# Construction of continuous wavelet transforms associated to unitary representations of semidirect product groups

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# Introduction

In this thesis, we study the notion of continuous wavelet transforms to the case of non-irreducible unitary representations.

The notion of continuous wavelet transforms can be stated as follows. Let $G$ be a locally compact group and $\pi$ a unitary representation of $G$ acting on a Hilbert space $\mathcal{H}_\pi$. If there exists a vector $f_0 \in \mathcal{H}_\pi$ such that

$$W_{f_0} f(g) := \langle f, \pi(g) f_0 \rangle \quad (f \in \mathcal{H}_\pi, g \in G) \tag{1}$$

defines an isometry from $\mathcal{H}_\pi$ into $L^2(G)$, then the vector $f_0$ is called an admissible vector, or a wavelet, and the isometry $W_{f_0}$ is called a continuous wavelet transform. Since $W_{f_0}$ is isometry, we have the following inversion formula of the continuous wavelet transform:

$$f = \int_G W_{f_0} f(g) \pi(g) f_0 \; d\mu_G(g),$$

where $\mu_G$ is a left Haar measure on $G$.

There have been attempts to study continuous wavelet transforms for non-irreducible unitary representation $\pi$. We study the case where $G$ is the semidirect product group $N \rtimes H$ of a locally compact abelian group $N$ with a closed subgroup $H$ of the automorphism group of $N$. The main example we have in mind is $N = \mathbb{R}^n$ with $H = SO(n) \times \mathbb{R}_+$. Let $\sigma$ be a unitary representation of $H$ on a Hilbert space $\mathcal{H}_\sigma$. We define the unitary representation $\pi$ of $G$ on the space $L^2(N, \mathcal{H}_\sigma)$ of $\mathcal{H}_\sigma$-valued square integrable functions on $N$ by

$$\pi(n,h) f(n_0) = \delta(h)^{-\frac{1}{2}} \sigma(h) f(h^{-1}(n^{-1} n_0)) \quad (n, n_0 \in N, h \in H), \tag{2}$$

where $\delta$ is a continuous homomorphism from $H$ to $\mathbb{R}_+$ (see Section 2). We discuss the existence of an admissible vector for this representation $\pi$. We first give a condition under which $\pi$ decomposes into a direct sum of square-integrable representations of $G$. Since $N$ is a normal subgroup of $G$, we have an action of $G$ on the unitary dual $\hat{N}$ defined by $g \cdot \nu(n) := \nu(g^{-1} n g)$ $(\nu \in \widehat{N}, g \in G, n \in N)$. We denote by $\mathcal{O}_\nu$ the $G$-orbit through $\nu \in \widehat{N}$, and by $G_\nu$ the stabilizer at $\nu$. Let us assume the following:

1. There exist elements $\nu_k$ $(k \in K)$ of $\widehat{N}$, indexed by some countable set $K$, such that $\mu(\mathcal{O}_{\nu_k}) > 0$, $\mathcal{O}_{\nu_k} \cap \mathcal{O}_{\nu_{k'}} = \emptyset$ $(k \neq k')$, and $\mu(\widehat{N} \setminus \bigsqcup_{k \in K} \mathcal{O}_{\nu_k}) = 0$, where $\mu$ is the Plancherel measure on $\widehat{N}$.

2. For every $k \in K$, the map $G/G_{\nu_k} \ni gG_{\nu_k} \mapsto g \cdot \nu_k \in \mathcal{O}_{\nu_k}$ is a homeomorphism.

3. For every $k \in K$, there exists an index set $\Lambda_k$ such that $\sigma|_{H_{\nu_k}} = \bigoplus_{\alpha\in\Lambda_k} \rho_\alpha$ ($\rho_\alpha \in \widehat{H}_{\nu_k}$), where $H_{\nu_k} = G_{\nu_k} \cap H$.

4. For $\alpha \in \Lambda_k$, the representation $\rho_\alpha$ of $H_{\nu_k}$ is square integrable.

Then, we get the following result.

**Theorem.** *Irreducible decomposition of the unitary representation $\pi$ is given by $\bigoplus_{k\in K} \bigoplus_{\alpha\in\Lambda_k} \mathrm{Ind}_{G_{\nu_k}}^G \nu_k \otimes \rho_\alpha$. Moreover, for each $k \in K$ and $\alpha \in \Lambda_k$, the induced representation $\mathrm{Ind}_{G_{\nu_k}}^G \nu_k \otimes \rho_\alpha$ is square-integrable. In particular, If $\sigma$ is trivial, the induced representation $\mathrm{Ind}_{G_{\nu_k}}^G \nu_k \otimes \rho_\alpha$ is square-integrable representation if and only if $H_{\nu_k}$ is compact for each $k \in K$ and $\alpha \in \Lambda_k$.*

Based on Theorem above, we can define the continuous wavelet transform associated to the representation $\pi$. For this purpose, we need one more technical assumption (A6) presented at Page 15, but it is automatically satisfied when the number of irreducible subrepresentations is finite (see Section 2 for the details). Under this condition, we obtain

**Theorem.** *One can construct an admissible vector $f_0 \in L^2(N, H_\sigma)$ for the representation $\pi$.*

A general construction of $f_0$ is explained in Section 2. We then study the case of similitude transformations on $\mathbb{R}^n$ in Section 3. Of special interest is the case $n = 3$, where we have explicit formulae for admissible vectors. We expect this formula to be especially useful for problems in 3 dimensions. For example, we find examples of admissible vector $f_0 \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ as follows:

$$f_0(x) = \frac{\sqrt{2}}{(2\pi)^6} e^{-\frac{|x|^2}{2}} \begin{pmatrix} (x_1 - \sqrt{3}x_2)(5 - |x|^2) \\ (x_2 + \sqrt{3}x_1)(5 - |x|^2) \\ x_3(5 - |x|^2) \end{pmatrix}.$$

and

$$f_0(z) = c\frac{1}{z_1^{k_1+1} z_2^{k_2+1} z_3^{k_3+1}} \begin{pmatrix} \frac{k_1+1}{z_1} - \frac{a(k_2+1)}{z_2} \\ \frac{k_2+1}{z_2} + \frac{a(k_1+1)}{z_1} \\ \frac{k_3+1}{z_3} \end{pmatrix} \quad (k_1, k_2, k_3 \in N)$$

where $z_j = x_j + iy_j (j = 1, 2, 3)$and $a, c$ are some constants. Furthermore, we find an admissible vector $f_0 \in L^2(\mathbb{R}^3, \mathbb{C}^5)$ as

$$f_0(x) = (\frac{1}{2\pi i})^3 \begin{pmatrix} P_{1,a,b}(D_x) \\ P_{2,a,b}(D_x) \\ P_{3,a,b}(D_x) \\ P_{4,a,b}(D_x) \\ P_{5,a,b}(D_x) \end{pmatrix} \bigotimes \frac{1}{(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)}$$

with

$$\begin{pmatrix} P_{1,a,b}(\xi) \\ P_{2,a,b}(\xi) \\ P_{3,a,b}(\xi) \\ P_{4,a,b}(\xi) \\ P_{5,a,b}(\xi) \end{pmatrix} = \begin{pmatrix} 2\xi_1\xi_2 - a(\xi_2^2 - \xi_1^2) + b(\xi_2^2 - \xi_1^2)\xi_3 \\ \xi_2^2 - \xi_1^2 + 2a\xi_1\xi_2 - 2b\xi_1\xi_2\xi_3 \\ 2\xi_1\xi_3 - a\xi_2\xi_3 - b\xi_2(\xi_1^2 + \xi_2^2) \\ 2\xi_2\xi_3 + a\xi_1\xi_3 + b\xi_1(\xi_1^2 + \xi_2^2) \\ \sqrt{3}(\xi_1^2 + \xi_2^2 - 2\xi_3^2) \end{pmatrix},$$

where $D_x = (\frac{1}{2\pi i}\frac{\partial}{\partial x_1}, \frac{1}{2\pi i}\frac{\partial}{\partial x_2}, \frac{1}{2\pi i}\frac{\partial}{\partial x_3})$ and $a, b$ are some constants, see Section 3.

In the rest of introduction, we first motivate the definition of continuous wavelet transforms by way of the Calderón reproducing formula. Next we give a brief survey of relevant literature, including generalizations to the non square-integrable case. We then give a detailed section by section description of this thesis.

## Calderón reproducing formula and wavelet transforms

Wavelet theory has about thirty years' history. Since the notion of wavelet is invented by J. Morlet for the purpose of oil protesting [26], the theory of wavelet has developed quite widely in connection with many areas of sciences such as physics, information science, and engineering as well as pure and applied mathematics [2, 22]. On the other hand, we can find a number of origins of wavelet before J. Morlet like coherent states in quantum mechanics and windowed Fourier transform or Gabor transform in signal analysis, etc. (we shall mention a little more about them later). Among such pre-wavelet objects, the Calderón reproducing formula is a notable prototype of wavelet theory in our perspective.

Let us recall the Calderón reproducing formula [11]. When a function $\phi \in L^2(\mathbb{R})$ satisfies the admissible condition

$$\int_{\mathbb{R}^*} \frac{|\widehat{\phi}(\xi)|^2}{|\xi|} d\xi = 1,$$

we define $\phi_{a,b} \in L^2(\mathbb{R})$ for $a \in \mathbb{R}^*, b \in \mathbb{R}$ by

$$\phi_{b,a}(x) = |a|^{-\frac{1}{2}} \phi(\frac{x-b}{a}).$$

Then for any $f \in L^2(\mathbb{R})$, we have

$$f = \int_{\mathbb{R}} \int_{\mathbb{R}^*} \langle f, \phi_{a,b} \rangle \, \phi_{a,b} \frac{dadb}{|a|^2}.$$

We can interpret and generalize the formula above in terms of unitary representation theory, which is essentially due to A. Grossmann, J. Morlet and T. Paul [29]. Let $G$ be the group $\mathrm{Aff}(\mathbb{R})$ of invertible affine transformations on the real line $\mathbb{R}$. Then $G$ is the described as the semideirect product group $\mathbb{R} \rtimes \mathbb{R}^*$. We define a unitary representation $\pi$ of $G$ on $L^2(\mathbb{R})$ by

$$\pi(b,a)f(x) = |a|^{-\frac{1}{2}} f(\frac{x-b}{a}) \quad (f \in L^2(\mathbb{R})). \tag{3}$$

Then $\pi$ is irreducible. If there exists $f_0 \in L^2(\mathbb{R})$ such that $L^2(\mathbb{R}) \ni f \mapsto \langle f, \pi(b,a)f_0 \rangle \in L^2(\mathbb{R} \rtimes \mathbb{R}^*)$ is isometry, we have

$$f = \int_{\mathbb{R} \rtimes \mathbb{R}^*} \langle f, \pi(b,a)f_0 \rangle \, \pi(b,a)f_0 \frac{dadb}{|a|^2}.$$

It is known that the isometry condition for $f_0$ is equivalent to the admissible condition, so that the formula above means exactly the Calderonón reproducing formula. Furthermore, a wide generalization of the Calderón reproducing formula is introduced in [29] using representation theory.

## Motivation

In the work mentioned above, a unitary representation is assumed to be irreducible. This assumption often can be weakened. Let us investigate one example. Let $G$ be a closed subgroup $\mathrm{Aff}_+(\mathbb{R}) = \{(b,a) : b \in \mathbb{R}, a > 0\}$ of $\mathrm{Aff}(\mathbb{R})$. Then $\pi$ in (1) restricted to $\mathrm{Aff}_+(\mathbb{R})$ is unitary, but not irreducible. The restriction $\pi|_{\mathrm{Aff}_+(\mathbb{R})}$ is decomposed into the direct sum of two irreducible unitary subrepresentations $\pi_+$ and $\pi_-$, which are defined on the Hardy spaces

$$H_+(\mathbb{R}) := \{f \in L^2(\mathbb{R}); \mathrm{supp}\, \widehat{f} \subset [0.\infty)\},$$

$$H_-(\mathbb{R}) := \{f \in L^2(\mathbb{R}); \mathrm{supp}\, \widehat{f} \subset (-\infty, 0)\},$$

respectively and $L^2(\mathbb{R}) = H_+(\mathbb{R}) \bigoplus H_-(\mathbb{R})$. If $f_{0,+} \in H_+$ (resp. $f_{0,-} \in H_-(\mathbb{R})$) satisfies admissible condition

$$\int_0^\infty \frac{|f_{0,+}(\xi)|^2}{\xi} d\xi = 1 \text{ (resp. } \int_{-\infty}^0 \frac{|f_{0,-}(\xi)|^2}{\xi} d\xi = 1),$$

we have a reconstruction formula for $H_\pm$ using the admissible vectors. Now, if we choose $f_0 = f_{0,+} + f_{0,-} \in L^2(\mathbb{R})$, then the map $W_{f_0}$ is an isometry from $L^2(\mathbb{R})$ into $L^2(\mathrm{Aff}_+(\mathbb{R}))$, that is, we obtain a reconstruction formula

$$f = \int_{\mathrm{Aff}_+(\mathbb{R})} \langle f, \pi(b,a) f_0 \rangle \, \pi(b,a) f_0 \frac{dadb}{a^2}$$

for any $f \in L^2(\mathbb{R})$.

A natural question is how to construct the wavelet transform for a unitary representation $\pi$ of higher dimensional cases. For instance, let us consider the 3-dimensional similitude group $\mathbb{R}^3 \rtimes (SO(3) \times \mathbb{R}_+)$. We denote by $(x, A, c)$ an element of this group. Let $\sigma_m$ be the $2m+1$-dimensional irreducible unitary representation of $SO(3)$ on $\mathbb{C}^{2m+1}$. We define a unitary representation $\pi$ of $\mathbb{R}^3 \rtimes (SO(3) \times \mathbb{R}_+)$ on the space $L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$ of square-integrable $\mathbb{C}^{2m+1}$-valued functions by

$$\pi(x, A, c) f(y) = c^{-\frac{3}{2}} \sigma_m(A) f(c^{-1} A^{-1}(y - x)) \quad (f \in L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})). \quad (4)$$

Then $\pi$ is not irreducible, which makes the situation more complicated. Does there exist an admissible vector for $\pi$, that is, an vector $f_0$ such that

$$f = \int_{\mathbb{R}^3} \int_{SO(3)} \int_{\mathbb{R}_+} \langle f, \pi(x, A, c) f_0 \rangle \, \pi(x, A, c) f_0 \frac{dxdAdc}{c^4}$$

for any $f \in L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$ ? In conclusion, we find examples of such $f_0 \in L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$, in Section 3.

## Wavelet transforms in disguise

S. Mallat and S. Zhong constructed transform related to the original continuous wavelet transform, called the dyadic wavelet transform [43]. Their result is interpreted as follows: Let $H = \{(b, 2^j) : b \in \mathbb{R}, j \in \mathbb{Z}\}$ be a closed subgroup of $\mathrm{Aff}_+(\mathbb{R})$. Then $\pi$ in (1) restricted to $H$ is not irreducible. If $f_0 \in L^2(\mathbb{R})$ satisfies the dyadic admissibility condition

$$\sum_{j \in \mathbb{Z}} |\widehat{f_0}(2^j \xi)|^2 = 1 \quad \text{for a.e. } \xi = \mathbb{R},$$

we obtain the inversion formula

$$f = \int_{\mathbb{R}} \sum_{j\in\mathbb{Z}} \left\langle f, \pi(b, 2^j) f_0 \right\rangle \pi(b, 2^j) f_0 2^{-j} db \quad (f \in L^2(\mathbb{R})).$$

The dyadic admissibility condition is the key in the discrete wavelet analysis and multiresolution analysis. In the discrete wavelet analysis and multiresolution analysis, basic problems are that construction of a frame and orthonormal basis for a Hilbert space. Let us consider $\mathcal{A} = \{2^{-\frac{j}{2}} f_0(2^{-j} \cdot -m) : j, m \in \mathbb{Z}\}$, which is called the affine system. If there exist $A$ and $B$ with $B \geq A > 0$ such that for any $f \in L^2(\mathbb{R})$

$$A\|f\|^2 \leq \sum_{m\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} | \left\langle f, 2^{-\frac{j}{2}} f_0(2^{-j} \cdot -m) \right\rangle |^2 \leq B\|f\|^2,$$

then $\mathcal{A}$ is called a frame. Moreover, if $A = B$, then $\mathcal{A}$ is called a tight. If $f_0 \in L^2(\mathbb{R})$ satisfies the dyadic admissible condition and following condition

$$\sum_{j\geq 0} \widehat{f_0}(2^j\xi)\overline{\widehat{f_0}(2^j(\xi+q))} = 0 \quad \text{for a.e. } \xi = \mathbb{R}, \text{for any odd integer } q,$$

then $A = B = 1$, and $\mathcal{A}$ is an orthonormal basis for $L^2(\mathbb{R})$. This approach is extended to higher dimensional cases, see [32]. In Section 3.2, we shall give a frame for $L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$.

