# LOW REGULARITY WELL-POSEDNESS FOR NONLINEAR DISPERSIVE EQUATIONS

SHINYA KINOSHITA

## Contents

1. Introduction 2
1.1. Introduction of (NLS) 2
1.2. Introduction of (KGZ) 5
1.3. Notations 6
Acknowledgement 6
2. The Cauchy problem of Hartree and pure power type nonlinear Schrödinger equations 7
2.1. Introduction 7
2.2. Preliminaries 11
2.3. (HNLS) 13
2.4. (PNLS) 20
2.5. (NLS) with inhomogeneous nonlinearities 25
3. Well-posedness of the Klein-Gordon-Zakharov system in 2D 29
3.1. Introduction 29
3.2. Preliminaries 32
3.3. Proof of Theorem 3.2for $\pm_1 = \pm_2$. 35
3.4. Proof of Theorem 3.2for $\pm_1 \neq \pm_2$. 39
4. Well-posedness of the Klein-Gordon-Zakharov system in $d \geq 5$ 57
4.1. Introduction 57
4.2. Notations and Preliminary Lemmas 58
4.3. Bilinear estimates 69
4.4. The proof of the main theorem 75
References 78

## 1. Introduction

We consider the Cauchy problem of the nonlinear Schrödinger equations (NLS):

$$\begin{cases} i\partial_t u + \Delta u = N(u), & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = \varphi \in \dot{H}^s(\mathbb{R}^d), \end{cases} \tag{1.1}$$

and the Cauchy problem of the Klein-Gordon-Zakharov system (KGZ):

$$\begin{cases} (\partial_t^2 - \Delta + 1)u = -nu, & (t,x) \in [-T,T] \times \mathbb{R}^d, \\ (\partial_t^2 - c^2\Delta)n = \Delta|u|^2, & (t,x) \in [-T,T] \times \mathbb{R}^d, \\ (u, \partial_t u, n, \partial_t n)|_{t=0} = (u_0, u_1, n_0, n_1) \\ \qquad \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d), \end{cases} \tag{1.2}$$

where $u, n$ are real valued functions, $0 < c < 1$. Our aim in this thesis is to prove the local or global in time well-posedness of (1.1) and (1.2) in low regularity Sovolev spaces. We first give an introduction and state our results on (NLS).

### 1.1. **Introduction of (NLS).**

In (1.1), we consider two types of nonlinearities. We first study the Cauchy problem of Hartree type nonlinear Schrödinger equations (HNLS):

$$\begin{cases} i\partial_t u + \Delta u = F(u), & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = \varphi \in \dot{H}^s(\mathbb{R}^d). \end{cases} \tag{1.3}$$

$F(u)$ is a nonlinear functional of Hartree type:

$$F(u) = (\lambda|x|^{-\gamma} * |u|^2)u, \quad \lambda \in \mathbb{C} \setminus \{0\},\ 0 < \gamma < d,$$

where $*$ denotes the convolution in $\mathbb{R}^d$. By the following scaling transformation:

$$u_\eta(t,x) = \eta^{\frac{d+2-\gamma}{2}} u(\eta^2 t, \eta x), \quad \eta > 0,$$

we see that (HNLS) has the scaling invariance in $\dot{H}^{s_c}$ with the critical index $s_c = \frac{\gamma-2}{2}$. The critical index is important in the sense that it is strongly believed that we cannot obtain the well-posedness of (1.1) for $s < s_c$. Therefore, it is natural that we aim to get the well-posedness of (1.3) with the scaling critical regularity $s = s_c$. Our first result is as follows.

**Theorem 1.1.** *Let $d \geq 3$, $4/3 < \gamma < 2$ and assume that $\varphi \in \dot{H}^{s_c}(\mathbb{R}^d)$ is radially symmetric and small, then (1.3) is globally well-posed in $\dot{H}^{s_c}(\mathbb{R}^d)$.*

We make a comment on Theorem 1.1. Seeing $s_c = \frac{\gamma-2}{2}$, the critical index is negative when $4/3 < \gamma < 2$. Generally speaking, it is difficult to show that (1.3) is well-posed in a negative regularity space since we need to recover the derivative (regularity) loss when we estimate the nonlinear term. The usual Strichartz estimates for Schrödinger equations, however, cannot work such recovery. To overcome this difficulty, we assume that an initial data is radially symmetric. It is known that we can get the better Strichartz estimates for radially symmetric functions. See [9]. However, the radial symmetricity of an initial data seems to be very restrictive. Next theorem ensures that the similar result holds for non-radial initial data.

**Theorem 1.2.** *Let $d \geq 3$, $4/3 < \gamma < 2$ and $\delta = \delta(d, \gamma) > 0$ be sufficiently small. Assume that $\varphi \in \dot{H}^{s_c} H_\omega^{\frac{3}{4}(2-\gamma)+\delta}$ is small, then (1.3) is globally well-posed in $\dot{H}^{s_c} H_\omega^{\frac{3}{4}(2-\gamma)+\delta}$.*

The function space $\dot{H}^s H_\omega^{\alpha,q}$ is defined as follows.

$$\dot{H}^s H_\omega^{\alpha,q} = \{f \in \mathcal{S}' \setminus \mathcal{P} : \|f\|_{\dot{H}^s H_\omega^{\alpha,q}} < \infty\}, \quad s, \alpha \in \mathbb{R},$$
$$\|f\|_{\dot{H}^s H_\omega^{\alpha,q}} = \||\nabla|^s D_\omega^\alpha f\|_{L_r^2 L_\omega^q},$$

where

$$\|f\|_{L_r^p L_\omega^q} = \left( \int_0^\infty \left( \int_{S^{d-1}} |f(r\omega)|^q d\omega \right)^{\frac{p}{q}} r^{d-1} dr \right)^{\frac{1}{p}}, \quad 1 \leq p, q < \infty.$$

Here $\mathcal{S}$ is the Schwartz space, $\mathcal{P}$ denotes the totality of polynomials. $|\nabla| = \sqrt{-\Delta}$, and $D_\omega = \sqrt{1 - \Delta_\omega}$ for the Laplace-Beltrami operator $\Delta_\omega$. The operator on the unit sphere $D_\omega$ is very similar to $\sqrt{1-\Delta}$. We refer to Appendix in [27] for the details of $D_\omega$. Roughly speaking, Theorem 1.2 says that if the initial datum $\varphi(r\omega)$ has some regularity with respect to the angular variable $\omega$ then the same result as Theorem 1.1 holds. We also consider the subcritical case, $s_c < s < 0$. The following two theorems show the local well-posedness in time for large initial data. The important difference from the sclaing critical case is that they include the case $d = 2$ and $0 < \gamma \leq \frac{4}{3}$ under the restriction $-\frac{\gamma}{4} < s$. They are established by the almost same proof as that of the scaling critical case.

**Theorem 1.3.** *Let $d \geq 2$, $0 < \gamma < 2$ and*

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < \ 0.$$

*Assume that $\varphi \in \dot{H}^s(\mathbb{R}^d)$ is radially symmetric, then (1.3) is locally well-posed in $\dot{H}^s(\mathbb{R}^d)$.*

**Theorem 1.4.** *Let $d \geq 2$, $0 < \gamma < 2$,*

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < \ 0,$$

*and suppose that $\delta = \delta(d, s, \gamma) > 0$ is sufficiently small. Then* (1.3) *is locally well-posed in $\dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}(\mathbb{R}^d)$.*

We next consider (1.1) with another nonlinearity called pure-power type nonlinearity.

$$\begin{cases} i\partial_t u + \Delta u = G(u), & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = \varphi \in \dot{H}^s(\mathbb{R}^d). \end{cases} \tag{1.4}$$

Here $G(u)$ is a nonlinear functional of pure power type:

$$G(u) = \lambda |u|^{p-1} u, \qquad \lambda \in \mathbb{C} \setminus \{0\}, \quad 1 < p.$$

Similarly to (HNLS) case, the following scaling transformation

$$\lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \qquad \lambda > 0,$$

shows that (PNLS) has the scaling invariance in $\dot{H}^{s_{c,p}}$ with the scale critical index $s_{c,p} = \frac{d}{2} - \frac{2}{p-1}$. Our result is as follows.

**Theorem 1.5.** *Let $3 \leq d \leq 14$, $p_0 < p < 1 + 4/d$ where $p_0$ is a unique solution of*

$$\begin{cases} 1 + \frac{4}{d+1} \leq p_0 < 1 + \frac{4}{d}, \\ 2p_0^3 + 6(d-2)p_0^2 + (d^2 - 13d + 10)p_0 - d(d-3) = 0, \end{cases}$$

*and suppose that $\delta = \delta(d, p) > 0$ is sufficiently small. Assume that $\varphi \in \dot{H}^{s_{c,p}} H_\omega^{s_0}(\mathbb{R}^d)$ is small, then* (1.4) *is globally well-posed in $\dot{H}^{s_{c,p}} H_\omega^{s_0}(\mathbb{R}^d)$ where*

$$s_0 = \begin{cases} \frac{1}{p-1}(7 - 3p) + \delta & (\text{if } d = 3), \\ \frac{1}{2(p-1)^2}\left(-(d+1)p^2 + (d+7)p - 2\right) + \delta & (\text{if } d \geq 4). \end{cases}$$

We make a comment on Theorem 1.5. In [21], Hidano proved the global existence for radially symmetric small initial data $\varphi \in \dot{H}^{s_{c,p}}$ if $d \geq 3$ and $1 + \frac{4}{d+1} < p < 1 + \frac{4}{d}$. Compared to radial case, the conditions for $d$ and $p$ in Theorem 1.5 seem to be very restrictive. This complicated restrictions is necessary when we apply the Moser type estimate on the unit sphere such that

$$\|D_\omega^s(|u|^{p-1}u)\|_{L_\omega^2} \lesssim \|u\|_{L_\omega^{q_0}}^{p-1} \|D_\omega^s u\|_{L_\omega^{q_1}}$$

where $1/2 = (p-1)/q_0 + 1/q_1$. The condition $p < 1 + 4/d$ means $p < 2$ when $d \geq 4$. Thus we need the condition $s \leq 1$ to verify the above estimate, which causes the restriction in Theorem 1.5.

1.2. **Introduction of (KGZ).** Next we consider the Klein-Gordon-Zakharov system. By the transformation $u_\pm := \omega_1 u \pm i\partial_t u, n_\pm := n \pm i(c\omega)^{-1}\partial_t n, \omega_1 := (1-\Delta)^{1/2}, \omega := (-\Delta)^{1/2}$, (1.2) can be written as follows;.

$$\begin{cases} (i\partial_t \mp \omega_1)u_\pm = \pm(1/4)(n_+ + n_-)(\omega_1^{-1}u_+ + \omega_1^{-1}u_-), & (t,x) \in [-T,T]\times\mathbb{R}^d, \\ (i\partial_t \mp c\omega)n_\pm = \pm(4c)^{-1}\omega|\omega_1^{-1}u_+ + \omega_1^{-1}u_-|^2, & (t,x)\in[-T,T]\times\mathbb{R}^d, \\ (u_\pm, n_\pm)|_{t=0} = (u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d)\times \dot{H}^s(\mathbb{R}^d). \end{cases} \tag{1.5}$$

We state our results for $d = 2$ and for $d \geq 5$.

**Theorem 1.6.** *Let $d = 2$ and $-3/4 < s < 0$. Then* (1.5) *is locally well-posed in $H^s(\mathbb{R}^2)\times\dot{H}^s(\mathbb{R}^2)$.*

**Theorem 1.7.** *Let $d \geq 5, s = s_c = d/2 - 2$ and assume the initial data $(u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d)\times\dot{H}^s(\mathbb{R}^d)$ is small. Then,* (1.5) *is globally well-posed in $H^s(\mathbb{R}^d)\times\dot{H}^s(\mathbb{R}^d)$.*

As a byproduct of Theorem 1.7, we can show that the obtained solution scatters.

**Corollary 1.8.** *The solution obtained in Theorem 1.7 scatters as $t \to \pm\infty$.*

We make a comment on the above theorems. Theorems 1.6 and 1.7 are both established by the Fourier restriction norm method introduced by Bourgain [5]. The Fourier restriction norm method, together with the function space $X^{s,b}$ called Bourgain space, has been applied to lots of dispersive equations and produced many remarkable results. We can find that the method also works effectively for (1.5). As an advantage of the Fourier restriction norm method, we can gain the extra regularity when we estimate the nonlinear term. Precisely speaking, we recover a half derivative loss. It should be emphasized that such recovery disappears in the case $c = 1$.

For the proof of Theorem 1.6, in fact, the Fourier restriction norm method is not enough to get the well-posedness for $s \leq -1/2$. Hence we employ the new estimates which was introduced in [3] and applied to Zakharov system in [1] and [2]. Zakharov system consists of two equations, wave equation and Schrödinger equation;

$$\begin{cases} (i\partial_t + \Delta)u = nu, & (t,x)\in\mathbb{R}\times\mathbb{R}^d, \\ (\partial_t^2 - \Delta)n = \Delta|u|^2, & (t,x)\in\mathbb{R}\times\mathbb{R}^d. \end{cases} \tag{1.6}$$

Roughly speaking, comparing (1.2) and (1.6), the two systems have similar structures, which suggests that we might get the well-posedness of (1.5) for $s \leq -1/2$ in the same way as in [1] and [2].

Theorem 1.7, which is joint work with I. Kato, is an improvement of his recent work [25]. Precisely speaking, in [25] he showed that (1.5) is globally well-posed if the the initial data $(u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$ is small and "radially symmetric". Theorem 1.7 says that the radial symmetry condition is not necessary for the well-posedness. In the proof, we apply $U^2$, $V^2$ type function spaces, which are similar to the Bourgain spaces in the sense that we can recover the derivative loss. It is known that the Bourgain spaces $X^{s,b}$ do not work well at the scaling critical regularity spaces. $U^2$, $V^2$ type spaces were developed in order to overcome such a weakness of the Fourier restriction norm method. In fact, $U^2$, $V^2$type spaces were already applied in [25]. In this thesis, the improvement is mainly done by using bilinear Strichartz estimates. See Propositions 4.19, 4.21 and 4.23 which hold true under the condition $c \neq 1$.

1.3. **Notations.** We introduce notations which will be utilized throughout the paper. $A \lesssim B$ means that there exists $C > 0$ such that $A \leq CB$. Also, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. Let $u = u(t, x)$. $\mathcal{F}_t u$, $\mathcal{F}_x u$ denote the Fourier transform of $u$ in time, space, respectively. $\mathcal{F}_{t,x} u = \widehat{u}$ denotes the Fourier transform of $u$ in space and time. $\chi_\Omega$ denotes the characteristic function of a set $\Omega$ and $\langle \cdot \rangle$ denotes $(1 + |\cdot|^2)^{1/2}$. We denote by $H^s$ and $\dot{H}^s$, $s \in \mathbb{R}$, the usual inhomogeneous Sobolev spaces and homogeneous Sobolev spaces, respectively. We denote the space $L^q(\mathbb{R}; X)$ by $L_t^q X$ and its norm by $\| \cdot \|_{L_t^q X}$ for some Banach space $X$, and also $L^q([0, T]; X)$ by $L_{I_T}^q X$ and its norm by $\| \cdot \|_{L_{I_T}^q X}$. We use the notations $C_b(\mathbb{R}; X) = C(\mathbb{R}; X) \cap L^\infty(\mathbb{R}; X)$ and $L_{x,t}^q = L_t^q L_x^q$, $L_{\xi,\tau}^q = L_\tau^q L_\xi^q$ for $\xi \in \mathbb{R}^d$, $\tau \in \mathbb{R}$.

The thesis is organized as follows. In Section 2, we consider the Cauchy problem of Nonlinear Schrödinger equations and establish Theorems 1.1-1.5. In Sections 3 and 4, we consider the Cauchy problem of Klein-Gordon-Zakharov system. Section 3 is devoted to the proof of Theorem 1.6 and finally we verify Theorem 1.7 in Section 4.

**Acknowledgement.** The author appreciates Professor Mitsuru Sugimoto, Professor Kotaro Tsugawa, Professor Isao Kato and Professor Tomoya Kato for giving many useful advices to the author.

## 2. The Cauchy problem of Hartree and pure power type nonlinear Schrödinger equations

2.1. **Introduction.** We consider the Cauchy problem of Hartree type nonlinear Schrödinger equations (HNLS):

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = F(u(t,x)), & \text{in } \mathbb{R}\times\mathbb{R}^d, \\ u(0,x) = \varphi(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{2.1}$$

Here $\Delta$ is the Laplacian in $\mathbb{R}^d$. $F(u)$ is a nonlinear functional of Hartree type:

$$F(u) = (\lambda|x|^{-\gamma} * |u|^2)u, \quad \lambda \in \mathbb{C}\setminus\{0\},\ 0<\gamma<d.$$

From Duhamel's formula, the solution $u$ of (2.1) can be written as

$$u(t,x) = U(t)(\varphi + \Phi_t)(x), \tag{2.2}$$

where

$$U(t) = e^{it\Delta}, \quad \Phi_t = \Phi_t(u) = -i\int_0^t U(-t')F(u)(t')dt'.$$

By the following scaling transformation:

$$u_\eta(t,x) = \eta^{\frac{d+2-\gamma}{2}} u(\eta^2 t, \eta x), \quad \eta > 0,$$

we see that (HNLS) has the scaling invariance in $\dot{H}^{s_c}$ with the critical index $s_c = \frac{\gamma-2}{2}$.

There are lots of works on the Cauchy problem of (HNLS). Almost all of them discussed the problem for $\varphi \in H^s$, $s \geq \max(0, s_c)$. As a fundamental result, Miao, Xu and Zhao [36] proved the local well-posedness in $H^s$ where $s > s_c$, $s \geq 0$. Furthermore for $s \geq 1$, by the energy conservation law, they proved the global well-posedness for $0 < \gamma \leq 2$, $\gamma < d$, $\lambda \geq 0$ and for $0 < \gamma < \min(2,d)$, $\lambda < 0$, and in particular, for $s = 1$, the global well-posedness was established for $2 < \gamma < 4$, $\gamma < d$ and $\lambda \geq 0$. In addition, the smallness condition of $\|\varphi\|_{\dot{H}^{s_c}_x}$ ensures the global existence in $H^s$, $s > s_c$ for $2 \leq \gamma < d$, $d \geq 3$. In [20], Hayashi and Ozawa proved the global well-posedness in $L^2$ for $0 < \gamma < \min(2,d)$ (see [6] for general nonlinearities). For the critical case, $s = s_c \geq 0$, (HNLS) is locally well-posed in $H^{s_c}$ for $2 \leq \gamma < d$, and globally well-posed and the solutions behave like linear ones in $H^{s_c}$ for $2 \leq \gamma < d$, $d \geq 3$ under the smallness condition of $\|\varphi\|_{\dot{H}^{s_c}_x}$ (see [36, 7, 6]). If initial data $\varphi$ has finite energy, it is known that (HNLS) is globally well-posed in $\dot{H}^1$ for $\gamma = 4$, $\lambda \geq 0$, $d \geq 5$ (see [37], and see also [35] for radially symmetric initial data).

As opposed to the case $s \geq \max(0, s_c)$, we have few results for $s_c \leq s < 0$ . Miao, Xu and Zhao [36] proved some ill-posedness results for $s < \max(0, s_c)$, while Cho, Hwang and Ozawa [8] proved the global well-posedness for radially symmetric small data $\varphi \in \dot{H}^{s_c}$, $\frac{8d-2}{6d-3} \leq \gamma < 2$:

**Theorem A** ([8] Theorem 5)**.** *Let $d \geq 2$, $\frac{8d-2}{6d-3} \leq \gamma < 2$. Then there exists a positive constant $\varepsilon = \varepsilon(d,\gamma)$ such that if $\varphi \in \dot{H}^{s_c}(\mathbb{R}^d)$ is radially symmetric and satisfies $\||\nabla|^{s_c}\varphi\|_{L^2_x} < \varepsilon$, then (2.2) has a unique radial solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_c}(\mathbb{R}^d)) \cap L^3(\mathbb{R}; L^r(\mathbb{R}^d)).$$

*Here $r$ satisfies $\frac{1}{r} = \frac{1}{2} - \frac{2}{3d} - \frac{s}{d}$. In addition, $u$ scatters in $\dot{H}^{s_c}(\mathbb{R}^d)$.*

They also discussed the problem of global well-posedness without assuming radial symmetry:

**Theorem B** ([8] Theorem 2)**.** *Let $d \geq 3$, $2 - \frac{3}{2d+2} < \gamma < 2$, $s_1 = \frac{d-1}{d+1} - \frac{\gamma-1}{2}$ and*

$$\max\left(\gamma - \frac{5d-3}{2d+2}, \frac{1}{2}\right) < s_2 < \min\left(\gamma - \frac{3d}{2d+2}, \frac{3(d-1)}{2d+2}\right).$$

*Then there exist a positive constant $\varepsilon = \varepsilon(d,\gamma)$ and $\alpha_1$, $\alpha_2 \in [2,\infty]$, $\beta_1$, $\beta_2 \in \mathbb{R}$, $\gamma_1$, $\gamma_2 \in (0,\infty)$ such that if $\varphi \in \dot{H}^{s_c}H^{s_1+s_2}_\omega(\mathbb{R}^d)$ satisfies $\||\nabla|^{s_c}D^{s_1+s_2}_\omega\varphi\|_{L^2_x} < \varepsilon$, then (2.2) has a unique solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_c}H^{s_1+s_2}_\omega(\mathbb{R}^d)) \cap L^{\alpha_1}(\mathbb{R}; |x|^{\beta_1}L^2_rH^{\gamma_1}_\omega) \cap L^{\alpha_2}(\mathbb{R}; |x|^{\beta_2}L^2_rH^{\gamma_2}_\omega).$$

*In addition, $u$ scatters in $\dot{H}^{s_c}H^{s_1+s_2}_\omega(\mathbb{R}^d)$.*

The main goal of this section is to widen the range of $\gamma$ in Theorems A and B in the case $d \geq 3$. That is, we improve the conditions $\frac{8d-2}{6d-3} \leq \gamma < 2$ in Theorem A and $2 - \frac{3}{2d+2} < \gamma < 2$ in Theorem B to $\frac{4}{3} < \gamma < 2$. To describe it precisely, we should introduce some function spaces. We define the norm

$$\|f\|_{L^p_rL^q_\omega} = \left(\int_0^\infty \left(\int_{S^{d-1}} |f(r\omega)|^q d\omega\right)^{\frac{p}{q}} r^{d-1}dr\right)^{\frac{1}{p}}, \quad 1 \leq p, q < \infty.$$

We also define the modified Sobolev space $\dot{H}^sH^\alpha_\omega$ and its norm by

$$\dot{H}^sH^{\alpha,q}_\omega = \{f \in \mathcal{S}' \setminus \mathcal{P} : \|f\|_{\dot{H}^sH^{\alpha,q}_\omega} < \infty\}, \quad s, \alpha \in \mathbb{R},$$
$$\|f\|_{\dot{H}^sH^{\alpha,q}_\omega} = \||\nabla|^sD^\alpha_\omega f\|_{L^2_rL^q_\omega}.$$

Here $\mathcal{S}$ is the Schwartz space, $\mathcal{P}$ denotes the totality of polynomials. $|\nabla| = \sqrt{-\Delta}$, and $D_\omega = \sqrt{1-\Delta_\omega}$ for the Laplace-Beltrami operator $\Delta_\omega$. We refer to [27], Appendix, [24] and [46] for the details of $D_\omega$. We denote $\dot{H}^0H^{\alpha,q}_\omega$ and $\dot{H}^sH^{\alpha,2}_\omega$ by $L^2_rH^{\alpha,q}_\omega$ and $\dot{H}^sH^\alpha_\omega$, respectively.

