mathbb{P}^1$ where at least any one of the coefficients of $x^i y^{p-1-i}$, $0 \leq i < \lceil \frac{p}{m} \rceil$ are nonzero,*
   - (c) *There are two multiple fibers $(0, \infty; 2, 2)$, $p > 2$ and global sections s of $\mathcal{L}_{n-1}^{\otimes(1-p)}$ with degree $(p-1)$ such that $s = (\alpha_1 x - \beta_1 y)^{n_1}(\alpha_2 x - \beta_2 y)^{n_2} \cdots (\alpha_r x - \beta_r y)^{n_r}$ on $\mathbb{P}^1$ where the coefficient of $(xy)^{\frac{p-1}{2}}$ is nonzero,*
3. *If $\deg(\mathcal{L}_{n-1}^{-1}) = 2$ then X is F-split if and only if the general fibers of π are ordinary Abelian varieties, there are no multiple fibers and a global section s of $\mathcal{L}_{n-1}^{\otimes(1-p)}$ of degree $2(p-1)$ such that $s = (\alpha_1 x - \beta_1 y)^{n_1}(\alpha_2 x - \beta_2 y)^{n_2} \cdots (\alpha_r x - \beta_r y)^{n_r}$ on $\mathbb{P}^1$ where the coefficient of $(xy)^{p-1}$ is nonzero.*

Remark 3.3.7. *We emphasize that the word "possible" means we do not know explicit examples realizing each of the permissible multiple fiber conditions listed. In characteristic 0, we can make multiple fibers via logarithmic transformations. However, if we use logarithmic transformations, we need to consider the category of the analytic varieties. Therefore, in our situation, i.e., in positive characteristic, we have no idea of how to make such examples of multiple fibers as an algebraic variety, especially in higher dimensional cases. However, in* *next section, we will discuss the existence of multiple fibers in the above list, in particular,* $\dim X = 2$.

Namely, in an elliptic surface, we give examples of elliptic surfaces realizing the some of the permissible multiple fiber conditions. We also prove that the some of the conditions are not realized for elliptic surfaces, see Corollary 3.4.13. This builds on work of Katsura and Ueno [KU, Theorem 3.3].

Proof. By Lemma 2.1.12, when a base curve is not F-split, X is never F-split. We assume that the base curve is F-split. Therefore, the base curve C is $\mathbb{P}^1$ or an ordinary elliptic curve by Remark 2.1.9. By Theorem 2.1.8, X is F-split if and only if the absolute Frobenius action

$$F_X^* : H^n(X, \omega_X) \to H^n(X, \omega_X^{\otimes p})$$

is nonzero. By Proposition 3.1.1, this condition is equivalent to the fact that the composition of the morphisms

$$H^1(C, R^{n-1}\pi_*\omega_X) \xrightarrow{F_C^*} H^1(C, F_C^* R^{n-1}\pi_*\omega_X) \xrightarrow{F_{X/C}^*} H^1(C, R^{n-1}\pi_*\omega_X^{\otimes p})$$

is nonzero.

By the relative Serre duality, we have the following isomorphisms:

$$(R^{n-1}\pi_*\omega_X)^\vee \cong \pi_*\mathcal{O}_X \otimes \omega_C^{-1} \cong \omega_C^{-1}$$

and

$$(R^{n-1}\pi_*\omega_X^{\otimes p})^\vee \cong \pi_*\omega_X^{\otimes(1-p)} \otimes \omega_C^{-1},$$

where $(-)^\vee$ means the $\mathcal{O}_C$-dual. By Lemma 3.3.1 and Theorem 2.2.6,

$$\begin{aligned} \pi_*\omega_X^{\otimes(1-p)} \otimes \omega_C^{-1} &\cong L_{n-1}^{\otimes(p-1)} \otimes \omega_C^{\otimes(-p)} \otimes \pi_*\mathcal{O}_X \left(\textstyle\sum_i (1-p)a_i \cdot P_i\right) \\ &\cong \mathcal{L}_{n-1}^{\otimes(p-1)} \otimes \omega_C^{\otimes(-p)} \otimes \mathcal{O}_C \left(\textstyle\sum_i \left\lfloor \frac{(1-p)a_i}{m_i} \right\rfloor \cdot b_i\right). \end{aligned}$$

Therefore, we have the following:

$$(R^{n-1}\pi_*\omega_X)^{\vee\vee} \cong \omega_C$$

and

$$(R^{n-1}\pi_*\omega_X^{\otimes p})^{\vee\vee} \cong \mathcal{L}_{n-1}^{\otimes(1-p)} \otimes \omega_C^{\otimes p} \otimes \mathcal{O}_C \left(\textstyle\sum_i \left\lceil \frac{(p-1)a_i}{m_i} \right\rceil \cdot b_i\right).$$

By Proposition 3.2.3, if the general fibers of π are ordinary Abelian varieties, then the relative Frobenius action

$$F_{X/C}^* : F_C^* R^{n-1}\pi_*\mathcal{O}_X \to R^{n-1}\pi_*\mathcal{O}_X$$

is isomorphic to the map

$$\times s : F_C^* R^{n-1}\pi_*\mathcal{O}_X \to R^{n-1}\pi_*\mathcal{O}_X, \tag{12}$$

where the nonzero section $s \in H^0(C, \mathcal{L}_{n-1}^{\otimes(1-p)})$. Note that this map is equal to $\times s : \mathcal{L}_{n-1}^{\otimes p} \to \mathcal{L}_{n-1}$ because of the local freeness of $R^{n-1}\pi_*\mathcal{O}_X$. Tensoring with $\mathcal{L}_{n-1}^{\otimes(-p)} \otimes \omega_C^{\otimes p}$ to (12), we have the following map:

$$\mathcal{L}_{n-1}^{\otimes p} \otimes \mathcal{L}_{n-1}^{\otimes(-p)} \otimes \omega_C^{\otimes p} \xrightarrow{\times s} \mathcal{L}_{n-1} \otimes \mathcal{L}_{n-1}^{\otimes(-p)} \otimes \omega_C^{\otimes p}.$$

