October 1993

Solutions to the $U_q(\widehat{sl_2})$ quantum Knizhnik-Zamolodchikov equation on $\mathbf{P}^1$

Hidetoshi AWATA[1] , Satoru ODAKE[2] and Jun'ichi SHIRAISHI[3]

[1] *Yukawa Institute for Theoretical Physics*
Kyoto University, Kyoto 606, Japan

[2] *Department of Physics, Faculty of Liberal Arts*
Shinshu University, Matsumoto 390, Japan

[3] *Research Institute for Mathematical Sciences*
Kyoto University, Kyoto 606-01, Japan

Abstract

Using the free field realization, we construct the Jackson-integral formulae for the $U_q(\widehat{sl_2})$ correlation functions due to the representation theory.

[1] e-mail address : awata@ps1.yukawa.kyoto-u.ac.jp, Fellow of Soryushi Shogakukai.
[2] e-mail address : odake@jpnyitp.yukawa.kyoto-u.ac.jp.
[3] e-mail address : shiraish@kurims.kyoto-u.ac.jp, On leave from Department of Physics, University of Tokyo, Tokyo 113, Japan.

1. Introduction

Using the free field realization, we construct the Jackson-integral formulae for the $U_q(\widehat{sl_2})$ correlation functions due to the representation theory.

2. Quantum affine algebra $U_q(\widehat{sl_2})$

2.1. First we fix some notations. The algebra $U_q(\widehat{sl_2})$ is generated by $E(z)$, $F(z)$ and $\psi_\pm(z)$ with relations:

$$
\begin{aligned}
\psi_\pm(z)\psi_\pm(w) &= \psi_\pm(w)\psi_\pm(z), \\
\psi_+(z)\psi_-(w) &= \psi_-(w)\psi_+(z)\frac{(z-wq^{k+2})(z-wq^{-k-2})}{(z-wq^{k-2})(z-wq^{-k+2})}, \\
\psi_+(z)E(w) &= E(w)\psi_+(z)\frac{z-wq^{-\frac{k}{2}-2}}{z-wq^{-\frac{k}{2}+2}}q^2, \\
\psi_+(z)F(w) &= F(w)\psi_+(z)\frac{z-wq^{\frac{k}{2}+2}}{z-wq^{\frac{k}{2}-2}}q^{-2}, \\
E(z)\psi_-(w) &= \psi_-(w)E(z)\frac{z-wq^{-\frac{k}{2}-2}}{z-wq^{-\frac{k}{2}+2}}q^2, \\
F(z)\psi_-(w) &= \psi_-(w)F(z)\frac{z-wq^{\frac{k}{2}+2}}{z-wq^{\frac{k}{2}-2}}q^{-2}, \\
(z-wq^2)E(z)E(w) &+ (w-zq^2)E(w)E(z) = 0, \\
(z-wq^{-2})F(z)F(w) &+ (w-zq^{-2})F(w)F(z) = 0, \\
[\,E(z), F(w)\,] &= \frac{1}{(q-q^{-1})zw}\left\{\delta\left(\frac{w}{z}q^k\right)\psi_+(wq^{\frac{k}{2}}) - \delta\left(\frac{w}{z}q^{-k}\right)\psi_-(wq^{-\frac{k}{2}})\right\},
\end{aligned}
$$

where $k \in \mathbf{C}$ is a center, $q \in \mathbf{C}$ and $\delta(z) := \sum_{n\in\mathbf{Z}} z^n$. Note that $1/(a-b)$ means following formal power series:

$$
\frac{1}{a-b} := \frac{1}{a}\sum_{n\geq 0}\left(\frac{b}{a}\right)^n \neq -\frac{1}{b-a}.
$$

Their mode expansion is defined as

$$
\begin{aligned}
\psi_\pm(z) &=: q^{H_0} exp\left\{\pm(q-q^{-1})\sum_{\mp n>0} H_n z^{-n}\right\}, \\
E(z) &=: \sum_{n\in\mathbf{Z}} E_n z^{-n-1}, \qquad F(z) =: \sum_{n\in\mathbf{Z}} F_n z^{-n-1}.
\end{aligned}
$$

The Chevalley generators e_i, f_i and invertible k_i $(i=0,1)$ are

$$e_1 := E_0, \qquad k_1 := q^{H_0}, \qquad f_1 := F_0,$$
$$e_0 := F_1 q^{-H_0}, \qquad k_0 := q^{k-H_0}, \qquad f_0 := q^{H_0} E_{-1}.$$

Let us introduce the comultiplication Δ for the Chevalley generators

$$\Delta(e_i) := e_i \otimes k_i + 1 \otimes e_i, \qquad \Delta(k_i) := k_i \otimes k_i, \qquad \Delta(f_i) := f_i \otimes 1 + k_i^{-1} \otimes f_i.$$

2.2. Let V_λ be the Verma module over $U_q(\widehat{sl_2})$, generated by the highest weight vector $|\lambda\rangle$, such that $e_i|\lambda\rangle = 0$ and $k_1|\lambda\rangle = q^\lambda|\lambda\rangle$ with $\lambda \in \mathbf{C}$. The dual module V_λ^* is generated by $\langle\lambda|$ which satisfies $\langle\lambda|f_i = 0$ and $\langle\lambda|k_1 = q^\lambda\langle\lambda|$. The bilinear form $V_\lambda^* \otimes V_\lambda \to \mathbf{C}$ is uniquely defined by $\langle\lambda|\lambda\rangle = 1$ and $\big(\langle u|X\big)|v\rangle = \langle u|\big(X|v\rangle\big)$ for any $\langle u| \in V_\lambda^*$, $|v\rangle \in V_\lambda$ and $X \in U_q(\widehat{sl_2})$.

Let $V_\ell(z)$ be the finite dimensional centerless representation of $U_q(\widehat{sl_2})$, which is defined by $V_\ell(z) := \oplus_{m\geq 0}\mathbf{C}(z)v_{\ell,m}$ and

$$e_1 v_{\ell,m} = [\ell-m]\, v_{\ell,m+1}, \qquad e_0 v_{\ell,m} = z[m]\, v_{\ell,m-1},$$
$$k_1 v_{\ell,m} = q^{-\ell+2m} v_{\ell,m}, \qquad k_0 v_{\ell,m} = q^{\ell-2m} v_{\ell,m},$$
$$f_1 v_{\ell,m} = [m]\, v_{\ell,m-1}, \qquad f_0 v_{\ell,m} = z^{-1}[\ell-m]\, v_{\ell,m+1},$$

where $[n] := (q^n - q^{-n})/(q-q^{-1})$. Throughout the paper, weight ℓ takes arbitrary complex value. If $\ell \in \mathbf{Z}_{\geq 0}$, then $\tilde{V}_\ell(z) := \oplus_{m=0}^{\ell}\mathbf{C}(z)v_{\ell,m}$ is a $(\ell+1)$-dimensional irreducible representation of $U_q(\widehat{sl_2})$.