Our results are the following. The first one is radially symmetric case, and the second is general case:

**Theorem 2.1.** *Let $d \geq 3$, $\frac{4}{3} < \gamma < 2$ and $\delta = \delta(d, \gamma) > 0$ be sufficiently small. Then there exist a positive constant $\varepsilon = \varepsilon(d, \gamma)$ and exponents $q_1$,$q_2$,$\ell \in [2, \infty]$ such that if $\varphi \in \dot{H}^{s_c}(\mathbb{R}^d)$ is radially symmetric and satisfies $\||\nabla|^{s_c}\varphi\|_{L^2_x} < \varepsilon$, then (2.2) has a unique radial solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_c}(\mathbb{R}^d)) \cap L^{q_1}(\mathbb{R}; |x|^{s_c-\delta}L^2(\mathbb{R}^d)) \cap L^{q_2}(\mathbb{R}; |x|^{s_c}L^{\ell}(\mathbb{R}^d)).$$

**Theorem 2.2.** *Let $d \geq 3$, $\frac{4}{3} < \gamma < 2$ and $\delta = \delta(d, \gamma) > 0$ be sufficiently small. Then there exist a positive constant $\varepsilon = \varepsilon(d, \gamma)$ and exponents $q_1$,$q_2$,$\ell$,$\sigma \in [2, \infty]$ such that if $\varphi \in \dot{H}^{s_c}H_\omega^{\frac{3}{4}(2-\gamma)+\delta}$ satisfies $\||\nabla|^{s_c}D_\omega^{\frac{3}{4}(2-\gamma)+\delta}\varphi\|_{L^2_x} < \varepsilon$, then (2.2) has a unique solution*

$$\begin{aligned} u \in C_b(\mathbb{R}; \dot{H}^{s_c}H_\omega^{\frac{3}{4}(2-\gamma)+\delta}) &\cap L^{q_1}(\mathbb{R}; |x|^{s_c-\delta}L^2_r H_\omega^{\frac{3}{4}(2-\gamma)+\frac{3}{2}\delta}) \\ &\cap L^{q_2}(\mathbb{R}; |x|^{s_c}L^{\ell}_r H_\omega^{\frac{3}{4}(2-\gamma)+(\frac{3}{2}-\frac{1}{d})\delta,\sigma}). \end{aligned}$$

*Remark* 2.1. Actually, the solutions of Theorems 2.1 and 2.2 scatter in $\dot{H}^{s_c}(\mathbb{R}^d)$ and $\dot{H}^{s_c}H_\omega^{\frac{3}{4}(2-\gamma)+\delta}$, respectively. See [8] for the details.

Next, we consider the subcritical case, $s_c < s < 0$. The following two theorems show the local well-posedness in time for large initial data. The important difference from the critical case is that they include the case $d = 2$ and $0 < \gamma \leq \frac{4}{3}$ under the restriction $-\frac{\gamma}{4} < s$.

**Theorem 2.3.** *Let $d \geq 2$, $0 < \gamma < 2$,*

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < 0,$$

*and suppose that $\delta = \delta(d, s, \gamma) > 0$ is sufficiently small. Then there exist a positive time $T$ and exponents $\alpha \in \mathbb{R}$, $q_1$,$q_2$,$\ell \in [2, \infty]$ such that if $\varphi \in \dot{H}^s(\mathbb{R}^d)$ is radially symmetric then (2.2) has a unique radial solution*

$$u \in C([0, T]; \dot{H}^s(\mathbb{R}^d)) \cap L^{q_1}([0, T]; |x|^{s-\delta}L^2(\mathbb{R}^d)) \cap L^{q_2}([0, T]; |x|^{\alpha}L^{\ell}(\mathbb{R}^d)).$$

**Theorem 2.4.** *Let $d \geq 2$, $0 < \gamma < 2$,*

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < 0,$$

*and suppose that $\delta = \delta(d, s, \gamma) > 0$ is sufficiently small. Then there exist a positive time $T$ and exponents $\alpha$,$\beta \in \mathbb{R}$, $q_1$,$q_2$,$\ell$,$\sigma \in [2, \infty]$ such that if $\varphi \in \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}(\mathbb{R}^d)$ then (2.2) has a unique solution*

$$u \in C([0, T]; \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}) \cap L^{q_1}([0, T]; |x|^{s-\delta}L^2_r H_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}) \cap L^{q_2}([0, T]; |x|^{\alpha}L^{\ell}_r H_\omega^{\beta,\sigma}).$$

*Remark* 2.2. If $-s$ is sufficiently close to 0 then the necessary angular regularity for $\varphi$ is sufficiently small. This seems to be natural since we do not need angular regularity assumption if $s \geq 0$.

Next, we study the Cauchy problem of pure power type nonlinear Schrödinger equations (PNLS):

$$\begin{cases} iu_t + \Delta u = G(u), & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = \varphi(x), & \text{in } \mathbb{R}^d. \end{cases}$$

Here $G(u)$ is a nonlinear functional of pure power type:

$$G(u) = \lambda |u|^{p-1} u, \qquad \lambda \in \mathbb{C} \setminus \{0\}, \quad 1 < p.$$

Similarly to (HNLS) case, the following scaling transformation

$$\lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \qquad \lambda > 0,$$

shows that (PNLS) has the scaling invariance in $\dot{H}^{s_{c,p}}$ with the scale critical index $s_{c,p} = \frac{d}{2} - \frac{2}{p-1}$. There exist a lot of works on the Cauchy problem of (PNLS). See [47, 7, 14, 41, 38, 39].

In [21], Hidano proved the global existence for radially symmetric small initial data $\varphi \in \dot{H}^{s_{c,p}}$ if $d \geq 3$ and $1 + \frac{4}{d+1} < p < 1 + \frac{4}{d}$. After that, Fang and Wang [13] proved the global existence for small initial data $\varphi \in \dot{H}^{s_{c,p}} H_\omega^{\frac{1}{p-1}}$ if $3 \leq d \leq 6$ and $1 + \sqrt{\frac{2}{d-1}} < p < 1 + \frac{4}{d}$. We relax the conditions of $n$ and $p$ in the general case. Our result is the following:

**Theorem 2.5.** *Let $3 \leq d \leq 14$, $p_0 < p < 1 + 4/d$ where $p_0$ is a unique solution of*

$$\begin{cases} 1 + \frac{4}{d+1} \leq p_0 < 1 + \frac{4}{d}, \\ 2p_0^3 + 6(d-2)p_0^2 + (d^2 - 13d + 10)p_0 - d(d-3) = 0, \end{cases}$$

*and suppose that $\delta = \delta(d,p) > 0$ is sufficiently small. Then there exist a positive constant $\varepsilon = \varepsilon(d,p)$ and exponents $\alpha \in \mathbb{R}$, $q, \ell, \sigma \in [2, \infty]$ such that if $\varphi \in \dot{H}^{s_{c,p}} H_\omega^{s_0}(\mathbb{R}^d)$ satisfies $\| |\nabla|^{s_{c,p}} D_\omega^{s_0} \varphi \|_{L_x^2} < \varepsilon$ where*

$$s_0 = \begin{cases} \frac{1}{p-1}(7 - 3p) + \delta & (\text{if } d = 3), \\ \frac{1}{2(p-1)^2}\left(-(d+1)p^2 + (d+7)p - 2\right) + \delta & (\text{if } d \geq 4), \end{cases}$$

*then the integral equation*

$$u(t,x) = U(t)(\varphi + \Phi_{t,p})(x), \tag{2.3}$$

*where*

$$\Phi_{t,p} = \Phi_{t,p}(u) = -i \int_0^t U(-t')G(u)(t')dt',$$

*has a unique solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_{c,p}} H_\omega^{s_0}) \cap L^q(\mathbb{R}; |x|^\alpha L^\ell H_\omega^{s_0,\sigma}).$$

*Remark* 2.3. Similarly to (HNLS) case, the solution of Theorem 2.5 scatters in $\dot{H}^{s_{c,p}} H_\omega^{s_0}(\mathbb{R}^d)$, and if $d = 3, 4$ the necessary angular regularity for $\varphi$ gets close to 0 as $-s_{c,p}$ approaches 0.

In Section 2.2, we introduce some estimates as preliminaries. In Section 2.3, we consider the Cauchy problem of (HNLS). To avoid redundancy, we only establish Theorem 2.4. In Section 2.4, we establish Theorem 2.5. Lastly in Section 2.5, we consider the Cauchy problem of inhomogeneous power type nonlinear Schrödinger equations.

2.2. **Preliminaries.** In this section, we introduce some estimates which will be used for the proof of the main results.

First, we introduce weighted Strichartz estimates for $U(t)$.

**Lemma 2.6** ([13] Theorem 1.15, [8] Lemma 2)**.** *Let $d \geq 2$, $2 \leq q \leq \infty$.*
(i) *If $c, \delta_1$ satisfy*

$$-\frac{d}{q} < c < -\frac{d}{q} + \frac{d-1}{2}, \quad \delta_1 \leq -\frac{d}{q} + \frac{d-1}{2} - c,$$

*then we have*

$$\| |x|^c |\nabla|^{c + \frac{d+2}{q} - \frac{d}{2}} D_\omega^{\delta_1} [U(t)\varphi] \|_{L_t^q L_r^q L_\omega^2} \lesssim \|\varphi\|_{L_x^2}. \tag{2.4}$$

(ii) *If $c, \delta_2$ satisfy*

$$-\frac{d}{q} < c < -\frac{1}{q}, \quad \delta_2 \leq -c - \frac{1}{q},$$

*then we have*

$$\| |x|^c |\nabla|^{c + \frac{2}{q}} D_\omega^{\delta_2} [U(t)\varphi] \|_{L_t^q L_x^2} \lesssim \|\varphi\|_{L_x^2}.$$

By interpolating between the inequality (2.4) and the classical Strichartz estimates, we immediately get the following weighted Strichartz estimates.

**Lemma 2.7.** *Let $d \geq 2$, $2 \leq \sigma \leq \ell \leq \infty$ and*

$$\begin{cases} \frac{1}{2} - \frac{1}{\sigma} \leq \frac{1}{q} < \frac{1}{2} + \frac{1}{\ell} - \frac{1}{\sigma} & (\text{if } d = 2), \\ \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\sigma}\right) \leq \frac{1}{q} \leq \frac{1}{2} + \frac{1}{\ell} - \frac{1}{\sigma} & (\text{if } d \geq 3). \end{cases}$$

*If $w$, $\delta$ satisfy*

$$\frac{d^2}{4} - \frac{d}{q} - \frac{d^2}{2\sigma} < w < \frac{d}{4} - \frac{1}{q} - \frac{d-1}{\ell} + \frac{d-2}{2\sigma},$$
$$\delta \le -w + \frac{d}{4} - \frac{1}{q} - \frac{d-1}{\ell} + \frac{d-2}{2\sigma},$$

*then we have*

$$\||x|^w |\nabla|^{w-\frac{d}{2}+\frac{2}{q}+\frac{d}{\ell}} D_\omega^\delta [U(t)\varphi]\|_{L_t^q L_r^\ell L_\omega^\sigma} \lesssim \|\varphi\|_{L_x^2}. \tag{2.5}$$

*Proof.* Suppose that

$$\theta = \frac{2}{q} - \frac{2}{\ell} + \frac{2}{\sigma} - d\left(\frac{1}{2} - \frac{1}{\sigma}\right),$$
$$\frac{1}{q_0} = \frac{d}{2(1-\theta)}\left(\frac{1}{2} - \frac{1}{\sigma}\right),$$
$$\frac{2}{q_0} = d\left(\frac{1}{2} - \frac{1}{r_0}\right),$$
$$\frac{1}{q_1} = \frac{1}{\theta}\left(\frac{1}{q} - \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\sigma}\right)\right).$$

It follows from the classical Strichartz estimates and Lemma 2.6 that

$$\|U(t)\varphi\|_{L_t^{q_0} L_x^{r_0}} \lesssim \|\varphi\|_{L_x^2}, \tag{2.6}$$
$$\||x|^c |\nabla|^{c+\frac{d+2}{q_1}-\frac{d}{2}} D_\omega^{\delta_1} [U(t)\varphi]\|_{L_t^{q_1} L_r^{q_1} L_\omega^2} \lesssim \|\varphi\|_{L_x^2}, \tag{2.7}$$

if

$$-\frac{d}{q_1} < c < -\frac{d}{q_1} + \frac{d-1}{2}, \quad \delta_1 \le -c + \frac{d-1}{2} - \frac{d}{q_1}.$$

By the complex interpolation between (2.6) and (2.7), we get (2.5). □

The following lemma is necessary to handle the nonlinear term.

**Lemma 2.8** ([10] Lemma 4.3)**.** *Let $p, q, q_1 \in [1, \infty]$, $0 \le \delta < \gamma < (d-1)/p'$,*

$$\frac{1}{q_1} \ge \frac{1}{q} - \frac{1}{p'} + \frac{\gamma}{d-1}, \quad \frac{\gamma}{d-1} \ne \frac{1}{q_1} - \frac{1}{p}.$$

*Then we have*

$$\||x|^\delta (|x|^{-\frac{d}{p}-\gamma} * f)\|_{L_r^p L_\omega^{q_1}} \lesssim \||x|^{-(\gamma-\delta)} f\|_{L_r^1 L_\omega^{q,1}},$$

*where $L_\omega^{q,1}$ is the Lorentz space on the unit sphere.*

The following lemma will be utilized for the time restriction $t' < t$. The general case was proved in [11], and see also [44].

**Lemma 2.9** ([11] Theorem 1.1)**.** *Let $1 \le r < q \le \infty$, and $X, Y$ be Banach spaces. If*

$$\|U(t)\varphi\|_{L_t^q(Y)} \lesssim \|\varphi\|_{L_x^2} \quad \text{and} \quad \|\int_{-\infty}^{\infty} U(-t')g(t')dt'\|_{L_x^2} \lesssim \|g\|_{L_t^r(X)},$$

*then we have*

$$\|\int_{-\infty}^{t} U(t-t')g(t')dt'\|_{L_t^q(Y)} \lesssim \|g\|_{L_t^r(X)}.$$

2.3. **(HNLS).** In this section, we consider the Cauchy problem of (HNLS). For convenience, we restate Theorems 2.1-2.4 with the explicit exponents.

**Theorem 2.10.** *Let $d \ge 3$, $\frac{4}{3} < \gamma < 2$ and $\delta = \delta(d,\gamma) > 0$ be sufficiently small. Then there exists a positive constant $\varepsilon = \varepsilon(d,\gamma)$ such that if $\varphi \in \dot{H}^{s_c}(\mathbb{R}^d)$ is radially symmetric and satisfies $\||\nabla|^{s_c}\varphi\|_{L_x^2} < \varepsilon$, then (2.2) has a unique radial solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_c}(\mathbb{R}^d)) \cap L^{2q_{1,s_c}}(\mathbb{R}; |x|^{s_c-\delta}L^2(\mathbb{R}^d)) \cap L^{q_{2,s_c}}(\mathbb{R}; |x|^{s_c}L^{\ell_1}(\mathbb{R}^d))$$

*where*

$$\frac{1}{q_{1,s_c}} = -2s_c + \delta, \qquad \frac{1}{q_{2,s_c}} = \frac{\gamma}{4} - \frac{\delta}{2},$$
$$\frac{1}{\ell_1} = \frac{1}{2} + \frac{2}{d} - \frac{3}{2d}\gamma + \frac{\delta}{d}.$$

**Theorem 2.11.** *Let $d \ge 3$, $\frac{4}{3} < \gamma < 2$ and $\delta = \delta(d,\gamma) > 0$ be sufficiently small. Then there exists a positive constant $\varepsilon = \varepsilon(d,\gamma)$ such that if $\varphi \in \dot{H}^{s_c}H_\omega^{\frac{3}{4}(2-\gamma)+\delta}$ satisfies $\||\nabla|^{s_c}D_\omega^{\frac{3}{4}(2-\gamma)+\delta}\varphi\|_{L_x^2} < \varepsilon$, then (2.2) has a unique solution*

$$u \in C_b(\mathbb{R}; \dot{H}^{s_c}H_\omega^{\frac{3}{4}(2-\gamma)+\delta}) \cap L^{2q_{1,s_c}}(\mathbb{R}; |x|^{s_c-\delta}L_r^2H_\omega^{\frac{3}{4}(2-\gamma)+\frac{3}{2}\delta})$$
$$\cap L^{q_{2,s_c}}(\mathbb{R}; |x|^{s_c}L_r^{\ell_1}H_\omega^{\frac{3}{4}(2-\gamma)+(\frac{3}{2}-\frac{1}{d})\delta,\sigma_1})$$

*where*

$$\frac{1}{q_{1,s_c}} = -2s_c + \delta, \qquad \frac{1}{q_{2,s_c}} = \frac{\gamma}{4} - \frac{\delta}{2},$$
$$\frac{1}{\ell_1} = \frac{1}{2} + \frac{2}{d} - \frac{3}{2d}\gamma + \frac{\delta}{d}, \quad \frac{1}{\sigma_1} = \frac{1}{2} + \frac{2}{d} - \frac{3}{2d}\gamma + \frac{2}{d}\delta.$$

**Theorem 2.12.** *Let $d \ge 2$, $0 < \gamma < 2$,*

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < \ 0,$$

*and suppose that $\delta = \delta(d,s,\gamma) > 0$ is sufficiently small. Then there exists a positive time $T$ such that if $\varphi \in \dot{H}^s(\mathbb{R}^d)$ is radially symmetric then (2.2) has a unique radial solution*

$$u \in C([0,T]; \dot{H}^s(\mathbb{R}^d)) \cap L^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}([0,T]; |x|^{s-\delta}L^2(\mathbb{R}^d)) \cap L^{q_2}([0,T]; |x|^{\alpha}L^{\ell_2}(\mathbb{R}^d)),$$

*where*

$$\frac{1}{q_1} = 1 - \frac{\gamma}{2} - s + \delta, \qquad \frac{1}{q_2} = \frac{\gamma}{4} - \frac{\delta}{2},$$

$$\alpha = \begin{cases} s - \delta & (\text{if } d = 2), \\ s & (\text{if } d \geq 3), \end{cases} \quad \frac{1}{\ell_2} = \begin{cases} \frac{1}{2} - \frac{\gamma}{4} - s + \delta & (\text{if } d = 2), \\ \frac{1}{2} - \frac{\gamma}{2d} - \frac{2}{d}s + \frac{\delta}{d} & (\text{if } d \geq 3). \end{cases}$$

**Theorem 2.13.** *Let $d \geq 2$, $0 < \gamma < 2$,*

$$\max\left(s_c, -\frac{\gamma}{4}\right) < s < 0,$$

*and suppose that $\delta = \delta(d, s, \gamma) > 0$ is sufficiently small. Then there exists a positive time $T$ such that if $\varphi \in \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}(\mathbb{R}^d)$ then (2.2) has a unique solution*

$$u \in C([0,T]; \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}) \cap L^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}([0,T]; |x|^{s-\delta} L_r^2 H_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}) \\ \cap L^{q_2}([0,T]; |x|^\alpha L_r^{\ell_2} H_\omega^{\beta,\sigma_0}),$$

*where*

$$\frac{1}{q_1} = 1 - \frac{\gamma}{2} - s + \delta, \qquad \frac{1}{q_2} = \frac{\gamma}{4} - \frac{\delta}{2},$$

$$\alpha = \begin{cases} s - \delta & (\text{if } d = 2), \\ s & (\text{if } d \geq 3), \end{cases} \quad \beta = \begin{cases} -\frac{3}{2}s + \frac{3}{2}\delta & (\text{if } d = 2), \\ -\frac{3}{2}s + \frac{3}{2}\delta - \frac{\delta}{d} & (\text{if } d \geq 3), \end{cases}$$

$$\frac{1}{\ell_2} = \begin{cases} \frac{1}{2} - \frac{\gamma}{4} - s + \delta & (\text{if } d = 2), \\ \frac{1}{2} - \frac{\gamma}{2d} - \frac{2}{d}s + \frac{\delta}{d} & (\text{if } d \geq 3), \end{cases} \quad \frac{1}{\sigma_0} = \frac{1}{2} - \frac{\gamma}{2d} - \frac{2}{d}s + \frac{2}{d}\delta.$$

Since the proofs of Theorems 2.10, 2.11 and 2.12 are analogous to that of Theorem 2.13, here we establish only Theorem 2.13. We should mention that if $d = 2$, as Theorems 2.10 and 2.11, we cannot prove the small data global existence for $\varphi \in \dot{H}^{s_c}(\mathbb{R}^d)$. See Remark 2.4 below for the details.

