Noname manuscript No.
(will be inserted by the editor)

Identification of building damage using vibrational eigenvalue and eigenmode pairs

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Received: date / Accepted: date

Abstract The present paper describes a solution to a problem of identifying damage in a building based on experimentally measured vibrational eigenvalue and eigenmode pairs. The healthy rate, which is defined as the stiffness rate with respect to a perfect material, is chosen as the design target to be identified. The range of the healthy rate is restricted to within the range of 0 to 1. In order to overcome this restriction, we define a function with no restriction on the range defined in the domain of a linear elastic body for a building as a design variable and assume that the healthy rate is given by a sigmoid function of the function of the design variable. The linear coupling of the mean squared errors of vibrational eigenvalues and eigenmodes with respect to the measured values are used as a cost function. The derivative of the cost function with respect to the design variable is evaluated by the adjoint variable method. In order to resolve the identification problem of the damaged area, we use an iterative algorithm based on the H^1 gradient method using the finite-element method to obtain numerical solutions. A numerical example using experimental data demonstrates that a damaged area can be identified by the proposed approach.

Keywords Inverse problem · Damage identification · Healthy rate · Eigenpair · H^1 gradient method

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1 Introduction

In order to prevent collapse of dilapidated historical buildings due to earthquakes, the development of non-destructive methods by which to identify damage in buildings is required [11]. A comparatively easy and accurate method of monitoring the stiffness of buildings is to measure vibrational eigenvalue and eigenmode pairs by observing ambient vibrations. Then, methods by which to identify damage by solving optimization problems in order to minimize the error of the vibrational eigenpairs of a numerical model from the measured data have been studied.

Methods that fit a numerical mode to experimental data are referred to as model updating methods [9, 10, 12, 15]. In these studies, the optimization problems are constructed by choosing material constants, such as the Young's modulus and density, of finite elements as the design variables and setting the squared error norm of the vibrational eigenpairs as the cost function to be minimized.

The applicability of this approach to real architectures has been demonstrated [6, 3, 5, 4]. The relation between the degree of damage to masonry buildings and the vibrational eigenpairs has been measured by shaking table tests [13, 1].

However, choosing the material constants of all of the finite elements as design variables complicates optimization problems, because doing so increases the dimension of the design vector space.

In order to address the above-described considerations, in the present study, we construct a non-parametric optimization problem to identify damage in buildings. In a previous study, we proposed a topology optimization method [7] of density type, referred to as the H^1 gradient method, and applied this method to the optimum design of mechanical parts.

Numerical solutions to topology optimization problems of density type have been investigated extensively [14, 8]. The H^1 gradient method differs from the other methods in two aspects. The first is the use of a function of free range as a design variable, which sets a topology optimization problem in the framework of a standard function optimization problem. The second is the use of a gradient method in a Hilbert space for functions of H^1 class, which secures the regularity of the solution to define the domain [7].

In the present paper, the healthy rate, which is defined as the stiffness rate with respect to a perfect material, is chosen as a design target, rather than the density in the topology optimization problem. The linear coupling of the mean squared errors of vibrational eigenvalues and eigenmodes with respect to the measured values are used as a cost function.

The remainder of the present paper is organized as follows. In Section 2, we define the healthy rate in a linear elastic body of a damaged building as a design target and introduce a set of functions θ as a design variable. For a given θ, in Section 3, we formulate the natural vibration problem. In Section 4, using the solution to the natural vibration problem, we formulate a damage identification problem using a cost function. The evaluation method for the

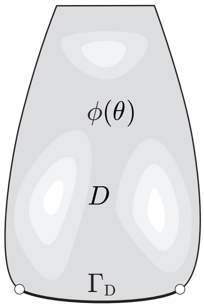


Fig. 1 Linear elastic body of a damaged building

Fréchet derivative of the cost function with respect to arbitrary variation of θ, which we refer to as the θ-derivative of the cost function, is shown in Section 5. Using the θ-derivative of the cost function, we present a method by which to obtain the variation of θ that decreases the cost function in Section 6. Finally, in Section 7, we present the numerical results for damage identification problems.

2 Set of design variables

In the present study, we assume that a building is a linear elastic body defined on $d \in \{2,3\}$-dimensional finite domain D, as shown in Fig. 1. Let ∂D be the boundary of D, and let $\Gamma_{\mathrm{D}} \subset \partial D$ be a homogeneous Dirichlet boundary. Moreover, let $\Gamma_{\mathrm{N}} \subset \partial D \setminus \bar{\Gamma}_{\mathrm{D}}$ ($\bar{\Gamma}_{\mathrm{D}}$ denotes $\Gamma_{\mathrm{D}} \cup \partial \Gamma_{\mathrm{D}}$) be a homogeneous Neumann boundary. We assume that Γ_{D} is not the empty set.

To define a set of design variables, we use the notation for function space as follows. Let $W^{s,p}(D;\mathbb{R})$ denote the Sobolev space for the set of functions defined in D and having values in $\mathbb{R}$ that are $s \in \{0,1,2,\cdots\}$ times differentiable and $p \in [1,\infty]$-th order Lebesgue integrable. The terms $L^p(D;\mathbb{R})$, $H^s(D;\mathbb{R})$, and $C^{s,\alpha}(D;\mathbb{R})$ for $\alpha \in (0,1]$ are used as $W^{0,p}(D;\mathbb{R})$, $W^{s,2}(D;\mathbb{R})$, and $W^{s+\alpha,\infty}(D;\mathbb{R})$.

Using the notation, let $\boldsymbol{C}_0 \in L^\infty\left(D;\mathbb{R}^{d\times d\times d\times d}\right)$ be the stiffness of a perfect model having ellipticity and boundedness. For a damaged building, we define the healthy rate $\phi : D \to [0,1]$, for which the stiffness is given as

$$\boldsymbol{C}(\phi) = \phi^\alpha \boldsymbol{C}_0, \tag{2.1}$$

where α is a positive constant, such as 1 or 2. The influence of the value of α will be checked in numerical examples (Section 7).