2.3. The type I vertex operator, $\Phi_\lambda^{\nu,\ell}(z) : V_\lambda \to V_\nu \otimes V_\ell(z)$ is defined as the intertwining operator such that

$$\Phi_\lambda^{\nu,\ell}(z)\, X = \Delta(X)\, \Phi_\lambda^{\nu,\ell}(z),$$

for any $X \in U_q(\widehat{sl_2})$.

We define the correlation functions as the matrix elements for the product of the vertex operators

$$\langle\lambda_\infty|\, \Phi_{\lambda_{n-1}}^{\lambda_\infty,\ell_n}(z_n) \cdots \Phi_{\lambda_0}^{\lambda_1,\ell_1}(z_1)\, |\lambda_0\rangle.$$

Then they satisfy the q-KZ equation [FR].

3. Free field realization

3.1. The free field algebra is generated by a_n, b_n, c_n, Q_a, Q_b and Q_c ($n \in \mathbf{Z}$) with relations

$$[\, a_n, a_m \,] = \delta_{n+m,0} \frac{[2n]\,[(k+2)n]}{n}, \qquad [\, a_0, Q_a \,] = 2(k+2),$$
$$[\, b_n, b_m \,] = -\delta_{n+m,0} \frac{[n]\,[n]}{n}, \qquad [\, b_0, Q_b \,] = -1,$$
$$[\, c_n, c_m \,] = +\delta_{n+m,0} \frac{[n]\,[n]}{n}, \qquad [\, c_0, Q_c \,] = +1.$$

The remaining commutators vanish.

Let us define the following generating functions

$$\left(\frac{M_1}{N_1} \cdots \frac{M_r}{N_r} a\right)(z;\beta) := -\sum_{n \neq 0} \frac{[M_1 n] \cdots [M_r n]}{[N_1 n] \cdots [N_r n]} \frac{a_n}{[n]} z^{-n} q^{|n|\beta} + \frac{M_1 \cdots M_r}{N_1 \cdots N_r} \left(a_0 \log z + Q_a\right),$$
$$\left(\frac{M_1}{N_1} \cdots \frac{M_r}{N_r} a_\pm\right)(z) := \pm(q - q^{-1}) \sum_{\mp n > 0} \frac{[M_1 n] \cdots [M_r n]}{[N_1 n] \cdots [N_r n]} a_n z^{-n} \pm \frac{M_1 \cdots M_r}{N_1 \cdots N_r} a_0 \log q.$$

The fields $b(z;\beta)$, $c(z;\beta)$, $b_\pm(z)$ and $c_\pm(z)$ are defined in the same way.

The quantum affine algebra $U_q(\widehat{sl_2})$ is realized by the free field algebra as follows [M2, Sh]:

$$E(z) := E_+(z) - E_-(z), \qquad F(z) := F_+(z) - F_-(z)$$

with

$$E_\pm(z) := \frac{-1}{(q-q^{-1})z} \,{}^{\circ}_{\circ} \exp\left\{ b_\pm(z) - (b+c)(zq^{\pm 1}; 0) \right\} {}^{\circ}_{\circ},$$
$$\psi_\pm(zq^{\pm\frac{k}{2}}) := \exp\left\{ a_\pm(zq^{\pm\frac{k+2}{2}}) + \left(\frac{2}{1} b_\pm\right)(zq^{\pm(k+1)}) \right\},$$
$$F_\pm(z) := \frac{1}{(q-q^{-1})z} \,{}^{\circ}_{\circ} \exp\left\{ a_\pm(zq^{\pm\frac{k+2}{2}}) + b_\pm(zq^{\pm(k+2)}) + (b+c)(zq^{\pm(k+1)}; 0) \right\} {}^{\circ}_{\circ}.$$

Here, normal order ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ mean the ordering such that a_n ($n \geq 0$) move to the right of a_n ($n < 0$) and Q_a, etc.

3.2. Let us define the screening current $S(z)$ as

$$S(z) := S_+(z) - S_-(z),$$

with

$$S_\pm(z) := \frac{1}{(q-q^{-1})z} \,{}^{\circ}_{\circ} \exp\left\{-b_\pm(z) - (b+c)(zq^{\mp 1};0)\right\} {}^{\circ}_{\circ}\, s(z),$$
$$s(z) := {}^{\circ}_{\circ} \exp\left\{-\left(\frac{1}{k+2}a\right)(z;-\frac{k+2}{2})\right\} {}^{\circ}_{\circ}.$$

Then it satisfies

$$[\,E(z), S(w)\,] = [\,\psi_\pm(z), S(w)\,] = 0, \qquad [\,F(z), S(w)\,] = \frac{1}{z}{}_{k+2}\partial_w \left\{\delta\left(\frac{w}{z}\right)\bar{s}(w)\right\},$$
$$\bar{s}(z) := {}^{\circ}_{\circ} \exp\left\{-\left(\frac{1}{k+2}a\right)(z;\frac{k+2}{2})\right\} {}^{\circ}_{\circ},$$

where ${}_a\partial_z$ is the following q difference operator

$${}_a\partial_z f(z) := \frac{f(zq^a) - f(zq^{-a})}{(q-q^{-1})z}.$$

Hence, if the Jackson integral of the screening currents

$$\int_0^{s\infty} d_p t S(t) := s(1-p)\sum_{n=-\infty}^{\infty} S(sp^n)p^n, \qquad p := q^{2(k+2)}$$

are convergent, then they commute with the action of $U_q(\widehat{sl_2})$ exactly.

3.3. Let us define the vertex operator $\phi_{\ell,m}(z) : V_\lambda \to V_{\lambda+\ell}$ as

$$\phi_{\ell,0}(z) := {}^{\circ}_{\circ} \exp\left\{\left(\frac{1}{2}\frac{1}{k+2}a\right)(zq^{k+2};\frac{k+2}{2})\right\} {}^{\circ}_{\circ},$$
$$\phi_{\ell,m+1}(z) := \oint dx \phi_{\ell,m+1}(z,x), \qquad \phi_{\ell,m+1}(z,x) := [\,\phi_{\ell,m}(z), F(x)\,]_{q^{\ell-2m}},$$

where $[A,B]_q := AB - qBA$. For $f(x) = \sum_{n\in\mathbf{Z}} f_n x^n$, the integral means $\oint \frac{dx}{x} f(x) = f_0$, i.e., the constant part of $f(x)$ in x. We will denote $\phi_\ell(z) := \phi_{\ell,0}(z)$.