We may often construct the wavelet transform associated to non-square-integrable representation $G$. Let us consider the Heisenberg group $G = \mathbb{R}^n \rtimes (\mathbb{R}^n \times \mathbb{R})$ and the Schrödinger representation $\pi_h (h \neq 0)$ of $G$ on $L^2(\mathbb{R})$ by

$$\pi_h(p, q, t) f(x) = e^{2\pi i h t + 2\pi i q\cdot x + \pi i h p\cdot q} f(x + hp) \quad (f \in L^2(\mathbb{R}^n)).$$

This representation is irreducible, but it is not square-integrable. If we restrict $\pi_h$ to $\mathbb{R}^n \times \mathbb{R}^n (= \{(p, q)\})$, we obtain that for (any) $f_0 \in \mathbb{R}^n$ with $\|f_0\| = 1$ the map

$$W_{f_0} : L^2(\mathbb{R}^n) \mapsto L^2(G/\mathbb{R})$$

is isometry. This means that $\pi$ is square-integrable with respect to the homogeneous space $\mathbb{R}^n \times \mathbb{R}^n$. We consider the case of $n = 1$. If $f_0(x) = e^{-\pi x^2}$, then $W_{f_0}$ coincides with the Gabor transform, which is a special case of the short-time Fourier transform. Therefore wavelet transform associated to the Schrödinger representation is interpreted as the short-time Fourier transform in time-frequency analysis.

Historically, the Heisenberg group is important in physics, especially in the classical and quantum kinematics of a single particle moving in $n$-dimensional space. The space $\mathbb{R}^n \times \mathbb{R}^n$ is called phase space with coordinate $(p, q)$ where $p$ is interpreted as the momentum vector of the particle and $q$ is interpreted as its position vector and $h$ is interpreted as Planck's constant. The wavelets obtained above is closely related to the canonical coherent states in quantum theory. The idea of coherent state were introduced by E. Schrödinger for studying quantum-to-classical translation. After that, by R. J. Glauber, J. R. Klauder and E. C. G. Sudarshan, a coherent state associated to the one-dimensional quantum harmonic oscillator was studied, which is so-called canonical coherent state. Furthermore, R. Gilmore [24] and A. M. Perelomov [45] constructed coherent states associated to other groups, independently. They considered a coherent state on a homogeneous space $G/H$, where $H$ is the isotropy group of some state. The wavelets associated to $\mathrm{Aff}(\mathbb{R})$ or $\mathrm{Aff}_+(\mathbb{R})$ are examples of Gilmore-Perelomov coherent state [29], see [30] for concrete expressions of admissible vectors. We find description of a connection between theory of wavelets and coherent states in [2].

In (4), when $m = 1$, a representation space $L^2(\mathbb{R}^3, \mathbb{C}^3)$ is decomposed into the space of divergence-free vector and the space of curl-free vector. Furthermore, the space of divergence-free vector is decomposed into the direct sum of two closed subspaces where $\pi$ acts irreducibly. In [6], G. Battle and P. Federbush constructed the divergence-free vector wavelets. Furthermore, K. Urban constructed the curl-free vector wavelet and presented the relationship between the space of divergence-free vector and the space of curl-free vector in terms of these wavelets [49]. Moreover, these wavelets have been adapted to the Navier-Stokes equation. We note that the symmetry of the vector fields in terms of the action of SO(3) is not considered at all in the works [6] and [48]. Therefore, our result (Proposition 13) may bring new insights to the study of such divergence-free or curl-free vector fields.

When $m = 0$, that is, $\sigma_0$ is trivial, then an admissible vector (it exists) associated to the 3-dimensional similitude group is an example of Glimore-Perelomov coherent states. When $m \geq 1$, an admissible vector associated to $\pi$ in (4) is closely related to the vector coherent states, which were introduced in [51].

Now, the wavelet transform is naturally extended to distributions as well as the Fourier transform. For example, using the Fourier transform, the wavelet transform associated to $\pi$ in (4) is described as

$$W_{f_0} f(x, A, c)) = \int_{\mathbb{R}^3} e^{2\pi i x \cdot \xi} \left\langle \widehat{f}(\xi), c^{\frac{3}{2}} \sigma_m(A) \widehat{f_0}(cA^{-1}\xi) \right\rangle d\xi.$$

This form is interpreted as a pseudo-differential operator with symbol

$$c^{\frac{n}{2}} \sigma_m(A) \widehat{f_0}(cA^{-1}\xi).$$

Hence wavelet theory may be developed in terms of pseudo-differential operator. In fact, when $m = 0$, the wavefront set of tempered distribution in the sense of Hörmander associated to the similitude group is characterized in [46], and the wavefront set for shearlet group were studied in [28, 39]. Moreover, in [18], J. Fell, H. Führ, and F. Voigtlaender characterized the wavefront set associated to $\mathbb{R}^n \rtimes H$, where $H$ is a closed subgroup of $GL(n, \mathbb{R})$. It will be an interesting subject in our future study to consider the wavefront set in our setting of vector-valued wavelets.

An admissible vector $f_0$ associated to $\pi$ in (4) is often called the vector-valued wavelet. A number of approaches to vector-valued wavelet transforms exist in the literature [7, 31, 33]. A vector-valued wavelet is applied to digitized stereo and color images, and wireless multiuser communication. Moreover, sampling of vector and tensor fields in medical imaging or geophysics give rise to true vector-valued wavelet.

## Organization of this thesis

This paper is organized as follows. In Section 1, we recall some fundamental concepts about induced representation of a locally compact group, the Fourier analysis of a locally compact abelian group, and continuous wavelet transform for a locally compact group.

In Section 2, we consider the unitary representation $\pi$ in (2) of the semidirect product group with a commutative closed normal subgroup. In general, the representation $\pi$ in (2) is not irreducible. Therefore $\pi$ is decomposed into a direct integral of irreducible unitary representations. The decomposition of $\pi$ is connected with the decomposition of $\sigma$ restricted to some subgroups of $H$. In this paper, we only consider the case where $\pi$ is decomposed into a discrete sum of irreducible unitary representations. Then the decomposition of $\sigma$ restricted to subgroups of $H$ have to be discrete. In general, in the representation theory of locally compact group, a basic problem is so-called a blanching law, that is, decomposition of the restrictions of a representation to subgroups. Recently, T. Kobayashi found many new examples of discrete branching law [37, 38]. One of the main task is to decompose $\pi$ into a direct sum of irreducible representations. To do so, we make use of the theory of induced representations. Actually, the representation $\pi$ is equivalent to the representation induced by $\sigma$. Since square-integrability of the irreducible unitary representations of $G$ depends only on equivalence classes, we can consider a convenient realization of the induced representations. The representation induced from trivial representation of $H$ is called a quasi-regular representation that acts on the space of square-integrable scalar-valued functions. In this case, continuous wavelet transforms arising from $\pi$ have been developed in various directions[1, 13, 14, 17, 34, 35].

In Section 3, we apply our method to the unitary representation of the similitude group $\mathbb{R}^n \rtimes (SO(n) \times \mathbb{R}_+)$. In the case of $n = 3$, we construct an admissible vector explicitly in the space of vector-valued square integrable functions. We expect the formulas to be useful for analysis of vector fields on $\mathbb{R}^3$, especially for divergence-free or curl-free vector fields. Note that, all irreducible unitary representation $\sigma$ of $SO(3)$ on $\mathbb{C}^{2m+1}$ is of class 1, and decomposed into 1-dimensional representations of $SO(2)$ as $\sigma = \bigoplus_{\alpha=-m}^{m} \rho_\alpha$ (see Section 3.2). Since our construction of an explicit admissible vector $f_0$ involves the Fourier transform, we need to find a function whose Fourier transform can be computable. For this, we investigate closely the matrix element of $\sigma$ expressed in terms of the Jacobi functions. Then we carry out the computation of the Fourier transform by applying differential operator to the Fourier transform of a radical function, which is computed by using a new formula [27].

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# 1 Preliminaries

## 1.1 Induced representation of locally compact group

In this section, we recall briefly the theory of induced representations of locally compact groups [19, 36]. Let $G$ be a locally compact group with a left Haar measure $\mu_G$. We denote by $\Delta_G$ the modular functions of $G$. Namely, the function $\Delta_G$ is a continuous homomorphism from $G$ to $\mathbb{R}_+$ such that $d\mu_G(gg^{'}) = \Delta_G(g^{'})d\mu_G(g)$ $(g, g^{'} \in G)$. The following construction of an induced representation of $G$ is due to Blattner [9]. Let $H$ be a closed subgroup of $G$ and $\sigma$ a unitary representation of $H$ defined on the Hilbert space $\mathcal{H}_\sigma$. We denote by $\langle \cdot, \cdot \rangle_\sigma$ and $\|\cdot\|_\sigma$ the inner product and the norm on $\mathcal{H}_\sigma$, respectively. Let $\mathcal{L}_\sigma$ be the space of continuous functions $F : G \to \mathcal{H}_\sigma$ such that

$$F(gh) = (\frac{\Delta_H(h)}{\Delta_G(h)})^{\frac{1}{2}}\sigma(h)^{-1}F(g) \quad (g \in G, h \in H),$$

$q(\text{supp } F)$ is compact, where $q$ is the canonical quotient map $G \ni g \mapsto gH \in G/H$. We denote by $C_c(G)$ the space of continuous functions of compact support on $G$. In order to define the inner product on $\mathcal{L}_\sigma$, we define the canonical map $P : C_c(G) \to C_c(G/H)$ by

$$P\phi(gH) = \int_H \phi(gh)d\mu_H(h).$$

This map is well-defined and supp $P\phi \subset q(\text{supp } \phi)$. For $F \in \mathcal{L}_\sigma$, we define the map $p_F$ from $C_c(G/H)$ to $\mathbb{C}$ by

$$p_F(P\phi) = \int_G \|F(g)\|_\sigma^2 \phi(g)d\mu_G(g) \quad (\phi \in C_c(G)).$$

Then $p_F$ is a well-defined positive linear functional and there exists a positive Radon measure $\mu_F$ on $G/H$ such that

$$\int_{G/H} P\phi(gH)d\mu_F(gH) = p_F(P\phi).$$

If supp $\phi \cap$ supp $F = \emptyset$, then the right-hand side equals zero, so supp $\mu_F$ is contained in $q(\text{supp } F)$. Therefore $\mu_F$ is compact. In particular, $\mu_F(G/H) < \infty$. In the same way, the pair of $F$ and $F^{'}$ defines a complex Radon measure $\mu_{F,F^{'}}$ on $G/H$. We define

$$\left\langle F, F^{'} \right\rangle = \mu_{F,F^{'}}(G/H). \tag{5}$$

Then $\left\langle F, F^{'} \right\rangle$ is an inner product on $\mathcal{L}_\sigma$ [19] and we define

$$\|F\|^2 = \langle F, F \rangle = \mu_F(G/H) \quad (F \in \mathcal{L}_\sigma).$$

**Proposition 1** ([36, Proposition 2.20])**.** *For $F, F^{'} \in \mathcal{L}_\sigma$, there exists $\phi \in C_c(G/H)$ such that $P\phi = 1$ on $q(\text{supp } F) \cup q(\text{supp } F^{'})$. Then we have*

$$\left\langle F, F^{'} \right\rangle = \int_G \left\langle F(g), F^{'}(g) \right\rangle_\sigma \phi(g) d\mu_G(g).$$

For $g \in G$ we define the operator $\Pi(g)$ on $\mathcal{L}_\sigma$ by

$$\Pi(g)F(g^{'}) = F(g^{-1}g^{'}).$$

Then $\Pi(g)$ is a bijection on $\mathcal{L}_\sigma$, and

$$\|\Pi(g)F\|_\sigma = \|F\|_\sigma.$$

In fact, we have

$$\begin{aligned}
\int_{G/H} P\phi(g^{'}H) d\mu_{\Pi(g)F}(g^{'}H) &= \int_G \|\Pi(g)F(g^{'})\|^2 \phi(g^{'}) d\mu_G(g^{'}) \\
&= \int_G \|F(g^{'})\|^2 L_{g^{-1}}\phi(g^{'}) d\mu_G(g^{'}) \\
&= \int_{G/H} P(L_{g^{-1}}\phi) d\mu_F(g^{'}H) \\
&= \int_{G/H} L_{g^{-1}} P(\phi)(g^{'}H) d\mu_F(g^{'}H),
\end{aligned}$$

where $L$ is the left translation operator and the fourth equality follows from the commutativity of $P$ and $L$. Therefore we have

$$\begin{aligned}
\|\Pi(g)F\|^2 &= \mu_{\Pi(g)F}(G/H) \\
&= \mu_F(g^{-1}G/H) \\
&= \mu_F(G/H) \\
&= \|F\|^2.
\end{aligned}$$

We denote by $\widetilde{\mathcal{L}_\sigma}$ the Hilbert completion of $\mathcal{L}_\sigma$ with respect to the inner product (5). The operator $\Pi(g)$ extends a unitary operator on $\widetilde{\mathcal{L}_\sigma}$. The resulting unitary representation is called *the induced representation* by $\sigma$. We denote it by $\text{Ind}_H^G \sigma$.

**Proposition 2** ([23, VI, Theorem 15]). *The Hilbert space $\mathcal{L}_\sigma$ of the induced representation $Ind_H^G\sigma$ can be identified with a subspace of $L^2(G, \mathcal{H}_\sigma)$.*

The following proposition states fundamental properties of induced representation.

**Proposition 3** ([19, Proposition 6.9]). *Let $H$ be a closed subgroup of $G$. If $\sigma$ and $\sigma^{'}$ are equivalent unitary representations of $H$, then $Ind_H^G\sigma$ and $Ind_H^G\sigma'$ are equivalent representations of $G$. If $\{\sigma_\beta\}$ is any family of unitary representations of $H$, then $\mathrm{Ind}_H^G(\bigoplus \sigma_\beta)$ is equivalent to $\bigoplus \mathrm{Ind}_H^G\sigma_\beta$.*

## 1.2 Our setting and the Mackey machine

We shall assume that $G$ is the semidirect product of a locally compact abelian group $N$ and a closed subgroup $H$ of $\mathrm{Aut}(N)$. Since $N$ is commutative, the irreducible unitary representations of $N$ are all one-dimensional. We define

$$\widehat{N} := \{\nu : G \to \mathbb{T} : \nu \text{ is a continuous homomorphism}\}.$$

Then $\widehat{N}$ is called the *unitary dual* of $N$. Since $G$ acts on $N$ by conjugation, thai is, $gn = gng^{-1}$, the dual action of $G$ on $\widehat{N}$ is defined by

$$g \cdot \nu(n) = \nu(g^{-1}ng) \quad (g \in G, \nu \in \widehat{N}, n \in N).$$

For each $\nu \in \widehat{N}$, we denote by $G_\nu$ the stabilizer of $\nu$, that is,

$$G_\nu = \{g \in G \ ; \ g \cdot \nu = \nu\},$$

which is a closed subgroup of $G$. We denote by $\mathcal{O}_\nu$ the $G$-orbit in $\widehat{N}$ through $\nu$ :

$$\mathcal{O}_\nu = \{g \cdot \nu, \ \ g \in G\}.$$

We define $H_\nu = G_\nu \cap H$. Since $N \subset G_\nu$, we have $G_\nu = N \rtimes H_\nu$. If $\nu \in \widehat{N}$ and $\rho$ is an irreducible representation of $H_\nu$, we define a unitary representation $\nu \otimes \rho$ of $G_\nu$ by

$$(\nu \otimes \rho)(n, h) = \nu(n)\rho(h) \quad (n \in N, h \in H_\nu).$$

Indeed, $\nu \otimes \rho$ is a representation because

$$\begin{aligned}(\nu \otimes \rho)((n,h)(n^{'},h^{'})) &= \nu(nhn^{'})\rho(hh^{'}) \\ &= \nu(n)\nu(hn')\rho(h)\rho(h') \\ &= \nu(n)\rho(h)\nu(n^{'})\rho(h^{'}) \\ &= (\nu \otimes \rho)(n,h)(\nu \otimes \rho)(n^{'},h^{'})\end{aligned}$$

where the third equally follows from $h \in H_\nu$. The following theorem tells us a condition where the induced representation $\mathrm{Ind}_{G_\nu}^G \nu \otimes \rho$ is irreducible.