However, since the range of ϕ is restricted to within 0 to 1, ϕ is not suitable as a design variable, because we cannot define the Fréchet derivatives of cost functions with respect to arbitrary variation of ϕ. Then, in the present paper, we assume that the design variable is defined in terms of a function θ belonging to

$$X = H^1(D;\mathbb{R}). \tag{2.2}$$

The healthy rate is given as a sigmoid function of $\theta \in X$. In the present paper, we use

$$\phi(\theta) = \frac{1}{\pi}\tan^{-1}\theta + \frac{1}{2}. \tag{2.3}$$

When we use (2.3), the value of $\phi(\theta)$ is limited to $(0,1)$ from $[0,1]$. It can be considered that this limitation is not critical by allowing sufficiently large absolute value for θ. Hence, we rewrite $\boldsymbol{C}(\phi)$ of (2.1) by $\boldsymbol{C}(\theta)$.

3 Main problem

Letting $\theta \in X$ be given, we formulate the main problem as follows. Let $\boldsymbol{u} \in U$ be the displacement of the linear elastic body, where

$$U = \left\{ \boldsymbol{u} \in H^1\left(D;\mathbb{R}^d\right) \;\middle|\; \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D}} \right\}. \tag{3.1}$$

In order to define the Fréchet derivative of the cost function including $\boldsymbol{u}$ with respect to an arbitrary variation of $\theta \in X$, we assume that $\boldsymbol{u}$ belongs to

$$\mathcal{S} = U \cap W^{1,2q}\left(D;\mathbb{R}^d\right) \tag{3.2}$$

for $q > d$. Using $\boldsymbol{u}$ and $\boldsymbol{C}(\theta)$, let

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{u}) &= \left(e_{ij}(\boldsymbol{u})\right)_{(i,j)\in\{1,\cdots,d\}^2} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)_{(i,j)\in\{1,\cdots,d\}^2}, \\
\boldsymbol{\Sigma}(\theta,\boldsymbol{u}) &= \left(\sigma_{ij}(\theta,\boldsymbol{u})\right)_{(i,j)\in\{1,\cdots,d\}^2} \\
&= \boldsymbol{C}(\theta)\boldsymbol{E}(\boldsymbol{u}) = \left(\sum_{(k,l)\in\{1,\cdots,d\}^2} c_{ijkl}(\theta)\, e_{kl}(\boldsymbol{u})\right)_{ij}
\end{aligned}$$

be the strain and the stress, respectively. Moreover, let $\rho \in L^\infty(D;\mathbb{R})$ be the density. In the present study, we assume that ρ is independent of θ.

In the present study, we assume vibrational eigenpairs of a building was measured by experiment. Let $\mathcal{M}$ be a set of the mode numbers of experimentally measured natural vibrations, and let $|\mathcal{M}|$ be the number of the elements of the set. We assume that the multiplicity of the eigenvalues is allowed and that the identification of the vibrational eigenmodes is checked by the modal assurance criterion (MAC) using the normalized inner products.

Using the above notation, we define the main problem as follows. In the present paper, $\boldsymbol{\nu}$ denotes the outer unit normal on the boundary.

Problem 1 (Natural vibration problem) Let $\theta \in X$ be given. For $i \in \mathcal{M}$, find $\eta_i \in \mathbb{R}$ and $\boldsymbol{u}_i : D \to \mathbb{R}^d$ such that

$$\begin{aligned}
&-\eta_i \rho \boldsymbol{u}_i^{\mathrm{T}} - \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\Sigma}\left(\theta, \boldsymbol{u}_i\right) = \boldsymbol{0}_{\mathbb{R}^d}^{\mathrm{T}} \quad \text{in } D, \\
&\boldsymbol{\Sigma}\left(\theta, \boldsymbol{u}_i\right) \boldsymbol{\nu} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \Gamma_{\mathrm{N}}, \\
&\boldsymbol{u}_i = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \Gamma_{\mathrm{D}}, \\
&\int_D \rho \left\| \boldsymbol{u}_i \right\|_{\mathbb{R}^d}^2 \, \mathrm{d}x = 1.
\end{aligned} \tag{3.3}$$

In Problem 1, we refer to $\eta_i = \omega_i^2$ as the i-th vibrational eigenvalue, ω_i as the i-th circular eigenfrequency, $\boldsymbol{u}_i$ as the i-th vibrational eigenmode, and $(\eta_i, \boldsymbol{u}_i)$ as the i-th vibrational eigenpair.

We define (for later use) the Lagrange function of Problem 1 for $i \in \mathcal{M}$ by

$$\mathscr{L}_{\mathrm{M}i}\left(\theta, \eta_i, \boldsymbol{u}_i, \boldsymbol{v}\right) = \int_D \left(\eta_i \rho \boldsymbol{u}_i \cdot \boldsymbol{v} - \boldsymbol{\Sigma}\left(\theta, \boldsymbol{u}_i\right) \cdot \boldsymbol{E}\left(\boldsymbol{v}\right)\right) \mathrm{d}x, \tag{3.4}$$

where $\boldsymbol{v} \in U$ is introduced as a Lagrange multiplier, and $\boldsymbol{\Sigma}\left(\theta, \boldsymbol{u}_i\right) \cdot \boldsymbol{E}\left(\boldsymbol{v}\right)$ denotes $\sum_{(i,j) \in \{1,\cdots,d\}^2} \sigma_{ij}\left(\theta, \boldsymbol{u}_i\right) e_{ij}\left(\boldsymbol{v}\right)$. If $(\eta_i, \boldsymbol{u}_i)$ is the solution of Problem 1,

$$\mathscr{L}_{\mathrm{M}i}\left(\theta, \eta_i, \boldsymbol{u}_i, \boldsymbol{v}\right) = 0$$

holds for all $\boldsymbol{v} \in U$.