Then the type I vertex operator $\Phi_\lambda^{\nu,\ell}(z) : V_\lambda \to V_\nu \otimes V_\ell(z)$ is realized as follows:

$$\Phi_\lambda^{\nu,\ell}(z) = \int_0^{s\infty} d_p\mathbf{t}\, S(t_1)\cdots S(t_r) \sum_{m\geq 0} \phi_{\ell,m}(z) \otimes v_{\ell,m},$$

where $\nu = \lambda + \ell - 2r$ and $d_p\mathbf{t} = d_pt_1\cdots d_pt_r$.

Therefore, the correlation functions are realized by the free fields as follows:

$$\sum_{m_1,\cdots,m_n\geq 0} \int_0^{s\infty} d_p\mathbf{t}\, \langle\, \lambda_\infty |\, S(t_1)\cdots S(t_m)\phi_{\ell_1,m_1}(z_1)\cdots\phi_{\ell_n,m_n}(z_n)\,|\lambda_0\,\rangle \otimes_{i=1}^n v_{\ell_i,m_i},$$

where $\lambda_\infty := \lambda_0 + \sum_i \ell_i - 2m$ with $m := \sum_a m_a$. Here we consider only the case that $\lambda_0 = 0$.

4. Factorization to one-point functions

4.1. Let us consider the integrand of our correlation function:

$$\Psi := \langle \prod_{i=1}^{m} S(t_i) \prod_{r=1}^{n} \phi_{\ell_r,m_r}(z_r) \rangle := \langle \lambda_\infty | \, S(t_1) \cdots S(t_m) \phi_{\ell_1,m_1}(z_1) \cdots \phi_{\ell_n,m_n}(z_n) \, |0 \rangle.$$

Here and after, we omit the highest weight λ_∞ and $\lambda_0 = 0$, and the product of the operators means the following order:

$$\prod_{i=a}^{b} \mathcal{O}(x_i) := \mathcal{O}(x_a)\mathcal{O}(x_{a+1}) \cdots \mathcal{O}(x_b).$$

Let $\mathbf{x} := (x_1, \cdots, x_m) := (\mathbf{x}_1, \cdots, \mathbf{x}_n)$, $\mathbf{x}_r := (x_{r,1}, \cdots, x_{r,m_r})$ and $d\mathbf{x} := \prod_{i=1}^{m} dx_i$, then

$$\Psi = \oint d\mathbf{x} \langle \prod_{i=1}^{m} S(t_i) \prod_{r=1}^{n} \phi_{\ell_r,m_r}(z_r, \mathbf{x}_r) \rangle,$$

$$\phi_{\ell,m}(z, \mathbf{x}) := [\cdots [\phi_\ell(z), F(x_1)]_{q^\ell} \cdots, F(x_m)]_{q^{\ell-2(m-1)}}.$$

Every rational function in this paper is a formal power series. But by the residue theorem, we can realize $\oint dx$ as a contour integral around the origin (counter clock wise) or the infinity (clock wise).

First, since $S_\pm(t)\{xF(x)|_{x=0}\} = 0$, $\phi_\ell(z)\{xF(x)|_{x=0}\} = \{xF(x)|_{x=0}\}\phi_\ell(z) = 0$ and $F_\pm(y)\{xF(x)|_{x=0}\} = \{xF(x)|_{x=0}\}F_\pm(y) = 0$, the residue at $x = 0$ cancels between $F_+(x)$ and $F_-(x)$, i.e., $\mathrm{res}_{x=0}\langle \cdots F(x) \cdots \rangle = 0$. Next, since $\langle S_+(t)S(u) \rangle = \langle S_-(t)S(u) \rangle$ and $S(t)\{xF_\pm(x)|_{x=\infty}\} = 0$, the residue at $x = \infty$ cancels between $S_+(t)$ and $S_-(t)$, i.e., $\mathrm{res}_{x=\infty}\langle \cdots S(t) \cdots \rangle = 0$. Thus we have,

Proposition. *The integrand of Ψ does not have residue at $x_i = 0$ nor ∞ $(i = 1, \cdots, m)$.*

4.2. As we prove in appendix B, one can show that: At first, by calculating the integral for the variables x's of the current $F(x)$'s, we find that the $F_-(x)$'s do not contribute to the results. Second, by the residue theorem, we can estimate the integral for x at the polls y such that $|y| > |x|$. Then we find that the $S_-(t)$'s contribute only for the residue canceling at $x = \infty$.

Therefore, for our correlation functions Ψ, only the $+$ parts of $F(x)$ and $S(z)$ are non-vanishing.

Proposition.

$$\Psi = \oint d\mathbf{x} \langle \prod_{i=1}^{m} S_+(t_i) \prod_{r=1}^{n} \phi^+_{\ell_r, m_r}(z_r, \mathbf{x}_r) \rangle - \mathrm{res}_\infty,$$

$$\phi^+_{\ell,m}(z, \mathbf{x}) := [\cdots [\phi_\ell(z), F_+(x_1)]_{q^\ell} \cdots, F_+(x_m)]_{q^{\ell-2(m-1)}}.$$

where res_∞ *means the residue at some of* x*'s are* ∞.

4.3. Now we denote

$$S_+(t) = s(t)\beta(t), \qquad F_+(x) = f(x)\gamma(x),$$

with

$$\beta(t) := \frac{1}{(q-q^{-1})t} \, {}^{\circ}_{\circ} \exp\left\{-b_+(t) - (b+c)(tq^{-1}; 0)\right\} {}^{\circ}_{\circ},$$

$$\gamma(x) := \frac{1}{(q-q^{-1})x} \, {}^{\circ}_{\circ} \exp\left\{b_+(xq^{k+2}) + (b+c)(xq^{k+1}; 0)\right\} {}^{\circ}_{\circ},$$

$$f(x) := \exp\left\{a_+(xq^{\frac{k+2}{2}})\right\}.$$

The correlation function decouple to the a-part Ψ_a and bc-part Ψ_{bc}

$$\Psi = \oint d\mathbf{x} \, \Psi_a \Psi_{bc} - \mathrm{res}_\infty,$$

where

$$\Psi_a := \langle \prod_{i=1}^{m} s(t_i) \prod_{r=1}^{n} \phi^{(m_r)}_{\ell_r}(z_r, \mathbf{x}_r) \rangle, \qquad \Psi_{bc} := \langle \prod_{i=1}^{m} \beta(t_i) \prod_{r=1}^{n} P^{(m_r)}_{\ell_r}(\mathbf{x}_r) \rangle,$$

and

$$\phi^{(m)}_\ell(z, \mathbf{x}) := [\cdots [\phi_\ell(z), f(x_1)]_{q^\ell}, \cdots, f(x_m)]_{q^{\ell-2(m-1)}},$$

$$P^{(m)}_\ell(\mathbf{x}) := [\cdots [\gamma(x_1), \gamma(x_2)]_{q^{\ell-2}}, \cdots, \gamma(x_m)]_{q^{\ell-2(m-1)}}.$$