Moreover, we have the following commutative diagram

$$
\begin{tikzcd}
F_C^*(R^{n-1}\pi_*\omega_X)^{\vee\vee} \arrow[d, equal] & \\
\mathcal{L}_{n-1}^{\otimes p} \otimes \mathcal{L}_{n-1}^{\otimes(-p)} \otimes \omega_C^{\otimes p} \arrow[r, "\times s"] \arrow[d, "\times t"'] & \mathcal{L}_{n-1} \otimes \mathcal{L}_{n-1}^{\otimes(-p)} \otimes \omega_C^{\otimes p} \arrow[d, "\times t"] \\
\mathcal{L}_{n-1}^{\otimes p} \otimes \mathcal{L}_{n-1}^{\otimes(-p)} \otimes \omega_C^{\otimes p} \otimes \mathcal{O}_C(D) \arrow[r, "\times s"'] & \mathcal{L}_{n-1} \otimes \mathcal{L}_{n-1}^{\otimes(-p)} \otimes \omega_C^{\otimes p} \otimes \mathcal{O}_C(D) \arrow[d, equal] \\
 & (R^{n-1}\pi_*\omega_X^{\otimes p})^{\vee\vee},
\end{tikzcd}
$$

where $D := \sum_i \left\lceil \frac{(p-1)a_i}{m_i} \right\rceil \cdot b_i$, and where t in $H^0(C, \mathcal{O}_C(D))$ is a nonzero section.

If all of the fibers of π are not ordinary Abelian varieties, then the relative Frobenius action $F_{X/C}^*$ is a zero map. This implies that X is not F-split. So we assume that the general fibers of π are ordinary Abelian varieties. Therefore, the absolute Frobenius action F_X^* is equivalent to the following composition map:

$$
\begin{aligned}
\times s \circ \times t \circ F_C^* : H^1(C, \omega_C) &\xrightarrow{F_C^*} H^1(C, \omega_C^{\otimes p}) \\
&\xrightarrow{\times t} H^1(C, \omega_C^{\otimes p} \otimes \mathcal{O}_C(D)) \\
&\xrightarrow{\times s} H^1(C, \mathcal{L}_{n-1}^{\otimes(1-p)} \otimes \omega_C^{\otimes p} \otimes \mathcal{O}_C(D)),
\end{aligned}
$$

where $D := \sum_i \left\lceil \frac{(p-1)(m_i-1)}{m_i} \right\rceil \cdot b_i$, s in $H^0(C, \mathcal{L}_{n-1}^{\otimes(1-p)})$ is a nonzero section, and t in $H^0(C, \mathcal{O}_C(D))$ is a nonzero section. Note that all values of a_i are $m_i - 1$ since all jumping numbers are 1, and all multiple fibers are tame fibers.

Since $\sum_i \left\lceil \frac{(p-1)(m_i-1)}{m_i} \right\rceil \cdot b_i = \sum_i \left\lfloor \frac{p(m_i-1)}{m_i} \right\rfloor \cdot b_i$ by Lemma 3.3.2, the above composition map is equivalent to the following:

$$
\begin{aligned}
\times s \circ F : H^1(C, \omega_C) &\xrightarrow{F} H^1\left(C, \omega_C^{\otimes p} \otimes \mathcal{O}_C\left(\sum_i \left\lfloor \frac{p(m_i-1)}{m_i} \right\rfloor \cdot b_i\right)\right) \\
&\xrightarrow{\times s} H^1\left(C, \mathcal{L}_{n-1}^{\otimes(1-p)} \otimes \omega_C^{\otimes p} \otimes \mathcal{O}_C\left(\sum_i \left\lfloor \frac{p(m_i-1)}{m_i} \right\rfloor \cdot b_i\right)\right),
\end{aligned}
$$

where the definition of F is given by (1) in Definition 3.3.3.

By Theorem 2.1.8, $(1-p)K_X$ is effective, we have the following effectivity:

$$
H^0\left(C, \left(\mathcal{L}_{n-1}^{-1} \otimes \omega_C \otimes \mathcal{O}_C\left(\sum_i \frac{m_i - 1}{m_i} \cdot b_i\right)\right)^{\otimes(1-p)}\right) > 0. \tag{13}
$$

The proof now breaks into cases depending on the isomorphism class of C and $\deg(\mathcal{L}_{n-1}^{-1})$. Firstly, we consider that the base curve is $\mathbb{P}^1$. We have the following inequality by (13):

$$
\deg(\mathcal{L}_{n-1}^{-1}) + \sum_i \frac{m_i - 1}{m_i} \leq 2.
$$

We claim that if X is F-split, then $\deg(\mathcal{L}_{n-1}^{-1}) \geq 0$. Indeed, if $\deg(\mathcal{L}_{n-1}^{-1}) < 0$, $\times s$ is a zero map. Therefore, we check the composition map $\times s \circ F$ case-by-case.