4 Damage identification problem

Let $\boldsymbol{\eta}$ and $\boldsymbol{U}$ denote $\{\eta_i\}_{i \in \mathcal{M}}$ and $\{\boldsymbol{u}_i\}_{i \in \mathcal{M}}$, respectively. Using $(\boldsymbol{\eta}, \boldsymbol{U})$, we define a damage identification problem. In the present paper, we assume that the i-th vibrational eigenpairs are given by $(\bar{\eta}_i, \bar{\boldsymbol{u}}_i)$, in which $\bar{\boldsymbol{u}}_i$ is given on an assigned domain or boundary $\bar{\Omega}_{\mathrm{M}} \subset \bar{D} \setminus \Gamma_{\mathrm{D}}$. We refer to

$$f_0\left(\boldsymbol{\eta}\right) = \sum_{i \in \mathcal{M}} \left|\eta_i - \bar{\eta}_i\right|^2 \tag{4.1}$$

as the error norm of vibrational eigenvalues. Moreover, assuming $\boldsymbol{\beta} = \{\beta_i\}_{i \in \mathcal{M}} \in \mathbb{R}^{|\mathcal{M}|}$ to be variables to control the magnitudes of the mode vectors, we refer to

$$f_1\left(\boldsymbol{U}, \boldsymbol{\beta}\right) = \sum_{i \in \mathcal{M}} h\left(\boldsymbol{u}_i, \beta_i\right) \tag{4.2}$$

as the error norm of vibrational eigenmodes, where

$$h\left(\boldsymbol{u}_i, \beta_i\right) = \bar{\eta}_i \left(\int_{\bar{\Omega}_{\mathrm{M}} \cap D} \left\| \boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i \right\|_{\mathbb{R}^d}^2 \, \mathrm{d}x + \int_{\bar{\Omega}_{\mathrm{M}} \cap \partial D} \left\| \boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i \right\|_{\mathbb{R}^d}^2 \, \mathrm{d}\gamma \right). \tag{4.3}$$

Using f_0 and f_1, we define

$$f(\boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}) = f_0(\boldsymbol{\eta}) + c_1 f_1(\boldsymbol{U}, \boldsymbol{\beta}) \tag{4.4}$$

as a cost function for damage identification, where c_1 is a constant that is appropriately determined depending on the method of taking $\bar{\Omega}_{\mathrm{M}}$. Using f, we define the damage identification problem as follows.

Problem 2 (Damage identification problem) Let X and U be defined as (2.2) and (3.1), respectively, and let f be defined as (4.4). Find θ such that

$$\min_{\theta \in X} \big\{ f(\boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}) \mid (\eta_i, \boldsymbol{u}_i) \in \mathbb{R} \times U,\ i \in \mathcal{M}, \\ \text{Problem 1} \big\}.$$

5 $\boldsymbol{\theta}$-derivative of $\boldsymbol{f}$

Since the solution $(\boldsymbol{\eta}, \boldsymbol{U})$ of Problem 1 is determined uniquely for $\theta \in X$, we denote (4.4) for θ as

$$\tilde{f}(\theta, \boldsymbol{\beta}) = \tilde{f}_0(\theta) + c_1 \tilde{f}_1(\theta, \boldsymbol{\beta}).$$

Hence, we refer to the partial Fréchet derivative of $\tilde{f}$ with respect to arbitrary variation $\vartheta \in X$ of θ denoted by

$$\tilde{f}_\theta(\theta, \boldsymbol{\beta})[\vartheta] = \tilde{f}_0'(\theta)[\vartheta] + c_1 \tilde{f}_{1\theta}(\theta, \boldsymbol{\beta})[\vartheta] = \int_D (g_0 + c_1 g_1)\,\vartheta \,\mathrm{d}x = \langle g, \vartheta \rangle \tag{5.1}$$

the partial θ-derivative of $\tilde{f}$, where $\langle \cdot, \cdot \rangle$ denotes the dual product. In the following, we present the evaluation methods of g_0 and g_1.

5.1 Evaluation of g_0

Since $f_0(\boldsymbol{\eta})$ includes the solution $\boldsymbol{\eta}$ of Problem 1, we define the Lagrange function of $f_0(\boldsymbol{\eta})$ as

$$\mathscr{L}_0(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0) = f_0(\boldsymbol{\eta}) + \sum_{i \in \mathcal{M}} \mathscr{L}_{\mathrm{M}i}(\theta, \eta_i, \boldsymbol{u}_i, \boldsymbol{v}_{i0}), \tag{5.2}$$

where $\boldsymbol{V}_0 = \{\boldsymbol{v}_{i0}\}_{i \in \mathcal{M}} \in U^{|\mathcal{M}|}$ is the Lagrange multiplier of Problem 1 for f_0. The Fréchet derivative of $\mathscr{L}_0$ with respect to arbitrary variation $\vartheta \in X$ of θ can be written as

$$\mathscr{L}_0'(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0)[\vartheta] = \mathscr{L}_{0\theta}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0)[\vartheta] + \mathscr{L}_{0\boldsymbol{\eta}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0)[\boldsymbol{\eta}'] \\ + \mathscr{L}_{0\boldsymbol{U}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0)[\boldsymbol{U}'] + \mathscr{L}_{0\boldsymbol{V}_0}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0)[\boldsymbol{V}_0']. \tag{5.3}$$

Here, $\boldsymbol{\eta}'$ denotes the Fréchet derivative $\boldsymbol{\eta}'(\theta)[\vartheta]$ of the solution $\boldsymbol{\eta}$ of Problem 1 with respect to arbitrary variation $\vartheta \in X$. In the same manner, $\boldsymbol{U}'$ and $\boldsymbol{V}_0'$ denote these Fréchet derivatives.

The forth term on the right-hand side of (5.3) becomes

$$\begin{aligned}
&\mathscr{L}_{0\boldsymbol{V}_0}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0)[\boldsymbol{V}_0'] = \langle \mathscr{L}_{0\boldsymbol{V}_0}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0), \boldsymbol{V}_0' \rangle \\
&\quad = \sum_{i \in \mathcal{M}} \mathscr{L}_{\mathrm{M}i}(\theta, \eta_i, \boldsymbol{u}_i, \boldsymbol{v}_{i0}'),
\end{aligned} \tag{5.4}$$

which agrees with the Lagrange function of Problem 1. Then, if $(\boldsymbol{\eta}, \boldsymbol{U})$ are weak solutions of Problem 1, the fourth term on the right-hand side of (5.3) becomes 0.