4.4. Here we analyze a term of Ψ such that

$$\Psi' := \oint d\mathbf{x} \, \Psi_a \langle \prod_{i=1}^{m} \beta(t_i) \prod_{r=1}^{m} \gamma(x_r) \rangle - \mathrm{res}_\infty.$$

Since Ψ_a has no polls about x, we have the following inductive formula:

$$\Psi' = \oint d\mathbf{x} \, \Psi_a \sum_j \langle \prod_{i=1, i\neq j}^{m} \beta(t_i) \prod_{r=2}^{m} \gamma(x_r) \rangle \frac{1}{t_j q^{-k-2}} \delta\left(\frac{x_1}{t_j q^{-k-2}}\right) \prod_{k>j} C_{jk} - \mathrm{res}_\infty,$$

here the symmetric factor C_{ij} is

$$C_{ij} := \frac{t_i - t_j q^2}{t_i q^2 - t_j}.$$

So we get

$$\Psi' = \oint d\mathbf{x} \Psi_a \sum_{\sigma \in S} \prod_{i=1}^{m} t_{\sigma(i)} q^{-k-2} \delta\left(\frac{x_i}{t_{\sigma(i)} q^{-k-2}}\right) \prod_{i>j, \sigma(i)<\sigma(j)} C_{\sigma(i)\sigma(j)}.$$

4.5. Similarly, we have other inductive formula as follows:

$$\Psi' = \oint d\mathbf{x}\, \Psi_a \sum_s \langle \prod_{i=1}^{m-1} \beta(t_i) \prod_{r=1, r\neq s}^{m} \gamma(x_r) \rangle \frac{1}{t_m q^{-k-2}} \delta\left(\frac{x_s}{t_m q^{-k-2}}\right) \prod_{k<s} G_{ks} - \mathrm{res}_\infty,$$

here the symmetric factor G_{rs} is

$$G_{rs} := \frac{x_r - x_s q^2}{x_r q^2 - x_s}.$$

From this property, we obtain the "screening current Ward identity"[ATY]:

$$\begin{aligned}
&\oint d\mathbf{x}\, \Psi_a \langle \prod_{i=1}^{m} \beta(t_i) \prod_{r=1}^{n} P_{\ell_r}^{(m_r)}(\mathbf{x}_r) \rangle - \mathrm{res}_\infty \\
=& \oint d\mathbf{x}\, \Psi_a \sum_s \langle \prod_{i=1}^{m-1} \beta(t_i) P_{\ell_1}^{(m_1)}(\mathbf{x}_1) \cdots (b(t_m) P_{\ell_s}^{(m_s)}(\mathbf{x}_s)) \cdots P_{\ell_n}^{(m_n)}(\mathbf{x}_n) \rangle \prod_{r<s} G_{rs}^{(m)} - \mathrm{res}_\infty,
\end{aligned}$$

with

$$G_{rs}^{(m)} := \prod_i \frac{x_{r_i} - t_m q^{-k}}{x_{r_i} q^2 - t_m q^{-k-2}}.$$

Here $(b(t_m) P_{\ell_s}^{(m_s)}(\mathbf{x}_s))$ means that $b(t_m)$ contract with some of $\gamma(x_{s_i})$ in $P_{\ell_s}^{(m_s)}(\mathbf{x}_s))$. From this identity, we have

$$\begin{aligned}
&\oint d\mathbf{x}\, \Psi_a \langle \prod_{i=1}^{m} \beta(t_i) \prod_{r=1}^{n} P_{\ell_r}^{(m_r)}(\mathbf{x}_r) \rangle - \mathrm{res}_\infty \\
=& \oint d\mathbf{x}\, \Psi_a \sum_{\mathrm{Part}} \prod_{r=1}^{n} \langle \prod_{i \in P_r} \beta(t_i) P_{\ell_r}^{(m_r)}(\mathbf{x}_r) \rangle \prod_{s<r} \prod_{j \in P_s} C_{ji}.
\end{aligned}$$

Here $\sum_{\mathrm{Part}}$ stands for the summation over all the partition of $P = 1, 2, \cdots, m$ into n disjoint union $P_1 \cup P_2 \cup \cdots \cup P_n$.

4.6. Form the fact that $\langle\, \phi_\ell(z) f(x)\,\rangle = 1$, a-part decouples to three parts as follows:

$$\Psi_a = \langle \prod_{i=1}^m s(t_i) \prod_{r=1}^n \phi_{\ell_r}(z_r)\,\rangle \prod_{r=1}^n \langle\, \phi_{\ell_r}^{(m_r)}(z_r, \mathbf{x}_r)\,\rangle \prod_{s<r} \langle \prod_{x\in \mathbf{x}_s} f(x)\phi_{\ell_r}(z_r)\,\rangle.$$

From this and

$$\langle \prod_i \beta(t_i) P_\ell^{(m)}(\mathbf{x})\,\rangle \langle\, \phi_\ell^{(m)}(z,\mathbf{x})\,\rangle = \frac{\langle \prod_{i\in P} S_+(t_i)\phi^+_{\ell,m}(z,\mathbf{x})\,\rangle}{\langle \prod_{i\in P} s(t_i)\phi_\ell(z)\,\rangle} = \frac{\langle \prod_{i\in P} S(t_i)\phi_{\ell,m}(z,\mathbf{x})\,\rangle}{\langle \prod_{i\in P} s(t_i)\phi_\ell(z)\,\rangle},$$

we obtain

<u>**Theorem**</u>.