- $\deg(\mathcal{L}_{n-1}^{-1}) = 2$

In this case, there are no multiple fibers. We fix the coordinates of $\mathbb{P}^1$ as $(\beta_1 : \alpha_1) \in \mathbb{P}^1$ for $\alpha_1, \beta_1 \in k$. Then F_X^* is equivalent to the following:

$$\times s \circ F : H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1}) \xrightarrow{F} H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes p}) \xrightarrow{\times s} H^1(\mathbb{P}^1, \mathcal{L}_{n-1}^{\otimes(1-p)} \otimes \omega_{\mathbb{P}^1}^{\otimes p}),$$

where

$$\frac{1}{xy} \longmapsto \frac{1}{(xy)^p} \longmapsto \frac{s}{(xy)^p},$$

$\{x, y\}$ are the homogeneous coordinates of $\mathbb{P}^1$ and nonzero section s forms $(\alpha_1 x - \beta_1 y)^{n_1} \cdots (\alpha_r x - \beta_r y)^{n_r}$. Thus, the composition $\times s \circ F$ is nonzero if and only if there are nonzero sections $s = (\alpha_1 x - \beta_1 y)^{n_1} \cdots (\alpha_r x - \beta_r y)^{n_r}$ with degree $2(p-1)$ such that the coefficient of $(xy)^{p-1}$ is nonzero.

- $\deg(\mathcal{L}_{n-1}^{-1}) = 1$

  In this case, we have three cases by the inequality in (13). If there are no multiple fibers, then we can find that the nonzero map $\times s \circ F$, similar to the case of $\deg(\mathcal{L}_{n-1}^{-1}) = 2$. Indeed,

$$\times s \circ F : H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1}) \xrightarrow{F} H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes p}) \xrightarrow{\times s} H^1(\mathbb{P}^1, \mathcal{L}_{n-1}^{\otimes(1-p)} \otimes \omega_{\mathbb{P}^1}^{\otimes p}),$$

where

$$\frac{1}{xy} \longmapsto \frac{1}{(xy)^p} \longmapsto \frac{s}{(xy)^p},$$

and $\{x, y\}$ are the homogeneous coordinates of $\mathbb{P}^1$. If we choose any nonzero section $s \in H^0(\mathbb{P}^1, \mathcal{L}_{n-1}^{\otimes(1-p)})$, then the composition map of these maps $\times s \circ F$ is nonzero.

If there is one multiple fiber, the situation is a little complicated. Let $F_b := \pi^{-1}(b)$ be the multiple fiber. We fix the coordinate of $\mathbb{P}^1$ as $b := (0 : 1) \in \mathbb{P}^1$. The map $\times s \circ F$ is as follows:

$$\begin{aligned} \times s \circ F : H^1\left(\mathbb{P}^1, \omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \tfrac{m-1}{m} \cdot b \right\rfloor\right)\right) \quad &\xrightarrow{F} \quad H^1\left(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes p} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor p\left(\tfrac{m-1}{m} \cdot b\right)\right\rfloor\right)\right) \\ &\xrightarrow{\times s} \quad H^1\left(\mathbb{P}^1, \mathcal{L}_{n-1}^{\otimes(1-p)} \otimes \omega_{\mathbb{P}^1}^{\otimes p} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor p\left(\tfrac{m-1}{m} \cdot b\right)\right\rfloor\right)\right), \end{aligned}$$

where

$$\frac{1}{xy} \times x^{\lfloor \frac{m-1}{m} \rfloor} \longmapsto \frac{1}{(xy)^p} \times x^{\lfloor p(\frac{m-1}{m}) \rfloor} \longmapsto \frac{s}{(xy)^p} \times x^{\lfloor p(\frac{m-1}{m}) \rfloor},$$

$\{x, y\}$ are the homogeneous coordinates of $\mathbb{P}^1$ and nonzero section $s = (\alpha_1 x - \beta_1 y)^{n_1} \cdots (\alpha_r x - \beta_r y)^{n_r}$. Since $\frac{x^{\lfloor p(\frac{m-1}{m}) \rfloor}}{(xy)^p} = \frac{1}{x^{\lceil \frac{p}{m} \rceil} y^p}$, the bases of the vector space

$$H^1\left(\mathbb{P}^1, \mathcal{L}_{n-1}^{\otimes(1-p)} \otimes \omega_{\mathbb{P}^1}^{\otimes p} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor p\left(\tfrac{m-1}{m} \cdot b\right)\right\rfloor\right)\right)$$

are

$$\frac{1}{x^{\lceil \frac{p}{m} \rceil} y}, \frac{1}{x^{\lceil \frac{p}{m} \rceil - 1} y^2}, \cdots \frac{1}{x^2 y^{\lceil \frac{p}{m} \rceil - 1}}, \frac{1}{x y^{\lceil \frac{p}{m} \rceil}}.$$

Setting s as follows,

$$\begin{aligned} s \quad &= \quad (\alpha_1 x - \beta_1 y)^{n_1} \cdots (\alpha_r x - \beta_r y)^{n_r} \\ &:= \quad \sum_{i \geq 0}^{p-1} a_{p-1-i} \cdot x^i y^{p-1-i}, \end{aligned}$$

then we have,

$$\frac{s}{(xy)^p} \times x^{\lfloor p(\frac{m-1}{m}) \rfloor} = \sum_{i \geq 0}^{p-1} \frac{a_{p-1-i}}{x^{\lceil \frac{p}{m} \rceil - i} y^{1+i}}.$$

Thus, the composition $\times s \circ F$ is nonzero if and only if there are nonzero sections s such that at least any one of the coefficients of $x^i y^{p-1-i}$, $0 \leq i < \lceil \frac{p}{m} \rceil$ are nonzero.