Throughout the section, we assume $d \geq 2$ and use the explicit exponents

$$\left(\frac{1}{q}, \frac{1}{q_1}, \frac{1}{q_2}\right) = \left(\frac{\gamma}{4} + s - \frac{\delta}{2},\ 1 - \frac{\gamma}{2} - s + \delta,\ \frac{\gamma}{4} - \frac{\delta}{2}\right),$$

$$\left(\frac{1}{\ell}, \frac{1}{\ell_1}, \frac{1}{\ell_2}\right) = \begin{cases} \left(\frac{1}{2} - \frac{\gamma}{4} + \delta, \frac{\gamma}{2} + s - 2\delta, \frac{1}{2} - \frac{\gamma}{4} - s + \delta\right) & (\text{if } d = 2), \\ \left(\frac{1}{2} - \frac{\gamma}{2d} + \frac{\delta}{d}, \frac{\gamma}{d} + \frac{2}{d}s - \frac{2}{d}\delta, \frac{1}{2} - \frac{\gamma}{2d} - \frac{2}{d}s + \frac{\delta}{d}\right) & (\text{if } d \geq 3), \end{cases}$$

$$\frac{1}{\sigma_0} = \frac{1}{2} - \frac{\gamma}{2d} - \frac{2}{d}s + \frac{2}{d}\delta,$$

$$\frac{d-1}{\sigma} = \begin{cases} \frac{\gamma}{2} + \frac{s}{2} - \frac{\delta}{2} & (\text{if } d = 2), \\ \frac{d-1}{d}\gamma + \frac{5}{2}s - \frac{4}{d}s - \frac{5}{2}\delta + \frac{3}{d}\delta & (\text{if } d \geq 3), \end{cases}$$

with sufficiently small $\delta=\delta(d,s,\gamma)>0$. Here $q'$ and $\ell'$ are given by $1/q+1/q'=1$ and $1/\ell+1/\ell'=1$, respectively. Note that

$$\frac{1}{q'}=\frac{1}{q_1}+\frac{1}{q_2},\qquad \frac{1}{\ell'}=\frac{1}{\ell_1}+\frac{1}{\ell_2}.$$

**Lemma 2.14.** *Let* $\max(s_c,-\gamma/4)<s<0$. *Then we have*

$$\||\nabla|^s D_\omega^{-\frac{3}{2}s+\delta}U(t)\Phi_t\|_{L^\infty_{I_T}L^2_x}+W_1(U(t)\Phi_t)+W_2(U(t)\Phi_t)\lesssim T^\theta[W_1(u)]^2W_2(u)$$

*where*

$$\begin{aligned}
\theta&=\frac{2+2s-\gamma}{2},\\
W_1(u)&=\||x|^{s-\delta}D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}u\|_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}L^2_x},\\
W_2(u)&=\begin{cases}\||x|^{s-\delta}D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}u\|_{L^{q_2}_{I_T}L^{\ell_2}_rL^{\sigma_0}_\omega} & (\text{if } d=2),\\ \||x|^{s}D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta-\frac{\delta}{d}}u\|_{L^{q_2}_{I_T}L^{\ell_2}_rL^{\sigma_0}_\omega} & (\text{if } d\geq 3).\end{cases}
\end{aligned}$$

*Proof.* (I) ($d\geq 3$)
First, we assume $d\geq 3$ and prove

$$\||\nabla|^s D_\omega^{-\frac{3}{2}s+\delta}U(t)\Phi_t\|_{L^\infty_{I_T}L^2_x}\lesssim T^\theta[W_1(u)]^2W_2(u). \tag{2.8}$$

Let us set

$$s_1=-\frac{d-2}{d}s+\frac{d-2}{2d}\delta,\qquad s_2=-\frac{d+4}{2d}s+\frac{d+2}{2d}\delta.$$

Note that $s_1+s_2=-\frac{3}{2}s+\delta$. Since $2\leq\sigma_0\leq\ell\leq\infty$,

$$\begin{aligned}
&\frac{d}{2}\left(\frac{1}{2}-\frac{1}{\sigma_0}\right)\leq\frac{1}{q}\leq\frac{1}{2}+\frac{1}{\ell}-\frac{1}{\sigma_0},\\
&\frac{d^2}{4}-\frac{d}{q}-\frac{d^2}{2\sigma_0}<-s<\frac{d}{4}-\frac{1}{q}-\frac{d-1}{\ell}+\frac{d-2}{2\sigma_0},
\end{aligned}$$

it follows from Lemma 2.7 that

$$\||x|^{-s}|\nabla|^sD_\omega^{s_1}[U(t)\varphi]\|_{L^q_tL^\ell_rL^{\sigma_0}_\omega}\lesssim\|\varphi\|_{L^2_x}.$$

By the dual estimate, we have

$$\|\int_{-\infty}^{\infty}U(-t')F(u)(t')dt'\|_{L^2_x}\lesssim\||x|^s|\nabla|^{-s}D_\omega^{-s_1}F(u)\|_{L^{q'}_tL^{\ell'}_rL^{\sigma_0'}_\omega}, \tag{2.9}$$

where $1/\sigma_0{}'=1-1/\sigma_0$. By applying Lemma 2.9 to $\|U(t)\varphi\|_{L^\infty_tL^2_x}=\|\varphi\|_{L^2_x}$ and (2.9), we have

$$\||\nabla|^sD_\omega^{s_1+s_2}U(t)\Phi_t\|_{L^\infty_{I_T}L^2_x}\lesssim\||x|^sD_\omega^{s_2}F(u)\|_{L^{q'}_{I_T}L^{\ell'}_rL^{\sigma_0'}_\omega}.$$

By Leibniz rule and Sobolev embedding on the unit sphere (see Appendix in [27]), we have

$$\begin{aligned}
&\||x|^s D_\omega^{s_2} F(u)\|_{L_{I_T}^{q'} L_r^{\ell'} L_\omega^{\sigma_0'}} \\
&\lesssim \|D_\omega^{s_2}(|x|^{-\gamma} * |u|^2)\|_{L_{I_T}^{q_1} L_r^{\ell_1} L_\omega^{\sigma}} \||x|^s u\|_{L_{I_T}^{q_2} L_r^{\ell_2} L_\omega^{\alpha}} \\
&\quad + \||x|^{-\gamma} * |u|^2\|_{L_{I_T}^{q_1} L_r^{\ell_1} L_\omega^{\frac{\sigma(d-1)}{d-1-\sigma s_2}}} \||x|^s D_\omega^{s_2} u\|_{L_{I_T}^{q_2} L_r^{\ell_2} L_\omega^{\beta}} \\
&\lesssim \|D_\omega^{s_2}(|x|^{-\gamma} * |u|^2)\|_{L_{I_T}^{q_1} L_r^{\ell_1} L_\omega^{\sigma}} \||x|^s D_\omega^{(d-1)(-1+\frac{2}{\sigma_0}+\frac{1}{\sigma})} u\|_{L_{I_T}^{q_2} L_r^{\ell_2} L_\omega^{\sigma_0}}.
\end{aligned}$$

Here the exponents $\alpha$, $\beta$ satisfy

$$\frac{1}{\alpha} = 1 - \frac{1}{\sigma_0} - \frac{1}{\sigma}, \qquad \frac{1}{\beta} = 1 - \frac{1}{\sigma_0} - \frac{1}{\sigma} + \frac{s_2}{d-1}.$$

We deduce from Lemma 2.8 that

$$\|D_\omega^{s_2}(|x|^{-\gamma} * |u|^2)\|_{L_r^{\ell_1} L_\omega^{\sigma}} \lesssim \||x|^{-\gamma+\frac{d}{\ell_1}} D_\omega^{s_2}(|u|^2)\|_{L_r^1 L_\omega^{\frac{d-1}{d-1-(\gamma-\frac{d-1}{\sigma}-\frac{1}{\ell_1})},1}}. \tag{2.10}$$

To estimate the right hand side of (2.10), we utilize Leibniz rule and Sobolev embedding in the Lorentz spaces on the unit sphere:

$$\|D_\omega^s(u\bar{u})\|_{L_\omega^{p,1}} \lesssim \|D_\omega^s u\|_{L_\omega^{p_0,2}} \|u\|_{L_\omega^{p_1,2}}, \tag{2.11}$$

for $s \in (0,1)$, $p$, $p_0$, $p_1 \in (1,\infty)$ and $1/p = 1/p_0 + 1/p_1$.

$$\|u\|_{L_\omega^{p,2}} \lesssim \|D_\omega^s u\|_{L_\omega^2}, \tag{2.12}$$

for $-\frac{d-1}{p} = s - \frac{d-1}{2}$, $s > 0$. The above two estimates are verified as follows. From the arguments in Appendix [27] and the general Marcinkiewicz interpolation theorem (Theorem 5.3.2 in [4]), (2.11) and (2.12) are easily transferred from the Euclidean case. Thus it suffices to prove the followings:

$$\||\nabla|^s(u\bar{u})\|_{L_x^{p,1}} \lesssim \||\nabla|^s u\|_{L_x^{p_0,2}} \|u\|_{L_x^{p_1,2}}, \tag{2.13}$$

for $s \in (0,1)$, $p$, $p_0$, $p_1 \in (1,\infty)$ and $1/p = 1/p_0 + 1/p_1$, and

$$\|u\|_{L_x^{q,2}} \lesssim \||\nabla|^s u\|_{L_x^2}, \tag{2.14}$$

for $-\frac{d}{q} = s - \frac{d}{2}$, $s > 0$. (2.13) is immediately verified by the proof of Leibniz rule in the Lebesgue spaces (see Proposition 3.3 in [12]), the simple inequality

$$\|u\bar{u}\|_{L_x^{p,1}} \lesssim \|u\|_{L_x^{p_0,2}} \|u\|_{L_x^{p_1,2}},$$

and the general Marcinkiewicz interpolation theorem. Similarly, (2.14) is proved by (real) interpolating Sobolev embedding in the Lebesgue spaces. By using (2.11) and (2.12), we get

$$\begin{aligned}
&\||x|^{2(s-\delta)}D_\omega^{s_2}(|u|^2)\|_{L_r^1 L_\omega^{\frac{d-1}{d-1-(\gamma-\frac{d-1}{\sigma}-\frac{1}{\ell_1})},1}}\\
&\lesssim \||x|^{s-\delta}D_\omega^{s_2}u\|_{L_r^2 L_\omega^{\frac{2(d-1)}{d-1-(\gamma-\frac{d-1}{\sigma}-\frac{1}{\ell_1}-s_2)},2}}\||x|^{s-\delta}u\|_{L_r^2 L_\omega^{\frac{2(d-1)}{d-1-(\gamma-\frac{d-1}{\sigma}-\frac{1}{\ell_1}+s_2)},2}}\\
&\qquad\lesssim \||x|^{s-\delta}D_\omega^{(s_2+\gamma-\frac{d-1}{\sigma}-\frac{1}{\ell_1})/2}u\|_{L_x^2}^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|D_\omega^{s_2}(|x|^{-\gamma}*|u|^2)\|_{L_{I_T}^{q_1}L_r^{\ell_1}L_\omega^\sigma} &\lesssim \||x|^{s-\delta}D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}u\|_{L_{I_T}^{2q_1}L_x^2}^2\\
&\lesssim T^\theta\||x|^{s-\delta}D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}u\|^2_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}L_x^2}.
\end{aligned}$$

This completes (2.8) if $d\geq 3$.

(II) ($d=2$)

Next, we assume $d=2$ and establish (2.8). The strategy is almost the same as in the case of $d\geq 3$ above. We set

$$s_3=\frac{\delta}{2},\qquad s_4=-\frac{3}{2}s+\frac{\delta}{2}.$$

We deduce from Lemma 2.7 that

$$\||x|^{-s-\delta}|\nabla|^s D_\omega^{s_3}[U(t)\varphi]\|_{L_t^q L_r^\ell L_\omega^{\sigma_0}}\lesssim \|\varphi\|_{L_x^2}.$$

By the similar argument as above, we get

$$\begin{aligned}
&\||\nabla|^s D_\omega^{-\frac{3}{2}s+\delta}U(t)\Phi_t\|_{L_{I_T}^\infty L_x^2}\lesssim \||x|^{s+\delta}D_\omega^{s_4}F(u)\|_{L_{I_T}^{q'}L_r^{\ell'}L_\omega^{\sigma_0'}}\\
&\quad\lesssim \||x|^{2\delta}D_\omega^{s_4}(|x|^{-\gamma}*|u|^2)\|_{L_{I_T}^{q_1}L_r^{\ell_1}L_\omega^\sigma}\||x|^{s-\delta}D_\omega^{-1+\frac{2}{\sigma_0}+\frac{1}{\sigma}}u\|_{L_{I_T}^{q_2}L_r^{\ell_2}L_\omega^{\sigma_0}}.
\end{aligned}$$

It follows from Lemma 2.8 that

$$\||x|^{2\delta}D_\omega^{s_4}(|x|^{-\gamma}*|u|^2)\|_{L_r^{\ell_1}L_\omega^\sigma}\lesssim \||x|^{-\gamma+\frac{2}{\ell_1}+2\delta}D_\omega^{s_4}(|u|^2)\|_{L_r^1 L_\omega^{\frac{1}{1-(\gamma-\frac{1}{\sigma}-\frac{1}{\ell_1})},1}}. \tag{2.15}$$

By Leibniz rule and Sobolev embedding in the Lorentz spaces on the unit sphere, we have

$$\||x|^{2(s-\delta)}D_\omega^{s_4}(|u|^2)\|_{L_r^1 L_\omega^{\frac{1}{1-(\gamma-\frac{1}{\sigma}-\frac{1}{\ell_1})},1}}\lesssim \||x|^{s-\delta}D_\omega^{(s_4+\gamma-\frac{1}{\sigma}-\frac{1}{\ell_1})/2}u\|_{L_x^2}^2.$$

Then we have

$$\begin{aligned}\||x|^{2\delta}D_\omega^{s_4}(|x|^{-\gamma}*|u|^2)\|_{L^{q_1}_{I_T}L^{\ell_1}_r L^{\sigma}_\omega} \lesssim & \||x|^{s-\delta}D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}u\|^2_{L^{2q_1}_{I_T}L^2_x}\\ \lesssim & T^\theta\||x|^{s-\delta}D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}u\|^2_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}L^2_x}.\end{aligned}$$

This completes (2.8).

*Remark* 2.4. It should be noted that to get the estimate (2.15) above we need the condition $1/\ell_1 > \gamma - 1$. This causes the exception of $d = 2$ in the scaling critical ($s = s_c$) results, that is Theorems 2.10 and 2.11.

(III) Lastly, we prove

$$W_1(U(t)\Phi_t) + W_2(U(t)\Phi_t) \lesssim T^\theta[W_1(u)]^2 W_2(u), \tag{2.16}$$

which completes the lemma. Here we only consider the case for $d \geq 3$. The same method can be utilized for $d = 2$. Since

$$-\frac{d}{2q_1} + \frac{2+2s-\gamma}{4}d < s-\delta < -\frac{1}{2q_1} + \frac{2+2s-\gamma}{4},$$

we deduce from Lemma 2.6 (ii) that

$$\||x|^{s-\delta}|\nabla|^{-s}D_\omega^{\frac{\delta}{2}}[U(t)\varphi]\|_{L_t^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}L^2_x} \lesssim \|\varphi\|_{L^2_x}. \tag{2.17}$$

Applying Lemma 2.9 to (2.9) and (2.17), we have

$$\||x|^{s-\delta}|\nabla|^{-s}D_\omega^{\frac{\delta}{2}}U(t)\Phi_t\|_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}L^2_x} \lesssim \||x|^{s}|\nabla|^{-s}D_\omega^{-s_1}F(u)\|_{L^{q'}_{I_T}L^{\ell'}_r L^{\sigma_0'}_\omega},$$

which implies

$$\||x|^{s-\delta}D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta}U(t)\Phi_t\|_{L_{I_T}^{\frac{4q_1}{2-q_1(2+2s-\gamma)}}L^2_x} \lesssim \||x|^{s}D_\omega^{s_2}F(u)\|_{L^{q'}_{I_T}L^{\ell'}_r L^{\sigma_0'}_\omega}.$$

As above, this estimate implies

$$W_1(U(t)\Phi_t) \lesssim T^\theta[W_1(u)]^2 W_2(u).$$

For $W_2$, since $2 \leq \sigma_0 \leq \ell_2 \leq \infty$,

$$\frac{d}{2}\left(\frac{1}{2}-\frac{1}{\sigma_0}\right) \leq \frac{1}{q_2} \leq \frac{1}{2} + \frac{1}{\ell_2} - \frac{1}{\sigma_0},$$

$$\frac{d^2}{4} - \frac{d}{q_2} - \frac{d^2}{2\sigma_0} < s < \frac{d}{4} - \frac{1}{q_2} - \frac{d-1}{\ell_2} + \frac{d-2}{2\sigma_0},$$

we deduce from Lemma 2.7 that

$$\||x|^{s}|\nabla|^{-s}D_\omega^{\frac{\delta}{2}-\frac{\delta}{d}}[U(t)\varphi]\|_{L^{q_2}_t L^{\ell_2}_r L^{\sigma_0}_\omega} \lesssim \|\varphi\|_{L^2_x}. \tag{2.18}$$

Applying Lemma 2.9 to (2.9) and (2.18), we have

$$\||x|^s D_\omega^{-\frac{3}{2}s+\frac{3}{2}\delta-\frac{\delta}{d}} U(t)\Phi_t\|_{L^{q_2}_{I_T} L^{\ell_2}_r L^{\sigma_0}_\omega} \lesssim \||x|^s D_\omega^{s_2} F(u)\|_{L^{q'}_{I_T} L^{\ell'}_r L^{\sigma_0'}_\omega},$$

which gives

$$W_2(U(t)\Phi_t) \lesssim T^\theta [W_1(u)]^2 W_2(u).$$

This completes (2.17). □

*Proof of Theorem 2.13.* We prove the existence by Banach's fixed-point theorem. Fix a positive constant $\rho$ and a positive time $T$, to be chosen later, and we define a complete metric space $(X_{\rho,T}, d_X)$ by

$$\begin{aligned} X_{\rho,T} =& \{u \in C([0,T]; \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}(\mathbb{R}^d)); \|u\|_{L^\infty_{I_T} \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} + W_1(u) + W_2(u) \le \rho\}, \\ d_X(u,v) =& \|u-v\|_{L^\infty_{I_T} \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} + W_1(u-v) + W_2(u-v), \end{aligned}$$

and the mapping

$$\mathcal{N}_X(u) = U(t)(\varphi + \Phi_t) \qquad \text{on } X_{\rho,T}.$$

Our strategy is to prove that $\mathcal{N}_X$ is a contraction mapping on $X_{\rho,T}$ for sufficiently small $T$.

It follows from $\|U(t)\varphi\|_{L^\infty_t L^2_x} = \|\varphi\|_{L^2_x}$, (2.17) and (2.18) (if $d=2$, (2.17) and $\||x|^{s-\delta}|\nabla|^{-s} D_\omega^{\frac{\delta}{2}}[U(t)\varphi]\|_{L^{q_2}_t L^{\ell_2}_r L^{\sigma_0}_\omega} \lesssim \|\varphi\|_{L^2_x}$) that there exists a positive constant $C_1$ such that

$$\|U(t)\varphi\|_{L^\infty_{I_T} \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} + W_1(U(t)\varphi) + W_2(U(t)\varphi) \le C_1 \|\varphi\|_{\dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}}. \tag{2.19}$$

For $u \in X_{\rho,T}$, we deduce from Lemma 2.14 that there exists a positive constant $C_2$ such that

$$\begin{aligned} \|U(t)\Phi_t\|_{L^\infty_{I_T} \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} + W_1(U(t)\Phi_t) + W_2(U(t)\Phi_t) &\le C_2 T^\theta [W_1(u)]^2 W_2(u) \\ &\le C_2 T^\theta \rho^3. \end{aligned} \tag{2.20}$$

For $u$, $v \in X_{\rho,T}$, we have

$$\begin{aligned} d_X(\mathcal{N}_X(u), \mathcal{N}_X(v)) =& \|U(t)(\Phi_t(u) - \Phi_t(v))\|_{L^\infty_{I_T} \dot{H}^s H_\omega^{-\frac{3}{2}s+\delta}} \\ &+ W_1(U(t)(\Phi_t(u) - \Phi_t(v))) + W_2(U(t)(\Phi_t(u) - \Phi_t(v))). \end{aligned}$$

By the arguments similar to the proof of Lemma 2.14, we have

$$d_X(\mathcal{N}_X(u), \mathcal{N}_X(v)) \lesssim \||x|^s D_\omega^{s_2}(F(u) - F(v))\|_{L^{q'}_{I_T} L^{\ell'}_r L^{\sigma_0'}_\omega}.$$

It follows from the following equality

$$\begin{aligned}F(u)-F(v)=&\lambda(|x|^{-\gamma}*|u|^2)u-\lambda(|x|^{-\gamma}*|v|^2)v\\=&\lambda(|x|^{-\gamma}*(u(\bar{u}-\bar{v})+(u-v)\bar{v}))u+\lambda(|x|^{-\gamma}*|v|^2)(u-v),\end{aligned}$$

and the same estimates as in Lemma 2.14 that

$$\begin{aligned}&d_X(\mathcal{N}_X(u),\mathcal{N}_X(v))\\&\quad\lesssim T^\theta(W_1(u)+W_2(u)+W_1(v)+W_2(v))^2(W_1(u-v)+W_2(u-v)).\end{aligned}$$

Then there exists a positive constant $C_3$ such that

$$d_X(\mathcal{N}_X(u),\mathcal{N}_X(v))\le C_3T^\theta\rho^2 d_X(u,v). \tag{2.21}$$

Now we define $C=\max(C_1,C_2,C_3)$ and choose $\rho$, $T$ such that

$$C\|\varphi\|_{\dot{H}^sH_\omega^{-\frac{3}{2}s+\delta}}\le\frac{\rho}{2},\quad CT^\theta\rho^2\le\frac{1}{2}.$$

Then, from (2.19)-(2.21), $\mathcal{N}_X$ is a contraction mapping on $X_{\rho,T}$. □

2.4. **(PNLS).** In this section, we establish Theorem 2.5. We then consider the problem in the scaling critical homogeneous Sobolev space $\dot{H}^{s_{c,p}}(\mathbb{R}^d)$. Let us recall that $s_{c,p}=\frac{d}{2}-\frac{2}{p-1}$. For convenience, we restate Theorem 2.5 with the explicit exponents.

**Theorem 2.15.** *Let $3\le d\le 14$, $p_0<p<1+4/d$ where $p_0$ is a unique solution of*

$$\begin{cases}1+\frac{4}{d+1}\le p_0<1+\frac{4}{d},\\ 2p_0^3+6(d-2)p_0^2+(d^2-13d+10)p_0-d(d-3)=0,\end{cases}$$

*and suppose that $\delta=\delta(d,p)$ is sufficiently small. Then there exists a positive constant $\varepsilon=\varepsilon(d,p)$ such that if $\varphi\in\dot{H}^{s_{c,p}}H_\omega^{s_0}(\mathbb{R}^d)$ satisfies $\||\nabla|^{s_{c,p}}D_\omega^{s_0}\varphi\|_{L_x^2}<\varepsilon$ where*

$$s_0=\begin{cases}\frac{1}{p-1}(7-3p)+\delta & (\text{if } d=3),\\ \frac{1}{2(p-1)^2}\left(-(d+1)p^2+(d+7)p-2\right)+\delta & (\text{if } d\ge 4),\end{cases}$$

*then (2.3) has a unique solution*

$$u\in C_b(\mathbb{R};\dot{H}^{s_{c,p}}H_\omega^{s_0})\cap L^{pq'}(\mathbb{R};|x|^\alpha L^{p\ell'}H_\omega^{s_0,\sigma})$$

*where*

$$\alpha = \begin{cases} \frac{7}{p} - \frac{4}{p-1} + \frac{2(p-1)}{7p}\delta & (\text{if } d=3), \\ \frac{d}{2} - \frac{2}{p-1} + \frac{(d-4)(p-1)}{5p}\delta & (\text{if } d \geq 4), \end{cases} \quad \frac{1}{q} = \begin{cases} \frac{p-1}{7}\delta & (\text{if } d=3), \\ 1 - \frac{p}{2} + \frac{2(p-1)}{5}\delta & (\text{if } d \geq 4), \end{cases}$$

$$\frac{1}{\ell} = \begin{cases} 2 - \frac{2}{p-1} & (\text{if } d=3), \\ 1 - \frac{4}{d} + \frac{p}{2} + \frac{p}{d} - \frac{4}{d(p-1)} + \frac{p-1}{5}\delta - \frac{8(p-1)}{5d}\delta & (\text{if } d \geq 4), \end{cases}$$

$$\frac{1}{\sigma} = \begin{cases} -\frac{3}{2} + \frac{4}{p} + \frac{4(p-1)}{7p}\delta & (\text{if } d=3), \\ \frac{1}{2d(p-1)}(-dp - 2p + d + 10) + \frac{8(p-1)}{5dp}\delta & (\text{if } d \geq 4). \end{cases}$$

Similarly to (HNLS) case, by using weighted Strichartz estimates, we establish the following crucial estimate.