The second and third terms on the right-hand side of (5.3) becomes

$$\begin{aligned}
&\mathscr{L}_{0\boldsymbol{\eta}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0)[\boldsymbol{\eta}'] + \mathscr{L}_{0\boldsymbol{U}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0)[\boldsymbol{U}'] \\
&\quad = \sum_{i \in \mathcal{M}} \Big(\langle \mathscr{L}_{0\eta_i}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0), \eta_i' \rangle + \langle \mathscr{L}_{\mathrm{M}\boldsymbol{u}_i}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0), \boldsymbol{u}_i' \rangle \Big),
\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
&\langle \mathscr{L}_{0\eta_i}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0), \eta_i' \rangle = f_{0\eta_i}(\boldsymbol{\eta})[\eta_i'] + \mathscr{L}_{\mathrm{M}i\eta_i}(\theta, \eta_i, \boldsymbol{u}_i, \boldsymbol{v}_{i0})[\eta_i'] \\
&\quad = 2(\eta_i - \bar{\eta}_i)\eta_i' + \eta_i' \int_D \rho \boldsymbol{u}_i \cdot \boldsymbol{v}_{i0} \, \mathrm{d}x, \\
&\langle \mathscr{L}_{\mathrm{M}\boldsymbol{u}_i}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0), \boldsymbol{u}_i' \rangle = \mathscr{L}_{\mathrm{M}i}(\theta, \eta_i, \boldsymbol{u}_i', \boldsymbol{v}_{i0}).
\end{aligned}$$

(5.5) agrees with the Lagrange function of the following problem. Then, if $\boldsymbol{V}_0$ is the weak solution of the following problem, the second and third terms on the right-hand side of (5.3) become 0.

Problem 3 (Adjoint problem for f_0) Let $\theta \in X$ be given. For $i \in \mathcal{M}$, let the solution $(\eta_i, \boldsymbol{u}_i)$ of Problem 1 be given. Find $\boldsymbol{v}_{i0} : D \to \mathbb{R}^d$ such that

$$\begin{aligned}
&-\eta_i \rho \boldsymbol{v}_{i0}^{\mathrm{T}} - \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\Sigma}(\theta, \boldsymbol{v}_{i0}) = \boldsymbol{0}_{\mathbb{R}^d}^{\mathrm{T}} \quad \text{in } D, \\
&\boldsymbol{\Sigma}(\theta, \boldsymbol{v}_{i0}) \boldsymbol{\nu} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \Gamma_{\mathrm{N}}, \\
&\boldsymbol{v}_{i0} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \Gamma_{\mathrm{D}}, \\
&\int_D \rho \boldsymbol{u}_i \cdot \boldsymbol{v}_{i0} \, \mathrm{d}x = -2(\eta_i - \bar{\eta}_i).
\end{aligned} \tag{5.6}$$

Problem 3 becomes the same eigenvalue problem as Problem 1, although the magnitude of $\boldsymbol{v}_{i0}$ is determined by the normalization condition of (5.6).

Moreover, the first term on the right-hand side of (5.3) becomes

$$\mathscr{L}_{0\theta}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_0)[\vartheta] = \int_D \sum_{i \in \mathcal{M}} g_{i0} \vartheta \, \mathrm{d}x = \langle g_0, \vartheta \rangle, \tag{5.7}$$

where

$$g_{i0} = -\frac{\partial \boldsymbol{\Sigma}(\theta, \boldsymbol{u}_i)}{\partial \theta} \cdot \boldsymbol{E}(\boldsymbol{v}_{i0}) = -\alpha \phi^{\alpha-1} \frac{\mathrm{d}\phi}{\mathrm{d}\theta} (\boldsymbol{C}_0 : \boldsymbol{E}(\boldsymbol{u}_i)) \cdot \boldsymbol{E}(\boldsymbol{v}_{i0})$$
$$= -\frac{\alpha}{\phi} \frac{\mathrm{d}\phi}{\mathrm{d}\theta} \boldsymbol{\Sigma}(\theta, \boldsymbol{u}_i) \cdot \boldsymbol{E}(\boldsymbol{v}_{i0}).$$

Based on these results, if $(\boldsymbol{\eta}, \boldsymbol{U})$ and $\boldsymbol{V}_0$ are weak solutions of Problems 1 and 3, respectively, since the Fréchet derivative of the second term on the right-hand side of (5.2) becomes 0, $\tilde{f}_0'(\theta)[\vartheta]$ in (5.1) agrees with (5.7). Here, if the conditions such that $\boldsymbol{u}_i$ and $\boldsymbol{v}_{i0}$ belong to $\mathcal{S}$ in (3.2) are satisfied, since g_0 belongs to $L^q(D; \mathbb{R}^d)$, the necessary condition $g_0 \in X'$ (X' denotes the dual space of X) for the Fréchet derivative with respect to arbitrary variation of $\theta \in X$ is satisfied.

5.2 Evaluation of g_1

Since $f_1(\boldsymbol{U}, \boldsymbol{\beta})$ includes the solution $\boldsymbol{U}$ of Problem 1, we define the Lagrange function for $f_1(\boldsymbol{U}, \boldsymbol{\beta})$ as

$$\mathscr{L}_1(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1) = f_1(\boldsymbol{U}, \boldsymbol{\beta}) + \sum_{i \in \mathcal{M}} \mathscr{L}_{\mathrm{M}i}(\theta, \eta_i, \boldsymbol{u}_i, \boldsymbol{v}_{i1}), \tag{5.8}$$

where $\boldsymbol{V}_1 = \{\boldsymbol{v}_{i1}\}_{i \in \mathcal{M}} \in U^{|\mathcal{M}|}$ is the Lagrange multiplier of Problem 1 for f_1. Here, let $\boldsymbol{\beta}' \in \mathbb{R}^{|\mathcal{M}|}$ denote the arbitrary variation of $\boldsymbol{\beta}$, and let $\vartheta \in X$ denote the arbitrary variation of θ. Then, the Fréchet derivative of $\mathscr{L}_1$ with respect to these variations can be written as

$$\begin{aligned}\mathscr{L}_1'(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\vartheta, \boldsymbol{\beta}'] &= \mathscr{L}_{1\theta}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\vartheta] \\ &+ \mathscr{L}_{1\boldsymbol{\eta}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\boldsymbol{\eta}'] + \mathscr{L}_{1\boldsymbol{U}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\boldsymbol{U}'] \\ &+ \mathscr{L}_{1\boldsymbol{V}_1}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\boldsymbol{V}_1'] + \mathscr{L}_{1\boldsymbol{\beta}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\boldsymbol{\beta}'].\end{aligned} \tag{5.9}$$