If there are two multiple fibers, then their multiplicities are 2 by the above inequality. Let $F_{b_1} := \pi^{-1}(b_1)$, $F_{b_2} := \pi^{-1}(b_2)$ be the two multiple fibers. We fix the coordinate of $\mathbb{P}^1$ as $b_1 := (0:1)$. By projective transformation, we may assume $b_1 = (0:1)$ and $b_2 = (1:0) \in \mathbb{P}^1$. The map $\times s \circ F$ is as follows:

$$\begin{aligned} \times s \circ F : H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1}) & \xrightarrow{F} H^1\left(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes p} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor p\left(\tfrac{1}{2} \cdot b_1\right)\right\rfloor + \left\lfloor p\left(\tfrac{1}{2} \cdot b_2\right)\right\rfloor\right)\right) \\ & \xrightarrow{\times s} H^1\left(\mathbb{P}^1, \mathcal{L}_{n-1}^{\otimes(1-p)} \otimes \omega_{\mathbb{P}^1}^{\otimes p} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor p\left(\tfrac{1}{2} \cdot b_1\right)\right\rfloor + \left\lfloor p\left(\tfrac{1}{2} \cdot b_2\right)\right\rfloor\right)\right), \end{aligned}$$

where

$$\frac{1}{xy} \longmapsto \frac{1}{(xy)^p} \times x^{\lfloor \frac{p}{2} \rfloor} y^{\lfloor \frac{p}{2} \rfloor} \longmapsto \frac{s}{(xy)^p} \times x^{\lfloor \frac{p}{2} \rfloor} y^{\lfloor \frac{p}{2} \rfloor},$$

$\{x, y\}$ are the coordinates of $\mathbb{P}^1$ and nonzero section $s = (\alpha_1 x - \beta_1 y)^{n_1} \cdots (\alpha_r x - \beta_r y)^{n_r}$. Note that if $p = 2$, X is not F-split because the dimension of the following vector space:

$$H^1\left(\mathbb{P}^1, \mathcal{L}_{n-1}^{\otimes(1-p)} \otimes \omega_{\mathbb{P}^1}^{\otimes p} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor p\left(\tfrac{1}{2} \cdot b_1\right)\right\rfloor + \left\lfloor p\left(\tfrac{1}{2} \cdot b_2\right)\right\rfloor\right)\right),$$

is zero.

For $p \neq 2$, the composition $\times s \circ F$ is nonzero if and only if there are nonzero sections $s = (\alpha_1 x - \beta_1 y)^{n_1} \cdots (\alpha_r x - \beta_r y)^{n_r}$ with degree $(p-1)$ such that the coefficient of $(xy)^{\frac{p-1}{2}}$ is nonzero.

- $\deg(\mathcal{L}_{n-1}^{-1}) = 0$

  In this case, F_X^* is isomorphic to F since the second map $\times s$ is just the multiplication of a constant. Therefore, we have the assertion by Theorem 3.3.4.

Finally, we consider the case where the base curve is an ordinary elliptic curve. Similar to the case for the projective line, we have the following inequality by the effectivity in (13):

$$\deg(\mathcal{L}_{n-1}^{-1}) + \sum_i \frac{m_i - 1}{m_i} \leq 0.$$

Therefore, we have no multiple fibers, and $\deg(\mathcal{L}_{n-1}^{-1}) = 0$. Note that if $\deg(\mathcal{L}_{n-1}^{-1}) < 0$, then the map $\times s$ is a zero map, similar to the case for a projective line. Then, ω_X is isomorphic to $\pi^* \mathcal{L}_{n-1}^{-1}$. If X is F-split, we have the following effectivity of $\mathcal{L}_{n-1}^{\otimes(p-1)}$ since $\omega_X^{\otimes(1-p)}$ has a global section by Theorem 2.1.8:

$$\begin{aligned} H^0(X, \omega_X^{\otimes(1-p)}) & \cong H^0(X, \pi^* \mathcal{L}_{n-1}^{\otimes(p-1)}) \\ & \cong H^0(C, \mathcal{L}_{n-1}^{\otimes(p-1)}) \neq 0. \end{aligned}$$

Since $\deg(\mathcal{L}_{n-1}^{\otimes(p-1)}) = 0$, $\mathcal{L}_{n-1}^{\otimes(p-1)}$ is isomorphic to $\mathcal{O}_C$. Therefore, $\omega_X^{\otimes(1-p)}$ is isomorphic to $\mathcal{O}_X$. This means that $\mathrm{ord}(K_X) \mid (p-1)$. Since C is ordinary and $\mathcal{L}_{n-1}^{\otimes(1-p)}$ is isomorphic to $\mathcal{O}_C$, then the following composition map is nonzero:

$$\times s \circ F : H^1(C, \mathcal{O}_C) \xrightarrow{F} H^1(C, \mathcal{O}_C) \xrightarrow{\times s} H^1(C, \mathcal{L}_{n-1}^{\otimes(1-p)}),$$

where the nonzero section $s \in H^0(C, \mathcal{L}_{n-1}^{\otimes(1-p)}) \cong H^0(C, \mathcal{O}_C)$. $\square$

Remark 3.3.8. *If we focus on the case of* $\dim X = 2$ *in Theorem 3.3.6, we can remove the assumption of jumping numbers and* $\deg(\mathcal{L}_{n-1}^{-1})$ *is* $\chi(\mathcal{O}_X)$.

3.4. **Examples.** In this section, we will see some examples of F-split elliptic surfaces. Especially, example 3.4.1 in char $k = 2$, is a new example of F-split elliptic surfaces. The other examples below are known by other method. We will show that these examples satisfy F-split properties by elliptic fibration structures.

In the previous section, we saw the possible list of multiple fibers of F-split Abelian fiber spaces in Theorem 3.3.6. We are interested in examples realizing the restricted conditions on the fibrations of F-split elliptic fibrations as described in Theorem 3.3.6. When there exists such elliptic surface, we will give an example; otherwise, we can prove that such multiple fibers does not exist by using the necessary condition for the algebraicity of the elliptic surface due to Katsura and Ueno [KU, Theorem 3.3]. By Remark 3.3.8, we replace $\deg(\mathcal{L}_1^{-1})$ with $\chi(\mathcal{O}_X)$.