$$\langle \prod_{i=1}^m S(t_i) \prod_{r=1}^n \phi_{\ell_r,m_r}(z_r,\mathbf{x}_r)\,\rangle = \langle \prod_{i=1}^m s(t_i) \prod_{r=1}^n \phi_{\ell_r}(z_r)\,\rangle$$
$$\times \sum_{\text{Part}} \prod_{r=1}^n \frac{\langle \prod_{i\in P_r} S(t_i)\phi_{\ell_r,m_r}(z_r,\mathbf{x}_r)\,\rangle}{\langle \prod_{i\in P_r} s(t_i)\phi_{\ell_r}(z_r)\,\rangle} \prod_{i\in P_r} \left\{ \prod_{s<r}\prod_{j\in P_s} C_{ji} \prod_{s>r} \langle\, f(t_i q^{-k-2})\phi_{\ell_r}(z_r)\,\rangle \right\},$$

where

$$\langle \prod_{i=1}^m s(t_i) \prod_{r=1}^n \phi_{\ell_r}(z_r)\,\rangle = \prod_{i<j}^m \langle\, s(t_i)s(t_j)\,\rangle \prod_{i=1}^m\prod_{r=1}^n \langle\, s(t_i)\phi_{\ell_r}(z_r)\,\rangle \prod_{r<s}^n \langle\, \phi_{\ell_r}(z_r)\phi_{\ell_s}(z_s)\,\rangle$$

and

$$\langle\, s(t_i)s(t_j)\,\rangle = t_i^{\frac{2}{k+2}} \frac{(\frac{t_j}{t_i}q^{-2};p)_\infty}{(\frac{t_j}{t_i}q^{-2};p)_\infty}, \qquad \langle\, s(t_i)\phi_{\ell_r}(z_r)\,\rangle = t_i^{-\frac{l_r}{k+2}} \frac{(\frac{z_r}{t_i}q^{\ell_r}p;p)_\infty}{(\frac{z_r}{t_i}q^{-\ell_r}p;p)_\infty},$$
$$\langle\, \phi_{\ell_r}(z_r)\phi_{\ell_s}(z_s)\,\rangle = q^{k+2} z_r^{\frac{\ell_r\ell_s}{2(k+2)}} \frac{(\frac{z_r}{z_s}q^{\ell_r+\ell_s+2}p;p,q^4)_\infty (\frac{z_r}{z_s}q^{-\ell_r-\ell_s+2}p;p,q^4)_\infty}{(\frac{z_r}{z_s}q^{\ell_r-\ell_s+2}p;p,q^4)_\infty (\frac{z_r}{z_s}q^{-\ell_r+\ell_s+2}p;p,q^4)_\infty},$$
$$\langle\, f(t_i q^{-k-2})\phi_{\ell_r}(z_r)\,\rangle = T_{ir} := \frac{t_i q^{\ell_r} - z_r p}{t_i - z_r q^{\ell_r} p}, \qquad C_{ij} = \frac{t_i - t_j q^2}{t_i q^2 - t_j},$$

with $p = q^{2(k+2)}$ *and*

$$(x;p)_\infty = \prod_{i\geq 0}(1 - xp^i), \qquad (x;p,q)_\infty = \prod_{i\geq 0, j\geq 0}(1 - xp^i q^j).$$

By this theorem, the n-point function in $\phi_{\ell_r,m_r}(z_r,\mathbf{x}_r)$ is factored to the one-point functions.

5. Jackson integral formulae from the free field realization

5.1. From the commutation relation for $F(x)$ and $S(t)$, we have following "current Ward identity":

$$\oint dx \langle\, S(t_1)\cdots S(t_m)[\phi_{\ell,m}(z,\mathbf{x}), F(x)]_{q^{\lambda-2m}}\,\rangle$$
$$= \sum_{i=1}^{m} {}_{k+2}\partial_{t_i} \langle\, S(t_1)\cdots \bar{s}(t_i)\cdots S(t_m)\phi_{\ell,m}(z,\mathbf{x})\,\rangle.$$

Finally we obtain

Theorem.

$$\oint d\mathbf{x}\, \frac{\langle\, S(t_1)\cdots S(t_m)\phi_{\ell,m}(z,\mathbf{x})\,\rangle}{\langle\, s(t_1)\cdots s(t_m)\phi_\ell(z)\,\rangle} = \sum_{\mathrm{Sym}} \prod_{i=1}^{m} \frac{{}_{k+2}\partial_{t_i}\langle\, \bar{s}(t_i)s(t_{i+1})\cdots s(t_m)\phi_\ell(z)\,\rangle}{\langle\, s(t_i)\cdots s(t_m)\phi_\ell(z)\,\rangle}.$$

By this theorem and that in sect. 4.6, we obtain the formula for Ψ.

5.2. There is another solution. If we symmetries for the integrable variable x of $F(x)$ then

$$\phi_{\ell,m}(z) = \oint d\mathbf{x} \,{}^\circ_\circ\, \phi_\ell(z) \prod_r^m F^+(x_r) \,{}^\circ_\circ\, \prod_r^m \frac{q^{\lambda-r+1}[-\lambda+r+1]}{x_r - zq^{\lambda+k+2}} \prod_{r<s} \frac{x_r - x_s}{x_r q - x_s q^{-1}}.$$

Hence we obtain

Theorem.

$$\langle \prod_{i=1}^{m} S(t_i)\phi_{\ell,m}(z)\,\rangle = \prod_{i=1}^{m} \frac{[i]q^{\lambda-i+1}[-\lambda+i+1]}{t_i - zq^\lambda p} \prod_{i<j} \frac{t_i - t_j}{t_i q - t_j q^{-1}}.$$

Here we use the identity

$$\prod_{i<j}^{m} T_{ij}^{-1} \sum_{\mathrm{Sym}} \prod_{i<j}^{m} T_{ij} = [m]! \prod_{i<j}^{m} T_{ji}.$$

By this theorem and that in sect. 4.6, we obtain another formula for Ψ.

Acknowledgments.

The authors would like to thank T. Inami, M. Jimbo, K. Kimura, A. Matsuo, T. Miwa, A. Tsuchiya and Y. Yamada for valuable discussions. H.A is supported by Soryushi-syougakkai.