**Lemma 2.16.** *Let* $3 \leq d \leq 14$, $p_\delta < p < 1 + 4/d$ *where* $p_\delta$ *satisfies the following:*

$$(\text{if } d=3) \qquad p_\delta = 2 + (p_\delta - 1)\sqrt{\frac{3}{14}\delta},$$

$$(\text{if } d \geq 4)$$

$$1 + \frac{4}{d+1} < p_\delta < 1 + \frac{4}{d},$$

$$2p_\delta^3 + 6(d-2)p_\delta^2 + (d^2 - 13d + 10)p_\delta - d(d-3) = \frac{2(p-1)^2}{5}(-3dp_\delta + 8p_\delta + 8d - 8)\delta.$$

*Then we have*

$$\||\nabla|^{s_{c,p}} D_\omega^{s_0} U(t)\Phi_{t,p}\|_{L_t^\infty L_x^2} + \||x|^\alpha D_\omega^{s_0} U(t)\Phi_{t,p}\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} \lesssim \||x|^\alpha D_\omega^{s_0} u\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}^p,$$

*where the exponents* $q$, $\ell$, $\sigma$, $s_0$, $\alpha$ *are same as in Theorem* 2.15.

*Remark* 2.5. Since $\delta$ is sufficiently small, it is easy to see that the above $p_\delta$ exists and is unique.

*Proof.* (I) $(d = 3)$

First, we assume $d = 3$ and prove

$$\||\nabla|^{s_{c,p}} D_\omega^{s_0} U(t)\Phi_{t,p}\|_{L_t^\infty L_x^2} \lesssim \||x|^\alpha D_\omega^{s_0} u\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}^p. \tag{2.22}$$

Let us set that

$$c_0 = -3 + \frac{4}{p-1} - \frac{2}{7}(p-1)\delta,$$

$$s_1 = \frac{p-1}{7}\delta, \quad s_2 = \frac{1}{p-1}(7 - 3p) + \frac{8-p}{7}\delta.$$

Note that $s_1 + s_2 = s_0$. Since $2 \le \ell \le q \le \infty$ and

$$-\frac{3}{q} < c_0 < 1 - \frac{1}{q} - \frac{2}{\ell},$$

we deduce from Lemma 2.7 that

$$\| |x|^{c_0} |\nabla|^{s_{c,p}} D_\omega^{s_1} [U(t)\varphi] \|_{L_t^q L_r^\ell L_\omega^2} \lesssim \|\varphi\|_{L_x^2}.$$

By the dual estimate, we have

$$\| \int_{-\infty}^{\infty} U(-t') G(u)(t') dt' \|_{L_x^2} \lesssim \| |x|^{-c_0} |\nabla|^{-s_{c,p}} D_\omega^{-s_1} G(u) \|_{L_t^{q'} L_r^{\ell'} L_\omega^2}. \qquad (2.23)$$

By applying Lemma 2.9 to $\|U(t)\varphi\|_{L_t^\infty L_x^2} = \|\varphi\|_{L_x^2}$ and (2.23), we have

$$\| |\nabla|^{s_{c,p}} D_\omega^{s_1+s_2} U(t) \Phi_{t,p} \|_{L_t^\infty L_x^2} \lesssim \| |x|^{-c_0} D_\omega^{s_2} (|u|^{p-1} u) \|_{L_t^{q'} L_r^{\ell'} L_\omega^2}.$$

Since $0 \le s_2 \le \min([p](=2), \frac{1}{p-1}(\frac{2p}{\sigma} - 1))$, where $[p]$ denotes the integral part of $p$, it follows from Moser type estimates and Sobolev embedding on the unit sphere that

$$\begin{aligned} \|D_\omega^{s_2}(|u|^{p-1}u)\|_{L_\omega^2} &\lesssim \|D_\omega^{s_2} u\|_{L_\omega^{\sigma_0}}^p \\ &\lesssim \|D_\omega^{s_0} u\|_{L_\omega^\sigma}^p. \end{aligned}$$

Here we have used the exponent

$$\frac{1}{\sigma_0} = \frac{1}{2p}(1 + (p-1)s_2).$$

This gives

$$\| |\nabla|^{s_{c,p}} D_\omega^{s_0} U(t) \Phi_{t,p} \|_{L_t^\infty L_x^2} \lesssim \| |x|^{-\frac{c_0}{p}} D_\omega^{s_0} u \|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}^p,$$

which completes the proof of (2.22) if $d = 3$.

(II) ($d \ge 4$)

Next we assume $d \ge 4$ and obtain (2.22). Similarly to the $d = 3$ case, we set

$$\begin{aligned} c_1 &= -\frac{d}{2}p + 2 + \frac{1}{p-1} - \frac{(d-4)(p-1)}{5}\delta, \\ s_1 &= \frac{p}{d} - \frac{d}{2} + \frac{3}{2} - \frac{4}{d} + \frac{2}{p-1} - \frac{4}{n(p-1)} + \frac{3(p-1)}{5}\delta - \frac{8(p-1)}{5d}\delta, \\ s_2 &= \frac{1}{p-1}\left( -\frac{p^2}{d} - 2p + \frac{5}{d}p - \frac{d}{2} + \frac{5}{2} + \frac{2}{p-1} \right) + \frac{8-3p}{5}\delta + \frac{8(p-1)}{5d}\delta. \end{aligned}$$

Note that $s_1 + s_2 = s_0$. Since $2 \le \ell \le q \le \infty$ and

$$-\frac{d}{q} < c_1 < \frac{d-1}{2} - \frac{1}{q} - \frac{d-1}{\ell},$$

we deduce from Lemma 2.7 that

$$\| |x|^{c_1} |\nabla|^{s_{c,p}} D_\omega^{s_1} [U(t)\varphi] \|_{L_t^q L_r^\ell L_\omega^2} \lesssim \|\varphi\|_{L_x^2}.$$

By the dual estimate, we have

$$\| \int_{-\infty}^{\infty} U(-t')G(u)(t')dt' \|_{L_x^2} \lesssim \| |x|^{-c_1} |\nabla|^{-s_{c,p}} D_\omega^{-s_1} G(u) \|_{L_t^{q'} L_r^{\ell'} L_\omega^2}, \tag{2.24}$$

which gives

$$\| |\nabla|^{s_{c,p}} D_\omega^{s_1+s_2} U(t) \Phi_{t,p} \|_{L_t^\infty L_x^2} \lesssim \| |x|^{-c_1} D_\omega^{s_2} (|u|^{p-1}u) \|_{L_t^{q'} L_r^{\ell'} L_\omega^2}.$$

Since $0 \le s_2 \le \min\bigl([p](=1), \frac{d-1}{p-1}\bigl(\frac{p}{\sigma} - \frac{1}{2}\bigr)\bigr)$, it follows from Moser type estimates and Sobolev embedding on the unit sphere that

$$\| D_\omega^{s_2} (|u|^{p-1}u) \|_{L_\omega^2} \lesssim \| D_\omega^{s_0} u \|_{L_\omega^\sigma}^p,$$

which completes (2.22).

(III)

Lastly, we establish

$$\| |x|^\alpha D_\omega^{s_0} U(t) \Phi_{t,p} \|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} \lesssim \| |x|^\alpha D_\omega^{s_0} u \|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}^p. \tag{2.25}$$

To avoid redundancy, here we assume $d \ge 4$. We can prove (2.25) in case of $d = 3$ by the same way as below. Since $2 \le \sigma \le p\ell' \le \infty$,

$$\begin{aligned} \frac{d}{2}\left(\frac{1}{2} - \frac{1}{\sigma}\right) \le & \frac{1}{pq'} \le \frac{1}{2} + \frac{1}{p\ell'} - \frac{1}{\sigma}, \\ \frac{d^2}{4} - \frac{d}{pq'} - \frac{d^2}{2\sigma} < & \alpha < \frac{d}{4} - \frac{1}{pq'} - \frac{d-1}{p\ell'} + \frac{d-2}{2\sigma}, \end{aligned}$$

we deduce from Lemma 2.7 that

$$\| |x|^\alpha |\nabla|^{-s_{c,p}} U(t) \varphi \|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} \lesssim \|\varphi\|_{L_x^2}. \tag{2.26}$$

By applying Lemma 2.9 to (2.24) and (2.26), we have

$$\| |x|^\alpha D_\omega^{s_0} U(t) \Phi_{t,p} \|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} \lesssim \| |x|^{-c_1} D_\omega^{s_2} (|u|^{p-1}u) \|_{L_t^{q'} L_r^{\ell'} L_\omega^2}.$$

By the same argument as above, this completes (2.25). □

*Proof of Theorem 2.15.* Obviously, $p_0$ in Theorem 2.15 is less than $p_\delta$ in Lemma 2.16, and if $\delta = \delta(d,p) > 0$ is sufficiently small then $p_\delta$ is sufficiently close to $p_0$. Thus it suffices to prove Theorem 2.15 for any $p$ such that $p_\delta < p < 1 + 4/d$. Similarly to the (HNLS) case, we prove Theorem 2.15 by the contraction mapping theorem. Let the exponents $s_0$, $s_1$, $\alpha$, $c_0$, $c_1$ be the same as in Lemma 2.16. Fix a positive constant $\varepsilon$, to be chosen later, and we define a complete metric space $(X_\varepsilon, d_X)$ by

$$\begin{aligned} X_\varepsilon = & \{u \in C(\mathbb{R}; \dot{H}^{s_{c,p}} H_\omega^{s_0}); \|u\|_{L_t^\infty \dot{H}^{s_{c,p}} H_\omega^{s_0}} + \| |x|^\alpha D_\omega^{s_0} u \|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} \le \varepsilon\}, \\ & d_X(u,v) = \|u - v\|_{L_t^\infty \dot{H}^{s_{c,p}}} + \| |x|^\alpha (u-v) \|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}, \end{aligned}$$

and the mapping

$$\mathcal{N}_X(u) = U(t)(\varphi + \Phi_{t,p}) \qquad \text{on } X_\varepsilon.$$

We show that $\mathcal{N}_X$ is a contraction mapping on $X_\varepsilon$ for sufficiently small $\varepsilon$. It follows from (2.26) and Lemma 2.16 that there exists a positive constant $C$ such that

$$\begin{aligned}
&\|U(t)\varphi\|_{L_t^\infty \dot{H}^{s_{c,p}} H_\omega^{s_0}} + \||x|^\alpha D_\omega^{s_0} U(t)\varphi\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} \le C\|\varphi\|_{\dot{H}^{s_{c,p}} H_\omega^{s_0}}, \\
&\|U(t)\Phi_{t,p}\|_{L_t^\infty \dot{H}^{s_{c,p}} H_\omega^{s_0}} + \||x|^\alpha D_\omega^{s_0} U(t)\Phi_{t,p}\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} \le \\
&\qquad\qquad\qquad\qquad C\||x|^\alpha D_\omega^{s_0} u\|^p_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}.
\end{aligned}$$

Next, we prove

$$\begin{aligned}
&d_X(\mathcal{N}_X(u), \mathcal{N}_X(v)) \\
&\quad \le (\||x|^\alpha D_\omega^{s_0} u\|^{p-1}_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} + \||x|^\alpha D_\omega^{s_0} v\|^{p-1}_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma}) d_X(u,v)
\end{aligned} \tag{2.27}$$

for any $u$, $v \in X_\varepsilon$.
Similarly to the proof of Lemma 2.16, we have

$$\begin{aligned}
&d_X(\mathcal{N}_X(u), \mathcal{N}_X(v)) \\
&\lesssim \||\nabla|^{s_{c,p}}(\int_0^t U(t-t')(|u(t')|^{p-1}u(t') - |v(t')|^{p-1}v(t'))dt')\|_{L_t^\infty L_x^2} \\
&\qquad + \||x|^\alpha(\int_0^t U(t-t')(|u(t')|^{p-1}u(t') - |v(t')|^{p-1}v(t'))dt')\|_{L_t^{pq'} L_r^{p\ell'} L_\omega^\sigma} \\
&\lesssim \||x|^{p\alpha} D_\omega^{-s_1}(|u|^{p-1}u - |v|^{p-1}v)\|_{L_t^{q'} L_r^{\ell'} L_\omega^2}.
\end{aligned}$$

Note that $p\alpha$ satisfies

$$p\alpha = \begin{cases} -c_0 & (\text{if } d = 3), \\ -c_1 & (\text{if } d \ge 4). \end{cases}$$

By Sobolev embedding on the unit sphere, we have

$$\||x|^{p\alpha} D_\omega^{-s_1}(|u|^{p-1}u - |v|^{p-1}v)\|_{L_t^{q'} L_r^{\ell'} L_\omega^2} \lesssim \||x|^{p\alpha}(|u|^{p-1} + |v|^{p-1})(u-v)\|_{L_t^{q'} L_r^{\ell'} L_\omega^{\sigma_0}}$$

where $\frac{1}{\sigma_0} = \frac{1}{2} + \frac{s_1}{d-1}$. By Holder's inequality with

$$\frac{1}{\sigma_0} = \left(\frac{1}{2} + \frac{s_1}{d-1} - \frac{1}{\sigma}\right) + \frac{1}{\sigma},$$

we have

$$\begin{aligned}
&\||x|^{p\alpha}(|u|^{p-1}+|v|^{p-1})(u-v)\|_{L^{\sigma_0}_\omega}\\
&\quad=\||x|^{(p-1)\alpha}(|u|^{p-1}+|v|^{p-1})|x|^\alpha(u-v)\|_{L^{\sigma_0}_\omega}\\
&\quad\lesssim\||x|^{(p-1)\alpha}(|u|^{p-1}+|v|^{p-1})\|_{L^{\sigma_1}_\omega}\||x|^\alpha(u-v)\|_{L^{\sigma}_\omega}\\
&\quad\lesssim(\||x|^\alpha u\|^{p-1}_{L^{\sigma_1(p-1)}_\omega}+\||x|^\alpha v\|^{p-1}_{L^{\sigma_1(p-1)}_\omega})\||x|^\alpha(u-v)\|_{L^{\sigma}_\omega},
\end{aligned}$$

where

$$\frac{1}{\sigma_1}=\frac12+\frac{s_1}{d-1}-\frac{1}{\sigma}.$$

Since

$$-\frac{d-1}{\sigma_1(p-1)}=s_0-\frac{d-1}{\sigma},$$

Sobolev embedding on the unit sphere gives

$$\||x|^\alpha u\|_{L^{\sigma_1(p-1)}_\omega}\lesssim\||x|^\alpha D^{s_0}_\omega u\|_{L^\sigma_\omega},$$

which completes (2.27).

From (2.27), there exists a positive constant $C'$ such that

$$d_X(\mathcal{N}_X(u),\mathcal{N}_X(v))\le C'\varepsilon^{p-1}d_X(u,v).$$

Now we choose $\varepsilon$ and an initial data $\varphi$ such that

$$\max(C,C')\varepsilon^{p-1}\le\frac12,\qquad C\|\varphi\|_{\dot H^{s_{c,p}}H^{s_0}_\omega}\le\frac{\varepsilon}{2},$$

then the functional $\mathcal{N}_X$ becomes a contraction mapping on $X_\varepsilon$. □

2.5. **(NLS) with inhomogeneous nonlinearities.** We consider the Cauchy problem of nonlinear Schrödinger equations with inhomogeneous nonlinearities:

$$\begin{cases} iu_t(t,x)+\Delta u(t,x)=w(x)|u(t,x)|^{p-1}u(t,x), & \text{in } \mathbb{R}\times\mathbb{R}^d,\\ u(0,x)=\varphi(x), & \text{in } \mathbb{R}^d.\end{cases}\tag{2.28}$$

Here we assume that $|w(x)|\lesssim|x|^{-a}$. Note that if $|w(x)|=|x|^{-a}$, the scale critical index for (2.28) is

$$s_{c,a}=\frac d2-\frac{2-a}{p-1}.$$

We prove that there exists a solution of (2.28) for $s_{c,a}<0$ and $\varphi\in\dot H^{s_{c,a}}(\mathbb{R}^d)$. In [8], the small data global well-posedness was established for each $a$ and $p$ if an initial data $\varphi$ is radially symmetric or under some angular regularity assumption. The following theorem shows that we can get the small data global well-posedness without angular conditions if the exponent $a$ is positive.

**Theorem 2.17.** *Let $d \geq 3$, $0 < a < 2$ and*

$$\begin{cases} p_0 < p < 1 + \frac{4-2a}{d} & (\text{If } 0 < a < 1 + \frac{2d-1}{d^2-4}), \\ 1 + \frac{4-2a}{d+1} < p < 1 + \frac{4-2a}{d} & (\text{If } 1 + \frac{2d-1}{d^2-4} \leq a < 2), \end{cases}$$

*where $p_0 \in (1, 1 + \frac{4-2a}{d})$ satisfies*

$$d(d-2){p_0}^2 - 2(d-4-2ad+4a)p_0 - d^2 - 4d + 4a = 0.$$

*Then there exists a positive constant $\varepsilon = \varepsilon(d,p,a)$ such that if $\varphi \in \dot{H}^{s_{c,a}}$ satisfies $\||\nabla|^{s_{c,a}}\varphi\|_{L^2_x} < \varepsilon$, then the integral equation*

$$u(t,x) = U(t)(\varphi + \Phi_{t,a})(x), \tag{2.29}$$

*where*

$$\Phi_{t,a} = \Phi_{t,a}(u) = -i \int_0^t U(-t')(w(x)|u(t')|^{p-1}u(t'))dt',$$

*has a unique solution*

$$u \in C(\mathbb{R}; \dot{H}^{s_{c,a}}) \cap L^{pq'}(\mathbb{R}; |x|^{-\frac{1}{p}(d+a-\frac{2-a}{p-1}-\frac{d+2}{q})}L^{pq'}).$$

*Here $q$ satisfies the condition in Lemma 2.19 below.*

*Remark* 2.6. (i) It should be noted that if $a$ is sufficiently small then $p_0$ is sufficiently close to $1 + \frac{4-2a}{d}$.
(ii) If we try to estimate the nonlinearity $w(x)|u|^{p-1}u$ with $a < 0$, loss of regularity on the sphere arises and we need some angular regularity condition for $\varphi$ to get the well-posedness. Precisely, the estimate (2.30) in Corollary 2.18 below for positive values of $d$ does not hold. Therefore, we assume $a > 0$ in Theorem 2.17.

Since we do not have to mind an angular condition, the proof of Theorem 2.17 is simple relatively. First we restate Lemma 2.7 with $q = \ell = \sigma$ for convenience.

**Corollary 2.18.** *Let $d \geq 2$ and*

$$\frac{d}{2(d+2)} \leq \frac{1}{q} \leq \frac{1}{2}, \quad \frac{d^2}{4} - \frac{d^2+2d}{2q} < w < \frac{d}{4} - \frac{d+2}{2q}.$$

*Then we have*

$$\||x|^w |\nabla|^{w-\frac{d}{2}+\frac{d+2}{q}}[U(t)\varphi]\|_{L^q_t L^q_x} \lesssim \|\varphi\|_{L^2_x}. \tag{2.30}$$

The following lemma can be established by simple calculation. We omit the details.

**Lemma 2.19.** *Let $d \geq 3$, $0 < a < 2$ and*

$$\begin{cases} p_0 < p < 1 + \frac{4-2a}{d} & (\text{If } 0 < a < 1 + \frac{2d-1}{d^2-4}), \\ 1 + \frac{4-2a}{d+1} < p < 1 + \frac{4-2a}{d} & (\text{If } 1 + \frac{2d-1}{d^2-4} \leq a < 2). \end{cases} \tag{2.31}$$

*Then there exists $q$ such that*

$$\max\left(\frac{d}{2(d+2)}, 1 - \frac{p}{2}\right) \leq \frac{1}{q} \leq \min\left(\frac{1}{2}, 1 - \frac{d}{2(d+2)}p\right),$$
$$\frac{2}{d^2-4}\left(\frac{d^2}{4} - d + \frac{2-a}{p-1}\right) < \frac{1}{q} < \frac{2}{d+2}\left(\frac{d}{4}p + \frac{d-2}{2} + a - \frac{2-a}{p-1}\right).$$

**Lemma 2.20.** *Let $d \geq 3$, $0 < a < 2$. Suppose that $p$ and $q$ satisfy the condition* (2.31) *and the conditions in Lemma* 2.19*, respectively. Then we have*

$$\||\nabla|^{s_{c,a}} U(t)\Phi_{t,a}\|_{L_t^\infty L_x^2} + \||x|^{-\frac{a+c}{p}} U(t)\Phi_{t,a}\|_{L_t^{pq'} L_x^{pq'}} \lesssim \||x|^{-\frac{a+c}{p}} u\|^p_{L_t^{pq'} L_x^{pq'}},$$

*where $c = d - \frac{2-a}{p-1} - \frac{d+2}{q}$.*

*Proof.* (I) First, we prove

$$\||\nabla|^{s_{c,a}} U(t)\Phi_{t,a}\|_{L_t^\infty L_x^2} \lesssim \||x|^{-\frac{a+c}{p}} u\|^p_{L_t^{pq'} L_x^{pq'}}. \tag{2.32}$$

If $\frac{2}{d^2-4}\left(\frac{d^2}{4} - d + \frac{2-a}{p-1}\right) < \frac{1}{q}$, the following inequality

$$\frac{d^2}{4} - \frac{d^2+2d}{2q} < c < \frac{d}{4} - \frac{d+2}{2q}$$

holds. Then we deduce from Corollary 2.18 that

$$\||x|^c |\nabla|^{s_{c,a}} U(t)\varphi\|_{L_t^q L_x^q} \lesssim \|\varphi\|_{L_x^2}.$$

By the dual estimate, we have

$$\|\int_{-\infty}^{\infty} U(-t')(w(x)|u(t')|^{p-1}u(t'))dt'\|_{L_x^2} \lesssim \||x|^{-c}|\nabla|^{-s_{c,a}}(w(x)|u|^{p-1}u)\|_{L_t^{q'} L_x^{q'}}, \tag{2.33}$$

which means

$$\begin{aligned} \||\nabla|^{s_{c,a}} U(t)\Phi_{t,a}\|_{L_t^\infty L_x^2} &\lesssim \||x|^{-(a+c)}|u|^{p-1}u\|_{L_t^{q'} L_x^{q'}} \\ &= \||x|^{-\frac{a+c}{p}} u\|^p_{L_t^{pq'} L_x^{pq'}}. \end{aligned}$$

This completes the proof of (2.32).