**Theorem 1** ([19, Theorem 6.39]). *Suppose for each $\nu \in \widehat{N}$ the map $G/G_\nu \ni gG_\nu \mapsto g \cdot \nu \in \mathcal{O}_\nu$ is a homeomorphism. If $\nu \in \widehat{N}$ and $\rho$ is an irreducible unitary representation of $H_\nu$, then $\mathrm{Ind}_{G_\nu}^G \nu \otimes \rho$ is an irreducible unitary representation of $G$. Moreover, $\mathrm{Ind}_{G_\nu}^G \nu \otimes \rho$ and $\mathrm{Ind}_{G_{\nu'}}^G \nu' \otimes \rho'$ are equivalent if and only if $\nu$ and $\nu'$ belong to the same orbit, say $\nu' = g \cdot \nu$. In this case $h \to \rho(h)$ and $h \to \rho'(g^{-1}hg)$ are equivalent representation of $H_\nu$.*

### 1.3 Plancherel measure for a locally compact abelian group

Let $N$ be a locally compact abelian group and $\widehat{N}$ the unitary dual of $N$. Since $N$ is abelian, a measure $\mu_N$ is both left and right invariant. We define the convolution on $L^1(N)$ by

$$f * f^{'}(n) = \int_N f(n')f^{'}(n^{'-1}n)d\mu_N(n'),$$

and involution on $L^1(N)$ by

$$f^*(n) = \overline{f(n^{-1})}$$

for $f, f^{'} \in L^1(N)$. Then $L^1(N)$ is a commutative Banach $*$-algebra, called $L^1$ *group algebra*. We identify $\widehat{N}$ with the spectrum of $L^1(N)$ and Gelfand theory of spectra can be applied to harmonic analysis on the group $N$ [19]. We define *the Fourier transform* $\mathcal{F}$ of $L^1(N)$ by

$$\mathcal{F}f(\nu) = \widehat{f}(\nu) = \int_N \overline{\nu(n)} f(n) d\mu_N(n).$$

Our goal in this subsection is to establish the reconstruction of the function in terms of characters. We denote by $M(N)$ the space of complex Radon measures on $N$. Then $M(N)$ is a Banach algebra with a unit. We can identify $L^1(N)$ with a subalgebra of $M(N)$, so the Fourier transform can be extend to complex Radon measures on $N$.

**Proposition 4** ([52, 1.3.3]). *If $\mu \in M(N)$, we can define the bounded continuous function $\widehat{\mu}$ on $\widehat{N}$ by*

$$\widehat{\mu}(\nu) = \int_N \overline{\nu(n)} d\mu(n) \quad (\nu \in \widehat{N}).$$

*$\widehat{\mu}$ is called the Fourier-Stieltjes transform of $\mu$.*

**Definition 1** ([19, p.76]). *A function $\phi \in L^\infty(N)$ is said to be of positive type if for any $f \in L^1(N)$,*

$$\int_N (f^* * f)(n)\phi(n)d\mu_N(n) \geq 0.$$

We denote by $\mathcal{P}(N)$ the set of all continuous functions of positive type on $N$. Since $\widehat{N}$ is a locally compact group, it possess a Haar measure. For $\mu \in M(\widehat{N})$, we define the bounded continuous functions $\phi_\mu$ on $N$ by

$$\phi_\mu(n) = \int_{\widehat{N}} \nu(n)d\mu(\nu) \quad (n \in N).$$

**Proposition 5** ([19, Theorem 4.18], [52, Theorem 1.4.3]). *A function $\phi$ is an element of $\mathcal{P}(N)$ if and only if there exists a positive measure $\mu \in M(\widehat{N})$ such that $\phi = \phi_\mu$.*

Proposition 5 is called Bochner's Theorem. We set $\mathcal{B}(N) = \{\phi_\mu : \mu \in M(\widehat{N})\}$. By Proposition 5, we obtain the following proposition, Fourier inversion formula.

**Proposition 6** ([19, Theorem 4.21], [52, 1.5.1]). *If $f \in \mathcal{B}(N) \cap L^1(N)$, then $\widehat{f} \in L^1(\widehat{N})$. If the Haar measure $\mu_{\widehat{N}}$ on $\widehat{N}$ on is suitably normalized relative to the given Haar measure $\mu_N$ on $N$, we have the formula*

$$f(n) = \int_{\widehat{N}} \nu(n)\widehat{f}(\nu)d\mu_{\widehat{N}}(\nu) \quad (f \in \mathcal{B}(N) \cap L^1(N)).$$

Proposition 6 tells us that, when a Haar measure $\mu_N$ on $N$ is given, the Haar measure $\mu_{\widehat{N}}$ on $\widehat{N}$ is uniquely determined. The measure $\mu(\widehat{N})$ is called *the Plancherel measure* on $\widehat{N}$. We introduce the fundamental theorem in the $L^2$ theory of the Fourier transform, called the Plancherel formula.

**Theorem 2** ([19, Theorem 4.25], [52, 1.6.1]). *The Fourier transform, restricted to $L^1(N) \cap L^2(N)$, extends uniquely to a unitary isomorphism from $L^2(N)$ onto $L^2(\widehat{N})$. Namely, we have*

$$\int_N |f(n)|^2 d\mu_N(n) = \int_{\widehat{N}} |\widehat{f}(\nu)|^2 d\mu_{\widehat{N}}(\nu). \tag{6}$$

## 1.4 Continuous wavelet transform associated to irreducible unitary representations

In this section, we recall basic notions about the wavelet transform associated to an irreducible unitary representation of a locally compact group. Let $G$ be a locally compact group and $\pi$ a unitary representation of $G$ defined on a complex separable Hilbert space $\mathcal{H}_\pi$. For a nonzero vector $\varphi \in \mathcal{H}_\pi$ we define the map $W_\varphi$ from $\mathcal{H}_\pi$ to the space $C(G)$ of continuous functions on $G$ by

$$W_\varphi\psi(g) = \langle \psi, \pi(g)\varphi \rangle \quad (\psi \in \mathcal{H}_\pi, g \in G).$$

**Definition 2.** *A nonzero vector $\varphi \in \mathcal{H}_\pi$ is said to be admissible if the image of $W_\varphi$ is contained in $L^2(G)$, that is,*

$$\int_G |W_\varphi\psi(g)|^2 d\mu_G(g) < \infty \quad (\psi \in L^2(G)).$$

**Definition 3.** *A unitary representation $\pi$ of $G$ is said to be square integrable representation if $\pi$ is irreducible and there exists at least one admissible vector in $\mathcal{H}_\pi$.*

**Theorem 3** ([29, Theorem 3.1])**.** *Suppose $\pi$ is a square integrable representation of $G$ defined on $\mathcal{H}_\pi$. There exists a unique positive self-adjoint operator $C$ whose domain coincides with the set of admissible vectors such that for any admissible vectors $\varphi_1$ and $\varphi_2$,*

$$\int_G \langle W_{\varphi_1}\psi_1(g), W_{\varphi_2}\psi_2(g)\rangle \, d\mu_G(g) = \langle \psi_1, \psi_2\rangle \, \langle C\varphi_2, C\varphi_1\rangle$$

*for $g \in G$ and $\psi_1, \psi_2 \in \mathcal{H}_\pi$.*

The self-adjoint operator $C$ is called the Duflo-Moore operator associated to $\pi$, while $C^{-2}$ is called the formal degree of $\pi$. If G is compact and $\mu_G$ is normalized in such a way that $\mu_G(G) = 1$, then $C^{-2}$ equals $(\dim\mathcal{H}_\pi)\mathrm{id}_{\mathcal{H}_\pi}$. If $G$ is unimodular, then $C$ equals a multiple of identity. Applying $\varphi_1 = \varphi_2 = \psi_1 = \psi_2 = \varphi$ in Theorem 3, we see that $\varphi \in \mathcal{H}_\pi$ is admissible if and only if

$$C_\varphi := \frac{1}{\langle \varphi, \varphi\rangle} \int_G |W_\varphi\varphi(g)|^2 d\mu_G(g) < \infty,$$

because the right-hand side equals $\langle C\varphi, C\varphi\rangle$. We consider mainly $\varphi$ such that $C_\varphi = 1$. In this case, the map $W_\varphi$ is isometry. Then for any $\psi \in \mathcal{H}_\pi$

we have

$$\psi = \int_G W_\varphi \psi(g) \pi(g) \varphi d\mu_G(g)$$

in the weak sense. Indeed, for any $\psi_1 \in \mathcal{H}_\pi$, we have

$$\begin{aligned}
\langle \psi, \psi_1 \rangle &= \langle W_\varphi \psi, W_\varphi \psi_1 \rangle \\
&= \int_G W_\varphi \psi(g) \overline{W_\varphi \psi_1(g)} d\mu_G(g) \\
&= \int_G W_\varphi \psi(g) \langle \pi(g)\varphi, \psi_1 \rangle \, d\mu_G(g) \\
&= \left\langle \int_G W_\varphi \psi(g) \pi(g) \varphi d\mu_G(g), \psi_1 \right\rangle .
\end{aligned}$$

In general, if $W_\varphi$ is isometry, then $W_\varphi$ is called a *continuous wavelet transform.* As is discussed in the next section, this definition makes sense for non-irreducible unitary representations.

# 2 Construction of the wavelet transforms associated to reducible unitary representations

## 2.1 Continuous wavelet transform for reducible unitary representations

In subsection 1.3, a unitary representation of $G$ is assumed to be irreducible. This assumption can be weakened. Before we discuss it, we shall recall some definition and basic properties of the representation theory of locally compact groups.

**Definition 4.** *If $\pi_\alpha$ and $\pi_\beta$ are unitary representations of $G$, a bounded linear map $T : \mathcal{H}_{\pi_\alpha} \to \mathcal{H}_{\pi_\beta}$ is called an intertwining operator if for all $g \in G$, $T\pi_\alpha(g) = \pi_\beta(g)T$ The set of all such operators is denoted by $\mathcal{C}(\pi_\alpha, \pi_\beta)$. If $\mathcal{C}(\pi_\alpha, \pi_\beta)$ contains a unitary operator, then $\pi_\alpha$ and $\pi_\beta$ are called (unitarily) equivalent.*

**Proposition 7** (The Schur orthogonality relation)**.** *Let $\pi_\alpha$ and $\pi_\beta$ be irreducible unitary representations of $G$. If $\pi_\alpha$ and $\pi_\beta$ are inequivalent, then we have*

$$\int_G \langle \psi_\alpha, \pi_\alpha(g)\varphi_\alpha \rangle \overline{\langle \psi_\beta, \pi_\alpha(g)\varphi_\beta \rangle} d\mu(g) = 0.$$

*for any $\psi_\alpha, \varphi_\alpha \in \mathcal{H}_{\pi_\alpha}$ and $\psi_\beta, \varphi_\beta \in \mathcal{H}_{\pi_\beta}$.*

**Proposition 8.** *Let $\pi_\alpha$ and $\pi_\beta$ be unitarily equivalent, square integrable representations of $G$ and $U$ a unitary intertwining operator for $\pi_\alpha$ and $\pi_\beta$. For admissible vectors $\varphi_\alpha \in \mathcal{H}_{\pi_\alpha}$ and $\varphi_\beta \in \mathcal{H}_{\pi_\beta}$ we have*

$$\int_G \langle \psi_\alpha, \pi_\alpha(g)\varphi_\alpha \rangle \overline{\langle \psi_\beta, \pi_\beta(g)\varphi_\beta \rangle} d\mu(g) = \langle U\psi_\alpha, \psi_\beta \rangle \left\langle C_{\pi_\beta}\varphi_\beta, C_{\pi_\beta} U\varphi_\alpha \right\rangle, \quad (7)$$

*where $C_{\pi_\beta}$ is the Duflo-Moore operator associated to $\pi_\beta$.*

Clearly, if $\varphi_\alpha \in \mathcal{H}_{\pi_\alpha}$ is an admissible vector, then $U\varphi_\alpha \in \mathcal{H}_{\pi_\beta}$ is admissible.

*Proof.* By direct computation, we have

$$\begin{aligned} &\int_G \langle \psi_1, \pi_1(g)\varphi_1 \rangle \overline{\langle \psi_2, \pi_2(g)\varphi_2 \rangle} d\mu(g) \\ &= \int_G \left\langle \psi_1, U^{-1}\pi_2(g)U\varphi_1 \right\rangle \overline{\langle \psi_2, \pi_2(g)\varphi_2 \rangle} d\mu(g) \\ &= \int_G \langle U\psi_1, \pi_2(g)U\varphi_1 \rangle \overline{\langle \psi_2, \pi_2(g)\varphi_2 \rangle} d\mu(g) \\ &= \langle U\psi_1, \psi_2 \rangle \langle C\varphi_2, CU\varphi_1 \rangle . \end{aligned}$$

□

**Theorem 4** ([22, Theorem 2.31])**.** *Let $\pi = \bigoplus_{\alpha\in\Lambda} \pi_\alpha$, where each $\pi_\alpha$ is a square-integrable representation. Denote by $P_\alpha$ the projection onto $\mathcal{H}_{\pi_\alpha}$, and by $C_{\pi_\alpha}$ the associated Duflo-Moore operators. Let $\varphi \in \mathcal{H}_\pi$. Then the following are equivalent:*

**(1)** *The map $W_\varphi : \mathcal{H}_\pi \to L^2(G)$ is isometry.*

**(2)** *The vector $\varphi_\alpha := P_\alpha\varphi$ is admissible such that $C_{\varphi_\alpha} = 1$ for any $\alpha$ and for all $\alpha, \beta$ with $\pi_\alpha \simeq \pi_\beta$*

$$\left\langle C_{\pi_\beta} U_{\alpha,\beta}\varphi_\alpha, C_{\pi_\beta}\varphi_\beta \right\rangle = 0, \quad (8)$$

*where $U_{\alpha,\beta}$ is an intertwining operator from $\mathcal{H}_{\pi_\alpha}$ to $\mathcal{H}_{\pi_\beta}$.*

*Proof.* Suppose that $W_\varphi$ is isometry. Then for any $\psi \in \mathcal{H}_\pi$, we have

$$\begin{aligned} & \int |W_\varphi f(g)|^2 d\mu(g) = \int |\langle \psi, \pi(g)\varphi\rangle|^2 d\mu(g) \\ = & \sum_\alpha \int |\langle P_\alpha \psi, \pi(g) P_\alpha \varphi\rangle|^2 d\mu(g) \\ + & \sum_{\substack{\alpha\neq\beta \\ \pi_\alpha \simeq \pi_\beta}} \int \langle P_\alpha\psi, \pi(g)P_\alpha\varphi\rangle \overline{\langle P_\beta\psi, \pi(g)P_\beta\varphi\rangle} d\mu(g) \\ + & \sum_{\substack{\alpha\neq\beta \\ \pi_\alpha \not\simeq \pi_\beta}} \int \langle P_\alpha\psi, \pi(g)P_\alpha\varphi\rangle \overline{\langle P_\beta\psi, \pi(g)P_\beta\varphi\rangle} d\mu(g). \end{aligned}$$

The first term equals to $\sum C_{\varphi_\alpha} \langle \psi_\alpha, \psi_\alpha\rangle = \langle \psi, \psi\rangle$ since $C_{\varphi_\alpha} = 1$ for any $\alpha$. The last terms equals to 0 by the Schur orthogonal relation. Therefore, the second term should be equal to 0. This is verified because for $\alpha, \beta$ such that $\pi_\alpha \simeq \pi_\beta$, we have (8). □

Theorem 4 implies that if $W_\varphi$ is isometry, then $\mathrm{mult}(\pi, \pi_\alpha) \leq \mathrm{dim}\mathcal{H}_{\pi_\alpha}$ holds for each $\alpha$.

## 2.2 Construction of the wavelet transform for reducible unitary representations of the semidirect product group.

In this Subsection, we construct the wavelet transforms associated to certain unitary representation $\pi$ defined below, which is not necessarily irreducible, after giving an irreducible decomposition of $\pi$. Let $G$ be the semidirect product group $N \rtimes H$ of a locally compact abelian group $N$ and a closed subgroup $H$ of $\mathrm{Aut}(N)$. An element $g \in G$ is written as $g = (n, h)$ with $n \in N$ and $h \in H$. The group law is given by

$$(n, h)(n', h') = (nhn', hh') \quad (n, n' \in N,\ h, h' \in H).$$

Let $d\mu_H(h)$ denote a left Haar measure of $H$ and $dn$ a Haar measure on $N$. We define the measure $d\mu_G$ of $G$ by

$$d\mu_G(g) = \delta(h)^{-1} dn d\mu_H(h), \quad g = (n, h) \in N \rtimes H,$$

where $\delta$ is the map from $H$ to $\mathbb{R}_+$ such that $d(hn) = \delta(h)dn$. Then $d\mu_G$ is a left Haar measure of $G$. Indeed, we have for $g = (n, h)$ and $g^{'} = (n', h')$

$$\begin{aligned} d\mu_G(g^{'}g) &= \delta(hh^{'})^{-1}d(n^{'}h^{'}n)d\mu_H(h^{'}h) \\ &= \delta(hh^{'})^{-1}\delta(h^{'})^{-1}dnd\mu_H(h^{'}h) \\ &= \delta(h)^{-1}dnd\mu_H(h) \\ &= d\mu_G(g). \end{aligned}$$

Let $\sigma$ be a unitary representation of $H$ on a Hilbert space $\mathcal{H}_\sigma$. We define the unitary representation $\pi$ of $G$ on the space $L^2(N, \mathcal{H}_\sigma)$ of $\mathcal{H}_\sigma$-valued square integrable functions on $N$ by

$$\pi(n, h)f(n_0) = \delta(h)^{-\frac{1}{2}}\sigma(h)f(h^{-1}(n^{-1}n_0)) \quad (n, n_0 \in N, h \in H).$$

This $\pi$ is equivalent to $\mathrm{Ind}_H^G\sigma$, which is not necessarily irreducible in general. For the study of irreducible subrepresentations of $\pi$, it is useful to introduce another unitary representation which is equivalent to $\pi$. We define the Fourier transform $\mathcal{F}$ from $L^2(N, \mathcal{H}_\sigma) = L^2(N) \otimes \mathcal{H}_\sigma$ onto $L^2(\widehat{N}, \mathcal{H}_\sigma) = L^2(\widehat{N}) \otimes \mathcal{H}_\sigma$ by

$$\mathcal{F}(f \otimes v) = \mathcal{F}f \otimes v \quad (f \in L^2(N), v \in \mathcal{H}_\sigma).$$

Taking the conjugate of $\pi$ by $\mathcal{F}$, we obtain the unitary representation $\widehat{\pi} = \mathcal{F} \circ \pi \circ \mathcal{F}^{-1}$ on $L^2(\widehat{N}, \mathcal{H}_\sigma)$.