Appendix A. Double loop algebra

A.1. We here show OPE relations used in this paper. For $F_\pm(x)$ and $S_\pm(t)$,

$$\begin{aligned}
F_\pm(x_1)F_\pm(x_2) &= \,{}^\circ_\circ F_\pm(x_1)F_\pm(x_2){}^\circ_\circ\, \frac{x_1-x_2}{x_1-x_2q^{-2}}q^{\mp1},\\
F_\pm(x_1)F_\mp(x_2) &= \,{}^\circ_\circ F_\pm(x_1)F_\mp(x_2){}^\circ_\circ\, \frac{x_1-x_2q^{\pm2}}{x_1-x_2q^{-2}}q^{\mp1},
\end{aligned}$$

$$\begin{aligned}
S_\pm(t)F_\pm(x) &= \,{}^\circ_\circ S_\pm(t)F_\pm(x){}^\circ_\circ\, \frac{t-xq^{\pm k}}{t-xq^{\pm k\pm2}}q^{\pm1},\\
S_\pm(t)F_\mp(x) &= \,{}^\circ_\circ S_\pm(t)F_\mp(x){}^\circ_\circ\, q^{\pm1},
\end{aligned}$$

$$\begin{aligned}
S_\pm(t_1)S_\pm(t_2) &= \,{}^\circ_\circ S_\pm(t_1)S_\pm(t_2){}^\circ_\circ\, \langle\, s(t_1)s(t_2)\,\rangle \frac{t_1-t_2}{t_1-t_2q^{-2}}q^{\mp1},\\
S_\pm(t_1)S_\mp(t_2) &= \,{}^\circ_\circ S_\pm(t_1)S_\mp(t_2){}^\circ_\circ\, \langle\, s(t_1)s(t_2)\,\rangle \frac{t_1-t_2q^{\pm2}}{t_1-t_2q^{-2}}q^{\mp1}.
\end{aligned}$$

For $F_\pm(x)$ and $\phi_\ell(z)$,

$$\begin{aligned}
F_\pm(x)\phi_\ell(z) &= \,{}^\circ_\circ F_\pm(x)\phi_\ell(z){}^\circ_\circ\, \frac{x-zq^{k+2\mp\ell}}{x-zq^{k+2+\ell}}q^{\pm\ell},\\
\phi_\ell(z)F_\pm(x) &= \,{}^\circ_\circ \phi_\ell(z)F_\pm(x){}^\circ_\circ\, \frac{z-xq^{-k-2\pm\ell}}{z-xq^{-k-2+\ell}}.
\end{aligned}$$

For $\phi_\ell(z)$, $s(t)$ and $\bar{s}(t)$,

$$\begin{aligned}
\phi_{\ell_r}(z_r)\phi_{\ell_s}(z_s) &= \,{}^\circ_\circ \phi_{\ell_r}(z_r)\phi_{\ell_s}(z_s){}^\circ_\circ\, q^{k+2} z_r^{\frac{\ell_r\ell_s}{2(k+2)}}\\
&\qquad\times \frac{(\frac{z_r}{z_s}q^{\ell_r+\ell_s+2}p;p,q^4)_\infty(\frac{z_r}{z_s}q^{-\ell_r-\ell_s+2}p;p,q^4)_\infty}{(\frac{z_r}{z_s}q^{\ell_r-\ell_s+2}p;p,q^4)_\infty(\frac{z_r}{z_s}q^{-\ell_r+\ell_s+2}p;p,q^4)_\infty},\\
s(t_i)s(t_j) &= \,{}^\circ_\circ s(t_i)s(t_j){}^\circ_\circ\, t_i^{\frac{2}{k+2}}\frac{(\frac{t_j}{t_i}q^{-2};p)_\infty}{(\frac{t_j}{t_i}q^{-2};p)_\infty},\\
s(t_i)\phi_{\ell_r}(z_r) &= \,{}^\circ_\circ s(t_i)\phi_{\ell_r}(z_r){}^\circ_\circ\, t_i^{-\frac{l_r}{k+2}}\frac{(\frac{z_r}{t_i}q^{\ell_r}p;p)_\infty}{(\frac{z_r}{t_i}q^{-\ell_r}p;p)_\infty},\\
\bar{s}(t)\phi_\ell(z) &= \,{}^\circ_\circ \bar{s}(t)\phi_\ell(z){}^\circ_\circ\, t^{-\frac{l}{k+2}}\frac{(\frac{z}{t}q^{\ell+k+2}p;p)_\infty}{(\frac{z}{t}q^{-\ell+k+2}p;p)_\infty}.
\end{aligned}$$

For $\beta(t)$, $\gamma(x)$ and $f(x)$,

$$\beta(t_1)\beta(t_2) = {}^{\circ}_{\circ}\beta(t_1)\beta(t_2){}^{\circ}_{\circ}\,\frac{t_1-t_2}{t_1-t_2q^{-2}}q^{-1},$$

$$\beta(t)\gamma(x) = {}^{\circ}_{\circ}\beta(t)\gamma(x){}^{\circ}_{\circ}\,\frac{t-xq^{k}}{t-xq^{k+2}}q,$$

$$\gamma(x_1)\gamma(x_2) = {}^{\circ}_{\circ}\gamma(x_1)\gamma(x_2){}^{\circ}_{\circ}\,\frac{x_1-x_2}{x_1-x_2q^{-2}}q^{-1},$$

$$f(x)\phi_\ell(z) = {}^{\circ}_{\circ}f(x)\phi_\ell(z){}^{\circ}_{\circ}\,\frac{x-zq^{k+2-\ell}}{x-zq^{k+2+\ell}}q^{\ell},$$

$$\phi_\ell(z)f(x) = {}^{\circ}_{\circ}\phi_\ell(z)f(x){}^{\circ}_{\circ}.$$

A.2. We here show the OPE relations of $E_\pm(z)$, $F_\pm(z)$ and $\psi_\pm(z)$.

$$\psi_\pm(z)\psi_\pm(w) = {}^{\circ}_{\circ}\psi_\pm(z)\psi_\pm(w){}^{\circ}_{\circ},$$

$$\psi_\pm(z)\psi_\mp(w) = {}^{\circ}_{\circ}\psi_\pm(z)\psi_\mp(w){}^{\circ}_{\circ}\,\frac{(z-wq^{k\pm2})(z-wq^{-k\mp2})}{(z-wq^{k-2})(z-wq^{-k+2})},$$

$$\psi_\pm(z)E_\alpha(w) = {}^{\circ}_{\circ}\psi_\pm(z)E_\alpha(w){}^{\circ}_{\circ}\,\frac{z-wq^{-\frac{k}{2}\mp2}}{z-wq^{-\frac{k}{2}+2}}q^{\pm2},$$

$$E_\alpha(z)\psi_\pm(w) = {}^{\circ}_{\circ}E_\alpha(z)\psi_\pm(w){}^{\circ}_{\circ}\,\frac{z-wq^{-\frac{k}{2}\pm2}}{z-wq^{-\frac{k}{2}+2}},$$

$$\psi_\pm(z)F_\alpha(w) = {}^{\circ}_{\circ}\psi_\pm(z)F_\alpha(w){}^{\circ}_{\circ}\,\frac{z-wq^{\frac{k}{2}\pm2}}{z-wq^{\frac{k}{2}-2}}q^{\mp2},$$