Example 3.4.1. There is an elliptic surface $\pi : X \to C$ with $\chi(\mathcal{O}_X) = 2$ and π has no multiple fibers. We assume that char $k \neq 2$. Let C_1 and C_2 be ordinary elliptic curves and C_2 is defined by $y^2 = x(x-1)(x-\lambda)$, $\lambda \neq 0, 1$. Each C_1 and C_2 has four 2-torsion points, and we have a quotient variety $X := C_1 \times C_2/(-1_{C_1}, -1_{C_2})$, where -1_{C_1} (resp. -1_{C_2}) means an involution map of C_1 (resp. C_2). The second projection $\mathrm{pr}_2 : C_1 \times C_2 \to C_2$ induces the morphism $f : X \to C_2/(-1_{C_2}) \cong \mathbb{P}^1$. Note that X has sixteen fixed points, and it is known that these points are A_1-singularities. Thus, if we take the minimal resolution of X, then we have a smooth variety $\widetilde{X}$, which is called a Kummer surface, and this $\widetilde{X}$ is $K3$ surface.

$$\begin{tikzcd}
X \arrow[r, "f"] & \mathbb{P}^1 \\
\widetilde{X} \arrow[u, "\text{minimal resolution}"] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

Therefore, the Euler characteristic $\chi(\mathcal{O}_{\widetilde{X}}) = 2$. Note that this $K3$ surface $\widetilde{X}$ is obtained from an ordinary Abelian surface. In general, Joshi and Rajan proved that the F-split property of Kummer surface is equivalent to the ordinarity of Abelian surface by [JR, Theorem 5.1.1 and Theorem 5.1.2] in char $k \neq 2$. In this example, we have an elliptic fibration $\pi : X \to \mathbb{P}^1$. We can also check the F-split property of $\widetilde{X}$ by this elliptic fibration structure. Indeed, $\pi : \widetilde{X} \to \mathbb{P}^1$ does not have any multiple fibers. Since both C_1 and C_2 are ordinary elliptic curves, we can easily check that the general fibers of π are ordinary elliptic curves. Moreover this elliptic fibration π has four singular fibers of type I_0^* at $\{0\}$, $\{1\}$, $\{\infty\}$, $\{\lambda\} \in \mathbb{P}^1$. Since the type I_0^* is additive type $\mathbb{G}_a$, then four points $\{0\}$, $\{1\}$, $\{\infty\}$, $\{\lambda\}$ are non-ordinary points with the order $\frac{p-1}{2}$. Thus, we have a nonzero section $s = \{xy(x-y)(x-\lambda y)\}^{\frac{p-1}{2}}$. Note that the coefficient of $(xy)^{p-1}$ is nonzero, since C_2 is ordinary. Therefore this $\widetilde{X}$ is F-split.

When the characteristic of k are 2, the situation is a little different. However, we have the following construction of "Kummer" elliptic fibration due to Shioda [Shi, Section 2]. We briefly recall his construction. Let C_1 and C_2 be elliptic curves, and the product $C_1 \times C_2$ is denoted by X. Note that in char $k = 2$, an elliptic curve C is defined by the following equation:

$$y^2 + axy + cy = x^3 + bx.$$

Moreover, the involution map $\iota : C \to C$ is given by the following:

$$(x, y) \mapsto (x, y + ax + c).$$

Obviously, X has the projection $\mathrm{pr}_1 : X \to C_1$, and this projection induces $f_1 : X/\iota \to C_1/\iota \cong \mathbb{P}^1$. We put

$$\Sigma := \begin{cases} \{\infty\} & if\ C_1 : \text{supersingular} \\ \{\infty, 0\} & if\ C_1 : \text{ordinary.} \end{cases}$$

For each $x \in \mathbb{P}^1 \setminus \Sigma$, the fiber $f_1^{-1}(x)$ is an elliptic curve. This f_1 induces a smooth elliptic surface $f : X \to \mathbb{P}^1$ with $f^{-1}(v) = C_v$ by a suitable configuration of C_v. Note that f has no multiple fibers since f has a section. Moreover, X is a rational surface if both C_1 and C_2 are supersingular; otherwise, X is a $K3$ surface. Therefore, if either C_1 or C_2 is an ordinary elliptic curve, then X has an elliptic fibration with no multiple fibers, and their Euler characteristic $\chi(\mathcal{O}_X) = 2$ because of $K3$ surface. Similarly, in the case of char $k \neq 2$, if both C_1 and C_2 are ordinary, the general fibers of f are ordinary elliptic curves.

Note that this elliptic fibration π has two singular fibers of type I_4^*. Since I_4^* is an additive type $\mathbb{G}_a$, then we have two non-ordinary points at $\{0\}, \{\infty\}$. Therefore the general fibers of π are ordinary and we have a nonzero section $s = (xy)$. This X is F-split.