**Lemma 1.** *The representation $\widehat{\pi}$ is described as*

$$\widehat{\pi}(n, h)\varphi(\nu) = \nu(n)^{-1}\delta(h)^{\frac{1}{2}}\sigma(h)\varphi(h^{-1} \cdot \nu) \quad (\varphi \in L^2(\widehat{N}, \mathcal{H}_\sigma)).$$

*Proof.* For any $f \in L^2(N, \mathcal{H}_\pi)$, we have

$$\begin{aligned} \mathcal{F}(\pi(n, h)f)(\nu) &= \int_N \pi(n, h)f(n')\overline{\nu(n')}dn' \\ &= \delta(h)^{-\frac{1}{2}} \int_N \sigma(h)f(h^{-1}(n^{-1}n'))\overline{\nu(n')}dn' \\ &= \delta(h)^{\frac{1}{2}}\nu(n)^{-1} \int_N \sigma(h)f(n')\overline{\nu(hn')}dn' \\ &= \delta(h)^{\frac{1}{2}}\nu(n)^{-1}\sigma(h)\mathcal{F}f(h^{-1} \cdot \nu). \end{aligned}$$

If we put $f = \mathcal{F}^{-1}\varphi$ ($\varphi \in L^2(\widehat{N}, \mathcal{H}_\sigma)$), we prove the statement. □

Our aim is to investigate when the image of the map $W_{f_0} : \mathcal{H}_\pi \ni f \mapsto \langle f, \pi(g) f_0 \rangle \in C(G)$ is in $L^2(G)$ for some nonzero vector $f_0 \in \mathcal{H}_\pi$. We assume the following conditions:

(A1) There exist elements $\nu_k$ $(k \in K)$ of $\widehat{N}$, indexed by some set $K$, such that $\mathcal{O}_{\nu_k} \cap \mathcal{O}_{\nu_{k'}} = \emptyset$ $(k \neq k')$, and $\mu(\widehat{N} \setminus \bigsqcup_{k \in K} \mathcal{O}_{\nu_k}) = 0$.

(A2) For each $\nu \in \widehat{N}$, the map $G/G_\nu \ni gG_\nu \mapsto g \cdot \nu \in \mathcal{O}_\nu$ is a homeomorphism.

(A3) For every $k \in K$, there exists an index set $\Lambda_k$ such that $\sigma|_{H_{\nu_k}} = \bigoplus_{\alpha \in \Lambda_k} \rho_\alpha (\rho_\alpha \in \widehat{H}_{\nu_k})$.

The assumption (A2) comes from the Macky theory in Section 1.1. The assumption (A1) and (A3) is essential in this paper.

We regard $L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$ as a subspace of $L^2(\widehat{N}, \mathcal{H}_\sigma)$ by zero extension. Thanks to Lemma 1, the space $L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$ is $G$-invariant. We denote by $\widehat{\pi}_k$ the subrepresentation $\widehat{\pi}|_{L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)}$. By the assumption (A1), we have $\widehat{\pi} = \bigoplus_{k \in K} \widehat{\pi}_k$.

**Proposition 9.** *The unitary representation $\widehat{\pi}_k$ is equivalent to* $\mathrm{Ind}_{G_{\nu_k}}^G \nu_k \otimes \sigma|_{H_{\nu_k}}$.

*Proof.* We denote by $\Pi_k$ the unitary representation $\mathrm{Ind}_{G_{\nu_k}}^G \nu_k \otimes \sigma|_{H_{\nu_k}}$. Since $\delta(h) = \Delta_G(h)^{-1} \Delta_H(h)$ $(h \in H)$ [4], as is seen in Subsection 2.1, $\Pi_k$ is the left-regular representation on the Hilbert space completion $\widetilde{\mathcal{L}}_{k,\sigma}$ of the space $\mathcal{L}_{k,\sigma}$ consisting of all the continuous functions $F : G \to \mathcal{H}_\sigma$ such that $q(\text{supp } F)$ is compact and for $n' \in N$ and $h' \in H_{\nu_k}$,

$$F((n,h)(n',h')) = \delta(h')^{-\frac{1}{2}} \nu_k(n')^{-1} \sigma(h')^{-1} F(n,h)$$

with the inner product

$$\langle F, F' \rangle = \int_{G/G_{\nu_k}} \langle F(g), F'(g) \rangle_\sigma \, \phi(g) d\mu_{G/G_{\nu_k}}(gG_{\nu_k}),$$

where $\phi \in C_c(G)$ such that $P\phi = 1$ on $q(\text{supp } F)$. We define the map $\Phi$ from $\mathcal{L}_{k,\sigma}$ to $C_c(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$ by

$$\Phi(F)(\nu) = \delta(h)^{\frac{1}{2}} \sigma(h) F(0,h) \quad (\nu = h \cdot \nu_k).$$

This map is well-defined. Indeed, if $h\cdot\nu_k = h^{'}\cdot\nu_k$, then there exists $h_0 \in H_{\nu_k}$ s.t. $h^{'} = hh_0$. Therefore we have

$$\begin{aligned}\delta(h^{'})^{\frac{1}{2}}\sigma(h^{'})F(0,h^{'}) &= \delta(hh_0)^{\frac{1}{2}}\sigma(hh_0)F(0,hh_0) \\ &= \delta(hh_0)^{\frac{1}{2}}\sigma(hh_0)\delta(h_0)^{-\frac{1}{2}}\sigma(h_0)^{-1}F(0,h) \\ &= \delta(h)^{\frac{1}{2}}\sigma(h)F(0,h).\end{aligned}$$

Since $q(\text{supp } F)$ is compact, assumption (A1) implies that supp $\Phi(F)$ is compact. The inverse map $\Phi^{-1}$ is given by

$$\Phi^{-1}\varphi(n,h) = \delta(h)^{-\frac{1}{2}}h\cdot\nu_k(n)^{-1}\sigma(h)^{-1}\varphi(h\cdot\nu_k).$$

Indeed, writing $F(n,h)$ for the right-hand above, we have

$$\begin{aligned}F((n,h)(n^{'},h^{'})) &= F(nhn^{'},hh^{'}) \\ &= \delta(hh^{'})^{-\frac{1}{2}}hh^{'}\cdot\nu_k(nhn^{'})^{-1}\sigma(hh^{'})^{-1}\varphi(hh^{'}\cdot\nu_k) \\ &= \delta(hh^{'})^{-\frac{1}{2}}h\cdot\nu_k(nhn^{'})^{-1}\sigma(hh^{'})^{-1}\varphi(h\cdot\nu_k) \\ &= \delta(h^{'})^{-\frac{1}{2}}\nu_k(n^{'})^{-1}\sigma(h^{'})^{-1}\delta(h)^{-\frac{1}{2}}h\cdot\nu_k(n)^{-1}\sigma(h)^{-1}\varphi(h\cdot\nu_k) \\ &= \delta(h^{'})^{-\frac{1}{2}}\nu_k(n^{'})^{-1}\sigma(h^{'})^{-1}F(n,h)\end{aligned}$$

where the fourth equality follows from $h^{'} \in H_{\nu_k}$. Since supp $\phi$ is compact, we see that $q(\text{supp } \Phi^{-1}\phi)$ is compact. The map $\Phi$ extends to a unitary operator from $\widetilde{\mathcal{L}}_{k,\sigma}$ onto $L^2(\mathcal{O}_{\nu_k},\mathcal{H}_\sigma)$. Therefore, it suffices to show that $\widehat{\pi}_k(n,h)\circ\Phi = \Phi\circ\Pi_k(n,h)$ for all $(n,h) \in G$. For any $F \in \mathcal{L}_{k,\sigma}$, we have

$$\widehat{\pi}_k(n,h)\circ\Phi F(\nu) = \nu(n)\delta(h)^{\frac{1}{2}}\sigma(h)\Phi(F)(h^{-1}\cdot\nu).$$

On the other hand, we have

$$\begin{aligned}\Phi\circ\Pi_k(n,h)F(\nu) &= \delta(h')^{\frac{1}{2}}\sigma(h')\Pi_k(n,h)F(0,h') \\ &= \delta(h')^{\frac{1}{2}}\sigma(h')F(h^{-1}n^{-1},h^{-1}h') \\ &= \delta(h')^{\frac{1}{2}}\sigma(h')\nu_k(h'^{-1}n^{-1})^{-1}F(0,h^{-1}h') \\ &= \delta(h')^{\frac{1}{2}}\sigma(h')h^{-1}\nu_k(h'^{-1}n) \\ &\times \delta(h^{-1}h')^{-\frac{1}{2}}\sigma(h^{-1}h')^{-1}\Phi(F)(h^{-1}h'\cdot\nu_k) \\ &= \delta(h)^{\frac{1}{2}}\nu(n)\sigma(h)\Phi(F)(h^{-1}\cdot\nu),\end{aligned}$$

where $\nu = h'\cdot\nu_k$. Therefore we see that $\Phi$ intertwines $\widehat{\pi}_k$ and $\Pi_k$. □

If $\sigma$ is trivial, by Theorem 1 and Proposition 9, all representations $\pi_k$ are irreducible and mutually inequivalent for $k \neq k'$. By Propositions 3 and 9, the unitary representation $\widehat{\pi}_k$ is equivalent to $\bigoplus_{\alpha\in\Lambda_k} \mathrm{Ind}_{G_{\nu_k}}^{G} \nu_k \otimes \rho_\alpha$. In conclusion, we obtaion the following theorem.

**Theorem 5.** *The irreducible decomposition of $\widehat{\pi}$ is given by*

$$\bigoplus_{k\in K} \bigoplus_{\alpha\in\Lambda_k} \mathrm{Ind}_{G_{\nu_k}}^{G} \nu_k \otimes \rho_\alpha.$$

Let us realize a representation space of $\mathrm{Ind}_{G_{\nu_k}}^{G} \nu_k \otimes \rho_\alpha$ as a subspace of $L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$. By the assumption (A3), the restriction $\sigma|_{H_{\nu_k}}$ is decomposed into $\bigoplus_{\alpha\in\Lambda_k} \rho_\alpha$. Therefore $\mathcal{H}_\sigma$ is a direct sum of irreducible $H_{\nu_k}$-invariant subspaces, that is,

$$\mathcal{H}_\sigma = \bigoplus_{\alpha\in\Lambda_k} \mathcal{H}_{\rho_\alpha}. \tag{9}$$

We recall the isomorphism $\Phi : \mathcal{L}_{k,\sigma} \to L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$ in the proof of Proposition 7, and define the invariant subspace $\mathcal{L}_{k,\sigma,\alpha}$ of $\mathcal{L}_{k,\sigma}$ by

$$\mathcal{L}_{k,\sigma,\alpha} = \{\varphi \in \mathcal{L}_{k,\sigma} \ ; \ \varphi(n,h) \in \mathcal{H}_{\rho_\alpha}, \ \text{a.a. } (n,h) \in G\}.$$

The Hilbert completion $\widetilde{\mathcal{L}}_{k,\sigma,\alpha}$ is the representation space of $\mathrm{Ind}_{G_{\nu_k}}^{G} \nu_k \otimes \rho_\alpha$. By (9), the space $\widetilde{\mathcal{L}}_{k,\sigma}$ is decomposed as $\bigoplus_{\alpha\in\Lambda_K} \widetilde{\mathcal{L}}_{k,\sigma,\alpha}$. Now we denote by $\mathcal{H}_{k,\sigma,\alpha}$ the subspace $\Phi(\widetilde{\mathcal{L}}_{k,\sigma,\alpha})$ of $L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$.

**Lemma 2.** *For any $\nu \in \mathcal{O}_{\nu_k}$, we define*

$$\mathcal{H}_{\alpha,\nu} = \sigma(h)\mathcal{H}_{\rho_\alpha},$$

*where $\nu = h \cdot \nu_k$ $(h \in H)$. Then $\mathcal{H}_{\alpha,\nu}$ is well-defined. Moreover $\mathcal{H}_{k,\sigma,\alpha}$ is described as*

$$\mathcal{H}_{k,\sigma,\alpha} = \{\varphi \in L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma) \ ; \ \varphi(\nu) \in \mathcal{H}_{\alpha,\nu} \ a.a. \ \nu\}.$$

*Proof.* For any element $\varphi \in \mathcal{H}_{k,\sigma,\alpha}$ there exists $F \in \widetilde{\mathcal{L}}_{k,\sigma,\alpha}$ such that $\varphi = \Phi(F)$. Then

$$\varphi(\nu) = \Phi(F)(\nu) = \delta(h)^{\frac{1}{2}}\sigma(h)\varphi(0,h) \in \sigma(h)\mathcal{H}_{\rho_\alpha},$$

so that we have

$$\mathcal{H}_{k,\sigma,\alpha} \subset \{\varphi \in L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma) \ ; \ \varphi(\nu) \in \mathcal{H}_{\alpha,\nu} \ a.a. \ \nu\}.$$

On the other hand, for any $\varphi \in L^2(\mathcal{O}_{\nu_k}, \mathcal{H}_\sigma)$ satisfying $\varphi(\nu) \in \mathcal{H}_{\alpha,\nu}$ *a.a.* $\nu$, we have

$$\Phi^{-1}\varphi(n,h) = \delta(h)^{-\frac{1}{2}} h \cdot \nu_k(n)\sigma(h)\varphi(h \cdot \nu_k) \in \mathcal{H}_{\rho_\alpha}.$$

Therefore we see that $\Phi^{-1}\varphi \in \widetilde{\mathcal{L}}_{k,\sigma,\rho}$, so that $\varphi \in \mathcal{H}_{k,\sigma,\alpha}$. □

**Theorem 6.** *The irreducible decomposition of* $L^2(\widehat{N}, \mathcal{H}_\sigma)$ *is given by*

$$\bigoplus_{k\in K} \bigoplus_{\alpha\in\Lambda_k} \mathcal{H}_{k,\sigma,\alpha}.$$

*Proof.* It follows from Theorem 5. □

In order to construct the continuous wavelet transform associated to $\widehat{\pi}$, we need to show that $\widehat{\pi}_{k,\alpha}$ is a square-integrable for any $k$ and $\alpha$.

**Proposition 10.** *The representation* $\widehat{\pi}_{k,\alpha}$ *has an admissible vector* $\varphi_{k,\alpha}$ *in* $\mathcal{H}_{k,\sigma,\alpha}$ *if and only if*

$$\int_H \left\langle \varphi_{k,\alpha}(h^{-1}\cdot\nu), \varphi_{k,\alpha}(h^{-1}\cdot\nu) \right\rangle d\mu(h) < \infty.$$

*Proof.* If $\pi_{k,\alpha}$ is square-integrable, then for some $\varphi_k \in \mathcal{H}_{k,\sigma,\alpha}$ we have

$$\begin{aligned} \infty &> \int |W_{\varphi_{k,\alpha}}\varphi_{k,\alpha}(g)|^2 d\mu(g) = \int_N \int_H |\left\langle \varphi_{k,\alpha}(\nu), \widehat{\pi}(g)\varphi_{k,\alpha}(\nu)\right\rangle|^2 d\mu(g) \\ &= \int_N \int_H |\left\langle \varphi_{k,\alpha}(\nu), \sigma(h)\varphi_{k,\alpha}(h^{-1}\cdot\nu)\right\rangle|^2 d\nu d\mu(h). \end{aligned}$$

The function $\sigma(h)\varphi_{k,\alpha}(h^{-1}\cdot\nu)$ does not depend on $\nu$. In fact, for a base point $\nu_k$, there exists $h_0 \in H$ such that $h_0^{-1}\cdot\nu_k = \nu$. Therefore, the right hand side equals

$$\begin{aligned} &\int_N \int_H |\left\langle \varphi_{k,\alpha}(\nu), \sigma(h_0 h)\varphi_{k,\alpha}(h^{-1}\cdot\nu_k)\right\rangle|^2 d\nu d\mu(h) \\ = \ &\langle \varphi_{k,\alpha}, \varphi_{k,\alpha}\rangle \int_H \left\langle \varphi_{k,\alpha}(h^{-1}\cdot\nu_k), \varphi_{k,\alpha}(h^{-1}\cdot\nu_k)\right\rangle d\mu(h). \end{aligned}$$

Hence it is proved. □

The square-integrability of $\widehat{\pi}_{k,\alpha}$ is equivalent to the square-integrability of $\mathrm{Ind}_{G_{\nu_k}}^G \nu_k \otimes \rho_\alpha$. The following theorem is useful in order to investigate whether such a representation is square-integrable.