Here, $\boldsymbol{\eta}'$ denotes the Fréchet derivative $\boldsymbol{\eta}'(\theta)[\vartheta]$ of the solution $\boldsymbol{\eta}$ of Problem 1. In the same manner, $\boldsymbol{U}'$ and $\boldsymbol{V}_1'$ denote these Fréchet derivatives.

The fifth term on the right-hand side of (5.9) becomes

$$\begin{aligned}\mathscr{L}_{1\boldsymbol{\beta}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\boldsymbol{\beta}'] &= \sum_{i \in \mathcal{M}} f_{1\beta_i}(\boldsymbol{U}, \boldsymbol{\beta})[\beta_i'] \\ = -\sum_{i \in \mathcal{M}} 2\beta_i' \bar{\eta}_i \Bigg(&\int_{D \cap \bar{\Omega}_{\mathrm{M}}} (\boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i) \cdot \bar{\boldsymbol{u}}_i \, \mathrm{d}x \\ &+ \int_{\Gamma_{\mathrm{N}} \cap \bar{\Omega}_{\mathrm{M}}} (\boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i) \cdot \bar{\boldsymbol{u}}_i \, \mathrm{d}\gamma \Bigg).\end{aligned}$$

Then, if we set

$$\beta_i = \frac{\displaystyle\int_{D\cap\bar{\Omega}_{\mathrm{M}}} \boldsymbol{u}_i \cdot \bar{\boldsymbol{u}}_i \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}\cap\bar{\Omega}_{\mathrm{M}}} \boldsymbol{u}_i \cdot \bar{\boldsymbol{u}}_i \,\mathrm{d}\gamma}{\displaystyle\int_{D\cap\bar{\Omega}_{\mathrm{M}}} \bar{\boldsymbol{u}}_i \cdot \bar{\boldsymbol{u}}_i \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}\cap\bar{\Omega}_{\mathrm{M}}} \bar{\boldsymbol{u}}_i \cdot \bar{\boldsymbol{u}}_i \,\mathrm{d}\gamma}, \tag{5.10}$$

the fifth term on the right-hand side of (5.9) becomes 0.

The forth term on the right-hand side of (5.9) agrees with (5.4), in which $\boldsymbol{V}_0$ is replaced by $\boldsymbol{V}_1$. Then, if $(\boldsymbol{\eta}, \boldsymbol{U})$ are weak solutions of Problem 1, the fourth term on the right-hand side of (5.9) becomes 0.

Moreover, the second and the third terms become

$$\begin{aligned}
&\mathscr{L}_{1\boldsymbol{\eta}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\boldsymbol{\eta}'] + \mathscr{L}_{1\boldsymbol{U}}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\boldsymbol{U}'] \\
&\quad = \sum_{i\in\mathcal{M}} \Big(\langle \mathscr{L}_{\mathrm{M}\eta_i}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_1), \eta_i' \rangle + \langle f_{1\boldsymbol{u}_i}(\boldsymbol{U}, \boldsymbol{\beta}), \boldsymbol{u}_i' \rangle \\
&\qquad + \langle \mathscr{L}_{\mathrm{M}\boldsymbol{u}_i}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_1), \boldsymbol{u}_i' \rangle \Big),
\end{aligned} \tag{5.11}$$

where

$$\begin{aligned}
&\langle \mathscr{L}_{\mathrm{M}\eta_i}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_1), \eta_i' \rangle = \eta_i' \int_D \rho \boldsymbol{u}_i \cdot \boldsymbol{v}_{i1} \,\mathrm{d}x, \\
&\langle f_{1\boldsymbol{u}_i}(\boldsymbol{U}, \boldsymbol{\beta}), \boldsymbol{u}_i' \rangle + \langle \mathscr{L}_{\mathrm{M}\boldsymbol{u}_i}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{V}_1), \boldsymbol{u}_i' \rangle \\
&\quad = 2\bar{\eta}_i \left(\int_{D\cap\bar{\Omega}_{\mathrm{M}}} (\boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i) \cdot \boldsymbol{u}_i' \,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}\cap\bar{\Omega}_{\mathrm{M}}} (\boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i) \cdot \boldsymbol{u}_i' \,\mathrm{d}\gamma \right) \\
&\qquad + \mathscr{L}_{\mathrm{M}i}(\theta, \eta_i, \boldsymbol{u}_i', \boldsymbol{v}_{i1}).
\end{aligned}$$

Here, (5.11) agrees with the Lagrange function of the following problem. Then, if $\boldsymbol{V}_1$ is the weak solution of the following problem, the second and third terms on the right-hand side of (5.9) become 0.