Example 3.4.2. There is an elliptic surface $\pi : X \to C$ with $\chi(\mathcal{O}_X) = 1$ and π has no multiple fibers. We recall Example 2.1.18. Let C_1 and C_2 be two distinct smooth cubic curves that are ordinary in $\mathbb{P}^2$. Note that the smooth cubic curve $C = \{g(x, y, z) = 0\}$ in $\mathbb{P}^2$ is ordinary if and only if the coefficient of $(xyz)^{p-1}$ in g^{p-1} is nonzero. We assume that the two curves C_1 and C_2 intersect at nine distinct points, which are in the general position. The pencil generated by C_1 and C_2 gives a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. If we blow-up $\mathbb{P}^2$ at these nine points, which is denoted by X, we obtain a morphism $\pi : X \to \mathbb{P}^1$, and the Euler characteristic $\chi(\mathcal{O}_X) = 1$ because $\chi(\mathcal{O}_X)$ is birational invariant.

$$\begin{tikzcd} \mathbb{P}^2 \arrow[r, dashed] & \mathbb{P}^1 \\ X \arrow[u, "f"] \arrow[ur, "\pi"'] & \end{tikzcd}$$

Note that this X is F-split by Proposition 2.1.17 ([LMP, Proposition 2.1]). Moreover $\pi : X \to \mathbb{P}^1$ is an elliptic fibration without multiple fibers. Indeed, the genus of general fibers are one by (1) in Example 2.2.5. Since C_1 and C_2 are reduced curves and the nine points are in the general position, we have no multiple fibers. We can easily check that the general fibers of π are ordinary elliptic curves. This follows from Proposition 3.2.3 since X is F-split.

Example 3.4.3. There is an elliptic surface $\pi : X \to C$ with $\chi(\mathcal{O}_X) = 1$ and π has only one multiple fibers. We recall (2) in Example 2.2.5 for arbitrary m by [CD, p.347]. Let C_1 be an ordinary smooth cubic curve in $\mathbb{P}^2$ such that it passes through the nine distinct points $p_1, \cdots, p_9$ that are in the general position and C_2 be a curve of degree $3m$ that passes through $p_1, \cdots, p_9$ with multiplicity m. A pencil generated by mC_1 and C_2 gives a rational map from $\mathbb{P}^2$ to $\mathbb{P}^1$. If we blow-up $\mathbb{P}^2$ at these nine points (denoted by X), then we have the morphism $\pi : X \to \mathbb{P}^1$, and the Euler characteristic $\chi(\mathcal{O}_X) = 1$. This X is also F-split by Proposition 2.1.17 ([LMP, Proposition 2.1]). In particular, the strict transform of mC_1 gives a multiple fiber of π with multiplicity m. Since C_1 is ordinary, the general fibers of π are ordinary elliptic curves, similar to Example 3.4.2.

Example 3.4.4. There is an elliptic surface $\pi : X \to C$ with $\chi(\mathcal{O}_X) = 1$ and π has two multiple fibers with multiplicities 2. Let X be a Kummer surface associated with two ordinary elliptic curves C_1 and C_2, which we constructed in Example 3.4.1. Let $a_1 \in C_1$ and $a_2 \in C_2$ be the 2-torsion points of C_1 and C_2. Then, $b := (a_1, a_2)$ is the 2-torsion point of $C_1 \times C_2$.

We can define the translation map t_b on $C_1 \times C_2$ as follows:

$$\begin{array}{rcl} t_b : C_1 \times C_2 & \to & C_1 \times C_2, \\ (x, y) & \longmapsto & (x + a_1, y + a_2). \end{array}$$

Since the involution map $(-1)_{C_1 \times C_2} := (-1_{C_1}, -1_{C_2})$ is commutative with this t_b, t_b induces the automorphism t'_b of X. We denote the automorphism of X that is induced by $(-1)_{C_2}$ by τ. Note that t'_b is commutative with τ, and the composition $\sigma := \tau \circ t'_b$ is also order of 2. In Example 3.4.1, we have the elliptic fibration

$$\pi : X \to \mathbb{P}^1.$$

Since τ acts on $\mathbb{P}^1$ trivially, all fibers are preserved. On the contrary, t'_b acts on $\mathbb{P}^1$, and this action has two fixed points at 0 and ∞. Therefore, we have an elliptic fibration induced by π as follows:

$$\pi' : X/\langle\sigma\rangle \to \mathbb{P}^1/\langle\sigma\rangle \cong \mathbb{P}^1.$$

This new variety $Y := X/\langle\sigma\rangle$ is an Enriques surface; thus, $\chi(\mathcal{O}_Y) = 1$. Note that this Y is obtained by the étale double quotient of X. Then the F-split property of X implies the same property of Y.

By the way, $\pi' : Y \to \mathbb{P}^1$ is an elliptic fibration. We can also check the F-split property of Y by elliptic fibration structure. Note that π' has two multiple fibers with multiplicities 2 and σ acting on the general fiber F_π of π by translation. Since F_π is ordinary, the general fibers of $F_{\pi'}$ are also ordinary elliptic curves. Moreover this elliptic fibration π' has two singular fibers of type I_0^*. Since C_2 is an ordinary elliptic curve, we have nonzero section s such that the coefficient of $(xy)^{\frac{p-1}{2}}$ is nonzero. Therefore Y is F-split.

Example 3.4.5. There is an elliptic surface $\pi : X \to C$ with $\chi(\mathcal{O}_X) = 0$ and π has no multiple fibers. Let C be an ordinary elliptic curve and X be a product $C \times \mathbb{P}^1$. The Euler characteristic is $\chi(\mathcal{O}_X) = 0$. X has no multiple fibers, and the general fibers are ordinary elliptic curves trivially. Therefore, X is F-split.

Example 3.4.6. There is an elliptic surface $\pi : X \to C$ with $\chi(\mathcal{O}_X) = 0$ and π has two multiple fibers with any multiplicities. Let C be an ordinary elliptic curve and $m_C : C \to C$ be a morphism defined by an m-times map. $\mathrm{Ker}(m_C)$ is isomorphic to the following:

$$\begin{cases} \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z} & if\ (m,p) = 1, \\ \mathbb{Z}/p^n\mathbb{Z} \oplus \mu_{p^n} \oplus G_n^0 & if\ m = p^n, \end{cases}$$

where G_n^0 is a local affine group scheme.