$$F_\alpha(z)\psi_\pm(w) = {}^{\circ}_{\circ}F_\alpha(z)\psi_\pm(w){}^{\circ}_{\circ}\,\frac{z-wq^{\frac{k}{2}\mp2}}{z-wq^{\frac{k}{2}-2}},$$

$$E_\pm(z)E_\pm(w) = {}^{\circ}_{\circ}E_\pm(z)E_\pm(w){}^{\circ}_{\circ}\,\frac{z-w}{z-wq^{2}}q^{\pm1},$$

$$E_\pm(z)E_\mp(w) = {}^{\circ}_{\circ}E_\pm(z)E_\mp(w){}^{\circ}_{\circ}\,\frac{z-wq^{\mp2}}{z-wq^{2}}q^{\pm1},$$

$$E_\pm(z)F_\pm(w) = {}^{\circ}_{\circ}E_\pm(z)F_\pm(w){}^{\circ}_{\circ}\,\frac{z-wq^{\pm k\pm2}}{z-wq^{\pm k}}q^{\mp1},$$

$$E_\pm(z)F_\mp(w) = {}^{\circ}_{\circ}E_\pm(z)F_\mp(w){}^{\circ}_{\circ}\,q^{\mp1},$$

$$F_\pm(z)E_\pm(w) = {}^{\circ}_{\circ}F_\pm(z)E_\pm(w){}^{\circ}_{\circ}\,\frac{z-wq^{\mp k\mp2}}{z-wq^{\mp k}}q^{\pm1},$$

$$F_\pm(z)E_\mp(w) = {}^{\circ}_{\circ}F_\pm(z)E_\mp(w){}^{\circ}_{\circ}\,q^{\pm1},$$

$$F_\pm(z)F_\pm(w) = {}^{\circ}_{\circ}F_\pm(z)F_\pm(w){}^{\circ}_{\circ}\,\frac{z-w}{z-wq^{-2}}q^{\mp1},$$

$$F_\pm(z)F_\mp(w) = {}^{\circ}_{\circ}F_\pm(z)F_\mp(w){}^{\circ}_{\circ}\,\frac{z-wq^{\pm2}}{z-wq^{-2}}q^{\mp1}.$$

A.3. We here show the algebra of $E_\pm(z)$, $F_\pm(z)$ and $\psi_\pm(z)$.

$$
\begin{aligned}
&\psi_\pm(z)\psi_\pm(w) = \psi_\pm(w)\psi_\pm(z),\\
&\psi_+(z)\psi_-(w) = \psi_-(w)\psi_+(z)\frac{(z-wq^{k+2})(z-wq^{-k-2})}{(z-wq^{k-2})(z-wq^{-k+2})},
\end{aligned}
$$

$$
\begin{aligned}
\psi_\pm^{\pm1}(z)E_\alpha(w)\psi_\pm^{\mp1}(z) &= E_\alpha(w)\frac{z-wq^{-\frac{k}{2}-2}}{z-wq^{-\frac{k}{2}+2}}q^2,\\
\psi_\pm^{\pm1}(z)F_\alpha(w)\psi_\pm^{\mp1}(z) &= F_\alpha(w)\frac{z-wq^{\frac{k}{2}+2}}{z-wq^{\frac{k}{2}-2}}q^{-2},\\
(z-wq^2)E_\pm(z)E_\pm(w) &+ (w-zq^2)E_\pm(w)E_\pm(z) = 0,\\
(z-wq^{-2})F_\pm(z)F_\pm(w) &+ (w-zq^{-2})F_\pm(w)F_\pm(z) = 0,\\
E_+(z)E_-(w) &= E_-(w)E_+(z)\frac{z-wq^{-2}}{z-wq^2}q^2,\\
F_+(z)F_-(w) &= F_-(w)F_+(z)\frac{z-wq^2}{z-wq^{-2}}q^{-2},\\
[E_\pm(z)F_\pm(w)] &= \frac{\pm1}{(q-q^{-1})zw}\delta\left(\frac{w}{z}q^{\pm k}\right)\psi_\pm(wq^{\pm\frac{k}{2}}),\\
E_\pm(z)F_\mp(w) &= F_\mp(w)E_\pm(z).
\end{aligned}
$$

Here we use the following relations:

$$
{}^\circ_\circ\, E_\pm(wq^{\pm k})F_\pm(w)\, {}^\circ_\circ = \frac{1}{(q-q^{-1})^2w^2q^{\pm k}}\psi_\pm(wq^{\pm\frac{k}{2}}),
$$

$$
\frac{1}{z-w} + \frac{1}{w-z} = \frac{1}{z}\delta\left(\frac{w}{z}\right).
$$

Appendix B. Proof of proposition in sect. 4.2

B.1. Let $\mathcal{O}\{A,B,\cdots,C\}$ be an any ordered product of $A,B,\cdots,C$, e.g., $AB\cdots C$, $BA\cdots C$, $CA\cdots B$, etc. Then our Ψ is a linear combination of

$$\oint d\mathbf{x}d\mathbf{y}\langle\prod_{i=1}^{m}S(t_i)\mathcal{O}\{F_+(x_1),\cdots,F_+(x_r),F_-(y_1),\cdots,F_-(y_s),\phi_{\ell_1}(z_1),\cdots,\phi_{\ell_n}(z_n)\}\rangle$$

with $r+s=m$.

At first, let us calculate the integral for the variables y's of the current $F_-(y)$'s, which can be realized as a counter clock wise contour integral around the origin. Since $F_-(y)F_+(x)$ and $F_-(y)\phi_\ell(z)$ do not have pole except for $y=0$, we have

Lemma. *Let $\mathcal{O}\{F_+(\mathbf{x}),\phi(\mathbf{z})\}$ be an arbitrary ordered product of $F_+(x_1),\cdots,F_+(x_r)$, $\phi(z_1),\cdots,\phi(z_n)$. Then*

$$\oint dyF_-(y)\mathcal{O}\{F_+(\mathbf{x}),\phi(\mathbf{z})\}|\lambda_0\rangle=\mathrm{res}_{y=0}F_-(y)\mathcal{O}\{F_+(\mathbf{x}),\phi(\mathbf{z})\}|\lambda_0\rangle.$$

Therefore, the most right $\oint dyF_-(y)$ in Ψ can be replaced with $yF_-(y)|_{y=0}$. Furthermore, since $F_-(y_1)\{yF_-(y)|_{y=0}\}$ also do not have pole except for $y=0$, every $\oint dyF_-(y)$ in Ψ can be replaced with $yF_-(y)|_{y=0}$.