(II) Next, we prove

$$\||x|^{-\frac{a+c}{p}} U(t)\Phi_{t,a}\|_{L_t^{pq'} L_x^{pq'}} \lesssim \||x|^{-\frac{a+c}{p}} u\|^p_{L_t^{pq'} L_x^{pq'}}.$$

From the inequalities $1-\frac{p}{2}\leq\frac{1}{q}\leq 1-\frac{d}{2(d+2)}p$ and

$$\frac{1}{q}<\frac{2}{d+2}\Big(\frac{d}{4}p+\frac{d-2}{2}+a-\frac{2-a}{p-1}\Big),$$

we have $\frac{d}{2(d+2)}\leq\frac{1}{pq'}\leq\frac{1}{2}$ and

$$\frac{d^2}{4}-\frac{d^2+2d}{2pq'}<-\frac{a+c}{p}<\frac{d}{4}-\frac{d+2}{2pq'}.$$

Then we deduce from Corollary 2.18 that

$$\||x|^{-\frac{a+c}{p}}|\nabla|^{-s_{c,a}}U(t)\varphi\|_{L_t^{pq'}L_x^{pq'}}\lesssim\|\varphi\|_{L_x^2}. \tag{2.34}$$

Applying Lemma 2.9 to (2.33) and (2.34), we have

$$\begin{aligned}\||x|^{-\frac{a+c}{p}}U(t)\Phi_{t,a}\|_{L_t^{pq'}L_x^{pq'}}&\lesssim\||x|^{-c}w(x)|u|^{p-1}u\|_{L_t^{q'}L_x^{q'}}\\&\lesssim\||x|^{-\frac{a+c}{p}}u\|^p_{L_t^{pq'}L_x^{pq'}}.\end{aligned}$$

This completes the proof. □

From Lemmas 2.19 and 2.20, Theorem 2.17 is established with the contraction mapping argument. The way of the proof is the same as in that of Theorems 2.13 and 2.15. We leave the details to the readers.

## 3. Well-posedness of the Klein-Gordon-Zakharov system in 2D

3.1. **Introduction.** We consider the Cauchy problem of the Klein-Gordon-Zakharov system:

$$\begin{cases} (\partial_t^2 - \Delta + 1)u = -nu, & (t,x) \in [-T,T] \times \mathbb{R}^d, \\ (\partial_t^2 - c^2\Delta)n = \Delta |u|^2, & (t,x) \in [-T,T] \times \mathbb{R}^d, \\ (u, \partial_t u, n, \partial_t n)|_{t=0} = (u_0, u_1, n_0, n_1) & \\ \qquad\qquad \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d), \end{cases} \tag{3.1}$$

where $u, n$ are real valued functions, $0 < c < 1$. As a physical model, (3.1) describes the interaction of the Langmuir wave and the ion acoustic wave in a plasma. The condition $0 < c < 1$, which plays an important role, comes from a physical phenomenon. There are some works on the Cauchy problem of (3.1) in low regularity Sobolev spaces. For 3D, Ozawa, Tsutaya and Tsutsumi [40] proved that (3.1) is globally well-posed in the energy space $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times \dot{H}^{-1}(\mathbb{R}^3)$. As they mentioned in [40] that if $c = 1$, (3.1) is very similar to the Cauchy problem of the following quadratic derivative nonlinear wave equation.

$$\begin{cases} (\partial_t^2 - \Delta)u = uDu, & (t,x) \in [-T,T] \times \mathbb{R}^3, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in H^{s+1}(\mathbb{R}^3) \times H^s(\mathbb{R}^3). \end{cases} \tag{3.2}$$

For $s > 0$, the local well-posedness of (3.2) was obtained from the iteration argument by using the Strichartz estimates. As opposed to that, it is known that (3.2) is ill-posed for $s \le 0$ by the works of Lindblad [32]-[33]. In [40], the authors showed that the difference between the propagation speeds of the two equations in (3.1) enable us to get the better result. That is, they applied the Fourier restriction norm method and obtained the local well-posedness of (3.1) in the energy space, and then, by the energy conservation law, they extended an existent time of a local solution globally in time. After that, for $d = 3$, Guo, Nakanishi and Wang [17] proved the scattering in the energy class with small, radial initial data. They applied the normal form reduction and the radial Strichartz estimates. For 4 and higher dimensions, I. Kato [25] recently proved that (3.1) is locally well-posed at $s = 1/4$ when $d = 4$ and $s = s_c + 1/(d+1)$ when $d \ge 5$ where $s_c = d/2 - 2$ is the critical exponent of (3.1). He also proved that if the initial data are radially symmetric then the small data globally well-posedness can be obtained at the scaling critical regularity for $d \ge 4$. He utilized the $U^2$, $V^2$ spaces introduced by Koch-Tataru [31]. As we mentioned in Section 1, we will improve his results in Section 4. We would like to emphasize that the above results hold under the condition $0 < c < 1$. Our aim in this section is to get the local well-posedness of (3.1) at very low regularity $s$ in 2 dimensions. Hereafter we assume $d = 2$.

By the transformation $u_\pm := \omega_1 u \pm i\partial_t u, n_\pm := n \pm i(c\omega)^{-1}\partial_t n, \omega_1 := (1-\Delta)^{1/2}, \omega := (-\Delta)^{1/2}$, (3.1) can be written as follows;.

$$\begin{cases} (i\partial_t \mp \omega_1)u_\pm = \pm(1/4)(n_+ + n_-)(\omega_1^{-1}u_+ + \omega_1^{-1}u_-), & (t,x) \in [-T,T] \times \mathbb{R}^2, \\ (i\partial_t \mp c\omega)n_\pm = \pm(4c)^{-1}\omega|\omega_1^{-1}u_+ + \omega_1^{-1}u_-|^2, & (t,x) \in [-T,T] \times \mathbb{R}^2, \\ (u_\pm, n_\pm)|_{t=0} = (u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^2) \times \dot{H}^s(\mathbb{R}^2). \end{cases} \tag{3.3}$$

We state our main result.

**Theorem 3.1.** *Let $-3/4 < s < 0$. Then* (3.3) *is locally well-posed in $H^s(\mathbb{R}^2) \times \dot{H}^s(\mathbb{R}^2)$.*

We make a comment on Theorem 3.1. Applying the iteration argument by the usual Strichartz estimates, we get the local well-posedness of (3.3) for $-1/4 \le s$. This suggests that if $c = 1$ the minimal regularity such that the well-posedness of (3.3) holds seems to be $-1/4$. If we utilize the condition $0 < c < 1$ in the same way as in [40] and [25] with minor modification, we can show that (3.3) is local well-posed only for $s > -1/2$. We find that the known arguments is not enough to get the well-posedness for $s \le -1/2$ which is the most difficult case. To overcome this, we employ a new estimate which was introduced in [3] and applied to Zakharov system in [1] and [2]. Zakharov system consists of two equations, wave equation and Schrödinger equation;

$$\begin{cases} (i\partial_t + \Delta)u = nu, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ (\partial_t^2 - \Delta)n = \Delta|u|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^d. \end{cases} \tag{3.4}$$

Roughly speaking, comparing (3.1) and (3.4), the two systems have the similar structure, which suggests that we might get the well-posedness of (3.3) for $s \le -1/2$ in the same way as in [1] and [2].

We will prove Theorem 3.1 by the iteration argument in the spaces $X_\pm^{s,b}(\mathbb{R}^3) \times X_{\pm,c}^{s,b}(\mathbb{R}^3)$. This spaces are defined as follows;

Let $N$, $L \ge 1$ be dyadic numbers. We define the dyadic decompositions of $\mathbb{R}^3$.

$$K_{N,L}^{\pm} := \{(\tau,\xi) \in \mathbb{R}^3 | N \le \langle \xi \rangle \le 2N, L \le \langle \tau \pm |\xi| \rangle \le 2L\},$$
$$K_{N,L}^{\pm,c} := \{(\tau,\xi) \in \mathbb{R}^3 | N \le \langle \xi \rangle \le 2N, L \le \langle \tau \pm c|\xi| \rangle \le 2L\}.$$

By using $K^{\pm}_{N,L}$, $K^{\pm,c}_{N,L}$, we define the dyadic decomposition in Fourier side;

$$P_{K^{\pm}_{N,L}} := \mathcal{F}^{-1}_{t,x}\chi_{K^{\pm}_{N,L}}\mathcal{F}_{t,x}, \quad P_{K^{\pm,c}_{N,L}} := \mathcal{F}^{-1}_{t,x}\chi_{K^{\pm,c}_{N,L}}\mathcal{F}_{t,x}.$$

We now introduce the solution spaces. Let $s, b \in \mathbb{R}$. We define $X^{s,b}_{\pm}(\mathbb{R}^3)$ and $X^{s,b}_{\pm,c}(\mathbb{R}^3)$ as follows;

$$\begin{aligned}
X^{s,b}_{\pm}(\mathbb{R}^3) &:= \{f \in \mathcal{S}'(\mathbb{R}^3) \mid \|f\|_{X^{s,b}_{\pm}} < \infty\}, \\
X^{s,b}_{\pm,c}(\mathbb{R}^3) &:= \{f \in \mathcal{S}'(\mathbb{R}^3) \mid \|f\|_{X^{s,b}_{\pm,c}} < \infty\}, \\
\|f\|_{X^{s,b}_{\pm}} &= \left(\sum_{N,L} N^{2s}L^{2b}\|P_{K^{\pm}_{N,L}}f\|^2_{L^2_{x,t}}\right)^{1/2}, \\
\|f\|_{X^{s,b}_{\pm,c}} &= \left(\sum_{N,L} N^{2s}L^{2b}\|P_{K^{\pm,c}_{N,L}}f\|^2_{L^2_{x,t}}\right)^{1/2}.
\end{aligned}$$

The key estimates to prove Theorem 3.1 are the following.

**Theorem 3.2.** *For any $s \in (-3/4,\ 0)$, there exist $b \in (1/2, 1)$ and $C$ which depend on $c$ such that*

$$\|u(\omega_1^{-1}v)\|_{X^{s,b-1}_{\pm_2}} \leq C\|u\|_{X^{s,b}_{\pm_0,c}}\|v\|_{X^{s,b}_{\pm_1}}, \tag{3.5}$$

$$\|\omega_1((\omega_1^{-1}u)(\omega_1^{-1}v))\|_{X^{s,b-1}_{\pm_0,c}} \leq C\|u\|_{X^{s,b}_{\pm_1}}\|v\|_{X^{s,b}_{\pm_2}}. \tag{3.6}$$

*regardless of the choice of signs $\pm_j$.*

*Remark* 3.1. In fact, the bilinear estimates naturally derived from (3.3) are slightly different from (3.5)-(3.6). They are described as follows;

$$\|u(\omega_1^{-1}v)\|_{X^{s,b-1}_{\pm_2}} \leq C\|\omega^s u\|_{X^{0,b}_{\pm_0,c}}\|v\|_{X^{s,b}_{\pm_1}}, \tag{3.7}$$

$$\|\omega^{s+1}((\omega_1^{-1}u)(\omega_1^{-1}v))\|_{X^{0,b-1}_{\pm_0,c}} \leq C\|u\|_{X^{s,b}_{\pm_1}}\|v\|_{X^{s,b}_{\pm_2}}. \tag{3.8}$$

It is easily confirmed that (3.5) and (3.6) are strict compared with (3.7) and (3.8), respectively. We also mention that it might be natural that we use $\langle \tau \pm \langle \xi \rangle \rangle$ instead of $\langle \tau \pm |\xi| \rangle$ in the definition of $K^{\pm}_{N,L}$. As was seen in [40], these two weights are equivalent and therefore $X^{s,b}_{\pm}$ does not depend on the choice of them in the definition of $K^{\pm}_{N,L}$.

Once Theorem 3.2 is verified, we can obtain Theorem 3.1 by the iteration argument given in [15] and many other papers. For example, see [30] and [45]. Therefore we focus on the proof of Theorem 3.2 in this section.

The sections are organized as follows. In Section 3.2, we introduce some fundamental estimates and property of the solution spaces as preliminary. In Section 3.3, we show (3.5) and (3.6) with $\pm_1 = \pm_2$ which is easier case compared to $\pm_1 \neq \pm_2$. Section 3.4 is devoted to the proof of (3.5) and (3.6) with $\pm_1 \neq \pm_2$, and complete the proof of Theorem 3.2.

3.2. **Preliminaries.** We first observe that fundamental properties of $X^{s,\,b}_{\pm}$ and $X^{s,\,b}_{\pm,\,c}$. A simple calculation gives the followings;

$$(i)\ \ \overline{X^{s,\,b}_{\pm}} = X^{s,\,b}_{\mp}, \qquad \overline{X^{s,\,b}_{\pm,\,c}} = X^{s,\,b}_{\mp,\,c}.$$
$$(ii)\ \ (X^{s,\,b}_{\pm})^* = X^{-s,\,-b}_{\mp}, \quad (X^{s,\,b}_{\pm,\,c})^* = X^{-s,\,-b}_{\mp,\,c},$$

for $s,\, b \in \mathbb{R}$. Next we define the angular decomposition of $\mathbb{R}^3$ in frequency. For a dyadic number $A \geq 64$ and an integer $j \in [-A,\ A-1]$, we define the sets $\{\mathfrak{D}^A_j\} \subset \mathbb{R}^3$ as follows;

$$\mathfrak{D}^A_j = \left\{ (\tau, |\xi|\cos\theta, |\xi|\sin\theta) \in \mathbb{R}\times\mathbb{R}^2 \mid \theta \in \left[\frac{\pi}{A}\,j,\ \frac{\pi}{A}(j+1)\right] \right\}.$$

For any function $u : \mathbb{R}^3 \to \mathbb{C}$, $\{\mathfrak{D}^A_j\}$ satisfy

$$\mathbb{R}^3 = \bigcup_{-A\leq j\leq A-1} \mathfrak{D}^A_j, \qquad u = \sum_{j=-A}^{A-1} \chi_{\mathfrak{D}^A_j} u \quad a.e.$$

Lastly we introduce the useful two estimates which are called the bilinear Strichartz estimates. The first one holds true regardless of $c$. As opposed to that, the second one is given by using the condition $0 < c < 1$. The first estimate is obtained by the same argument as in the proof of Theorem 2.1 in [42]. We omit the proof.

**Proposition 3.3** (Theorem 2.1. [42])**.**

$$\begin{aligned}\|P_{K^{\pm_0}_{N_0,L_0}(K^{\pm_0,c}_{N_0,L_0})}((P_{K^{\pm_1}_{N_1,L_1}(K^{\pm_1,c}_{N_1,L_1})} f)(P_{K^{\pm_2}_{N_2,L_2}(K^{\pm_2,c}_{N_2,L_2})} g))\|_{L^2_{x,t}} \\ \lesssim (N^{012}_{\min} L^{12}_{\min})^{1/2} (N^{12}_{\min} L^{12}_{\max})^{1/4} \|f\|_{L^2_{x,t}} \|g\|_{L^2_{x,t}},\end{aligned} \tag{3.9}$$

*regardless of the choice of signs* $\pm_j$. *Here* $N^{012}_{\min} := \min(N_0, N_1, N_2)$, *and* $N^{12}_{\min}$, $L^{12}_{\min}$, $L^{12}_{\max}$ *are defined similarly.*

**Proposition 3.4.**

$$\|P_{K^{\pm_0}_{N_0,L_0}}((P_{K^{\pm_1,c}_{N_1,L_1}} f)(P_{K^{\pm_2}_{N_2,L_2}} g))\|_{L^2_{x,t}} \lesssim (N^{012}_{\min} L_1 L_2)^{1/2} \|f\|_{L^2_{x,t}} \|g\|_{L^2_{x,t}} \tag{3.10}$$

*holds regardless of the choice of* $\pm_j$.

*Proof.* Let $A = 2^{10}(1-c)^{-1/2}$. From Plancherel theorem, we observe that

$$\begin{aligned}&\|P_{K^{\pm_0}_{N_0,L_0}} \left( (P_{K^{\pm_1,c}_{N_1,L_1}} f)(P_{K^{\pm_2}_{N_2,L_2}} g) \right) \|_{L^2_{x,t}} \\ &\qquad \sim \|\chi_{K^{\pm_0}_{N_0,L_0}} \left( \left( \chi_{K^{\pm_1,c}_{N_1,L_1}} \widehat{f} \right) * \left( \chi_{K^{\pm_2}_{N_2,L_2}} \widehat{g} \right) \right) \|_{L^2_{\xi,\tau}}\end{aligned} \tag{3.11}$$

where $*$ denotes the convolution of $\mathbb{R}^3$. It follows from the finiteness of $A$ and

$$\widehat{g} = \sum_{j=-A}^{A-1} \chi_{\mathfrak{D}_j^A} \widehat{g} \quad a.e.$$

that we can replace $\widehat{g}$ with $\chi_{\mathfrak{D}_j^A}\widehat{g}$ in (3.10) for fixed $j$. After applying rotation in space, we may assume that $j=0$. Also we can assume that there exists $\xi' \in \mathbb{R}^2$ such that the support of $\chi_{\mathfrak{D}_j^A}\widehat{g}$ is contained in the column

$$C_{N_{\min}^{012}}(\xi') := \{(\tau,\xi) \in \mathbb{R}^3 \mid |\xi - \xi'| \leq N_{\min}^{012}\}.$$

We sketch the validity of the above assumption roughly. See [45] for more details. If $N_2 \sim N_{\min}^{012}$ the above assumption is harmless obviously. Therefore we may assume that $N_0 = N_{\min}^{012} \ll N_2$ or $N_1 = N_{\min}^{012} \ll N_2$. Since both are treated similarly, we here consider only the former case. Note that the condition $N_0 \ll N_2$ means $N_2/2 \leq N_1 \leq 2N_2$, otherwise (3.11) vanishes. We can choose the two sets $\{C_{N_{\min}^{012}}(\xi_k')\}_k$ and $\{C_{N_{\min}^{012}}(\xi_\ell'')\}_\ell$ such that

$$\#k \sim \left(\frac{N_1}{N_0}\right)^2, \qquad \operatorname{supp} \chi_{\mathfrak{D}_0^A}\widehat{g} \subset \bigcup_k C_{N_{\min}^{012}}(\xi_k'), \qquad |\xi_k' - \xi_{k'}'| \geq N_{\min}^{012} \text{ for any } k, k',$$

$$\#\ell \sim \left(\frac{N_1}{N_0}\right)^2, \qquad \operatorname{supp} \widehat{f} \subset \bigcup_\ell C_{N_{\min}^{012}}(\xi_\ell''), \qquad |\xi_\ell'' - \xi_{\ell'}''| \geq N_{\min}^{012} \text{ for any } \ell, \ell',$$

where $\#k$ and $\#\ell$ denote the numbers of $k$ and $\ell$, respectively. We see that for fixed $k$, independently of $N_0$, $N_1$, $N_2$, there is only a finite number of $\ell$ which satisfy

$$\left\| \chi_{K_{N_0,L_0}^{\pm_0}} \left( \left( \chi_{K_{N_1,L_1}^{\pm_1,c} \cap C_{N_{\min}^{012}}(\xi_k')} \widehat{f} \right) * \left( \chi_{K_{N_2,L_2}^{\pm_2} \cap C_{N_{\min}^{012}}(\xi_\ell'') \cap \mathfrak{D}_0^A} \widehat{g} \right) \right) \right\|_{L^2_{\xi,\tau}} > 0,$$

and vice versa. This means that $k$ and $\ell$ depend on each other. Once we obtain

$$\begin{aligned} &\left\| \chi_{K_{N_0,L_0}^{\pm_0}} \left( \left( \chi_{K_{N_1,L_1}^{\pm_1,c} \cap C_{N_{\min}^{012}}(\xi_k')} \widehat{f} \right) * \left( \chi_{K_{N_2,L_2}^{\pm_2} \cap C_{N_{\min}^{012}}(\xi_{\ell(k)}'') \cap \mathfrak{D}_0^A} \widehat{g} \right) \right) \right\|_{L^2_{\xi,\tau}} \\ &\qquad \lesssim (N_{\min}^{012} L_1 L_2)^{1/2} \|f\|_{L^2_{x,t}} \|g\|_{L^2_{x,t}} \end{aligned}$$

for fixed $k$, from Minkowski inequality and $\ell^2$ almost orthogonality, we confirm

$$\begin{aligned} &\|\chi_{K_{N_0,L_0}^{\pm_0}} \left( \left( \chi_{K_{N_1,L_1}^{\pm_1,c}} \widehat{f} \right) * \left( \chi_{K_{N_2,L_2}^{\pm_2}} \widehat{g} \right) \right) \|_{L^2_{\xi,\tau}} \\ \lesssim &\sum_{k,\ell} \left\| \chi_{K_{N_0,L_0}^{\pm_0}} \left( \left( \chi_{K_{N_1,L_1}^{\pm_1,c} \cap C_{N_{\min}^{012}}(\xi_k')} \widehat{f} \right) * \left( \chi_{K_{N_2,L_2}^{\pm_2} \cap C_{N_{\min}^{012}}(\xi_{\ell(k)}'') \cap \mathfrak{D}_0^A} \widehat{g} \right) \right) \right\|_{L^2_{\xi,\tau}} \\ \lesssim &(N_{\min}^{012} L_1 L_2)^{1/2} \sum_{k,\ell} \|\chi_{C_{N_{\min}^{012}}(\xi_k')} \widehat{f}\|_{L^2_{\xi,\tau}} \|\chi_{C_{N_{\min}^{012}}(\xi_{\ell(k)}'')} \widehat{g}\|_{L^2_{\xi,\tau}} \\ \lesssim &(N_{\min}^{012} L_1 L_2)^{1/2} \|f\|_{L^2_{x,t}} \|g\|_{L^2_{x,t}}, \end{aligned}$$

which verify the validity of the assumption. Hereafter, we call the above argument "$\ell^2$ almost orthogonality".