Firstly, we consider the case of $(m, p) = 1$. In this case, $\mathbb{Z}/m\mathbb{Z}$ acts on C by translation and also acts on $\mathbb{P}^1$ as follows:

$$\begin{array}{rcl} \sigma : \mathbb{P}^1 & \to & \mathbb{P}^1, \\ (x : y) & \longmapsto & (x : \xi y), \end{array}$$

where σ is a generator of $\mathbb{Z}/m\mathbb{Z}$, and ξ is a primitive the m-th root of unity. Note that this cover ramifies at $\{0\}, \{\infty\}$ such that the quotient variety $\mathbb{P}^1/(\mathbb{Z}/m\mathbb{Z})$ is also $\mathbb{P}^1$. Then, $\mathbb{Z}/m\mathbb{Z}$ acts on $C \times \mathbb{P}^1$. Therefore, the second projection $p : C \times \mathbb{P}^1 \to \mathbb{P}^1$ induces an elliptic fibration $f : (C \times \mathbb{P}^1)/G \to \mathbb{P}^1/G \cong \mathbb{P}^1$, where G is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. Note that the Euler characteristic $\chi(\mathcal{O}_X)$ are 0 since $X := (C \times \mathbb{P}^1)/G$ is relatively minimal elliptic ruled surface. Note that this ruled surface is obtained from the quotients $C \times \mathbb{P}^1$ by G acting freely. Then the quotient is also F-split.

We will check the F-split property of X by elliptic fibration. Indeed, f has two multiple fibers at $\{0\}, \{\infty\}$ with multiplicity m. Note that the quotient map $C \to C' := C/(\mathbb{Z}/m\mathbb{Z})$

gives an isogeny between C and C'. Since an elliptic curve that is isogeneous with another ordinary elliptic curves is also ordinary, C' is an ordinary elliptic curve because of the ordinarity of C. Therefore X is F-split.

$$\begin{tikzcd} C \times \mathbb{P}^1 \arrow[r, "p"] \arrow[d] & \mathbb{P}^1 \arrow[d] \\ X := (C \times \mathbb{P}^1)/G \arrow[r, "f"] & \mathbb{P}^1/G \cong \mathbb{P}^1 \end{tikzcd}$$

Secondly, we consider the case of $m = p^n$. The following example was given by Katsura and Ueno in [KU, Example 4.9]. In this case, the situation is little different. Let G be a local group scheme μ_{p^n} and G_m be $k^\times = k \setminus \{0\}$, where k is a base field. Since μ_{p^n} is a subset of G_m, μ_{p^n} acts on G_m, and this action extends to $\mathbb{P}^1$. Note that this action has two fixed points. Let C be an ordinary elliptic curve. Since this C has μ_{p^n} as a subgroup, μ_{p^n} acts on C. Then, μ_{p^n} acts on $C \times \mathbb{P}^1$. The second projection induces the morphism

$$f : (C \times \mathbb{P}^1)/\mu_{p^n} \to \mathbb{P}^1/\mu_{p^n} \cong \mathbb{P}^1.$$

Therefore, this morphism has two multiple fibers with multiplicity p^n with $\chi(\mathcal{O}_X) = 0$. Note that the general fibers of f are elliptic curves, similar to the case of $(m, p) = 1$.

Example 3.4.7. There is an elliptic surface $\pi : X \to C$ with $\chi(\mathcal{O}_X) = 0$ and π has three multiple fibers with multiplicities 2. Let C be an ordinary elliptic curve and X be a quotient variety $(C \times C)/S_2$, where S_2 acts by $\sigma(x, y) := (y, x)$. Moreover, let $g : C \times C \to C$ be a morphism defined by $g(x, y) := y - x$. Then, we have the following commutative diagram:

$$\begin{tikzcd} C \times C \arrow[r, "g"] \arrow[d, "\pi"'] & C \arrow[d] \\ X := (C \times C)/S_2 \arrow[r, "f"] & C/G \cong \mathbb{P}^1, \end{tikzcd}$$

where G is an involution group acting on C by $p(x) := -x$. We can easily check that the general fibers of f are ordinary elliptic curves. Note that X is birationally isomorphic to $C \times \mathbb{P}^1$; thus, the Euler characteristic $\chi(\mathcal{O}_X)$ is equal to 0. Moreover, this elliptic fibration f has three multiple fibers with multiplicity 2 that correspond to the three nonzero points of order 2 of C. Therefore this X is F-split. Note that a symmetric product of F-split varieties is also F-split by [KT, Lemma 14].

Example 3.4.8. There are elliptic surfaces $\pi : X \to C$ with $\chi(\mathcal{O}_X) = 0$ and π has three multiple fibers with multiplicities $(3, 3, 3)$, $(2, 3, 6)$, $(2, 4, 4)$ and four multiple fibers with multiplicities 2. We briefly recall the construction of hyperelliptic surfaces due to Bombieri and Mumford [BM]. Since such surfaces are obtained by étale quotients of Abelian surfaces, we have F-split hyperelliptic surfaces. In this example, we will show the F-split property of hyperelliptic surfaces by their elliptic fibration structures.