Since $[\phi_\ell(z),\{yF_-(y)|_{y=0}\}]_{q^\ell}=0$ and $[F_+(x),\{yF_-(y)|_{y=0}\}]_{q^{-2}}=0$ we have

Lemma.

$$[\phi^+_{\ell,m}(z),\{yF_-(y)|_{y=0}\}]_{q^{\ell-2m}}=0.$$

Therefore, the nearest $yF_-(y)|_{y=0}$ to $\phi_\ell(z)$ vanishes. Thus

$$\prod_{i=1}^{r}\phi_{\ell_i,m_i}(z_i)|0\rangle=\prod_{i=1}^{r}\phi^+_{\ell_i,m_i}(z_i)|0\rangle.$$

This means that the $F_-(y)$'s do not contribute to the results, thus our Ψ is a linear combination of

$$\oint d\mathbf{x}\langle\prod_{i=1}^{m}S(t_i)\mathcal{O}\{F_+(x_1),\cdots,F_+(x_m),\phi_{\ell_1}(z_1),\cdots,\phi_{\ell_n}(z_n)\}\rangle.$$

B.2. Second, let us calculate the integral for the variables x's of the current $F_+(x)$'s, which can be realized as a clock wise counture integral around the infinity, by the residue theorem. Since

$$S_+(t)F_+(x)\prod_{i=1}^{r}F_+(x_i)={}^{\circ}_{\circ}\,S_+(t)F_+(x)\prod_{i=1}^{r}F_+(x_i)\,{}^{\circ}_{\circ}\,q\frac{t-xq^k}{t-xq^{k+2}}\prod_{i=1}^{r}\frac{t-x_iq^k}{t-x_iq^{k+2}}\frac{x-x_i}{x-x_iq^{-2}},$$

we have

$$\mathrm{res}_{x=tq^{-k-2}}\langle\,\lambda_\infty|S_+(t)F_+(x)\prod_{i=1}^{r}F_+(x_i)=\langle\,\lambda_\infty|\,{}^{\circ}_{\circ}\,S_+(t)F_+(q^{-k-2})\prod_{i=1}^{r}F_+(x_i)\,{}^{\circ}_{\circ}\,t(q-q^{-1}).$$

Thus this residue does not have extra poles at $x_i=tq^{**}$. Similarly

$$\mathrm{res}_{x=tq^{-k-2}}\langle\,\lambda_\infty|S_+(t)\mathcal{O}\left\{\phi(\mathbf{z})\right\}F_+(x)\mathcal{O}\left\{F_+(\mathbf{x}),\phi(\mathbf{w})\right\}$$

is also the same. Therefore, the residue at $x=t_iq^{-k-2}$ of the most left $\oint dxF_+(x)$ in Ψ does not produce extra poles.

Let $d\mathbf{x}:=\prod_{i=1}^{m}dx_i$ and $F_+(\mathbf{x}):=\{F_+(x_1),\cdots,F_+(x_m)\}$, and let us consider the following m-integral, which contain at least one $S_-(t)$:

$$\oint d\mathbf{x}\langle\,\lambda_\infty|\prod_{i=1}^{k-1}S(t_i)\cdot S_-(t_k)\prod_{j=k+1}^{m}S(t_j)\cdot\mathcal{O}\left\{F_+(\mathbf{x}),\phi(\mathbf{z})\right\}.$$

Since $S_-(t)F_+(x)$ do not have pole except for $x=\infty$, some variable x_i of our m-integral should localize to ∞. Therefore, we have

Lemma.

$$\oint d\mathbf{x}\langle\,\lambda_\infty|\prod_{i=1}^{m}S(t_i)\mathcal{O}\left\{F_+(\mathbf{x}),\phi(\mathbf{z})\right\}=\oint d\mathbf{x}\langle\,\lambda_\infty|\prod_{i=1}^{m}S_+(t_i)\mathcal{O}\left\{F_+(\mathbf{x}),\phi(\mathbf{z})\right\}+\cdots,$$

here $\cdots$ means residues at some x_i's are ∞.

But Ψ has no contribution from the residue at some x_i's are ∞. This means that the $S_-(t)$'s contribute only for the residue canceling at $x=\infty$. Therefore, for our correlation functions, it is enough to consider only the + parts of $F(x)$ and $S(z)$.

References

[ABG] A. Abada, A. Bougourzi and M. El Gradechi, *Deformation of the Wakimoto construction,* (hep-th/9209009) preprint CRM-1829 (1992).

[ATY] H. Awata, A. Tsuchiya and Y. Yamada, *Integral formulas for the WZNW correlation functions,* Nucl. Phys. **B365** (1991) 680-696.

H. Awata, *Screening current Ward identity and integral formulas for the WZNW correlation functions,* Prog. Theor. Phys. Supplement **110** (1992) 303-319.

[DFJMN] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki, *Diagonalization of the XXZ Hamiltonian by vertex operators,* Commun. Math. Phys. **151** (1993) 89-154.

[FR] I. Frenkel and N. Reshetikhin, *Quantum affine algebras and holonomic difference equations,* Commun. Math. Phys. **146** (1992) 1-60.

[JMMN] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, *Correlation functions of the XXZ model for $\Delta < -1$,* Phys. Lett. **A168** (1992) 256-263.

[KQS] A. Kato, Y. Quano and J. Shiraishi, *Free Boson Representation of q-Vertex Operators and their Correlation Functions,* Tokyo Univ. preprint UT-618 (1992).

[M1] A. Matsuo, *Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations,* Nagoya Univ. preprint (1992);*Quantum algebra structure of certain Jackson integrals,* Nagoya Univ. preprint (1992).

[M2] A. Matsuo, *Free field representation of quantum affine algebra $U_q(\widehat{sl_2})$,* (hep-th/9208079) Nagoya Univ. preprint (1992).

[M3] A. Matsuo, *Free field representation of q-deformed primary fields for $U_q(\widehat{sl_2})$,* (hep-th/9212040) Nagoya Univ. preprint (1992).

[R] N. Reshetikhin, *Jackson-type integrals, Bethe vectors, and solutions to a difference analog of the Knizhnik-Zamolodchikov system,* Lett. Math. Phys. **26** (1992) 153-165.

[Sh] J. Shiraishi, *Free boson representation of $U_q(\widehat{sl_2})$,* Phys. Lett. **A171** (1992) 243-248.

[Sm] F. Smirnov, *Dynamical symmetries of massive integrable models,* Int. J. Mod. Phys. **A7** Supplement 1 (1992) 813-837;839-858.

[SV] V. Schechtman and A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology,* Invent. Math. **106** (1991) 139-194.