Now we turn to the proof of (3.10).

$$\begin{aligned}
&\|\chi_{K^{\pm_0}_{N_0,L_0}}\left(\left(\chi_{K^{\pm_1,c}_{N_1,L_1}}\widehat{f}\right)*\left(\chi_{K^{\pm_2}_{N_2,L_2}\cap\mathfrak{D}^A_0\cap C_{N^{012}_{\min}}(\xi')}\widehat{g}\right)\right)\|_{L^2_{\xi,\tau}}\\
\lesssim&\|\chi_{K^{\pm_0}_{N_0,L_0}}(\tau,\xi)\int\left(\chi_{K^{\pm_1,c}_{N_1,L_1}}\widehat{f}\right)(\tau-\tau_1,\xi-\xi_1)\left(\chi_{K^{\pm_2}_{N_2,L_2}\cap\mathfrak{D}^A_0\cap C_{N^{012}_{\min}}(\xi')}\widehat{g}\right)(\tau_1,\xi_1)d\tau_1d\xi_1\|_{L^2_{\xi,\tau}}\\
\lesssim&\|\chi_{K^{\pm_0}_{N_0,L_0}}(\tau,\xi)\left(\int|\widehat{f}|^2(\tau-\tau_1,\xi-\xi_1)|\widehat{g}|^2(\tau_1,\xi_1)d\tau_1d\xi_1\right)^{1/2}(E(\tau,\xi))^{1/2}\|_{L^2_{\xi,\tau}}\\
\lesssim&\sup_{(\tau,\xi)\in K^{\pm_0}_{N_0,L_0}}|E(\tau,\xi)|^{1/2}\||\widehat{f}|^2*|\widehat{g}|^2\|^{1/2}_{L^1_{\xi,\tau}}\\
\lesssim&\sup_{(\tau,\xi)\in K^{\pm_0}_{N_0,L_0}}|E(\tau,\xi)|^{1/2}\|f\|_{L^2_{x,t}}\|g\|_{L^2_{x,t}}
\end{aligned}$$

where

$$E(\tau,\xi):=\{(\tau_1,\xi_1)\in C_{N^{012}_{\min}}(\xi')\cap\mathfrak{D}^A_0\mid\langle\tau-\tau_1\pm c|\xi-\xi_1|\rangle\sim L_1,\langle\tau_1\pm|\xi_1|\rangle\sim L_2\}.$$

Thus it suffices to show that

$$\sup_{\tau,\xi}|E(\tau,\xi)|\lesssim N^{012}_{\min}L_1L_2. \tag{3.12}$$

From $\langle\tau-\tau_1\pm c|\xi-\xi_1|\rangle\sim L_1$ and $\langle\tau_1\pm|\xi_1|\rangle\sim L_2$, for fixed $\xi_1$,

$$|\{\tau_1\mid(\tau_1,\xi_1)\in E(\tau,\xi)\}|\lesssim L^{12}_{\min}. \tag{3.13}$$

It follows from $(\tau_1,\xi_1)\in\mathfrak{D}^A_0$ that

$$\begin{aligned}
|\partial_1(\tau\pm|\xi_1|\pm c|\xi-\xi_1|)|&\geq\frac{(\xi_1)_1}{|\xi_1|}-c\\
&\geq\left(\frac{(\xi_1)_1}{|\xi_1|}\right)^2-c\\
&=1-c-\left(\frac{(\xi_1)_2}{|\xi_1|}\right)^2\\
&\geq(1-c)/2,
\end{aligned}\tag{3.14}$$

where $(\xi_1)_1$ is the first component of $\xi_1$ and $\partial_1$ is the derivative with respect to $(\xi_1)_1$. Combining $|\tau\pm|\xi_1|\pm c|\xi-\xi_1||\lesssim L^{12}_{\max}$ with (3.14), for fixed $(\xi_1)_2$ we have

$$|\{(\xi_1)_1\mid(\tau_1,\xi_1)\in E(\tau,\xi)\}|\lesssim L^{12}_{\max}. \tag{3.15}$$

Collecting (3.13), (3.15) and $\xi_1\in C_{N^{012}_{\min}}(\xi')$, we get (3.12). □

3.3. **Proof of Theorem 3.2 for $\pm_1 = \pm_2$.** In (3.5)-(3.6), replacing $u$ and $n$ with its complex conjugates $\bar{u}$ and $\bar{v}$ respectively, we easily find that there is no difference between the case $(\pm_0, \pm_1, \pm_2)$ and $(\mp_0, \mp_1, \mp_2)$. Here $\mp_j$ denotes a different sign to $\pm_j$. Therefore we assume $\pm_1 = -$ in (3.5)-(3.6) hereafter. By the dual argument, we observe that

$$
\begin{aligned}
(3.5) \iff & \left| \int f(\omega_1^{-1} g_1) g_2 dtdx \right| \le C \|f\|_{X^{s,\,b}_{\pm_0,\,c}} \|g_1\|_{X^{s,\,b}_{-}} \|g_2\|_{X^{-s,\,1-b}_{\pm_2}}. \\
\Longleftarrow & \sum_{N_j, L_j (j=0,1,2)} \left| N_1^{-1} \int (P_{K^{\pm,c}_{N_0,L_0}} f)(P_{K^{-}_{N_1,L_1}} g_1)(P_{K^{\pm}_{N_2,L_2}} g_2) dtdx \right| \\
& \qquad \lesssim \|f\|_{X^{s,\,b}_{\pm_0,\,c}} \|g_1\|_{X^{s,\,b}_{-}} \|g_2\|_{X^{-s,\,1-b}_{\pm_2}}. \\
\iff & \left( \sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} + \sum_{N_0} \sum_{1 \le N_1 \lesssim N_0 \sim N_2} + \sum_{N_0} \sum_{1 \le N_2 \lesssim N_0 \sim N_1} \right) I_1 \\
& \qquad \lesssim \|f\|_{X^{s,\,b}_{\pm_0,\,c}} \|g_1\|_{X^{s,\,b}_{-}} \|g_2\|_{X^{-s,\,1-b}_{\pm_2}},
\end{aligned} \tag{3.16}
$$

where

$$
I_1 := \sum_{L_j} \left| N_1^{-1} \int (P_{K^{\pm_0,c}_{N_0,L_0}} f)(P_{K^{-}_{N_1,L_1}} g_1)(P_{K^{\pm_2}_{N_2,L_2}} g_2) dtdx \right|.
$$

Similarly, (3.6) is verified by the following estimate.

$$
\begin{aligned}
& \left( \sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} + \sum_{N_0} \sum_{1 \le N_1 \lesssim N_0 \sim N_2} + \sum_{N_0} \sum_{1 \le N_2 \lesssim N_0 \sim N_1} \right) I_2 \\
& \qquad \lesssim \|f\|_{X^{-s,\,1-b}_{\pm_0,\,c}} \|g_1\|_{X^{s,\,b}_{-}} \|g_2\|_{X^{s,\,b}_{\pm_2}},
\end{aligned} \tag{3.17}
$$

where

$$
I_2 := \sum_{L_j} \left| N_0 N_1^{-1} N_2^{-1} \int (P_{K^{\pm_0,c}_{N_0,L_0}} f)(P_{K^{-}_{N_1,L_1}} g_1)(P_{K^{\pm_2}_{N_2,L_2}} g_2) dtdx \right|.
$$

We now try to establish (3.16) and (3.17). First we assume that $\pm_2 = -$. In this case, we can obtain (3.16) and (3.17) by using the bilinear Strichartz estimates Propositions 3.3, 3.4 and the following estimate:

**Lemma 3.5.** *Let $\tau = \tau_1 + \tau_2$, $\xi = \xi_1 + \xi_2$. Then we have*

$$
\max(\langle \tau \pm_0 c|\xi| \rangle,\ \langle \tau_1 - |\xi_1| \rangle,\ \langle \tau_2 - |\xi_2| \rangle) \gtrsim \max(|\xi_1|,\ |\xi_2|). \tag{3.18}
$$

*Proof.*

$$\begin{aligned}\max(\langle \tau \pm_0 c|\xi|\rangle,\ \langle \tau_1 - |\xi_1|\rangle,\ \langle \tau_2 - |\xi_2|\rangle) &\geq |\tau \pm_0 c|\xi| - (\tau_1 - |\xi_1|) - (\tau_2 - |\xi_2|)| \\ &\geq ||\xi_1| + |\xi_2| - c|\xi|| \\ &\geq |\xi_1| + |\xi_2| - c(|\xi_1| + |\xi_2|) \\ &= (1-c)(|\xi_1| + |\xi_2|)\end{aligned}$$

□

For simplicity, we use $f^{\pm,c} := P_{K^{\pm,c}_{N_0,L_0}} f$, $g_k^- := P_{K^-_{N_k,L_k}} g$ for $k = 1, 2$.

**Theorem 3.6.** *For any $s \in (-3/4,\ 0)$, there exists $b \in (1/2,\ 1)$ such that for $f, g_1, g_2 \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$, the following estimates hold:*

$$\left(\sum_{N_1} \sum_{1 \leq N_0 \lesssim N_1 \sim N_2} + \sum_{N_0} \sum_{1 \leq N_1 \lesssim N_0 \sim N_2} + \sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1}\right) I_1^- \lesssim \|f\|_{X^{s,\, b}_{\pm,\, c}} \|g_1\|_{X^{s,\, b}_{-}} \|g_2\|_{X^{-s,\, 1-b}_{-}}, \tag{3.19}$$

$$\left(\sum_{N_1} \sum_{1 \leq N_0 \lesssim N_1 \sim N_2} + \sum_{N_0} \sum_{1 \leq N_1 \lesssim N_0 \sim N_2} + \sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1}\right) I_2^- \lesssim \|f\|_{X^{-s,\, 1-b}_{\pm,\, c}} \|g_1\|_{X^{s,\, b}_{-}} \|g_2\|_{X^{s,\, b}_{-}}, \tag{3.20}$$

*where*

$$\begin{aligned} I_1^- &:= \sum_{L_j} \left| N_1^{-1} \int (P_{K^{\pm,c}_{N_0,L_0}} f)(P_{K^-_{N_1,L_1}} g_1)(P_{K^-_{N_2,L_2}} g_2) dtdx \right|, \\ I_2^- &:= \sum_{L_j} \left| N_0 N_1^{-1} N_2^{-1} \int (P_{K^{\pm,c}_{N_0,L_0}} f)(P_{K^-_{N_1,L_1}} g_1)(P_{K^-_{N_2,L_2}} g_2) dtdx \right|. \end{aligned}$$

*Proof.* Since the proof of (3.20) is analogous to that of (3.19), we establish only (3.19). From Lemma 3.5, it holds that $L^{012}_{\max} \gtrsim N^{12}_{\max}$. We decompose the proof into the three cases:

(I) $1 \leq N_0 \lesssim N_1 \sim N_2$, (II) $1 \leq N_1 \lesssim N_0 \sim N_2$, (III) $1 \leq N_2 \lesssim N_0 \sim N_1$.

First we consider the case (I). Considering that $L^{012}_{\max} \gtrsim N^{12}_{\max}$, we subdivide the cases further:

(Ia) $N_1 \lesssim L_0$. We deduce from Hölder inequality and Proposition 3.3 that

$$\begin{aligned}
&\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} \sum_{L_j} \left| N_1^{-1} \int f^{\pm,c} g_1^- g_2^- dtdx \right| \\
\lesssim &\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} \sum_{L_j} N_1^{-1} \| f^{\pm,c} \|_{L^2_{x,t}} \| P_{K^{\pm,c}_{N_0,L_0}} (g_1^- \ g_2^-) \|_{L^2_{x,t}} \\
\lesssim &\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} \sum_{L_1, L_2} N_1^{-1} N_0^{1/2} N_1^{1/4} L_1^{1/2} L_2^{1/4} N_1^{-b} \| f^{\pm,c} \|_{X^{0,b}_{\pm,c}} \| g_1^- \|_{L^2_{x,t}} \| g_2^- \|_{L^2_{x,t}} \\
\lesssim &\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} N_0^{1/2-s} N_1^{-3/4-b} N_0^s \| f^{\pm,c} \|_{X^{0,b}_{\pm,c}} N_1^s \| g_1^- \|_{X^{0,b}_-} N_2^{-s} \| g_2^- \|_{X^{0,1-b}_-} \\
\lesssim &\| f \|_{X^{s,b}_{\pm,c}} \| g_1 \|_{X^{s,b}_-} \| g_2 \|_{X^{-s,1-b}_-}.
\end{aligned}$$

(Ib) $N_1 \lesssim L_1$. Similarly, from Hölder inequality and Proposition 3.3 we get

$$\begin{aligned}
&\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} \sum_{L_j} \left| N_1^{-1} \int f^{\pm,c} g_1^- g_2^- dtdx \right| \\
\lesssim &\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} \sum_{L_j} N_1^{-1} \| P_{K^-_{N_1,L_1}} (f^{\pm,c} g_2^-) \|_{L^2_{x,t}} \| g_1^- \|_{L^2_{x,t}} \\
\lesssim &\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} \sum_{L_0, L_2} N_1^{-1} N_0^{3/4} L_0^{1/2} L_2^{1/4} \| f^{\pm,c} \|_{L^2_{x,t}} N_1^{-b} \| g_1^- \|_{X^{0,b}_-} \| g_2^- \|_{L^2_{x,t}} \\
\lesssim &\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} N_0^{3/4-s} N_1^{-1-b} N_0^s \| f^{\pm,c} \|_{X^{0,b}_{\pm,c}} N_1^s \| g_1^- \|_{X^{0,b}_-} N_2^{-s} \| g_2^- \|_{X^{0,1-b}_-} \\
\lesssim &\| f \|_{X^{s,b}_{\pm,c}} \| g_1 \|_{X^{s,b}_-} \| g_2 \|_{X^{-s,1-b}_-}.
\end{aligned}$$

(Ic) $N_1 \lesssim L_2$.

$$\begin{aligned}
&\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} \sum_{L_j} \left| N_1^{-1} \int f^{\pm,c} g_1^- g_2^- dtdx \right| \\
\lesssim &\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} \sum_{L_j} N_1^{-1} \| P_{K^-_{N_2,L_2}} (f^{\pm,c} g_1^-) \|_{L^2_{x,t}} \| g_2^- \|_{L^2_{x,t}} \\
\lesssim &\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} \sum_{L_0, L_1} N_1^{-1} N_0^{3/4} L_0^{1/2} L_1^{1/4} \| f^{\pm,c} \|_{L^2_{x,t}} \| g_1^- \|_{L^2_{x,t}} N_1^{-1+b} \| g_2^- \|_{X^{0,1-b}_-} \\
\lesssim &\sum_{N_1} \sum_{1 \le N_0 \lesssim N_1 \sim N_2} N_0^{3/4-s} N_1^{-2+b} N_0^s \| f^{\pm,c} \|_{X^{0,b}_{\pm,c}} N_1^s \| g_1^- \|_{X^{0,b}_-} N_2^{-s} \| g_2^- \|_{X^{0,1-b}_-} \\
\lesssim &\| f \|_{X^{s,b}_{\pm,c}} \| g_1 \|_{X^{s,b}_-} \| g_2 \|_{X^{-s,1-b}_-}.
\end{aligned}$$

For the case (II), we can show (3.19) in the same manner as above. We omit the proof. Lastly, we consider the case (III).

(IIIa) $N_0 \lesssim L_0$. We deduce from Hölder inequality and Proposition 3.3 that

$$\begin{aligned}
&\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} \sum_{L_j} \left| N_1^{-1} \int f^{\pm,c} g_1^- g_2^- dtdx \right| \\
\lesssim &\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} \sum_{L_j} N_0^{-1} \|f^{\pm,c}\|_{L^2_{x,t}} \|P_{K^{\pm}_{N_0,L_0}} (g_1^- \ g_2^-)\|_{L^2_{x,t}} \\
\lesssim &\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} \sum_{L_1,L_2} N_0^{-1} N_2^{3/4} L_1^{1/2} L_2^{1/4} N_0^{-b} \|f^{\pm,c}\|_{X^{0,b}_{\pm,c}} \|g_1^-\|_{L^2_{x,t}} \|g_2^-\|_{L^2_{x,t}} \\
\lesssim &\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} N_0^{-1-b-2s} N_2^{3/4+s} N_0^s \|f^{\pm,c}\|_{X^{0,b}_{\pm,c}} N_1^s \|g_1^-\|_{X^{0,b}_-} N_2^{-s} \|g_2^-\|_{X^{0,1-b}_-} \\
\lesssim &\|f\|_{X^{s,b}_{\pm,c}} \|g_1\|_{X^{s,b}_-} \|g_2\|_{X^{-s,1-b}_-}.
\end{aligned}$$

(IIIb) $N_0 \lesssim L_1$. Similarly,

$$\begin{aligned}
&\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} \sum_{L_j} \left| N_1^{-1} \int f^{\pm,c} g_1^- g_2^- dtdx \right| \\
\lesssim &\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} \sum_{L_j} N_0^{-1} \|P_{K^-_{N_1,L_1}} (f^{\pm,c} g_2^-)\|_{L^2_{x,t}} \|g_1^-\|_{L^2_{x,t}} \\
\lesssim &\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} \sum_{L_0,L_2} N_0^{-1} N_2^{3/4} L_0^{1/2} L_2^{1/4} \|f^{\pm,c}\|_{L^2_{x,t}} N_0^{-b} \|g_1^-\|_{X^{0,b}_-} \|g_2^-\|_{L^2_{x,t}} \\
\lesssim &\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} N_0^{-1-b-2s} N_2^{3/4+s} N_0^s \|f^{\pm,c}\|_{X^{0,b}_{\pm,c}} N_1^s \|g_1^-\|_{X^{0,b}_-} N_2^{-s} \|g_2^-\|_{X^{0,1-b}_-} \\
\lesssim &\|f\|_{X^{s,b}_{\pm,c}} \|g_1\|_{X^{s,b}_-} \|g_2\|_{X^{-s,1-b}_-}.
\end{aligned}$$

(IIIc) $N_0 \lesssim L_2$. In this case, we need to utilize Proposition 3.4 instead of Proposition 3.3.

$$\begin{aligned}
&\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} \sum_{L_j} \left| N_1^{-1} \int f^{\pm,c} g_1^- g_2^- dtdx \right| \\
\lesssim &\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} \sum_{L_j} N_0^{-1} \|P_{K^-_{N_2,L_2}} (f^{\pm,c} g_1^-)\|_{L^2_{x,t}} \|g_2^-\|_{L^2_{x,t}} \\
\lesssim &\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} \sum_{L_0,L_1} N_0^{-1} N_2^{1/2} L_0^{1/2} L_1^{1/4} \|f^{\pm,c}\|_{L^2_{x,t}} \|g_1^-\|_{L^2_{x,t}} N_0^{-1+b} \|g_2^-\|_{X^{0,1-b}_-} \\
\lesssim &\sum_{N_0} \sum_{1 \leq N_2 \lesssim N_0 \sim N_1} N_0^{-2+b-2s} N_2^{1/2+s} N_0^s \|f^{\pm,c}\|_{X^{0,b}_{\pm,c}} N_1^s \|g_1^-\|_{X^{0,b}_-} N_2^{-s} \|g_2^-\|_{X^{0,1-b}_-} \\
\lesssim &\|f\|_{X^{s,b}_{\pm,c}} \|g_1\|_{X^{s,b}_-} \|g_2\|_{X^{-s,1-b}_-}.
\end{aligned}$$

□

3.4. **Proof of Theorem 3.2 for $\pm_1 \neq \pm_2$.** In this section, we establish (3.16) and (3.17) with $\pm_2 = +$. Note that if one of the inequalities $|\xi_2| \leq \frac{1-c}{2(1+c)}|\xi_1|$ and $|\xi_1| \leq \frac{1-c}{2(1+c)}|\xi_2|$ holds, then we observe that for $\tau = \tau_1 + \tau_2$, $\xi = \xi_1 + \xi_2$,

$$\begin{aligned}\max(\langle \tau \pm_0 c|\xi|\rangle,\ \langle \tau_1 - |\xi_1|\rangle,\ \langle \tau_2 + |\xi_2|\rangle) &\geq | \pm c|\xi| - |\xi_1| + |\xi_2|| \\ &\geq ||\xi_1| - |\xi_2|| - c|\xi_1| - c|\xi_2| \\ &\geq \frac{1-c}{2}\max(|\xi_1|,\ |\xi_2|)\end{aligned}$$

and we can verify (3.16) and (3.17) by the same proof as in the case $\pm_2 = -$. To avoid redundancy, we omit the proof.

**Proposition 3.7.** *For any $s \in (-3/4, 0)$, there exists $b \in (1/2, 1)$ such that for $f, g_1, g_2 \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$, the following estimates hold:*

$$\begin{aligned}&\left(\sum_{N_0}\sum_{1\leq N_1 \ll N_0 \sim N_2} + \sum_{N_0}\sum_{1\leq N_2 \ll N_0 \sim N_1}\right) I_1^+ \\ &\qquad\qquad\qquad\lesssim \|f\|_{X^{s,b}_{\pm,c}} \|g_1\|_{X^{s,b}_{-}} \|g_2\|_{X^{-s,1-b}_{-}}, \\ &\left(\sum_{N_0}\sum_{1\leq N_1 \ll N_0 \sim N_2} + \sum_{N_0}\sum_{1\leq N_2 \ll N_0 \sim N_1}\right) I_2^+ \\ &\qquad\qquad\qquad\lesssim \|f\|_{X^{-s,1-b}_{\pm,c}} \|g_1\|_{X^{s,b}_{-}} \|g_2\|_{X^{s,b}_{-}},\end{aligned}$$

*where*

$$\begin{aligned}I_1^+ &:= \sum_{L_j} \left| N_1^{-1} \int (P_{K^{\pm,c}_{N_0,L_0}} f)(P_{K^{-}_{N_1,L_1}} g_1)(P_{K^{+}_{N_2,L_2}} g_2) dtdx \right|, \\ I_2^+ &:= \sum_{L_j} \left| N_0 N_1^{-1} N_2^{-1} \int (P_{K^{\pm,c}_{N_0,L_0}} f)(P_{K^{-}_{N_1,L_1}} g_1)(P_{K^{+}_{N_2,L_2}} g_2) dtdx \right|.\end{aligned}$$

Thanks to Proposition 3.7, we may assume that $1 \leq N_0 \lesssim N_1 \sim N_2$. In this case, we no longer make use of the useful estimate such as (3.18) and, as we mentioned in Introduction, it appears that the bilinear Strichartz estimates Propositions 3.3, 3.4 are not enough to show (3.16) and (3.17). Thus we employ new estimate developed by Bejenaru-Herr-Tataru [3] and applied to Zakharov system in [1]. To describe it precisely, we introduce the decomposition of $\mathbb{R}^3 \times \mathbb{R}^3$ utilized in [3]. For dyadic numbers $M_0$, $M_1$, to be chosen later, we decompose $\mathbb{R}^3\times\mathbb{R}^3$ by the sets $\{\mathfrak{D}_j^A\}$.

$$\begin{aligned}\mathbb{R}^3\times\mathbb{R}^3 &= \left\{\angle(\xi_1,\ \xi_2)\le\frac{16}{M_0}\pi\right\}\cup\bigcup_{64\le A\le M_0}\left\{\frac{16}{A}\pi\le\angle(\xi_1,\ \xi_2)\le\frac{32}{A}\pi\right\}\\ &\cup\left\{\pi-\frac{16}{M_1}\pi\le\angle(\xi_1,\ \xi_2)\right\}\cup\bigcup_{64\le A\le M_1}\left\{\pi-\frac{32}{A}\pi\le\angle(\xi_1,\ \xi_2)\le\pi-\frac{16}{A}\pi\right\}\\ &=\bigcup_{\substack{-M_0\le j_1,j_2\le M_0-1\\ |j_1-j_2|\le16}}\mathfrak{D}_{j_1}^{M_0}\times\mathfrak{D}_{j_2}^{M_0}\cup\bigcup_{64\le A\le M_0}\bigcup_{\substack{-A\le j_1,j_2\le A-1\\ 16\le|j_1-j_2|\le32}}\mathfrak{D}_{j_1}^{A}\times\mathfrak{D}_{j_2}^{A}\\ &\cup\bigcup_{\substack{-M_1\le j_1,j_2\le M_1-1\\ |j_1-j_2\pm M_1|\le16}}\mathfrak{D}_{j_1}^{M_1}\times\mathfrak{D}_{j_2}^{M_1}\cup\bigcup_{64\le A\le M_1}\bigcup_{\substack{-A\le j_1,j_2\le A+1\\ 16\le|j_1-j_2\pm A|\le32}}\mathfrak{D}_{j_1}^{A}\times\mathfrak{D}_{j_2}^{A},\end{aligned}$$

where $\angle(\xi_1,\ \xi_2)\in[0,\ \pi]$ is the smaller angle between $\xi_1$ and $\xi_2$.