Let C_1 and C_2 be ordinary elliptic curves and A be a finite subgroup of C_1. Further, A acts on $C_1 \times C_2$ by

$$k(u, v) := (u + k, \alpha(k)(v))$$

for some injective homomorphism

$$\alpha : A \to \mathrm{Aut}(C_2).$$

Note that $\mathrm{Aut}(C_2)$ is described as the semiproduct:

$$\mathrm{Aut}(C_2) = C_2 \cdot \mathrm{Aut}(C_2, 0),$$

where C_2 is a normal subgroup of translations, and $\mathrm{Aut}(C_2, 0)$ is a finite subgroup of automorphisms, fixing 0. By [BM, p.36],

$$\alpha(A) = A_0 \cdot \mathbb{Z}/n\mathbb{Z},$$

where A_0 is a finite group scheme of the translation $A_0 \subset C_2$, and $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group ($n = 2, 3, 4$, or 6). We have two elliptic fibrations for X that are given by

$$\begin{array}{rcl} (C_1 \times C_2)/A & \to & C_1/A, \\ (C_1 \times C_2)/A & \to & C_2/\alpha(A) \cong \mathbb{P}^1. \end{array}$$

We have several cases of elliptic surfaces that depend on the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and A_0. We write the two fibrations f and g as follows:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P}^1 & & C \cong C_1/A. \end{tikzcd}$$

For example, we have the following:

- $\mathbb{Z}/2\mathbb{Z}$

  In this case, the first fibration f has four multiple fibers whose multiplicities are $(2, 2, 2, 2)$. The Euler characteristic $\chi(\mathcal{O}_X) = 0$, and their order $\mathrm{ord}(K_X) = 2$. Then, the Kodaira dimension of X is zero. Therefore, the second fibration g has no multiple fibers.
- $\mathbb{Z}/3\mathbb{Z}$

  In this case, the first fibration f has three multiple fibers whose multiplicities are $(3, 3, 3)$. The Euler characteristic $\chi(\mathcal{O}_X) = 0$, and their order $\mathrm{ord}(K_X) = 3$. Then, the Kodaira dimension of X is zero. Therefore, the second fibration g has no multiple fibers.
- $\mathbb{Z}/4\mathbb{Z}$

  In this case, the first fibration f has three multiple fibers whose multiplicities are $(2, 4, 4)$. The Euler characteristic $\chi(\mathcal{O}_X) = 0$, and their order $\mathrm{ord}(K_X) = 4$. Then, the Kodaira dimension of X is zero. Therefore, the second fibration g has no multiple fibers.
- $\mathbb{Z}/6\mathbb{Z}$

  In this case, the first fibration f has three multiple fibers whose multiplicities are $(2, 3, 6)$. The Euler characteristic $\chi(\mathcal{O}_X) = 0$, and their order $\mathrm{ord}(K_X) = 6$. Then, the Kodaira dimension of X is zero. Therefore, the second fibration g has no multiple fibers.

Note that the general fibers are ordinary elliptic curves in all cases since both C_1 and C_2 are ordinary elliptic curves.

We can construct some examples of elliptic fibrations. However, can we construct the other examples of elliptic surfaces? Katsura and Ueno [KU, Theorem 3.3] provide an answer to this question. We use their necessary condition of the existence for the algebraicity of elliptic surfaces with given multiple fibers.

Definition 3.4.9. [KU, Definition 3.1] *The relatively minimal elliptic fibration $f : X \to \mathbb{P}^1$ with $\chi(\mathcal{O}_X) = 0$ is of type $(m_1, \cdots, m_\lambda \mid \nu_1, \cdots, \nu_\lambda)$ if each multiple fiber has multiplicity m_i and $\nu_i := \mathrm{ord}(E_i|_{E_i})$, where E_i is the indecomposable curve of canonical type of a multiple fiber. In the case where all multiple fibers are tame, i.e., $\nu_i = m_i$, $i = 1, 2, \cdots, \lambda$, such an elliptic surface is of type $(m_1, \cdots, m_\lambda)$.*

Definition 3.4.10. [KU, Definition 3.2] *For a fixed i, $1 \le i \le \lambda$, it is said that $(m_1, \cdots, m_\lambda \mid \nu_1, \cdots, \nu_\lambda)$ satisfies the condition U_i if there exist integers $n_1, n_2, \cdots, n_\lambda$ such that*

$$\begin{cases} n_i \equiv 1 \pmod{\nu_i}, \\ \dfrac{n_1}{m_1} + \dfrac{n_2}{m_2} + \cdots + \dfrac{n_\lambda}{m_\lambda} \in \mathbb{Z}. \end{cases}$$

Theorem 3.4.11. [KU, Theorem 3.3] *Let $f : X \to \mathbb{P}^1$ be a relatively minimal elliptic fibration of type $(m_1, \cdots, m_\lambda \mid \nu_1, \cdots, \nu_\lambda)$. Then, $(m_1, \cdots, m_\lambda \mid \nu_1, \cdots, \nu_\lambda)$ satisfies the condition U_i, $i = 1, 2, \cdots, \lambda$.*

Lemma 3.4.12. *There are no relatively minimal elliptic fibration $\pi : X \to \mathbb{P}^1$ such that $\chi(\mathcal{O}_X) = 0$ with only tame fibers and type $(2, 2, d)$ $(d \ge 3)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. Further, there are no two multiple fibers with different multiplicities, and only one multiple fiber.*

Proof. The above types of multiple fibers do not satisfy condition U_i; see [KU, Definition 3.2]. By [KU, Theorem 3.3], there are no such multiple fibers. □

By Remark 3.3.8, we have similar list of F-split elliptic surfaces with only tame fibers. This contains possible multiple fibers, but now we have the list of F-split elliptic surfaces with only tame fibers without possibilities.

Corollary 3.4.13. *For relatively minimal elliptic fibrations over algebraically closed field of positive characteristic $p > 0$, every case in Theorem 3.3.6 arises except where $\chi(\mathcal{O}_X) = 0$. In this case one has an F-split pair $(\mathbb{P}^1, D)$ where D describes the structure of the multiplicities of the multiple fibers of π and Theorem 3.3.4 controls the possibilities for D. Denoting by r the number of the fibers in the fibration $\pi : X \to C$, the only cases from Theorem 3.3.4 arising are $r = 0$, $r = 2$ and both fibers have the same multiplicities, 1.(a) with $d = 2$, and all of the cases in 2.*

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