**Theorem 7** ([4, Theorem 2]). *Let $\tau$ be an irreducible unitary representation of $G$ and $\mathcal{O}_\nu$ the associated orbit and an element $\rho$ of $\widehat{H}_\nu$ s.t. $\tau = \mathrm{Ind}_{G_\nu}^G \nu \otimes \rho$. The representation $\tau$ is square-integrable if and only if $\mu(\mathcal{O}_\nu) > 0$ and $\rho$ is square-integrable.*

Therefore, in order to consider the wavelet transform associated to $\pi$, in addition to the assumptions (A1), (A2) and (A3), we require the following assumptions:

(A4) For $k \in K$, $\mu(\mathcal{O}_{\nu_k}) > 0$.

(A5) For $\alpha \in \Lambda_k$, the representation $\rho_\alpha$ of $H_{\nu_k}$ is square-integrable

If $\sigma$ is trivial, the assumption (A5) is equivalent to

(A5)' For $k \in K$, the stabilizer $H_{\nu_k}$ is compact.

It immediately follows from Theorem 7. Now we give an explicit form for the formal degree of $\widehat{\pi}_{k,\alpha}$. Let us construct the wavelet transforms associated to $\pi$. We take an admissible vector $\varphi_{k,\alpha} \in \mathcal{H}_{k,\sigma,\alpha}$ such that $C_{\varphi_{k,\alpha}}$=1 for each $k$ and $\alpha$, and for $\alpha, \beta$ with $\widehat{\pi}_{k,\alpha} \simeq \widehat{\pi}_{k,\beta}$

$$\left\langle C_{\widehat{\pi}_{k,\beta}} U_{k,\alpha,\beta} \varphi_{k,\alpha}, C_{\widehat{\pi}_{k,\beta}} \varphi_{k,\beta} \right\rangle = 0,$$

where $C_{\widehat{\pi}_{k,\beta}}$ is the Doflo-Moore operator associated to $\widehat{\pi}_{k,\beta}$ and $U_{k,\alpha,\beta}$ is unitary intertwining operator for $\widehat{\pi}_{k,\alpha}$ and $\widehat{\pi}_{k,\beta}$. Moreover, we assume that

(A6) $\varphi = \sum_{k \in K} \sum_{\alpha \in \Lambda_k} \varphi_{k,\alpha}$ converges in $L^2(\widehat{N}, \mathcal{H}_\sigma)$.

We note that if $K$ and all $\Lambda_k$ ($k \in K$) are finite sets, (A6) is trivial.

**Theorem 8.** *Put $f_0 = \mathcal{F}^{-1}\varphi \in L^2(N, \mathcal{H}_\sigma)$. We can define the map $W_{f_0}$ from $L^2(N, \mathcal{H}_\sigma)$ to $L^2(G)$ by*

$$W_{f_0} f(g) = \langle f_1, \pi(g) f \rangle \quad (f \in L^2(N, \mathcal{H}_\sigma)).$$

*Then $W_{f_0}$ is isometry. For any $f \in L^2(N, \mathcal{H}_\sigma)$, we have*

$$f = \int_G W_{f_0} f(g) \pi(g) f_0 \, d\mu_G(g)$$

*in the weak sense.*

*Proof.* For any $f = \mathcal{F}^{-1}\phi \in L^2(N, \mathcal{H}_\sigma)$ ($\phi \in L^2(\widehat{N}, \mathcal{H}_\sigma)$), we have

$$\begin{aligned} \int_G |W_{f_0} f(g)|^2 d\mu_G(g) &= \int_G |\langle f, \pi(n,h) f_0 \rangle|^2 d\mu_G(g) \\ &= \int_G |\langle \phi, \widehat{\pi}(n,h) \varphi \rangle|^2 d\mu_G(g). \end{aligned}$$

By Proposition and the Schur orthogonality formula, the last term equals

$$\sum_{k\in K}\sum_{\alpha\in\Lambda_k}\int_G |\langle\phi_{k,\alpha},\widehat{\pi}(n,h)\varphi_{k,\alpha}\rangle|^2 d\mu_G(g)$$
$$+\sum_{k\in K}\sum_{\substack{\alpha,\beta\in\Lambda_k\\ \widehat{\pi}_{k,\alpha}\simeq\widehat{\pi}_{k,\beta}}}\int_G |\langle\phi_{k,\alpha},\widehat{\pi}(n,h)\varphi_{k,\alpha}\rangle|\overline{\langle\phi_{k,\beta},\widehat{\pi}(n,h)\varphi_{k,\beta}\rangle}d\mu_G(g).$$

where $\phi=\sum_{k\in K}\sum_{\alpha\in\Lambda_k}\phi_{k,\alpha}$ ($\phi_{k,\alpha}\in\mathcal{H}_{k,\sigma,\alpha}$). By the assumption, the second term equals 0. Theorem 3 tells us that the expression above equals

$$\sum_{k\in K}\sum_{\alpha\in\Lambda_k} C_{\varphi_{k,\alpha}}\langle\phi_{k,\alpha},\phi_{k,\alpha}\rangle=\langle\phi,\phi\rangle=\langle f,f\rangle$$

since $C_{\varphi_{k,\alpha}}=1$. Therefore we have

$$\int_G |W_{f_0}f(g)|^2 d\mu_G(g)=\langle f,f\rangle$$

for any $f\in L^2(N,\mathcal{H}_\sigma)$. Thus $W_{f_0}$ is isometry. The latter part is shown in the same argument as in Subsection 1.3. Hence, Theorem 8 is proved. □

# 3 Admissible vector associated to unitary representations of similitude group

## 3.1 The similitude group $\mathbb{R}^{\mathbf{n}}\rtimes(\mathbf{SO(n)}\times\mathbb{R}_+)$ with $\mathbf{n}\geq 4$

In this section, we consider the similitude group $\mathbb{R}^n\rtimes(SO(n)\times\mathbb{R}_+)$ with $n\geq 4$. Let $H^m(\mathbb{R}^n)$ be the space of harmonic polynomials on $\mathbb{R}^n$ of degree $m$. The dimension of $H^m(\mathbb{R}^n)$ is equal to $d(n,m)=\frac{(2m+n-2)(n+m-3)!}{(n-2)m!}$, that is, $H^m(\mathbb{R}^3)\simeq\mathbb{C}^{d(n,m)}$. Let $\sigma_{n,m}$ be the unitary representation of $SO(n)$ defined on $H^m(\mathbb{R}^n)$ by

$$\sigma_{n,m}(A)P(x)=P(A^{-1}x)\quad(P\in H^m(\mathbb{R}^n),A\in SO(n),x\in\mathbb{R}^n).$$

Then $\sigma_{n,m}$ is irreducible and of class 1 [53]. Let us construct the wavelet transform associated to $\pi$, which is defined on $L^2(\mathbb{R}^n,\mathbb{C}^{d(n,m)})$ by

$$\pi_{n,m}(x,A,c)f(y)=c^{-\frac{n}{2}}\sigma_{n,m}(A)f(c^{-1}A^{-1}(y-x))$$

for $x\in\mathbb{R}^n, A\in SO(n)$ and $c\in\mathbb{R}_+$. Let us apply the argument in Section 2 to our case with $N=\mathbb{R}^n$ and $H=SO(n)\times\mathbb{R}_+$. We identify $\nu\in\widehat{\mathbb{R}}^n$ with

$\xi \in \mathbb{R}^n$ by $\nu_\xi(x) = e^{2\pi i x\cdot\xi}$ $(x \in \mathbb{R}^n)$. It is easy to see that the action of $G$ on $\widehat{\mathbb{R}}^n \simeq \mathbb{R}^n$ is given by $(x, A, c)\cdot\nu_\xi = \nu_{c^{-1}\,{}^tA\xi}$ where ${}^tA$ is the transpose of $A$. Therefore the orbits are $\mathbb{R}^n\setminus\{0\}$ and $\{0\}$. The latter orbit has zero Lebesgue measure, so that representations of $G$ whose associated orbit is $\{0\}$ are not square integrable by Theorem 7. We consider the orbit $\mathbb{R}^n \setminus \{0\}$ and a base point $\mathbf{e}_n \in \mathbb{R}^n \setminus \{0\}$. Then $H_{\mathbf{e}_n} = SO(n-1)$, where $SO(n-1)$ is identified with the closed subgroup of $SO(n)$ that leaves the $n$-th coordinate fixed. It is known that the unitary representation $\sigma_{n,m}$ restricted to $SO(n-1)$ is decomposed as

$$\sigma_{n,m}|_{SO(n-1)} = \bigoplus_{\alpha=0}^{m} \sigma_{n-1,\alpha}.$$

Since $\sigma_{n-1,\alpha}$ and $\sigma_{n-1,\alpha'}$ are inequivalent for $\alpha \neq \alpha'$, the decomposition above is multiplicity free [53]. Therefore,Theorem 5 tells us that the irreducible decomposition $\pi_{n,m}$ is given by

$$\pi_{n,m} \simeq \mathrm{Ind}_H^G \sigma_{n,m} = \bigoplus_{\alpha=0}^{m} \mathrm{Ind}_{G_{\mathbf{e}_n}}^G \nu_{\mathbf{e}_n} \otimes \sigma_{n-1,\alpha}.$$

Since $SO(n-1)$ is compact and $\mathbb{R}^n \setminus \{0\}$ has positive Lesbegue measure, every irreducible unitary representation $\mathrm{Ind}_{G_{\mathbf{e}_n}}^G \nu_{\mathbf{e}_n} \otimes \sigma_{n-1,\alpha}$ is a square integrable representation by Theorem 7. The Fourier transform $\mathcal{F}_n$ on $L^2(\mathbb{R}^n, \mathbb{C}^{d(n,m)})$ is defined by

$$\mathcal{F}_n f(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)dx.$$

By Lemma 1 $\widehat{\pi}$ is described as

$$\widehat{\pi}_{n,m}(x, A, c)\varphi(\xi) = e^{-2\pi i x\cdot\xi} c^{\frac{n}{2}} \sigma_{n,m}(A)\varphi(cA^{-1}\xi) \quad (\varphi \in L^2(\mathbb{R}^n, \mathbb{C}^{d(n,m)})).$$

We denote by $\widehat{\pi}_{n,m,\alpha}$ the irreducible representation equivalent to $\mathrm{Ind}_{G_{\mathbf{e}_n}}^G \nu_{\mathbf{e}_n} \otimes \sigma_{n-1,\alpha}$. By Lemma 2, the representation space $\mathcal{H}_{n,m,\alpha}$ of $\widehat{\pi}_{n,m,\alpha}$ is given by

$$\mathcal{H}_{n,m,\alpha} = \{\varphi \in L^2(\mathbb{R}^n, \mathbb{C}^{d(n,m)}); \varphi(\xi) \in \sigma_{n,m}(A)\mathcal{H}_{\sigma_{n-1,\alpha}} \text{ a.a. } \xi, \xi = c^{-1}A\mathbf{e}_n\}.$$

An element $\varphi_{n,m,\alpha}$ of $\mathcal{H}_{n,m,\alpha}$ is admissible vector if and only if

$$C_{\varphi_{n,m,\alpha}} = \int_{\mathbb{R}^n\setminus\{0\}} \frac{\langle \varphi_{n,m,\alpha}(\xi), \varphi_{n,m,\alpha}(\xi)\rangle}{|\xi|^n} d\xi < \infty, \tag{10}$$

see [3]. Therefore, we take $\varphi_{n,m,\alpha} \in \mathcal{H}_{n,m,\alpha}$ satisfying admissibly condition (10) and $C_{\varphi_{n,m,\alpha}} = 1$ for each $\alpha$, we can define the continuous wavelet transform associated to $\pi$ by Theorem 8.

### 3.2 Construction of admissible vector in the case of $\mathbf{n = 3}$

It is known that any irreducible unitary representation of $SO(3)$ is of class 1. Let $\sigma_m = \sigma_{3,m}$ be the irreducible unitary representation of $SO(3)$. Then dim $\sigma_m$ is $d(3, m) = 2m + 1$. As in Subsection 3.1, we identify $SO(2)$ with the closed subgroup of $SO(3)$ in the following way:

$$SO(2) \simeq \left\{ A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \theta \in [0, 2\pi) \right\}.$$

In the case of $n = 3$, the representation $\sigma_{2,\alpha}$ is reducible. Indeed since the dimension $d(2, \alpha) = 2$, the representation $\sigma_{2,\alpha}$ is 2-dimensional representation, while every irreducible unitary representation of $SO(2)$ is 1-dimensional because $SO(2)$ is commutative. Therefore the representation $\sigma_{2,\alpha}$ is decomposed into two irreducible representations. We can take $v_\alpha$ $(-m \le \alpha \le m)$ such that $\sigma_m(A_\theta)v_\alpha = e^{i\alpha\theta}v_\alpha$ $(\theta \in [0, 2\pi))$ and $\|v_\alpha\| = 1$. Then $\{v_\alpha\}_{-m\le\alpha\le m}$ forms a basis of $\mathbb{C}^{2m+1}$. In other words, the representation $\sigma_m$ is decomposed into

$$\sigma_m = \bigoplus_{\alpha=-m}^{m} \rho_\alpha,$$

where $\rho_\alpha(A_\theta) = e^{i\alpha\theta}$. Therefore,

$$\pi_m \simeq \bigoplus_{\alpha=-m}^{m} \mathrm{Ind}_{G_{\mathbf{e}_3}}^{G} \nu_{e_3} \otimes \rho_\alpha.$$

The representation space $\mathcal{H}_{m,\alpha}$ of $\widehat{\pi}_\alpha$, which is equivalent to $\mathrm{Ind}_{G_{\nu_{\mathbf{e}_3}}} G\nu_{\mathbf{e}_3} \otimes \rho_\alpha$, is given by

$$\mathcal{H}_{m,\alpha} = \{\varphi \in L^2(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}^{2m+1})\ ;\ \varphi(\xi) \in \mathbb{C}\sigma(A)v_\alpha\ a.a.\ \xi = c^{-1}A\mathbf{e}_3\}.$$

For any $\xi \in \mathbb{R}^3 \backslash \mathbb{R}\mathbf{e}_3$, there exist a unique $c > 0$ and $A = A_\theta B_\eta$ in $SO(3)$ such that $\xi = c^{-1}A\mathbf{e}_3$, where $B_\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & -\sin\eta \\ 0 & \sin\eta & \cos\eta \end{pmatrix}$ $(0 < \eta < \pi)$. In what follows, we assume these relations between $\xi, c$ and $A$ without comment. For

$\xi \in \mathbb{R}^3 \setminus \mathbb{R}\mathbf{e}_3$, we have

$$\begin{aligned}
\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} &= c^{-1} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & -\sin\eta \\ 0 & \sin\eta & \cos\eta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= c^{-1} \begin{pmatrix} \sin\theta\sin\eta \\ -\cos\theta\sin\eta \\ \cos\eta \end{pmatrix},
\end{aligned}$$

so that

$$c^{-1} = |\xi|, \quad \cos\eta = \frac{\xi_3}{|\xi|}, \quad \sin\eta = \frac{\sqrt{\xi_1^2+\xi_2^2}}{|\xi|}, \quad e^{i\theta} = \frac{-\xi_2 + i\xi_1}{\sqrt{\xi_1^2+\xi_2^2}}. \tag{11}$$

We define a $\mathbb{C}^{2m+1}$-valued function $u_\alpha$ on $\mathbb{R}^3 \setminus \mathbb{R}\mathbf{e}_3$ by

$$u_\alpha(\xi) = \sigma(A)v_\alpha. \tag{12}$$

Then we have

$$\begin{aligned}
u_\alpha(\xi) &= \sum_\beta \langle u_\alpha(\xi), v_\beta\rangle v_\beta \\
&= \sum_\beta \langle \sigma(A_\theta B_\eta)v_\alpha, v_\beta\rangle v_\beta \\
&= \sum_\beta \left\langle \sigma(B_\eta)v_\alpha, \sigma(A_\theta)^{-1}v_\beta\right\rangle v_\beta \\
&= \sum_\beta e^{i\beta\theta} \langle \sigma(B_\eta)v_\alpha, v_\beta\rangle v_\beta.
\end{aligned}$$