First we assume that $\pi/2\le\angle(\xi_1,\xi_2)\le\pi$. We find that if $\angle(\xi_1,\ \xi_2)$ is sufficiently close to $\pi$ , then the following helpful inequality holds true.

**Lemma 3.8.** *Let $\tau=\tau_1+\tau_2$, $\xi=\xi_1+\xi_2$ and $M_1$ be the minimal dyadic number which satisfies*

$$M_1\ge 2^7(1-c)^{-\frac12}\frac{(|\xi_1||\xi_2|)^{\frac12}}{|\xi|}, \tag{3.21}$$

*then for any $(\tau_1,\xi_1)\in\mathfrak{D}_{j_1}^{M_1}$, $(\tau_2,\xi_2)\in\mathfrak{D}_{j_2}^{M_1}$ where $|j_1-j_2\pm M_1|\le16$, the following inequality holds:*

$$\max(\langle\tau\pm c|\xi|\rangle,\ \langle\tau_1-|\xi_1|\rangle,\ \langle\tau_2+|\xi_2|\rangle)\gtrsim|\xi|$$

*Proof.* After rotation, we may assume $\xi_1=(|\xi_1|,\ 0)$, and then $|j_2\pm M_1|\le16$. It follows from the inequality

$$\max(\langle\tau\pm c|\xi|\rangle,\ \langle\tau_1-|\xi_1|\rangle,\ \langle\tau_2+|\xi_2|\rangle)\ge||\xi_1|-|\xi_2||-c|\xi|,$$

it suffices to show $||\xi_1|-|\xi_2||>\sqrt{\frac{1+c}{2}}|\xi|$. Indeed,

$$\sqrt{\frac{1+c}{2}}-c>\frac14(1-c)(1+2c)>\frac{1-c}{4}.$$

From $|j_2\pm M_1|\le16$, we obtain

$$\begin{aligned}|\xi|^2&=(|\xi_1|+|\xi_2|\cos(\angle(\xi_1,\ \xi_2)))^2+(N_2\sin(\angle(\xi_1,\ \xi_2)))^2\\ &<(|\xi_1|-|\xi_2|)^2+2|\xi_1||\xi_2|(1+\cos(\angle(\xi_1,\ \xi_2)))\\ &<(|\xi_1|-|\xi_2|)^2+4|\xi_1||\xi_2|(\angle(\xi_1,\ \xi_2))^2\\ &<(|\xi_1|-|\xi_2|)^2+\frac{1-c}{2}|\xi|^2,\end{aligned}$$

which gives

$$\frac{1+c}{2}|\xi|^2 < (|\xi_1| - |\xi_2|)^2.$$

This completes the proof. □

Next we consider the case $64 \leq A \leq M_1$ and $16 \leq |j_1 - j_2 \pm A| \leq 32$.

**Proposition 3.9.** *Let $\tau = \tau_1 + \tau_2$, $\xi = \xi_1 + \xi_2$ and $f$, $g_1$, $g_2 \in L^2$ be satisfy*

$$\operatorname{supp} f \subset K^{\pm,c}_{N_0,L_0}, \quad \operatorname{supp} g_1 \subset \mathfrak{D}^A_{j_1} \cap K^-_{N_1,L_1}, \quad \operatorname{supp} g_2 \subset \mathfrak{D}^A_{j_2} \cap K^+_{N_2,L_2},$$

*and $64 \leq N_0 \lesssim N_1 \sim N_2$, $64 \leq A \leq M_1$, $16 \leq |j_1 - j_2 \pm A| \leq 32$. Then the following estimate holds:*

$$\left| \int f(\tau,\xi) g_1(\tau_1,\xi_1) g_2(\tau_2,\xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \lesssim A^{\frac{7}{8}} (L_0 L_1 L_2)^{\frac{1}{2}} \|f\|_{L^2_{\xi,\tau}} \|g_1\|_{L^2_{\xi,\tau}} \|g_2\|_{L^2_{\xi,\tau}}.$$

For the proof of the above proposition, we introduce the important estimate. See [2] for more general case.

**Proposition 3.10** ([3] Corollary 1.5)**.** *Assume that the surface $\tilde{S}_i$ $(i = 1, 2, 3)$ is an open and bounded subset of $\tilde{S}_i^*$ which satisfies the following conditions* (Assumption 1.1 in [3]).

(i) *$\tilde{S}_i^*$ is defined as*

$$\tilde{S}_i^* = \{\tilde{\sigma}_i \in U_i \mid \Phi_i(\tilde{\sigma}_i) = 0, \nabla\Phi_i \neq 0, \Phi_i \in C^{1,1}(U_i)\},$$

*for a convex $U_i \subset \mathbb{R}^3$ such that* $\operatorname{dist}(\tilde{S}_i, U_i^c) \geq \operatorname{diam}(\tilde{S}_i)$*;*

(ii) *the unit normal vector field $\tilde{\mathfrak{n}}_i$ on $\tilde{S}_i^*$ satisfies the Hölder condition*

$$\sup_{\tilde{\sigma},\tilde{\sigma}' \in \tilde{S}_i^*} \frac{|\tilde{\mathfrak{n}}_i(\tilde{\sigma}) - \tilde{\mathfrak{n}}_i(\tilde{\sigma}')|}{|\tilde{\sigma} - \tilde{\sigma}'|} + \frac{|\tilde{\mathfrak{n}}_i(\tilde{\sigma})(\tilde{\sigma} - \tilde{\sigma}')|}{|\tilde{\sigma} - \tilde{\sigma}'|^2} \lesssim 1;$$

(iii) *the matrix $\tilde{N}(\tilde{\sigma_1}, \tilde{\sigma_2}, \tilde{\sigma_3}) = (\tilde{\mathfrak{n}}_1(\tilde{\sigma_1}), \tilde{\mathfrak{n}}_2(\tilde{\sigma_2}), \tilde{\mathfrak{n}}_3(\tilde{\sigma_3}))$ satisfies the transversality condition*

$$\frac{1}{2} \leq \det \tilde{N}(\tilde{\sigma_1}, \tilde{\sigma_2}, \tilde{\sigma_3}) \leq 1$$

*for all $(\tilde{\sigma_1}, \tilde{\sigma_2}, \tilde{\sigma_3}) \in \tilde{S}_1^* \times \tilde{S}_2^* \times \tilde{S}_3^*$.*

*We also assume* $\operatorname{diam}(\tilde{S}_i) \leq 1$. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an invertible, linear map and $S_i = T\tilde{S}_i$. Then for functions $f \in L^2(S_1)$ and $g \in L^2(S_2)$, the restriction of the convolution $f * g$ to $S_3$ is a well-defined $L^2(S_3)$-function which satisfies*

$$\|f * g\|_{L^2(S_3)} \lesssim \frac{1}{\sqrt{d}} \|f\|_{L^2(S_1)} \|g\|_{L^2(S_2)},$$

*where*

$$d = \inf_{\sigma_i \in S_i} |\det N(\sigma_1, \sigma_2, \sigma_3)|$$

*and* $N(\sigma_1, \sigma_2, \sigma_3)$ *is the matrix of the unit normals to* $S_i$ *at* $(\sigma_1, \sigma_2, \sigma_3)$.

*Remark* 3.2. As was mentioned in [3], the condition of $S_i^*$ (i) is used only to ensure the existence of a global representation of $S_i$ as a graph. In the proof of Proposition 3.9, the implicit function theorem and the other conditions may show the existence of such a graph. Thus we will not treat the condition (i) in the proof of Proposition 3.9.

By utilizing Proposition 3.10, we verify Proposition 3.9.

*Proof of Proposition 3.9.* Let $\theta_0^{\pm} \in (0, \pi)$ be defined as $\cos\theta_0^{\pm} = \pm c$. We divide the proof into the following two cases:

(I) $|\angle(\xi, \xi_1) - \theta_0^+| > 2^{10}(1-c)^{-1}A^{-3/4}$ and $|\angle(\xi, \xi_1) - \theta_0^-| > 2^{10}(1-c)^{-1}A^{-3/4}$,

(II) $|\angle(\xi, \xi_1) - \theta_0^+| \leq 2^{10}(1-c)^{-1}A^{-3/4}$ or $|\angle(\xi, \xi_1) - \theta_0^-| \leq 2^{10}(1-c)^{-1}A^{-3/4}$,

where $\angle(\xi, \xi_1) \in (0, \pi)$ is the smaller angle between $\xi$ and $\xi_1$. We here assume that $A > 2^{20}(1-c)^{-2}$. If $A \leq 2^{20}(1-c)^{-2}$, the proposition is verified by the almost same proof as that for the case (II) below.

We first consider the case (I). The proof is very similar to that for $\pm_1 = \pm_2$. We utilize the following two estimates.

**Lemma 3.11.** *Let* $\tau = \tau_1 + \tau_2$, $\xi = \xi_1 + \xi_2$, $2^{20}(1-c)^{-2} < A \leq M_1$ *and* $\angle(\xi, \xi_1)$ *satisfies* (I)*. Then the following inequality holds:*

$$\max(\langle \tau \pm c|\xi|\rangle, \ \langle \tau_1 - |\xi_1|\rangle, \ \langle \tau_2 + |\xi_2|\rangle) \gtrsim A^{-3/4}|\xi|$$

*Proof.* After rotation, we may assume that $\xi_1 = (|\xi_1|, 0)$ and $\xi = (|\xi|\cos\theta, |\xi|\sin\theta)$ with $\theta \in (0, \pi)$. By the simple calculation, we have

$$\begin{aligned}
&\max(\langle \tau \pm c|\xi|\rangle, \ \langle \tau_1 - |\xi_1|\rangle, \ \langle \tau_2 + |\xi_2|\rangle) \\
\geq & \big| \pm c|\xi| + |\xi_1| - |\xi_2| \big| \\
\geq & \left| \pm c|\xi| + |\xi_1| - \sqrt{|\xi_1|^2 - 2|\xi||\xi_1|\cos\theta + |\xi|^2} \right| \\
\geq & \big| \pm c|\xi| + |\xi|\cos\theta \big| - \left| \frac{2|\xi||\xi_1|\cos\theta - |\xi|^2}{|\xi_1| + \sqrt{|\xi_1|^2 - 2|\xi||\xi_1|\cos\theta + |\xi|^2}} - |\xi|\cos\theta \right| \\
=: & K_1 - K_2.
\end{aligned}$$

From $\theta_0^{\pm}, \theta \in (0,\pi)$ and (I), we get

$$\begin{aligned} K_1 &= |\xi| |\cos\theta_0^{\mp} - \cos\theta| \\ &\geq |\xi| \ \frac{\sqrt{1-c}}{4} \ |\theta_0^{\mp} - \theta| \\ &\geq 2^8 (1-c)^{-\frac{1}{2}} |\xi| A^{-\frac{3}{4}}. \end{aligned}$$

From $2^{20}(1-c)^{-2} < A \leq M_1$, we have

$$\begin{aligned} K_2 &= \left| \frac{2|\xi||\xi_1|\cos\theta - |\xi|^2}{|\xi_1| + \sqrt{|\xi_1|^2 - 2|\xi||\xi_1|\cos\theta + |\xi|^2}} - |\xi|\cos\theta \right| \\ &\leq \left| \frac{2|\xi||\xi_1|\cos\theta}{|\xi_1| + \sqrt{|\xi_1|^2 - 2|\xi||\xi_1|\cos\theta + |\xi|^2}} - \frac{2|\xi||\xi_1|\cos\theta}{2|\xi_1|} \right| + \frac{|\xi|^2}{|\xi_1|} \\ &\leq \frac{|\xi||\xi_1||\cos\theta|}{|\xi_1|(\sqrt{|\xi_1|^2 - 2|\xi||\xi_1|\cos\theta + |\xi|^2})} \left| |\xi_1| - \sqrt{|\xi_1|^2 - 2|\xi||\xi_1|\cos\theta + |\xi|^2} \right| + \frac{|\xi|^2}{|\xi_1|} \\ &\leq 4\frac{|\xi|^2}{|\xi_1|} \leq 2^{10}(1-c)^{-1}|\xi|A^{-1} \\ &\leq 2^5(1-c)^{-\frac{1}{2}}|\xi|A^{-\frac{3}{4}}. \end{aligned}$$

From above, we have

$$K_1 - K_2 \gtrsim |\xi| A^{-\frac{3}{4}}.$$

This completes the proof. □

**Lemma 3.12.** *Let $g_1,\ g_2 \in L^2$ be satisfy*

$$\operatorname{supp} g_1 \subset \mathfrak{D}_{j_1}^A \cap K_{N_1,L_1}^-, \quad \operatorname{supp} g_2 \subset \mathfrak{D}_{j_2}^A \cap K_{N_2,L_2}^+,$$

*and* $64 \leq N_0 \lesssim N_1 \sim N_2,\ \ 64 \leq A \leq M_1,\ \ 16 \leq |j_1 - j_2 \pm A| \leq 32.$ *Then the following estimate holds:*

$$\|\chi_{K_{N_0,L_0}^{\pm_0}}(g_1 * g_2)\|_{L^2_{\xi,\tau}} \lesssim (AN_0L_1L_2)^{\frac{1}{2}} \|g_1\|_{L^2_{\xi,\tau}} \|g_2\|_{L^2_{\xi,\tau}}$$

*Proof.* By the same way as in the proof of Proposition 3.4, we observe that the desired estimate is proved by

$$\sup_{\tau,\xi} |E(\tau,\xi)| \lesssim AN_0L_1L_2 \tag{3.22}$$

where

$$E(\tau,\xi) := \left\{ (\tau_1,\xi_1) \in \mathfrak{D}_0^A \cap C_{N_0}(\xi') \ \middle| \ \begin{array}{l} \langle \tau - \tau_1 - |\xi - \xi_1| \rangle \sim L_1,\ \langle \tau_1 + |\xi_1| \rangle \sim L_2, \\ (\tau - \tau_1, \xi - \xi_1) \in \mathfrak{D}_{j_2}^A. \end{array} \right\}$$

with $16 \le |j_2 \pm A| \le 32$ and fixed $\xi' \in \mathbb{R}^2$. From $\langle \tau - \tau_1 - |\xi - \xi_1| \rangle \sim L_1$ and $\langle \tau_1 + |\xi_1| \rangle \sim L_2$, for fixed $\xi_1$,

$$|\{\tau_1 \mid (\tau_1, \xi_1) \in E(\tau, \xi)\}| \lesssim L_{\min}^{12}. \tag{3.23}$$

It follows from $(\tau_1, \xi_1) \in \mathfrak{D}_0^A$ and $(\tau - \tau_1, \xi - \xi_1) \in \mathfrak{D}_{j_2}^A$ that

$$\begin{aligned} |\partial_2(\tau - |\xi_1| + |\xi - \xi_1|)| &\geq \left| \frac{(\xi_1)_2}{|\xi_1|} + \frac{(\xi - \xi_1)_2}{|\xi - \xi_1|} \right| \\ &\gtrsim A^{-1}. \end{aligned} \tag{3.24}$$

Combining $|\tau - |\xi_1| + |\xi - \xi_1|| \lesssim L_{\max}^{12}$ with (3.24), for fixed $(\xi_1)_1$ we have

$$|\{(\xi_1)_2 \mid (\tau_1, \xi_1) \in E(\tau, \xi)\}| \lesssim A L_{\max}^{12}. \tag{3.25}$$

Collecting (3.23), (3.25) and $\xi_1 \in C_{N_0}(\xi')$, we get (3.22). $\square$

We now prove Proposition 3.9 for the case (I). From Lemma 3.11, it holds that $L_{\max}^{012} \gtrsim A^{-\frac{3}{4}} N_0$. We decompose the proof into the three cases:

(Ia) $A^{-\frac{3}{4}} N_0 \lesssim L_0$, (Ib) $A^{-\frac{3}{4}} N_0 \lesssim L_1$, (Ic) $A^{-\frac{3}{4}} N_0 \lesssim L_2$.

(Ia) From Hölder inequality and Lemma 3.12, we have

$$\begin{aligned} \left| \int f(\tau, \xi) g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| &\lesssim \|f\|_{L^2_{\xi,\tau}} \|g_1 * g_2\|_{L^2_{\xi,\tau}} \\ &\lesssim A^{\frac{7}{8}} (L_0 L_1 L_2)^{\frac{1}{2}} \|f\|_{L^2_{\xi,\tau}} \|g_1\|_{L^2_{\xi,\tau}} \|g_2\|_{L^2_{\xi,\tau}}. \end{aligned}$$

(Ib) From Hölder inequality and Lemma 3.4, we have

$$\begin{aligned} \left| \int f(\tau, \xi) g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| &\lesssim \|g_1\|_{L^2_{\xi,\tau}} \|f * g_{2,-}\|_{L^2_{\xi,\tau}} \\ &\lesssim A^{\frac{3}{8}} (L_0 L_1 L_2)^{\frac{1}{2}} \|f\|_{L^2_{\xi,\tau}} \|g_1\|_{L^2_{\xi,\tau}} \|g_2\|_{L^2_{\xi,\tau}}. \end{aligned}$$

(Ic) From Hölder inequality and Lemma 3.4, we have

$$\begin{aligned} \left| \int f(\tau, \xi) g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| &\lesssim \|g_2\|_{L^2_{\xi,\tau}} \|f * g_{1,-}\|_{L^2_{\xi,\tau}} \\ &\lesssim A^{\frac{3}{8}} (L_0 L_1 L_2)^{\frac{1}{2}} \|f\|_{L^2_{\xi,\tau}} \|g_1\|_{L^2_{\xi,\tau}} \|g_2\|_{L^2_{\xi,\tau}}. \end{aligned}$$

Here $g_{j,-}$ is defined as $g_{j,-}(\cdot) = g_j(-\cdot)$.

We next consider the case (II). We apply the same strategy as that of the proof of Proposition 4.4 in [1]. Applying the transformation $\tau_1 = |\xi_1| + c_1$ and $\tau_2 = -|\xi_2| + c_2$ and Fubini's theorem, we find that it suffices to prove

$$\begin{aligned} &\left| \int f(\phi_{c_1}^+(\xi_1) + \phi_{c_2}^-(\xi_2)) g_1(\phi_{c_1}^+(\xi_1)) g_2(\phi_{c_2}^-(\xi_2)) d\xi_1 d\xi_2 \right| \\ &\qquad \lesssim A^{\frac{7}{8}} \|g_1 \circ \phi_{c_1}^+\|_{L^2_\xi} \|g_2 \circ \phi_{c_2}^-\|_{L^2_\xi} \|f\|_{L^2_{\xi,\tau}}, \end{aligned} \tag{3.26}$$

where $f(\tau,\xi)$ is supported in $c_0 \le \tau \pm c|\xi| \le c_0+1$ and

$$\phi^{\pm}_{c_k}(\xi) = (\pm|\xi| + c_k, \xi) \quad \text{for } k=1,2.$$

First we decompose $f$ by angular localization characteristic functions $\left\{\chi_{\mathfrak{D}^{A_1}_j}\right\}_{j_1=-A_1}^{A_1+1}$ where $A_1$ is the minimal dyadic number which satisfies $A_1 \ge 2^{20}(1-c)^{-2}A$ and thickened circular localization characteristic functions $\left\{\chi_{\mathbb{S}^{N_0+k\delta}_\delta}\right\}_{k=-\left[\frac{N_0}{2\delta}\right]}^{\left[\frac{N_0}{\delta}\right]+1}$ where $[s]$ denotes the maximal integer which is not greater than $s \in \mathbb{R}$ and $\mathbb{S}^{\xi^0}_\delta = \{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^2 \mid \xi^0 \le |\xi| \le \xi^0+\delta\}$ with $\delta = 2^{-20}(1-c)^2 N_0 A^{-1/2}$ as follows:

$$f = \sum_{k=-\left[\frac{N_0}{2\delta}\right]}^{\left[\frac{N_0}{\delta}\right]+1} \sum_{j_1=-A_1}^{A_1+1} \chi_{\mathbb{S}^{N_0+k\delta}_\delta}\chi_{\mathfrak{D}^{A_1}_j} f.$$

From the assumption (II), we see that the sum of $(k,\ j_1)$ is $\sim A^{\frac{3}{4}}$. Therefore we only need to verify

$$\left|\int f(\phi^+_{c_1}(\xi_1)+\phi^-_{c_2}(\xi_2))g_1(\phi^+_{c_1}(\xi_1))g_2(\phi^-_{c_2}(\xi_2))d\xi_1 d\xi_2\right| \lesssim A^{\frac{1}{2}}\|g_1\circ\phi^+_{c_1}\|_{L^2_\xi}\|g_2\circ\phi^-_{c_2}\|_{L^2_\xi}\|f\|_{L^2_{\xi,\tau}}, \quad (3.27)$$

for $\operatorname{supp} f \subset \mathfrak{D}^{A_1}_j \cap \mathbb{S}^{N_0+k\delta}_\delta$ with fixed $k,\ j_1$. We use the scaling $(\tau,\ \xi) \to (N_0\tau,\ N_0\xi)$ to define

$$\tilde{f}(\tau,\xi) = f(N_0\tau, N_0\xi), \quad \tilde{g_k}(\tau_k,\xi_k) = g_k(N_0\tau_k, N_0\xi_k).$$

If we set $\tilde{c_k} = N_0^{-1}c_k$, inequality (3.27) reduces to

$$\left|\int \tilde{f}(\phi^+_{\tilde{c_1}}(\xi_1)+\phi^-_{\tilde{c_2}}(\xi_2))\tilde{g_1}(\phi^+_{\tilde{c_1}}(\xi_1))\tilde{g_2}(\phi^-_{\tilde{c_2}}(\xi_2))d\xi_1 d\xi_2\right| \lesssim A^{\frac{1}{2}}N_0^{-\frac{1}{2}}\|\tilde{g_1}\circ\phi^+_{\tilde{c_1}}\|_{L^2_\xi}\|\tilde{g_2}\circ\phi^-_{\tilde{c_2}}\|_{L^2_\xi}\|\tilde{f}\|_{L^2_{\xi,\tau}}. \quad (3.28)$$