Problem 4 (Adjoint problem for f_1) Let $\theta \in X$ be given. For $i \in \mathcal{M}$, let the solution $(\eta_i, \boldsymbol{u}_i)$ of Problem 1 be given. Find $\boldsymbol{v}_{i1} : D \to \mathbb{R}^d$ such that

$$\begin{aligned}
&-\eta_i \rho \boldsymbol{v}_{i1}^{\mathrm{T}} - \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{\Sigma}(\theta, \boldsymbol{v}_{i1}) \\
&\quad = \begin{cases} 2\bar{\eta}_i (\boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i)^{\mathrm{T}} & \text{in } D \cap \bar{\Omega}_{\mathrm{M}}, \\ \boldsymbol{0}_{\mathbb{R}^d}^{\mathrm{T}} \text{ in } D \setminus \bar{\Omega}_{\mathrm{M}}, \end{cases} \\
&\boldsymbol{\Sigma}(\theta, \boldsymbol{v}_{i1}) \boldsymbol{\nu} = \begin{cases} 2\bar{\eta}_i (\boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i) \text{ on } \Gamma_{\mathrm{N}} \cap \bar{\Omega}_{\mathrm{M}}, \\ \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \Gamma_{\mathrm{N}} \setminus \bar{\Omega}_{\mathrm{M}}, \end{cases} \\
&\boldsymbol{v}_{i1} = \boldsymbol{0}_{\mathbb{R}^d} \quad \text{on } \Gamma_{\mathrm{D}}, \\
&\int_D \rho \boldsymbol{u}_i \cdot \boldsymbol{v}_{i1} \,\mathrm{d}x = 0.
\end{aligned} \tag{5.12}$$

Problem 4 can be solved as follows. Let $\mathcal{N}$ be a set of all of the mode numbers of eigenfrequencies in a frequency range that is sufficiently large, which covers all of the eigenfrequencies of $\mathcal{M}$, and $\boldsymbol{u}_j$ for all $j \in \mathcal{N}$ be the solutions of Problem 1. If $|\mathcal{N}|$ is sufficiently large, we can write the following approximation:

$$\boldsymbol{v}_{i1} = \sum_{j \in \mathcal{N} \setminus \{i\}} \xi_{ij} \boldsymbol{u}_j. \tag{5.13}$$

Here, $\boldsymbol{\xi}_i = \left(\xi_{ij}\right)_j \in \mathbb{R}^{|\mathcal{N}|-1}$ is referred to as the mode coordinates of $\boldsymbol{v}_{i1}$ with respect to $\{\boldsymbol{u}_j\}_{j \in \mathcal{N}}$. Since $\boldsymbol{v}_{i1}$ is the weak solution of Problem 4, the Lagrange function of Problem 4 given by (5.11) becomes 0. Using this condition, we have

$$\xi_{ij} = \frac{2\bar{\eta}_i}{\eta_j - \eta_i} \left(\int_{D \cap \bar{\Omega}_{\mathrm{M}}} (\boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i) \cdot \boldsymbol{u}_j \, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}} \cap \bar{\Omega}_{\mathrm{M}}} (\boldsymbol{u}_i - \beta_i \bar{\boldsymbol{u}}_i) \cdot \boldsymbol{u}_j \, \mathrm{d}\gamma \right). \tag{5.14}$$

Here, we used the following conditions. Since $j \in \mathcal{N} \setminus \{i\}$ is assumed in (5.13), the first term on the right-hand side of (5.11) becomes 0. Moreover, substituting (5.13) into the second and third terms on the right-hand side of (5.11), setting $\boldsymbol{u}_i' = \boldsymbol{u}_j$, considering $\mathscr{L}_{\mathrm{M}j}(\theta, \eta_j, \boldsymbol{u}_j, \boldsymbol{u}_j) = 0$, and using the normalization condition (3.3) of $\boldsymbol{u}_j$, we have (5.14). Then, we can compute the weak solution $\boldsymbol{v}_{i1}$ of Problem 4 by (5.13) using ξ_{ij} in (5.14).

Moreover, the first term on the right-hand side of (5.9) becomes

$$\mathscr{L}_{1\theta}(\theta, \boldsymbol{\eta}, \boldsymbol{U}, \boldsymbol{\beta}, \boldsymbol{V}_1)[\vartheta] = \int_D \sum_{i \in \mathcal{M}} g_{i1} \vartheta \, \mathrm{d}x = \langle g_1, \vartheta \rangle, \tag{5.15}$$

where

$$\begin{aligned} g_{i1} &= -\frac{\partial \boldsymbol{\Sigma}(\theta, \boldsymbol{u}_i)}{\partial \theta} \cdot \boldsymbol{E}(\boldsymbol{v}_{i1}) = -\alpha \phi^{\alpha-1} \frac{\mathrm{d}\phi}{\mathrm{d}\theta} \boldsymbol{C}_0 \boldsymbol{E}(\boldsymbol{u}_i) \cdot \boldsymbol{E}(\boldsymbol{v}_{i1}) \\ &= -\frac{\alpha}{\phi} \frac{\mathrm{d}\phi}{\mathrm{d}\theta} \boldsymbol{\Sigma}(\theta, \boldsymbol{u}_i) \cdot \boldsymbol{E}(\boldsymbol{v}_{i1}). \end{aligned}$$

From these results, if $(\boldsymbol{\eta}, \boldsymbol{U})$ and $\boldsymbol{V}_1$ are weak solutions of Problems 1 and 4, respectively, since the Fréchet derivative of the second term on the right-hand side of (5.8) becomes 0, $\tilde{f}_{1\theta}(\theta, \boldsymbol{\beta})[\vartheta]$ in (5.1) agrees with (5.15). Here, if the conditions for $\boldsymbol{u}_i$ and $\boldsymbol{v}_{i1}$ to belong to $\mathcal{S}$ in (3.2) are satisfied, since g_1 belongs to $L^q\left(D; \mathbb{R}^d\right)$, $g_1 \in X'$ is satisfied.

6 H^1 gradient method

Since θ-derivative g of f is evaluated, the variation of θ, which belongs to X and minimizes f, can be found by the H^1 gradient method for topology optimization problem of θ type [7].

Problem 5 (H^1 gradient method for θ type) Let X be defined as (2.2), $a_X : X \times X \to \mathbb{R}$ be a coercive bilinear form on X, and the θ-derivative $g \in X'$ of f. Find $\vartheta_g \in X$ such that

$$a_X(\vartheta_g, z) = -\langle g, z \rangle$$

for all $z \in X$.