It is known that the matrix coefficient$\langle \sigma(B_\eta)v_\alpha, v_\beta\rangle$ is expressed as

$$\langle \sigma(B_\eta)v_\alpha, v_\beta\rangle = P^m_{\alpha,\beta}(\cos\eta),$$

where the Jacobi function $P^m_{\alpha\beta}$ is given by

$$P^m_{\alpha,\beta}(x) = c^m_{\alpha,\beta}(1+x)^{\frac{-\alpha-\beta}{2}}(1-x)^{\frac{\alpha-\beta}{2}}\frac{d^{m-\beta}}{dx^{m-\beta}}[(1-x)^{m-\alpha}(1+x)^{m+\alpha}],$$

and $c^m_{\alpha,\beta}$ is a constant depending on a normalization of $\{v_\alpha\}$. Therefore, we have

$$u_\alpha(\xi) = \sum_\beta e^{i\beta\theta}P^m_{\alpha,\beta}(\cos\eta)v_\beta. \tag{13}$$

If we take a suitable normalization of $\{v_\alpha\}$, then $P^m_{\alpha,\beta} = P^m_{\beta,\alpha}$ and $c^m_{\alpha,\beta}$ is equal to

$$c^m_{\alpha,\beta} = \frac{(-1)^{m-\alpha} i^{\beta-\alpha}}{2^m (m-\alpha)!} \sqrt{\frac{(m-\alpha)!(m+\beta)!}{(m+\alpha)!(m-\beta)!}}.$$

See [53, III. 3]. Since $\|u_\alpha(\xi)\| = 1$ for $\xi$, by (12) we have

$$\mathcal{H}_{m,\alpha} = \{\varphi = \phi u_\alpha : \phi \in L^2(\mathbb{R}^3)\}.$$

The function $\varphi = \phi u_\alpha \in \mathcal{H}_{m,\alpha}$ is admissible if and only if

$$\begin{aligned} C_\varphi &= \int \frac{\langle \varphi(\xi), \varphi(\xi) \rangle}{|\xi|^3} d\xi \\ &= \int \frac{|\phi(\xi)|^2}{|\xi|^3} d\xi < \infty. \end{aligned}$$

Therefore, if $\phi \in L^2(\mathbb{R}^3)$ satisfies the condition above, then $\mathcal{F}_3^{-1}(\phi u_\alpha)(x)$ is admissible vector in $\mathcal{F}_3^{-1}(\mathcal{H}_{m,\alpha})$. Now, we define the $\mathbb{C}^{2m+1}$-valued function $\psi_{m,\alpha}$ on $\mathbb{R}^3 \setminus \mathbb{R}\mathbf{e}_3$ by

$$\psi_{m,\alpha}(\xi) = |\xi|^m (\xi_1^2 + \xi_2^2)^{\frac{|\alpha|}{2}} u_\alpha(\xi). \tag{14}$$

**Proposition 11.** *There exist polynomials $u_{\alpha,\beta,l}(\xi)$ such that*

$$\psi_{m,\alpha}(\xi) = \sum_\beta \sum_l |\xi|^l u_{\alpha,\beta,l}(\xi) v_\beta. \tag{15}$$

By (13), the function $\psi_{m,\alpha}$ is expressed as

$$\psi_{m,\alpha}(\xi) \quad = \quad \sum_\beta |\xi|^m (\xi_1^2 + \xi_2^2)^{\frac{|\alpha|}{2}} e^{i\beta\theta} P^m_{\alpha,\beta}(\cos\eta) v_\beta.$$

Let $u_{\alpha,\beta}$ be a function given by

$$u_{\alpha,\beta}(\xi) = |\xi|^m (\xi_1^2 + \xi_2^2)^{\frac{|\alpha|}{2}} e^{i\beta\theta} P^m_{\alpha,\beta}(\cos\eta).$$

For the proof of Proposition 11, it suffices to show that $u_{\alpha,\beta}$ is a polynomial of $\xi_1, \xi_2, \xi_3$ and $|\xi|$.

**Lemma 3.** *The function $u_{\alpha,\beta}$ is a polynomial of $\xi_1, \xi_2, \xi_3$ and $|\xi|$.*

*Proof.* (i) In the case of $\alpha \geq 0$ and $\beta \leq 0$.
By (8), we obtain

$$\begin{aligned}
u_{\alpha,\beta}(\xi) &= |\xi|^m(\xi_1^2+\xi_2^2)^{\frac{|\alpha|}{2}} e^{i\beta\theta} c^m_{\alpha,\beta} P^m_{\alpha,\beta}(\cos\eta) \\
&= |\xi|^m(\xi_1^2+\xi_2^2)^{\frac{|\alpha|}{2}} e^{i\beta\theta} (1-\cos^2\eta)^{\frac{-\alpha-\beta}{2}} Q^{1,m}_{\alpha,\beta}(\cos\eta) \\
&= |\xi|^{m+\alpha+\beta} e^{i\beta\theta}(\xi_1^2+\xi_2^2)^{\frac{|\alpha|-\alpha-\beta}{2}} Q^{1,m}_{\alpha,\beta}(\frac{\xi_3}{|\xi|}) \\
&= |\xi|^{m+\alpha+\beta}(-\xi_2-i\xi_1)^{-\beta} Q^{1,m}_{\alpha,\beta}(\frac{\xi_3}{|\xi|}),
\end{aligned}$$

where $Q^{1,m}_{\alpha,\beta}(x) = c^m_{\alpha,\beta}(1-x)^{\alpha}\frac{d^{m-\beta}}{dx^{m-\beta}}[(1-x)^{m-\alpha}(1+x)^{m+\alpha}]$. Since $\alpha \geq 0$, the function $Q^{1,m}_{\alpha,\beta}(x)$ is a polynomial of $x$ of degree $m+\alpha+\beta$. Therefore $|\xi|^{m+\alpha+\beta}Q^{1,m}_{\alpha,\beta}(\frac{\xi_3}{|\xi|})$ is a polynomial of $|\xi|$ and $\xi_3$ of degree $m+\alpha+\beta$. Since $\beta \leq 0$, then $(-\xi_2-i\xi_1)^{-\beta}$ is a polynomial of $\xi_1$ and $\xi_2$. Therefore we have

$$u_{\alpha,\beta}(\xi) = \sum_{l=0}^{m+\alpha+\beta} |\xi|^l u_{\alpha,\beta,l}(\xi),$$

where $u_{\alpha,\beta,l}$ is a polynomial of $\xi$.
(ii) In the case of $\alpha \geq 0$ and $\beta \geq 0$.
The Jacobi function $P^m_{\alpha,\beta}(x)$ is expressed as

$$(1-x^2)^{\frac{-\alpha+\beta}{2}} Q^{2,m}_{\alpha,\beta}(x),$$

where $Q^{2,m}_{\alpha,\beta}(x) = c^m_{\alpha,\beta}(1-x)^{\alpha-\beta}(1+x)^{-\beta}\frac{d^{m-\beta}}{dx^{m-\beta}}[(1-x)^{m-\alpha}(1+x)^{m+\alpha}]$. Since $\alpha \geq 0$, the function $Q^{2,m}_{\alpha,\beta}(x)$ is a polynomial of $x$ of degree $m+\alpha-\beta$. Then we have

$$\begin{aligned}
u_{\alpha,\beta}(\xi) &= |\xi|^{m+\alpha-\beta} e^{i\beta\theta}(\xi_1^2+\xi_2^2)^{\frac{|\alpha|-\alpha+\beta}{2}} Q^{2,m}_{\alpha,\beta}(\frac{\xi_3}{|\xi|}) \\
&= |\xi|^{m+\alpha-\beta}(-\xi_2+i\xi_1)^{\beta} Q^{2,m}_{\alpha,\beta}(\frac{\xi_3}{|\xi|}).
\end{aligned}$$

Therefore we obtain the same result as (i).
(iii) In the case $\alpha \leq 0$ and $\beta \geq 0$.
The Jacobi function $P^m_{\alpha,\beta}(x)$ is expressed as

$$(1-x^2)^{\frac{\alpha+\beta}{2}} Q^{3,m}_{\alpha,\beta}(x),$$

where $Q^{3,m}_{\alpha,\beta}(x) = c^m_{\alpha,\beta}(1-x)^{-\beta}(1+x)^{-\alpha-\beta}\frac{d^{m-\beta}}{dx^{m-\beta}}[(1-x)^{m-\alpha}(1+x)^{m+\alpha}]$. Since $\alpha \leq 0$, the function $Q^{3,m}_{\alpha,\beta}(x)$ is a polynomial of $x$ of degree $m-\alpha-\beta$. Then we have

$$\begin{aligned} u_{\alpha,\beta}(\xi) &= |\xi|^{m-\alpha-\beta}e^{i\beta\theta}(\xi_1^2+\xi_2^2)^{\frac{|\alpha|+\alpha+\beta}{2}}Q^{3,m}_{\alpha,\beta}(\frac{\xi_3}{|\xi|}) \\ &= |\xi|^{m-\alpha-\beta}(-\xi_2+i\xi_1)^{\beta}Q^{3,m}_{\alpha,\beta}(\frac{\xi_3}{|\xi|}). \end{aligned}$$

Therefore we obtain the same result as (i).
(iv) In the case $\alpha \leq 0$ and $\beta \leq 0$.
The Jacobi function $P^m_{\alpha,\beta}(x)$ is expressed as

$$(1-x^2)^{\frac{\alpha-\beta}{2}}Q^{4,m}_{\alpha,\beta}(x),$$

where $Q^{4,m}_{\alpha,\beta}(x) = c^m_{\alpha,\beta}(1-x)^{-\alpha}\frac{d^{m-\beta}}{dx^{m-\beta}}[(1-x)^{m-\alpha}(1+x)^{m+\alpha}]$. Since $\alpha \leq 0$, the function $Q^{4,m}_{\alpha,\beta}(x)$ is a polynomial of $x$ of degree $m-\alpha+\beta$. Then we have

$$\begin{aligned} u_{\alpha,\beta}(\xi) &= |\xi|^{m-\alpha+\beta}e^{i\beta\theta}(\xi_1^2+\xi_2^2)^{\frac{|\alpha|+\alpha-\beta}{2}}Q^{4,m}_{\beta,\alpha}(\frac{\xi_3}{|\xi|}) \\ &= |\xi|^{m-\alpha+\beta}(-\xi_2-i\xi_1)^{-\beta}Q^{4,m}_{\beta,\alpha}(\frac{\xi_3}{|\xi|}). \end{aligned}$$

It follows from $\beta \leq 0$ that $(-\xi_2-i\xi_1)^{-\beta}$ is a polynomial. Then we obtain the same result as (i). □

Hence, we have shown that for any $m$, $\alpha$, the function $\psi_{m,\alpha}$ is described as (15). If $\phi$ is an element of the Schwartz class $\mathcal{S}(\mathbb{R}^3)$ and satisfies admissibility condition, then $\phi\psi_{m,\alpha}$ is an admissible vector. In general, for a polynomial $P(\xi_1,\xi_2,\xi_3)$ of $\xi$, we denote $P(D_x)$ by the differential operator $P(\frac{1}{2\pi i}\frac{\partial}{\partial x_1}, \frac{1}{2\pi i}\frac{\partial}{\partial x_2}, \frac{1}{2\pi i}\frac{\partial}{\partial x_3})$. If

$$\phi_l = |\xi|^l\phi \in L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3) \tag{16}$$

for $l$, then we obtain

$$\mathcal{F}_3^{-1}\phi\psi_{m,\alpha}(x) = \sum_{\beta}\sum_{l} u_{\alpha,\beta,l}(D_x)\mathcal{F}_3^{-1}\phi_l(x)v_{\beta}.$$

**Theorem 9.** *Let $\phi\psi_{m,\alpha} \in \mathcal{H}_{m,\alpha}$ be an admissible vector for any $m$ and $\alpha$. We set $\varphi_{m,\alpha} = C_{\phi\psi_{m,\alpha}}^{-\frac{1}{2}} \phi\psi_{m,\alpha}$ and $\varphi = \sum \varphi_{m,\alpha}$. Then an admissible vector $\mathcal{F}_3^{-1}\varphi = f_0 \in L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$ is given by*

$$f_0(x) = \mathcal{F}_3^{-1}\varphi(x) = \sum_{\alpha}\sum_{\beta}\sum_{l} C_{\phi\psi_{m,\alpha}}^{-\frac{1}{2}} u_{\alpha,\beta,l}(D_x)\mathcal{F}_3^{-1}\phi_l(x)v_\beta.$$

Based on Theorem 9, we can construct directional wavelets in the same way as [5, 3.3] and [2, 14.2]. A function $f \in L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$ is said to be directional if the effective support of its Fourier transform $\widehat{f}$ is contained in a convex and proper cone $C^*$ in the $\xi$-space, with vertex at the origin, or a finite union of disjoint such cones [5, 3.3.1.2]. According to [5, 3.3.4], in order to reach a genuinely directional wavelet $f_0$, it suffices to consider a smooth function $\widehat{f_0}$, that is,

$$\widehat{f_0} = \begin{pmatrix} P_1(\xi) \\ \vdots \\ P_{2m+1}(\xi) \end{pmatrix} \bigotimes e^{-\xi\cdot y} 1_{C^*}(\xi),$$

where $y$ is an element of the dual cone $C = (C^*)^*$ of $C^*$, and $P_j$ is a polynomial. The wavelet $f_0$ constructed in this way is called a conical. It is known that a directional wavelet is extended to a holomorphic function of tube domain $\{z = x + iy : x \in \mathbb{R}^3, y \in C\} \subset \mathbb{C}^3$. We give directional wavelets in the case of $m = 1$ and $m = 2$ in Pages 27 and 34 respectively. As a result, we obtain explicit wavelets analogous to the one called “ Cauchy wavelets ” in [2, 14.2]

At the end of this section, we consider a frame for $L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$.

**Definition 5.** *Let $\mathcal{A} = \{f_j\}_{j\in J}$ be a subset of $L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$, where $J$ is an index set. The set $\mathcal{A}$ is called a (wavelet) frame if each $f_j$ is an admissible vector in $L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$ such that for some $D \geq C > 0$, the inequality*

$$C\|f\|^2 \leq \sum_{j\in J} |\langle \pi(x, A, c)f, f_j\rangle|^2 \leq D\|f\|^2$$

*holds for all $f \in L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$. If $A = B$, then $\mathcal{A}$ is called tight.*

**Proposition 12.** *Suppose $\phi^l \in L^2(\mathbb{R}^3)$ is an admissible vector for $l = 1, \ldots, L$. For fixed $c' \in \mathbb{R}_+$ and $A' \in SO(3)$, we define*

$$\mathcal{A} = \{\phi_{j,k}^l = c^{-\frac{3j}{2}} \phi^l((c'A')^{-j}x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}^3, l = 1, \ldots, L\}$$

*and*

$$\mathcal{A}_m = \{f^l_{j,k} = \sum_\alpha \sigma(A'^j)\mathcal{F}_3^{-1}(\widehat{\phi}^l_{j,k} u_\alpha(c'^j A'^{-j}\xi)) : j \in \mathbb{Z}, k \in \mathbb{Z}^3, l = 1, \ldots, L\}.$$

*If $\mathcal{A}$ is a frame for $L^2(\mathbb{R}^3)$, then $\mathcal{A}_m$ is a frame for $L^2(\mathbb{R}^3, \mathbb{C}^{2m+1})$.*

*Proof.* It suffices to show that $\mathcal{A}_{m,\alpha} := \{f^l_{\alpha,j,k}(x) = P_\alpha f^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^3, l \in L\}$, where $P_\alpha$ is the orthogonal projection on $\mathcal{F}_3^{-1}(\mathcal{H}_{m,\alpha})$, is a frame for $\mathcal{F}_3^{-1}(\mathcal{H}_{m,\alpha})$ for each $\alpha$. We show that for some $D \geq C > 0$,

$$C\|f\|^2 \leq \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3} |\left\langle f, f^l_{\alpha,j,m} \right\rangle|^2 \leq D\|f\|^2$$

holds for all $f \in \mathcal{F}_3^{-1}(\mathcal{H}_{m,\alpha})$. To do this, we estimate $\sum_l \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3} |\left\langle f, f^l_{\alpha,j,m} \right\rangle|^2$, that is,

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3} |\int_{\mathbb{R}^3} c^{\frac{3j}{2}} e^{2\pi i k\cdot\xi} \left\langle \widehat{f}(\xi), \widehat{f}^l_{\alpha,j,k}(c'^j A'^{-j}\xi) \right\rangle d\xi|^2.$$