For example, for the case in which we use

$$a_X(\theta, \vartheta) = c_a \int_D (\boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\vartheta + c_\Omega \theta\vartheta)\, \mathrm{d}x, \tag{6.1}$$

the strong form of Problem 5 becomes as follows, where c_a is a positive constant to control the magnitude of ϑ_g, and c_Ω is a positive constant to control the smoothness of ϑ_g. Here, let ∂_ν denote $\boldsymbol{\nabla} \cdot \boldsymbol{\nu}$.

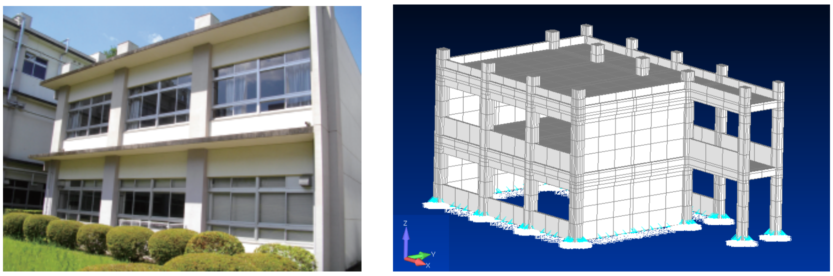
Problem 6 (Strong form of Problem 5) Let $g \in X'$ be given. Find $\vartheta_g \in X$ such that

$$c_a(-\Delta\vartheta_g + c_\Omega\vartheta_g) = g \quad \text{in } D,$$
$$c_a \partial_\nu \vartheta_g = 0 \quad \text{on } \partial D.$$

7 Numerical example

We developed a program in which g_0 and g_1 were computed using the numerical solutions of Problems 1, 3, and 4 by the finite-element method, and θ was renewed using the numerical solution ϑ_g of Problem 6 by the finite-element method. The correspondence of mode numbers between the finite-element model and the measurement data was checked using the modal assurance criterion. In (2.1), $\alpha = 2$ was used, while similar results were obtained using $\alpha = 1$. c_1 in (4.4) was determined to hold $f_0 \approx f_1$ in the initial model. $\mathcal{N} = \{1, \cdots, 150\}$ and $c_\Omega = 10$ [1/m^2] were used in (5.13) and (6.1), respectively.

A School of Pharmacy building of Nagoya City University that is to be demolished was used as a target object. Figure 2 shows a photograph of the building and the finite-element model used in the present analyses. The bottom plane was assumed to be Γ_{D}. The hexahedral second-order element was used for the finite-element model. The numbers of nodes and elements were 32,410 and 5,706, respectively. Moreover, 21.1 [GPa], 0.2, and 2,210 [kg/m^3] were used as the Young's modulus, Poisson's ratio, and the density, which were determined by sampling based on material experiments.



(a) Photograph (b) FE model

Fig. 2 Building used in the experiment

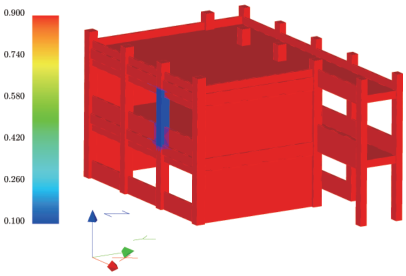
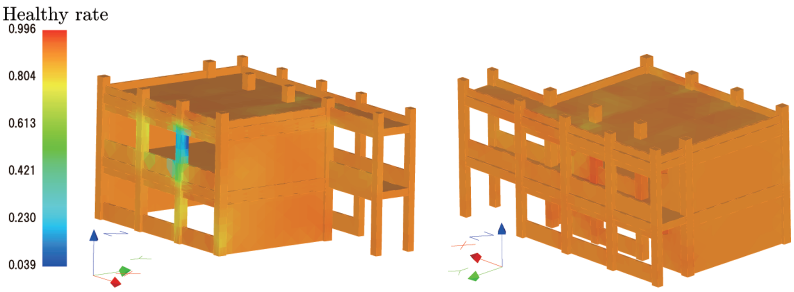


Fig. 3 FE model assuming imaginary damage

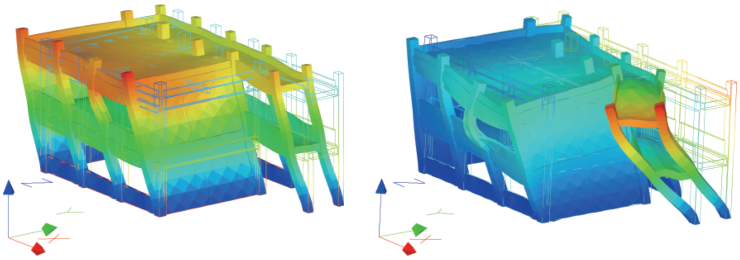


(a) Front view (b) Back view

Fig. 4 Healthy rate identified from 1st and 2nd mode pairs with respect to the FE model of Fig. 3

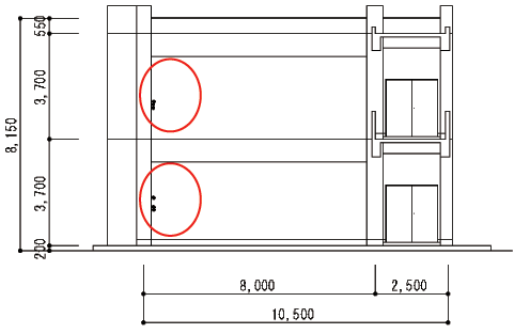
7.1 Theoretical damage

First, a problem of identifying theoretical damage was solved. Figure 3 shows this finite-element model, in which healthy rates of 0.1 and 0.9 were assumed for the damaged area and the remainder, respectively. In Problem 2, we assumed that $\mathcal{M} = \{1, 2\}$ and $\bar{\Omega}_{\mathrm{M}} = \bar{D} \setminus \Gamma_{\mathrm{D}}$. The initial healthy rate was 0.9 uniformly. Figures 4 and 5 show the identified healthy rate and its eigenmodes, respectively.