For $c'$ and $A'$, we have

$$u_\alpha(c'^j A'^{-j}\xi) = \sigma(A'^{-j}) u_\alpha(\xi)$$

for any $\alpha$. Since any $\varphi \in \mathcal{H}_{m,\alpha}$ is written by $\psi u_\alpha$ for some $\psi \in L^2(\mathbb{R}^3)$, we set $\widehat{f} = \psi u_\alpha$. Then we have

$$\begin{aligned}
& \int c^{\frac{3j}{2}} e^{2\pi i k\cdot\xi} \left\langle \widehat{f}(\xi), \widehat{f}^l_{\alpha,j,k}(c'^j A'^{-j}\xi) \right\rangle d\xi \\
= & \int c^{\frac{3j}{2}} e^{2\pi i k\cdot\xi} \left\langle \psi(\xi) u_\alpha(\xi), \sigma(A'^j)\widehat{\phi}^l_{j,k}(c'^j A'^{-j}\xi) u_\alpha(c'^j A'^{-j}\xi) \right\rangle d\xi \\
= & \int c^{\frac{3j}{2}} e^{2\pi i k\cdot\xi} \left\langle \psi(\xi) u_\alpha(\xi), \sigma(A^j)\widehat{\phi}^l_{j,k}(c'^j A'^{-j}\xi)\sigma(A'^{-j}) u_\alpha(\xi) \right\rangle d\xi \\
= & \int c^{\frac{3j}{2}} e^{2\pi i k\cdot\xi} \psi(\xi) \overline{\widehat{\phi}^l_{j,m}(c^j A^{-j}\xi)} d\xi.
\end{aligned}$$

By assumption, for some $D \geq C > 0$, we have

$$C\|\mathcal{F}_3^{-1}\psi\|^2 \leq \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3} |\left\langle \mathcal{F}_3^{-1}\psi, \phi^l_{j,k} \right\rangle|^2 \leq D\|\mathcal{F}_3^{-1}\psi\|^2,$$

while since $\|\mathcal{F}_3^{-1}\psi\| = \|f\|$, we obtain

$$C\|f\|^2 \leq \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^3} |\left\langle f, f^l_{\alpha,j,k}\right\rangle|^2 \leq D\|f\|^2.$$

Therefore $\mathcal{A}_{m,\alpha}$ is a frame for $\mathcal{F}_3^{-1}(\mathcal{H}_{m,\alpha})$. $\square$

### 3.3 $\mathbb{C}^3$-valued admissible vectors

The irreducible unitary representation $\sigma_1$ of $SO(3)$ is the ordinary action on $\mathbb{R}^3$, namely, $\sigma(A)x = Ax$. We choose $v_{-1}, v_0$ and $v_1$ as

$$v_{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ 0 \end{pmatrix}, v_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Then every $\mathcal{H}_{1,\alpha}$ is given by

$$\mathcal{H}_{1,0} = \left\{\varphi \in L^2(\mathbb{R}^3,\mathbb{C}^3) : \varphi(\xi) \in \mathbb{C}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \text{a.a } \xi\right\}$$

and

$$\mathcal{H}_{1,\pm 1} = \left\{\varphi \in L^2(\mathbb{R}^3,\mathbb{C}^3) : \varphi(\xi) \in \mathbb{C}\begin{pmatrix} \xi_2|\xi| \mp i\xi_1\xi_3 \\ \xi_1|\xi| \mp i\xi_2\xi_3 \\ \pm i(\xi_1^2+\xi_2^2) \end{pmatrix} \text{a.a. } \xi\right\}.$$

The following proposition is verified by a direct calculation.

**Proposition 13.** *The subspace $\mathcal{F}_3^{-1}(\mathcal{H}_{1,0})$ of $L^2(\mathbb{R}^3,\mathbb{C}^3)$ is the space of curl-free vector, that is,*

$$\mathcal{F}_3^{-1}(\mathcal{H}_{1,0}) = \{f \in L^2(\mathbb{R}^3,\mathbb{C}^3); \nabla \wedge f = 0\},$$

*and the subspace $\mathcal{F}_3^{-1}(\mathcal{H}_{1,-1}\bigoplus\mathcal{H}_{1,1})$ of $L^2(\mathbb{R}^3,\mathbb{C}^3)$ is the space of divergence-free vector, namely,*

$$\mathcal{F}_3^{-1}(\mathcal{H}_{1,-1}\bigoplus\mathcal{H}_{1,1}) = \{f \in L^2(\mathbb{R}^3,\mathbb{C}^3); \nabla\cdot f = 0\}.$$

Now let us give two examples of admissible vectors, which are associated to derivative of the Gaussian function and the directional wavelet. The second derivative of Gaussian function associated to affine group is typical, classical and the most widely used wavelet, see [2, Secition 12.2] for example. For positive integer $k$, we set

$$\phi_k(\xi) = |\xi|^k e^{-2\pi^2|\xi|^2} \tag{17}$$

and

$$\varphi_{1,0}(\xi) = 2\sqrt{\pi}(\frac{1}{2\pi})^k \phi_k(\xi)|\xi| u_0(\xi),$$

and

$$\varphi_{1,\pm1}(\xi) = 2\sqrt{\pi}(\frac{1}{2\pi})^k \frac{1}{\sqrt{2}} \phi_{k-1}(\xi)|\xi|(\xi_1^2+\xi_2^2)^{\frac{1}{2}} u_{\pm1}(\xi).$$

Then each $\varphi_{1,\alpha}$ is admissible vector in $\mathcal{H}_{1,\alpha}$, respectively. In fact, we have

$$C_{\varphi_{1,0}} = 1, \quad C_{\varphi_{1,\pm1}} = \frac{1}{3}.$$

Each $\varphi_{1,\alpha}$ is expressed as

$$\varphi_{1,0}(\xi) = 2\sqrt{\pi}(\frac{1}{2\pi})^k \phi_k(\xi) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

and

$$\varphi_{1,\pm1}(\xi) = 2\sqrt{\pi}(\frac{1}{2\pi})^k \frac{1}{\sqrt{2}} \phi_k(\xi) \begin{pmatrix} -\frac{\xi_2}{\sqrt{2}} \\ \frac{\xi_1}{\sqrt{2}} \\ 0 \end{pmatrix} + i2\sqrt{\pi}(\frac{1}{2\pi})^k \frac{1}{\sqrt{2}} \phi_{k-1}(\xi) \begin{pmatrix} \mp\frac{\xi_1\xi_3}{\sqrt{2}} \\ \mp\frac{\xi_2\xi_3}{\sqrt{2}} \\ \pm\frac{(\xi_1^2+\xi_2^2)}{\sqrt{2}} \end{pmatrix}.$$

Let $\varphi = \sum C_{\varphi_{1,\alpha}}^{-\frac{1}{2}} \varphi_{1,\alpha}$. Then $\varphi$ is admissible vector and

$$\begin{aligned} \varphi(\xi) &= 2\sqrt{\pi}(\frac{1}{2\pi})^k \varphi_{1,0}(\xi) + \frac{4\sqrt{3\pi}}{\sqrt{2}}(\frac{1}{2\pi})^k \sqrt{2}\mathrm{Re}\varphi_{1,1}(\xi) \\ &= 2\sqrt{\pi}(\frac{1}{2\pi})^k |\xi|^k e^{-2\pi^2|\xi|^2} \begin{pmatrix} \xi_1 - \sqrt{3}\xi_2 \\ \xi_2 + \sqrt{3}\xi_1 \\ \xi_3 \end{pmatrix}. \end{aligned}$$

Therefore the inverse Fourier transform of $\varphi$ is given by

$$f_0(x) = \mathcal{F}_3^{-1}\varphi(x) = 2\sqrt{\pi}(\frac{1}{2\pi})^k \begin{pmatrix} D_{x_1} - \sqrt{3}D_{x_2} \\ D_{x_2} + \sqrt{3}D_{x_1} \\ D_{x_3} \end{pmatrix} \bigotimes \mathcal{F}_3^{-1}\phi_k(x).$$

If $k = 2s$, then $f_0$ is the function associated to $k$-derivative of Gaussian function. Put $k = 2$. Then we have

$$\begin{aligned} f_0(x) &= \mathcal{F}_3^{-1}\varphi(x) = 2\sqrt{\pi}(\frac{1}{2\pi})^2 \begin{pmatrix} D_{x_1} - \sqrt{3}D_{x_2} \\ D_{x_2} + \sqrt{3}D_{x_1} \\ D_{x_3} \end{pmatrix} \bigotimes \mathcal{F}_3^{-1}\phi_2(x) \\ &= -\sqrt{2}(\frac{1}{2\pi})^5 \begin{pmatrix} D_{x_1} - \sqrt{3}D_{x_2} \\ D_{x_2} + \sqrt{3}D_{x_1} \\ D_{x_3} \end{pmatrix} \bigotimes (-3 + |x|^2)e^{-\frac{|x|^2}{2}} \\ &= i\sqrt{2}(\frac{1}{2\pi})^6 e^{-\frac{|x|^2}{2}} \begin{pmatrix} (x_1 - \sqrt{3}x_2)(5 - |x|^2) \\ (x_2 + \sqrt{3}x_1)(5 - |x|^2) \\ x_3(5 - |x|^2) \end{pmatrix}. \end{aligned}$$

Next, we consider the directional wavelet. Let $C^* = \{\xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \xi_3\mathbf{e}_3 : \xi_1, \xi_2, \xi_3 > 0\}$. Then dual cone $C$ of $C^*$ coincides with $C^*$, that is, $C = \{x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 : x_1, x_2, x_3 > 0\}$. Let $\phi(\xi) = \xi_1^{k_1}\xi_2^{k_2}\xi_3^{k_3}e^{-2\pi\xi\cdot y}, y \in C, k_1, k_2, k_3 \in \mathbb{N}$ and supp $\phi = C^*$. If $k_1 + k_2 + k_3$ is large enough, we set

$$\varphi_{1,0}(\xi) = C_{\varphi_{1,0}}^{-\frac{1}{2}}\phi(\xi) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \quad \mathrm{Re}\varphi_{1,\pm1}(\xi) = C_{\varphi_{1,1}}^{-\frac{1}{2}}\frac{1}{\sqrt{2}}\phi(\xi) \begin{pmatrix} -\xi_2 \\ -\xi_1 \\ 0 \end{pmatrix},$$

and $\varphi(\xi) = \varphi_{1,0}(\xi) + 2\mathrm{Re}\varphi_{1,\pm1}(\xi)$. Then we have

$$\begin{aligned} f_0(x) &= \mathcal{F}_3^{-1}\varphi(x) = C_{\varphi_{1,0}}^{-\frac{1}{2}}(\frac{i}{2\pi})^{k_1+k_2+k_3+4}k_1!k_2!k_3! \\ &\times \frac{1}{(x_1+iy_1)^{k_1+1}(x_2+iy_2)^{k_2+1}(x_3+iy_3)^{k_3+1}} \begin{pmatrix} \frac{k_1+1}{x_1+iy_1} - \frac{a(k_2+1)}{x_2+iy_2} \\ \frac{k_2+1}{x_2+iy_2} + \frac{a(k_1+1)}{x_1+iy_1} \\ \frac{k_3+1}{x_3+iy_3} \end{pmatrix}, \end{aligned}$$

where $a = \sqrt{\frac{C\varphi_{1,1}}{C\varphi_{1,0}}}$. An interesting property of the directional wavelets are their analyticity (see [5]). We define the tube domain $T(C)$ by

$$T(C) = \{z = x + iy \in \mathbb{C}^3; x \in \mathbb{R}^3, y \in C\}.$$

Then $f_0$ is expressed as

$$f_0(z) = const.\frac{1}{z_1^{k_1+1}z_2^{k_2+1}z_3^{k_3+1}} \begin{pmatrix} \frac{k_1+1}{z_1} - \frac{a(k_2+1)}{z_2} \\ \frac{k_2+1}{z_2} + \frac{a(k_1+1)}{z_1} \\ \frac{k_3+1}{z_3} \end{pmatrix}.$$

Therefore, $f_0(z)$ is a holomorphic function on the Hardy space for the upper half plane $T(C)$ and $f_0(x)$ is the boundary value of $f_0(z)$. Thus, the directional wavelet can be extended to a holomorphic function on $T(C)$ for the cone $C$. Let $\varphi$ be admissible vector such that $\varphi$ is a Schwartz function and $\mathrm{supp}(\varphi)$ is contained convex and proper cone $C^*$. In addition, we assume that $\widehat{\pi}(0, A, 1)\varphi = \varphi$ for any $A \in SO(3)$. Let $f_0 = \mathcal{F}_3^{-1}\varphi$. Then for any $f \in L^2(\mathbb{R}^3, \mathbb{C}^3)$, we have

$$W_{f_0} f(x, A, c) = \left\langle \widehat{f}, \widehat{\pi}(x, A, c)\varphi \right\rangle = \int e^{2\pi x \cdot \xi} \left\langle \widehat{f}(\xi), c^{\frac{3}{2}} \varphi(c\xi) \right\rangle d\xi.$$

Therefore, the Fourier transform of $W_{f_0} f$ is expressed as

$$\mathcal{F}_3(W_{f_0} f)(\xi) = \left\langle \widehat{f}(\xi), c^{\frac{3}{2}} \varphi(c\xi) \right\rangle.$$

Since $cC^* \subset C^*$ for $c > 0$, $\mathrm{supp}(\mathcal{F}_3(W_{f_0} f)) = \mathrm{supp}(\varphi) \subset C^*$. Hence the wavelet transform $W_{f_0} f$ is naturally extensible as an analytic function on $T(C)$ in the sense of distribution. The directional wavelet is applied to phase problem [44] and Maxwell's equation [21]. Consideration above implies that we always take an admissible vector and suggests the following proposition:

**Proposition 14.** *Let $\phi$ be a function on $\mathbb{R}^3$ such that $|\xi|\phi \in L^1(\mathbb{R}^3)$ and*

$$\varphi_{1,0}(\xi) = \phi(\xi)|\xi|^2 u_0(\xi), \;\; \varphi_{1,\pm 1}(\xi) = |\xi|\phi(\xi)(\xi_1^2 + \xi_2^2)^{\frac{1}{2}} u_{\pm 1}(\xi)$$

*are admissible vectors in $\mathcal{H}_{1,\alpha}$, respectively. Then admissible vector $\varphi = \sum C_{\varphi_{1,\alpha}}^{-\frac{1}{2}} \varphi_{1,\alpha}$ is expressed as*

$$\varphi(\xi) = C_{\varphi_{1,0}}^{-\frac{1}{2}} |\xi|\phi(\xi) \begin{pmatrix} \xi_1 - a\xi_2 \\ \xi_2 + a\xi_1 \\ \xi_3 \end{pmatrix},$$

*where $a = \sqrt{\frac{C\varphi_{1,1}}{C\varphi_{1,0}}}$. Hence we have*

$$f_0(x) = \mathcal{F}_3^{-1}\varphi(x) = C_{\varphi_{1,0}}^{-\frac{1}{2}} \begin{pmatrix} D_{x_1} - aD_{x_2} \\ D_{x_2} + aD_{x_1} \\ D_{x_3} \end{pmatrix} \bigotimes \mathcal{F}_3^{-1}(|\xi|\phi)(x).$$

In Proposition 14, if $\phi$ is radial, then $a$ is uniquely determined, $a = \sqrt{3}$. In both examples above, we can compute an admissible vector $f_0$, while $\mathcal{F}_3^{-1}(\mathrm{Im}\varphi_{1,\pm 1})$ cannot be computed explicitly. If $|\xi|\phi$ is radial, then we may compute $\mathcal{F}_3^{-1}(\mathrm{Im}\varphi_{1,\pm 1})$ and $\mathcal{F}_3^{-1}(\mathrm{Re}\varphi_{1,\pm 1})$. Now let us suppose $f \in L^1(\mathbb{R}^3)$ is radial, that is, $f(x) = h(|x|)$ for some function $h$ on the line. We use the notation

$$\mathcal{F}_n f(\xi) = F_n h(r)$$

where $r = |\xi|$. The following theorem give the formula relating the Fourier transform of a radial function on $\mathbb{R}^n$ with the Fourier transform of the same function on $\mathbb{R}^{n+2}$.