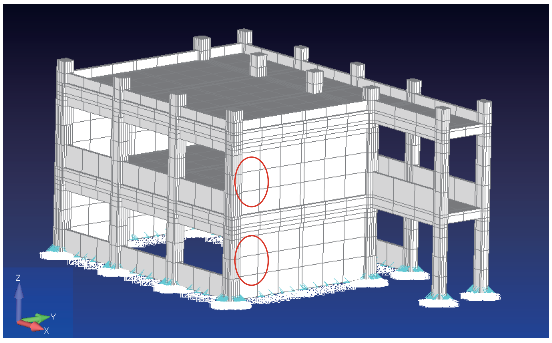


(a) 1st mode 2.74 [Hz] 2nd mode 3.10 [Hz]

Fig. 5 Eigenmodes of the identified FE model of Fig. 4



(a) Diagram [unit: m]



(b) FE model

Fig. 6 Artificially damaged part

7.2 Experimental data

Aoki et al. [2] obtained the data of the eigenpairs by measuring ambient vibration before and after damage was artificially generated. Figure 6 shows the damaged area. Figure 7 shows 12 points at which ambient vibration was measured. The eigenfrequencies from the first mode to the eighth mode are listed in Table 1. The eigenmodes before and after the damage are shown in Figs. 8 and 9. The healthy rate identified from these experimental data is shown in

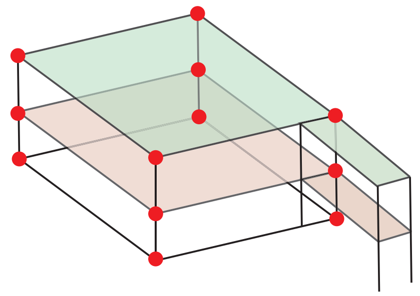


Fig. 7 Measurement points of ambient vibration

Table 1 Measured eigenfrequencies

Mode	Pre-damage [Hz]	Post-damage [Hz]
1st	3.02	2.95
2nd	4.76	4.05
3rd	6.28	4.69
4th	7.41	6.11
5th	9.65	6.76
6th	10.10	7.39
7th	11.00	8.69
8th	11.30	9.50

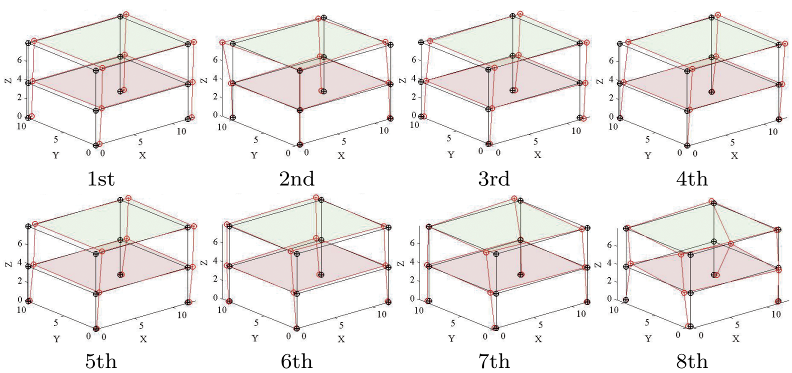


Fig. 8 Measured eigenmodes of pre-damage

Fig. 10. From this figure, it is confirmed that the healthy rate is low in the neighborhood of the damaged area.

In the case using the experimental data, since the measuring points of vibration and the number of the eigenpairs were limited to 12 points and 8 modes, respectively, it is considered that the resolution of the identified damaged area was not enough. However, considering that the accurate result was obtained in the case of using the theoretical damage, it is expected that more accurate identification is realized by using sufficient experimental data.

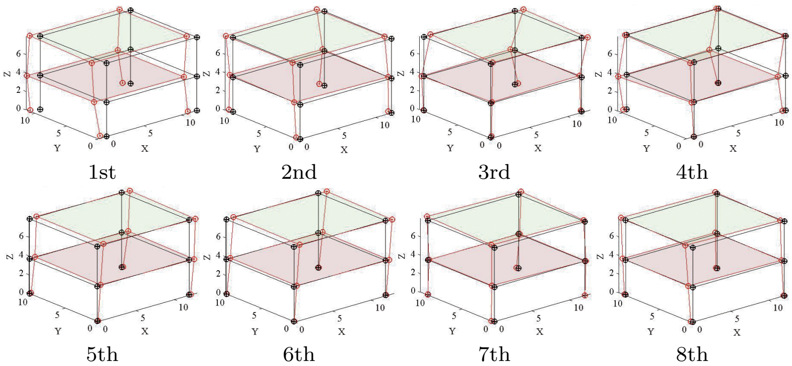
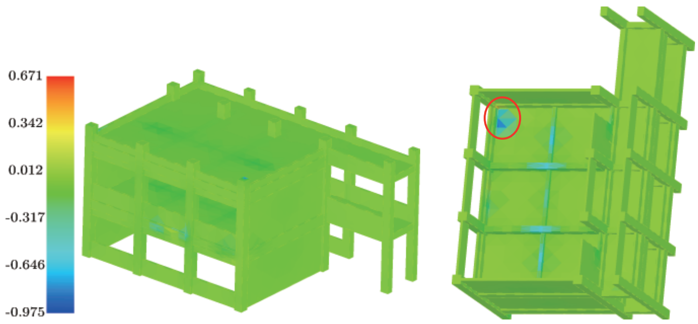


Fig. 9 Measured eigenmodes of post-damage



(a) Front view (b) Ground-up view

Fig. 10 Healthy rate identified from experimental data

8 Conclusions

The present paper described a solution to a problem of identifying damage in buildings based on experimentally measured vibrational eigenvalue and eigenmode pairs. The healthy rate, which was defined as the stiffness rate with respect to a perfect material, was chosen as an identification variable. The linear coupling of the mean squared errors of vibrational eigenvalues and eigenmodes with respect to the measured values was used as a cost function. The derivative of the cost function with respect to the design variable was evaluated by the adjoint variable method. A numerical solution using the H^1 gradient method was presented, and numerical examples were presented.

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