**Theorem 10** ([27, Theorem 1.1])**.** *Suppose that $f$ is a function on the real line such that the functions $f(|\cdot|)$ are in $L^1(\mathbb{R}^{n+2})$ and also in $L^1(\mathbb{R}^n)$. Then we have*

$$F_{n+2}f(r) = -\frac{1}{2\pi r}\frac{d}{dr}F_n f(r) \quad r > 0.$$

*Moreover, the following formula is valid for all even Schwartz functions $f$ on the real line:*

$$F_{n+2}f(r) = \frac{1}{2\pi r^2}F_n(s^{-n+1}\frac{d}{ds}(f(s)s^n)) \quad r > 0.$$

**Proposition 15.** *For $m$ and $\alpha$, if $\phi$ is a radial function, the function $\phi\psi_{m,\alpha} \in \mathcal{H}_{m,\alpha}$ is an admissible vector if and only if*

$$\frac{2\Gamma(|\alpha|+1)\pi^{\frac{3}{2}}}{\Gamma(|\alpha|+1+\frac{1}{2})}\int_0^\infty |\phi(r)|^2 r^{2m+2|\alpha|-1}dr < \infty,$$

*where $\Gamma$ is the gamma function.*

*Proof.* Let the function $\phi\psi_{m,\alpha} \in \mathcal{H}_{m,\alpha}$ be an admissible vector. Then we have

$$\begin{aligned}
\int \frac{\langle \phi\psi_{m,\alpha}(\xi), \phi\psi_{m,\alpha}(\xi)\rangle}{|\xi|^3}d\xi &= \int |\phi(\xi)|^2|\xi|^{2m-3}(\xi_1^2+\xi_2^2)^{|\alpha|}d\xi \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^\infty |\phi(r)|^2 r^{2m-3}(r^2\sin^2\eta)^{|\alpha|}r^2\sin\eta\, dr\, d\eta\, d\theta \\
&= 2\pi\int_0^\pi(\sin\eta)^{2|\alpha|+1}d\eta\int_0^\infty |\phi(r)|^2 r^{2m+2|\alpha|-1}dr \\
&= 2\pi\frac{\Gamma(|\alpha|+1)\pi^{\frac{1}{2}}}{\Gamma(|\alpha|+1+\frac{1}{2})}\int_0^\infty |\phi(r)|^2 r^{2m+2|\alpha|-1}dr < \infty.
\end{aligned}$$

□

Let us present one more example of admissible vector where not only $f_0 = \mathcal{F}_3^{-1}(C_{\varphi_{1,0}}^{-\frac{1}{2}}\varphi_{1,0} + 2C_{\varphi_{1,1}}^{-\frac{1}{2}}\mathrm{Re}\varphi_{1,1})$ but also $\mathcal{F}_3^{-1}\varphi_{1,\pm 1}$ is computable explicitly. For a positive integer $k$, we set $\phi_k(\xi) = |\xi|^k e^{-2\pi|\xi|}$ in $L^1(\mathbb{R}^3)$. For $\alpha = \pm 1$ we choose an admissible vector $\varphi_{1,\pm 1} \in \mathcal{H}_{1,\pm 1}$ as

$$\varphi_{1,\pm 1}(\xi) = \phi_{k+1}(\xi)\begin{pmatrix} \xi_2 \\ -\xi_1 \\ 0 \end{pmatrix} + i\phi_k(\xi)\begin{pmatrix} \mp\xi_1\xi_3 \\ \mp\xi_2\xi_3 \\ \pm(\xi_1^2 + \xi^2) \end{pmatrix}.$$

We set $h_k(\xi) = |\xi|^k e^{-2\pi|\xi|} \in L^1(\mathbb{R})$. Then $\phi_k(\xi) = h_k(|\xi|)$. The inverse Fourier transform of $h$ is given by

$$\begin{aligned} \mathcal{F}_1^{-1}h(x) &= \int_{\mathbb{R}} |\xi|^k e^{-2\pi|\xi|} e^{2\pi i x\xi} d\xi \\ &= \begin{cases} (\frac{1}{2\pi i})^{k+1} \frac{d^k}{dx^k} \frac{2i}{1+x^2} & \text{(if } k \text{ is even)} \\ (\frac{1}{2\pi i})^{k+1} \frac{d^k}{dx^k} \frac{-2x}{1+x^2} & \text{(if } k \text{ is odd).} \end{cases} \end{aligned}$$

Put $k = 1$. First, in order to compute of the inverse Fourier transform of $\varphi_{1,\pm 1}$, we compute the inverse Fourier transform $\phi_1$ and $\phi_2$. Since the statement above, we have

$$\mathcal{F}_1^{-1}h_1(x) = \frac{1}{2\pi^2}\frac{1-x^2}{(1+x^2)^2}, \quad \mathcal{F}_1^{-1}h_1(x) = \frac{1}{2\pi^3}\frac{1-3x^2}{(1+x^2)^3}.$$

Therefore, by Theorem 10, we have

$$F_3^{-1}\phi_1(r) = \frac{1}{2\pi r}\frac{d}{dr}F_1^{-1}h_1(r) = \frac{1}{4\pi^3 r}\frac{d}{dr}\frac{1-r^2}{(1+r^2)^2} = \frac{1}{2\pi^3}\frac{r^2-3}{(1+r^2)^3}$$

and

$$F_3^{-1}\phi_2(r) = \frac{1}{2\pi r}\frac{d}{dr}F_1^{-1}h_2(r) = \frac{1}{4\pi^4 r}\frac{d}{dr}\frac{1-3r^2}{(1+r^2)^3} = \frac{1}{\pi^4}\frac{3(r^2-1)}{(1+r^2)^4}.$$

Hence, we obtain

$$\mathcal{F}_3^{-1}\phi_1(x) = \frac{1}{2\pi^3}\frac{|x|^2-3}{(1+|x|^2)^3}, \quad \mathcal{F}_3^{-1}\phi_2(x) = \frac{3(|x|^2-1)}{\pi^4(1+|x|^2)^4}.$$

The inverse Fourier transform of $\varphi_{1,\pm 1}$ is given by

$$\begin{aligned}
\mathcal{F}_3^{-1}(\varphi_{1,\pm 1})(x) &= \begin{pmatrix} -D_{x_2} \\ D_{x_1} \\ 0 \end{pmatrix} \bigotimes \mathcal{F}_3^{-1}\phi_2(x) \\
&+ i \begin{pmatrix} \mp D_{x_1}D_{x_3} \\ \mp D_{x_2}D_{x_3} \\ \pm D_{x_1}^2 + D_{x_2}^2 \end{pmatrix} \bigotimes \mathcal{F}_3^{-1}\phi_1(x) \\
&= \frac{-i}{\pi^5(1+|x|^2)^5} \begin{pmatrix} 3(5x_1 - 3|x|^2) \mp 3x_1x_3(|x|^2-7) \\ 3(5x_2 - 3|x|^2) \mp 3x_2x_3(|x|^2-7) \\ \pm 3(x_1^2+x_2^2) \pm (1+|x|^2)(|x|^2-5) \end{pmatrix}.
\end{aligned}$$

On the other hand, the admissible wavelet $f_0$ in this case(i.e.$\phi(\xi) = |\xi|e^{-2\pi|\xi|}$) can be computed by Proposition 14.

### 3.4 $\mathbb{C}^5$-valued admissible vectors

We consider the case of $m = 2$. We identify the $\mathbb{C}^5$ with $3 \times 3$ complex symmetric traceless matrices in the following way [10]:

$$\mathbb{C}^5 \ni z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} \mapsto s(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{z_5}{\sqrt{3}} - z_2 & z_1 & z_3 \\ z_1 & \frac{z_5}{\sqrt{3}} + z_2 & z_4 \\ z_3 & z_4 & -\frac{2z_5}{\sqrt{3}} \end{pmatrix}.$$

We define 5-dimensional irreducible unitary representation $\sigma_2$ of $SO(3)$ on $\mathbb{C}^5$ by

$$\sigma(A)z := s^{-1}(As(z)^tA) \quad (A \in SO(3), z \in \mathbb{C}^5).$$

Then for any $A_\theta$ we have

$$\sigma(A_\theta) = \begin{pmatrix} \cos 2\theta & -\sin 2\theta & 0 & 0 & 0 \\ \sin 2\theta & \cos 2\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and for any $B_\eta$ we have

$$\sigma(B_\eta) = \begin{pmatrix} \cos\eta & 0 & -\sin\eta & 0 & 0 \\ 0 & \frac{2\cos^2\eta+\sin^2\eta}{2} & 0 & -\cos\eta\sin\eta & -\frac{\sqrt{3}\sin^2\eta}{2} \\ \sin\eta & 0 & \cos\eta & 0 & 0 \\ 0 & \cos\eta\sin\eta & 0 & \cos^2\eta-\sin^2\eta & \sqrt{3}\cos\eta\sin\eta \\ 0 & -\frac{\sqrt{3}\sin^2\eta}{2} & 0 & -\sqrt{3}\cos\eta\sin\eta & \frac{2\cos^2\eta-\sin^2\eta}{2} \end{pmatrix}.$$

Then $\sigma(A)\mathbb{R}^3 \subset \mathbb{R}^3$. We take a basis $\{v_{\pm 2}, v_{\pm 1}, v_0\}$ of $\mathbb{C}^5$ as

$$v_{\pm 2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \mp\frac{i}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_{\pm 1} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \mp\frac{i}{\sqrt{2}} \\ 0 \end{pmatrix}, v_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then every $\mathcal{H}_{2,\alpha}$ $(-2 \le \alpha \le 2)$ is given by

$$\mathcal{H}_{2,0} = \left\{\varphi \in L^2(\mathbb{R}^3, \mathbb{C}^5) : \varphi \in \mathbb{C} \begin{pmatrix} -\sqrt{3}\xi_1\xi_2 \\ -\frac{\sqrt{3}}{2}(\xi_2^2-\xi_1^2) \\ -\sqrt{3}\xi_1\xi_3 \\ -\sqrt{3}\xi_2\xi_3 \\ \frac{1}{2}(2\xi_3^2-\xi_1^2-\xi_2^2) \end{pmatrix} \text{ a.a. } \xi\right\}$$

and

$$\mathcal{H}_{2,\pm 1} = \left\{\varphi \in L^2(\mathbb{R}^3, \mathbb{C}^5) : \varphi \in \mathbb{C} \begin{pmatrix} -(\xi_2^2-\xi_1^2)|\xi| \pm i2\xi_1\xi_2\xi_3 \\ 2\xi_1\xi_2|\xi| \pm i(\xi_2^2-\xi_1^2)\xi_3 \\ -\xi_2\xi_3|\xi| \pm i\xi_1(\xi_3^2-\xi_1^2-\xi_2^2) \\ \xi_1\xi_3|\xi| \pm i\xi_2(\xi_3^2-\xi_1^2-\xi_2^2) \\ \pm i\sqrt{3}\xi_3(\xi_1^2+\xi_2^2) \end{pmatrix} \text{ a.a } \xi\right\}$$

and

$$\mathcal{H}_{2,\pm 2} = \left\{\varphi \in L^2(\mathbb{R}^3, \mathbb{C}^5) : \varphi \in \mathbb{C} \begin{pmatrix} (\xi_2^2-\xi_1^2)\xi_3|\xi| \mp i\xi_1\xi_2(|\xi|^2+\xi_3^2) \\ -2\xi_1\xi_2\xi_3|\xi| \mp \frac{i}{2}(\xi_2^2-\xi_1^2)(\xi_3^2+|\xi|^2) \\ -\xi_2(\xi_1^2+\xi_2^2)|\xi| \pm i\xi_1\xi_3(\xi_1^2+\xi_2^2) \\ \xi_1(\xi_1^2+\xi_2^2)|\xi| \pm i\xi_2\xi_3(\xi_1^2+\xi_2^2) \\ \pm\frac{i\sqrt{3}}{2}(\xi_1^2+\xi_2^2)^2 \end{pmatrix} \text{ a.a } \xi\right\}.$$

We consider a directional wavelet. As in the case $m = 1$, a convex and proper cone $C^* = \{\xi \in \mathbb{R}^3 : \xi_1, \xi_2, \xi_3 > 0\}$ and for positive $k$ we set $\phi_k =$

$|\xi|^k e^{-2\pi\xi\cdot y}, y \in C$. For $\alpha$, we define $\varphi_{2,\alpha}$ by

$$\varphi_{2,\alpha}(\xi) = \begin{cases} -\frac{2}{\sqrt{3}}\phi_2(\xi)\psi_{2,0}(\xi)u_0(\xi) & (\alpha = 0) \\ \frac{1}{\sqrt{2}}\phi_1(\xi)\psi_{2,\alpha}(\xi)u_\alpha(\xi) & (\alpha \neq 0). \end{cases}$$

Then each $\varphi_{2,\alpha}$ is expressed as

$$\varphi_{2,0} = \phi_2(\xi)\begin{pmatrix} 2\xi_1\xi_2 \\ \xi_2^2 - \xi_1^2 \\ 2\xi_1\xi_3 \\ 2\xi_2\xi_3 \\ \sqrt{3}(\xi_1^2 + \xi_2^2 - 2\xi_3^2) \end{pmatrix}$$

and

$$\varphi_{2,\pm 1}(\xi) = \frac{1}{2}\phi_2(\xi)\begin{pmatrix} -(\xi_2^2 - \xi_1^2) \\ 2\xi_1\xi_2 \\ -\xi_2\xi_3 \\ \xi_1\xi_3 \\ 0 \end{pmatrix} + \frac{i}{2}\phi_1(\xi)\begin{pmatrix} \pm 2\xi_1\xi_2\xi_3 \\ \pm(\xi_2^2 - \xi_1^2)\xi_3 \\ \pm\xi_1(\xi_3^2 - \xi_1^2 - \xi_2^2) \\ \pm\xi_2(\xi_3^2 - \xi_1^2 - \xi_2^2) \\ \pm\sqrt{3}\xi_3(\xi_1^2 + \xi_2^2) \end{pmatrix}$$

and

$$\varphi_{2,\pm 2}(\xi) = \frac{1}{2}\phi_2(\xi)\begin{pmatrix} (\xi_2^2 - \xi_1^2)\xi_3 \\ -2\xi_1\xi_2\xi_3 \\ -\xi_2(\xi_1^2 + \xi_2^2) \\ \xi_1(\xi_1^2 + \xi_2^2) \\ 0 \end{pmatrix} + \frac{i}{2}\phi_1(\xi)\begin{pmatrix} \mp\xi_1\xi_2(|\xi|^2 + \xi_3^2) \\ \mp\frac{1}{2}(\xi_2^2 - \xi_1^2)(|\xi|^2 + \xi_3^2) \\ \pm\xi_1\xi_3(\xi_1^2 + \xi_2^2) \\ \pm\xi_2\xi_3(\xi_1^2 + \xi_2^2) \\ \pm\frac{\sqrt{3}}{2}(\xi_1^2 + \xi_2^2)^2 \end{pmatrix},$$

respectively. We set $\varphi = \sum C_{\varphi_{2,\alpha}}^{-\frac{1}{2}}\varphi_{2,\alpha}$, and

$$\begin{aligned} \varphi &= \sum C_{\varphi_{2,\alpha}}^{-\frac{1}{2}}\varphi_{2,\alpha} \\ &= C_{\varphi_{2,0}}^{-\frac{1}{2}}\varphi_{2,0} + 2C_{\varphi_{2,1}}^{-\frac{1}{2}}\mathrm{Re}\ \varphi_{2,1} + 2C_{\varphi_{2,2}}^{-\frac{1}{2}}\mathrm{Re}\ \varphi_{2,2} \\ &= C_{\varphi_{2,0}}^{-\frac{1}{2}}\phi_2(\xi)\begin{pmatrix} 2\xi_1\xi_2 - a(\xi_2^2 - \xi_1^2) + b(\xi_2^2 - \xi_1^2)\xi_3 \\ \xi_2^2 - \xi_1^2 + 2a\xi_1\xi_2 - 2b\xi_1\xi_2\xi_3 \\ 2\xi_1\xi_3 - a\xi_2\xi_3 - b\xi_2(\xi_1^2 + \xi_2^2) \\ 2\xi_2\xi_3 + a\xi_1\xi_3 + b\xi_1(\xi_1^2 + \xi_2^2) \\ \sqrt{3}(\xi_1^2 + \xi_2^2 - 2\xi_3^2) \end{pmatrix}, \end{aligned}$$

where $a = \sqrt{\frac{C_{\varphi_{2,1}}}{C_{\varphi_{2,0}}}}$ and $b = \sqrt{\frac{C_{\varphi_{2,2}}}{C_{\varphi_{2,0}}}}$. Therefore, writing

$$\begin{pmatrix} P_{1,a,b}(\xi) \\ P_{2,a,b}(\xi) \\ P_{3,a,b}(\xi) \\ P_{4,a,b}(\xi) \\ P_{5,a,b}(\xi) \end{pmatrix} = \begin{pmatrix} 2\xi_1\xi_2 - a(\xi_2^2 - \xi_1^2) + b(\xi_2^2 - \xi_1^2)\xi_3 \\ \xi_2^2 - \xi_1^2 + 2a\xi_1\xi_2 - 2b\xi_1\xi_2\xi_3 \\ 2\xi_1\xi_3 - a\xi_2\xi_3 - b\xi_2(\xi_1^2 + \xi_2^2) \\ 2\xi_2\xi_3 + a\xi_1\xi_3 + b\xi_1(\xi_1^2 + \xi_2^2) \\ \sqrt{3}(\xi_1^2 + \xi_2^2 - 2\xi_3^2) \end{pmatrix},$$

we have

$$f_0(x) = \mathcal{F}_3^{-1}\varphi(x) = (\frac{1}{2\pi i})^3 \begin{pmatrix} P_{1,a,b}(D_x) \\ P_{2,a,b}(D_x) \\ P_{3,a,b}(D_x) \\ P_{4,a,b}(D_x) \\ P_{5,a,b}(D_x) \end{pmatrix} \bigotimes \frac{1}{(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)}.$$

As in Section 3.3, in this case also it is difficult to compute $\mathcal{F}_3^{-1}(\mathrm{Im}\varphi_{2,\alpha})$. If we replace $e^{-2\pi\xi\cdot y}$ by $e^{-2\pi|\xi|}$, we can compute each admissible vector $\varphi_{2,\alpha}$.


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