Note that $\tilde{f}$ is supported in $S^{\mp}_3(N_0^{-1})$ where

$$S^{\mp}_3(N_0^{-1}) = \left\{(\tau,\xi) \in \mathfrak{D}^{A_1}_j \cap \mathbb{S}^{1+k\tilde{\delta}}_{\tilde{\delta}} \mid \mp c|\xi| + \frac{c_0}{N_0} \le \tau \le \mp c|\xi| + \frac{c_0+1}{N_0}\right\}$$

with $\tilde{\delta} = N_0^{-1}\delta$. Thus from the $\ell^2$ almost orthogonality, we may assume that there exist $\xi^0_1,\ \xi^0_2$ such that

$$\frac{N_1}{2N_0} \le |\xi^0_1| \le 4\frac{N_1}{N_0}, \quad \frac{N_2}{2N_0} \le |\xi^0_2| \le 4\frac{N_2}{N_0} \quad (3.29)$$

such that space variables of $\operatorname{supp} \tilde{g}_1 \circ \phi^+_{\tilde{c}_1}$ and $\operatorname{supp} \tilde{g}_2 \circ \phi^-_{\tilde{c}_2}$ are contained in the balls $B_{\tilde{\delta}}(\xi_1^0)$ and $B_{\tilde{\delta}}(\xi_2^0)$, respectively. By density and duality it suffices to show for continuous $\tilde{g}_1$ and $\tilde{g}_2$ that

$$\|\tilde{g}_1|_{S_1} * \tilde{g}_2|_{S_2}\|_{L^2(S_3^{\pm}(N_0^{-1}))} \lesssim A^{\frac{1}{2}} N_0^{-\frac{1}{2}} \|\tilde{g}_1\|_{L^2(S_1)} \|\tilde{g}_2\|_{L^2(S_2)} \tag{3.30}$$

where $S_1$, $S_2$ denote the following surfaces

$$S_1 = \{\phi^+_{\tilde{c}_1}(\xi_1) \in \mathbb{R}^3 \mid \xi_1 \in B_{\tilde{\delta}}(\xi_1^0)\},$$
$$S_2 = \{\phi^-_{\tilde{c}_2}(\xi_2) \in \mathbb{R}^3 \mid \xi_2 \in B_{\tilde{\delta}}(\xi_2^0)\}.$$

(3.30) is immediately established from

$$\|\tilde{g}_1|_{S_1} * \tilde{g}_2|_{S_2}\|_{L^2(S_3^{\pm})} \lesssim A^{\frac{1}{2}} \|\tilde{g}_1\|_{L^2(S_1)} \|\tilde{g}_2\|_{L^2(S_2)} \tag{3.31}$$

where

$$S_3^{\mp} = \{(\psi^{\mp}(\xi), \xi) \in \mathfrak{D}_j^{A_1} \cap \mathbb{S}_{\tilde{\delta}}^{1+k\tilde{\delta}} \mid \psi^{\mp}(\xi) = \mp c|\xi| + \frac{c_0}{N_0}\}.$$

For any $\sigma_i \in S_i$, $i = 1, 2, 3$, there exist $\xi_1$, $\xi_2$, $\xi$ such that

$$\sigma_1 = \phi^+_{\tilde{c}_1}(\xi_1), \quad \sigma_2 = \phi^+_{\tilde{c}_2}(\xi_2), \quad \sigma_3 = (\psi(\xi), \xi),$$

and the unit normals $\mathfrak{n}_i$ on $\sigma_i$ are written as

$$\mathfrak{n}_1(\sigma_1) = \frac{1}{\sqrt{2}} \left(-1,\ \frac{(\xi_1)_1}{|\xi_1|},\ \frac{(\xi_1)_2}{|\xi_1|}\right),$$
$$\mathfrak{n}_2(\sigma_2) = \frac{1}{\sqrt{2}} \left(1,\ \frac{(\xi_2)_1}{|\xi_2|},\ \frac{(\xi_2)_2}{|\xi_2|}\right),$$
$$\mathfrak{n}_3(\sigma_3) = \frac{1}{\sqrt{c^2+1}} \left(\pm 1,\ c\frac{(\xi)_1}{|\xi|},\ c\frac{(\xi)_2}{|\xi|}\right).$$

We deduce from $1 \lesssim |\xi|$ and (3.29) that the surfaces $S_1$, $S_2$, $S_3^{\mp}$ satisfy the following Hölder condition.

$$\sup_{\sigma_i, \sigma_i' \in S_i} \frac{|\mathfrak{n}_i(\sigma_i) - \mathfrak{n}_i(\sigma_i')|}{|\sigma_i - \sigma_i'|} + \frac{|\mathfrak{n}_i(\sigma_i)(\sigma_i - \sigma_i')|}{|\sigma_i - \sigma_i'|^2} \lesssim 1, \tag{3.32}$$

$$\sup_{\sigma_3, \sigma_3' \in S_3^{\pm}} \frac{|\mathfrak{n}_3(\sigma_3) - \mathfrak{n}_3(\sigma_3')|}{|\sigma_3 - \sigma_3'|} + \frac{|\mathfrak{n}_3(\sigma_3)(\sigma_3 - \sigma_3')|}{|\sigma_3 - \sigma_3'|^2} \lesssim 1. \tag{3.33}$$

We may assume that there exist $\xi_1', \xi_2', \xi' \in \mathbb{R}^2$ such that

$$\xi_1' + \xi_2' = \xi', \quad \phi^+_{\tilde{c}_1}(\xi_1') \in S_1,\ \phi^-_{\tilde{c}_2}(\xi_2') \in S_2,\ (\psi^{\mp}(\xi'), \xi') \in S_3^{\mp},$$

otherwise the left-hand side of (3.30) vanishes. Let $\sigma_1' = \phi^+_{\tilde{c}_1}(\xi_1')$, $\sigma_2' = \phi^-_{\tilde{c}_2}(\xi_2')$, $\sigma_3' = (\psi^\mp(\xi'), \xi')$. For any $\sigma_1 = \phi^+_{\tilde{c}_1}(\xi_1) \in S_1$, we deduce from $\xi_1, \xi_1' \in B_{\tilde{\delta}}(\xi_1^0)$ and $A \le M_1 \le 2^{10}(1-c)^{-1}N_1/N_0$ that

$$|\mathfrak{n}_1(\sigma_1) - \mathfrak{n}_1(\sigma_1')| \le 2^{-18}\frac{N_0}{N_1}(1-c)^2 A^{-\frac{1}{2}} \le 2^{-8}(1-c)A^{-\frac{3}{2}}. \tag{3.34}$$

Similarly, for any $\sigma_2 = \phi^-_{\tilde{c}_2}(\xi_2) \in S_2$ we have

$$|\mathfrak{n}_2(\sigma_2) - \mathfrak{n}_2(\sigma_2')| \le 2^{-18}\frac{N_0}{N_2}(1-c)^2 A^{-\frac{1}{2}} \le 2^{-8}(1-c)A^{-\frac{3}{2}}. \tag{3.35}$$

For any $\sigma_3 \in S_3^\mp$, it follows from $S_3^\mp \subset \mathfrak{D}_j^{A_1}$ that

$$|\mathfrak{n}_3(\sigma_3) - \mathfrak{n}_3(\sigma_3')| \le 2^{-10}(1-c)A^{-1}. \tag{3.36}$$

It is obvious that $|\sigma_1 - \sigma_1'|$, $|\sigma_2 - \sigma_2'| \le 2\tilde{\delta} \le 2^{-10}(1-c)^2 A^{-1/2}$, then we get from (3.34) and (3.35) that

$$|(\sigma_1 - \sigma_1') \cdot \mathfrak{n}_1(\sigma_1')| \le 2^{-15}(1-c)^2 A^{-2}, \tag{3.37}$$

$$|(\sigma_2 - \sigma_2') \cdot \mathfrak{n}_2(\sigma_2')| \le 2^{-15}(1-c)^2 A^{-2}. \tag{3.38}$$

Similarly, we deduce from $\left|\sigma_3 - \frac{|\sigma_3|}{|\sigma_3'|}\sigma_3'\right| \le 2^{-10}(1-c)^2 A^{-1}$ and (3.36) that

$$|(\sigma_3 - \sigma_3') \cdot \mathfrak{n}_3(\sigma_3')| = \left|\left(\sigma_3 - \frac{|\sigma_3|}{|\sigma_3'|}\sigma_3'\right) \cdot \mathfrak{n}_3(\sigma_3')\right| \le 2^{-15}(1-c)^2 A^{-2}. \tag{3.39}$$

(3.37) means that $S_1$ is contained in an slab of thickness $2^{-15}(1-c)^2 A^{-2}$ with respect to the $\mathfrak{n}_1(\sigma_1')$ direction. From $\ell^2$ orthogonality, we may assume that $S_2$ and $S_3$ are also contained in similar $2^{-15}(1-c)^2 A^{-2}$ thick slabs;

$$|(\sigma_2 - \sigma_2') \cdot \mathfrak{n}_1(\sigma_1')| \le 2^{-15}(1-c)^2 A^{-2},$$

$$|(\sigma_3 - \sigma_3') \cdot \mathfrak{n}_1(\sigma_1')| \le 2^{-15}(1-c)^2 A^{-2}.$$

Similarly, we may assume that surfaces $S_1$, $S_2$ are contained in slabs of thickness $2^{-15}(1-c)^2 A^{-2}$ with respect to the $\mathfrak{n}_2(\sigma_2')$ direction and the surfaces $S_1$, $S_2$ are contained in slabs of thickness $2^{-15}(1-c)^2 A^{-2}$ with respect to the $\mathfrak{n}_3(\sigma_3')$ direction. Collection the above assumptions, for $i, j = 1, 2, 3$,

$$|(\sigma_i - \sigma_i') \cdot \mathfrak{n}_j(\sigma_j')| \le 2^{-15}(1-c)^2 A^{-2}. \tag{3.40}$$

We define $T : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$T = 2^{-10}(1-c)^2 A^{-2}(N^\top)^{-1}, \qquad N = N(\sigma_1', \sigma_2', \sigma_3').$$

If the following conditions are established, we immediately obtain the desired estimate (3.31) by applying Proposition 3.10 with $T$ and $\tilde{S}_i := T^{-1}S_i$ $(i = 1, 2, 3)$.

(I) $\frac{1-c}{2}A^{-1} \leq |\mathrm{det}N(\sigma_1, \sigma_2, \sigma_3)|$ for any $\sigma_i \in S_i$.

(II) $\mathrm{diam}(\tilde{S}_i) < 1$.

(III) $\frac{1}{2} \leq \det(\tilde{\mathfrak{n}}_1(\tilde{\sigma_1}), \tilde{\mathfrak{n}}_2(\tilde{\sigma_2}), \tilde{\mathfrak{n}}_3(\tilde{\sigma_3})) \leq 1$ for any $\tilde{\sigma}_i \in \tilde{S}_i$.

(IV) $\sup\limits_{\tilde{\sigma}_i, \tilde{\sigma}_i^0 \in \tilde{S}_i} \frac{|\tilde{\mathfrak{n}}_i(\tilde{\sigma}_i) - \tilde{\mathfrak{n}}_i(\tilde{\sigma}_i^0)|}{|\tilde{\sigma}_i - \tilde{\sigma}_i^0|} + \frac{|\tilde{\mathfrak{n}}_i(\tilde{\sigma}_i^0) \cdot (\tilde{\sigma}_i - \tilde{\sigma}_i^0)|}{|\tilde{\sigma}_i - \tilde{\sigma}_i^0|^2} \leq 1$ for the unit normals $\tilde{\mathfrak{n}}_i$ on $\tilde{S}_i$.

We first show (I). From (3.34)-(3.36) it suffices to show

$$(1-c)A^{-1} \leq |\mathrm{det}N(\sigma_1', \sigma_2', \sigma_3')|. \tag{3.41}$$

Seeing that $\sigma_1' = \phi_{\tilde{c}_1}^{+}(\xi_1')$, $\sigma_2' = \phi_{\tilde{c}_2}^{-}(\xi_2')$, $\sigma_3' = (\psi^{\mp}(\xi'), \xi')$ and $\xi_1' + \xi_2' = \xi'$, we get

$$\begin{aligned}
|\mathrm{det}N(\sigma_1', \sigma_2', \sigma_3')| \geq & \frac{1}{4}\left|\det\begin{pmatrix} -1 & 1 & \pm 1 \\ \frac{(\xi_1')_1}{|\xi_1'|} & \frac{(\xi_2')_1}{|\xi_2'|} & c\frac{(\xi')_1}{|\xi'|} \\ \frac{(\xi_1')_2}{|\xi_1'|} & \frac{(\xi_2')_2}{|\xi_2'|} & c\frac{(\xi')_2}{|\xi'|} \end{pmatrix}\right| \\
\geq & \frac{1}{4}\left|\frac{(\xi_1')_1(\xi_2')_2 - (\xi_1')_2(\xi_2')_2}{|\xi_1'||\xi_2'|}\right| \left(1 - c\left|\frac{|\xi_2'|}{|\xi'|} - \frac{|\xi_1'|}{|\xi'|}\right|\right) \\
\geq & (1-c)A^{-1}.
\end{aligned} \tag{3.42}$$

(II) is established from (3.40).

$$\begin{aligned}
|T^{-1}(\sigma_i - \sigma_i)| = 2^{10}(1-c)^{-2}A^2 & \left|\begin{pmatrix} \mathfrak{n}_1(\sigma_1') \cdot (\sigma_i - \sigma_i') \\ \mathfrak{n}_2(\sigma_2') \cdot (\sigma_i - \sigma_i') \\ \mathfrak{n}_3(\sigma_3') \cdot (\sigma_i - \sigma_i') \end{pmatrix}\right| \\
\leq 2^{-3} < \frac{1}{2}. &
\end{aligned}$$

Next we show (III). Note that the unit normals $\tilde{\mathfrak{n}}_i$ on $\tilde{S}_i$ are written as follows.

$$\tilde{\mathfrak{n}}_i(\tilde{\sigma}_i) = \frac{(T^{-1})^{\top}\mathfrak{n}_i(T\tilde{\sigma}_i)}{|(T^{-1})^{\top}\mathfrak{n}_i(T\tilde{\sigma}_i)|} = \frac{N^{-1}\mathfrak{n}_i(T\tilde{\sigma}_i)}{|N^{-1}\mathfrak{n}_i(T\tilde{\sigma}_i)|}.$$

In particular, the unit normals on $T^{-1}\sigma_i'$ are the unit vectors $e_i$;

$$\tilde{\mathfrak{n}}_i(T^{-1}\sigma_i') = N^{-1}\mathfrak{n}_i(\sigma_i') = e_i. \tag{3.43}$$

From (3.42), we get

$$\|N^{-1}\| = \|(N^{\top})^{-1}\| \leq 2|\mathrm{det}N^{\top}|^{-1}\|N^{\top}\|^2 \leq 12(1-c)^{-1}A. \tag{3.44}$$

Thus we obtain

$$\|T\| \leq 2^{-6}(1-c)A^{-1}. \tag{3.45}$$

We deduce from (3.34)-(3.36), (3.43), (3.44) that

$$|N^{-1}\mathfrak{n}_i(T\tilde{\sigma}_i) - e_i| = |N^{-1}(\mathfrak{n}_i(T\tilde{\sigma}_i) - \mathfrak{n}_i(\sigma_i'))| \leq 2^{-7}. \tag{3.46}$$

This gives $|\tilde{\mathfrak{n}}_i(\tilde{\sigma}_i) - e_i| \leq 2^{-5}$ and (III) is now obtained. Finally we show (IV). It follows from (3.44)-(3.46) that

$$\begin{aligned}\frac{|\tilde{\mathfrak{n}}_i(\tilde{\sigma}_i) - \tilde{\mathfrak{n}}_i(\tilde{\sigma}_i^0)|}{|\tilde{\sigma}_i - \tilde{\sigma}_i^0|} &\leq 3\frac{|N^{-1}(\mathfrak{n}_i(T\tilde{\sigma}_i) - \mathfrak{n}_i(T\tilde{\sigma}_i^0))|}{|\tilde{\sigma}_i - \tilde{\sigma}_i^0|} \\ &\leq 3\|N^{-1}\|\|T\|\frac{|\mathfrak{n}_i(T\tilde{\sigma}_i) - \mathfrak{n}_i(T\tilde{\sigma}_i^0)|}{|T\tilde{\sigma}_i - T\tilde{\sigma}_i^0|} \lesssim 1.\end{aligned}$$

The last inequality is verified from (3.32) and (3.33). Similarly, from (3.45) and $(T^{-1})^\top N^{-1} = 2^{10}(1-c)^{-2}A^2E$ we have

$$\begin{aligned}\frac{|\tilde{\mathfrak{n}}_i(\tilde{\sigma}_i^0)\cdot(\tilde{\sigma}_i - \tilde{\sigma}_i^0)|}{|\tilde{\sigma}_i - \tilde{\sigma}_i^0|^2} &\leq 2\|T\|^2\frac{|N^{-1}\mathfrak{n}_i(T\tilde{\sigma}_i^0)\cdot(T^{-1}T\tilde{\sigma}_i - T^{-1}T\tilde{\sigma}_i^0)|}{|T\tilde{\sigma}_i - T\tilde{\sigma}_i^0|^2} \\ &\leq 2\|T\|^2\frac{|(T^{-1})^\top N^{-1}\mathfrak{n}_i(T\tilde{\sigma}_i^0)\cdot(T\tilde{\sigma}_i - T\tilde{\sigma}_i^0)|}{|T\tilde{\sigma}_i - T\tilde{\sigma}_i^0|^2} \\ &\leq \frac{1}{2}\frac{|\mathfrak{n}_i(T\tilde{\sigma}_i^0)\cdot(T\tilde{\sigma}_i - T\tilde{\sigma}_i^0)|}{|T\tilde{\sigma}_i - T\tilde{\sigma}_i^0|^2} \lesssim 1.\end{aligned}$$

This completes (IV). □

We now consider $0 \leq \angle(\xi_1,\ \xi_2) \leq \pi/2$. First we show the estimate which is similar to Proposition 3.9 for $64 \leq A \leq N_0^{\frac{1}{2}}$ and $16 \leq |j_1 - j_2| \leq 32$. In this case, thanks to $0 \leq \angle(\xi_1,\ \xi_2) \leq \pi/2$, $N_0 \sim N_1 \sim N_2$ always holds true and we can obtain the better estimates compared to Proposition 3.9.

**Proposition 3.13.** *Let $f,\ g_1,\ g_2 \in L^2$ be satisfy*

$$\operatorname{supp} f \subset K^{\pm,c}_{N_0,L_0}, \quad \operatorname{supp} g_1 \subset \mathfrak{D}^A_{j_1} \cap K^-_{N_1,L_1}, \quad \operatorname{supp} g_2 \subset \mathfrak{D}^A_{j_2} \cap K^+_{N_2,L_2},$$

*and $N_0 \sim N_1 \sim N_2$, $64 \leq A \leq N_0^{\frac{1}{2}}$, $16 \leq |j_1 - j_2| \leq 32$. Then the following estimate holds:*

$$\left|\int f(\tau,\xi)g_1(\tau_1,\xi_1)g_2(\tau_2,\xi_2)d\tau_1 d\tau_2 d\xi_1 d\xi_2\right| \lesssim A^{\frac{1}{2}}(L_0L_1L_2)^{\frac{1}{2}}\|f\|_{L^2_{\xi,\tau}}\|g_1\|_{L^2_{\xi,\tau}}\|g_2\|_{L^2_{\xi,\tau}}. \tag{3.47}$$

*Proof.* The proof is almost analogous to that of Proposition 3.9. Difference between them is a step of decomposition. Precisely, in the proof of Proposition 3.9, we decomposed $f$ into $\sim A^{\frac{3}{4}}$ pieces. We here decompose functions into finite pieces.

From $\operatorname{supp} g_1 \subset \mathfrak{D}^A_{j_1}$, $\operatorname{supp} g_2 \subset \mathfrak{D}^A_{j_2}$ and $16 \leq |j_1 - j_2| \leq 32$, after suitable and harmless decomposition, we can assume that there exists $j$ such that $16 \leq |j_1 - j| \leq 32$ and $\operatorname{supp} f \in \mathfrak{D}^A_j$. Furthermore we decompose $f$, $g_1$, $g_2$ into finite pieces as follows;

$$f = \sum_{j'=j^0}^{j^0+k} \chi_{\mathfrak{D}^{A_1}_{j'}} f, \qquad g_1 = \sum_{j_1'=j_1^0}^{j_1^0+k} \chi_{\mathfrak{D}^{A_1}_{j_1'}} g_1, \qquad g_2 = \sum_{j_2'=j_2^0}^{j_2^0+k} \chi_{\mathfrak{D}^{A_1}_{j_2'}} g_2$$

where $k$ is the minimal dyadic number which satisfies $k \geq 2^{20}(1-c)^{-2}$, $A_1 := kA$ and $j^0, j_1^0, j_2^0$ satisfy

$$\bigcup_{j^0 \leq j' \leq j^0+k} \mathfrak{D}^{A_1}_{j'} = \mathfrak{D}^A_j, \qquad \bigcup_{j_1^0 \leq j_1' \leq j_1^0+k} \mathfrak{D}^{A_1}_{j_1'} = \mathfrak{D}^A_{j_1}, \qquad \bigcup_{j_2^0 \leq j_2' \leq j_2^0+k} \mathfrak{D}^{A_1}_{j_2'} = \mathfrak{D}^A_{j_2}.$$

Thanks to the finiteness of $k$, it suffices to prove the desired estimate (3.47) for

$$\operatorname{supp} f \subset \mathfrak{D}^{A_1}_{j'}, \quad \operatorname{supp} g_1 \subset \mathfrak{D}^{A_1}_{j_1'}, \quad \operatorname{supp} g_2 \subset \mathfrak{D}^{A_1}_{j_2'}$$

with fixed $j' \in [j^0, j^0+k]$, $j_1' \in [j_1^0, j_1^0+k]$, $j_2' \in [j_2^0, j_2^0+k]$.

We utilize the same notations as in the proof of Proposition 3.9. By the same argument as of the proof of Proposition 3.9, we only need to verify the following estimate;

$$\|\tilde{g_1}|_{S_1} * \tilde{g_2}|_{S_2}\|_{L^2(S_3)} \lesssim A^{\frac{1}{2}} \|\tilde{g_1}\|_{L^2(S_1)} \|\tilde{g_2}\|_{L^2(S_2)} \tag{3.48}$$

where

$$S_1 = \left\{ \phi^+_{\tilde{c_1}}(\xi_1) \in \mathfrak{D}^{A_1}_{j_1'} \mid \frac{1-c}{4} \leq |\xi_1| \leq 2 \right\},$$
$$S_2 = \left\{ \phi^-_{\tilde{c_2}}(\xi_2) \in \mathfrak{D}^{A_1}_{j_2'} \mid \frac{1-c}{4} \leq |\xi_2| \leq 2 \right\},$$
$$S_3 = \left\{ (\psi^{\mp}(\xi), \xi) \in \mathfrak{D}^{A_1}_{j'} \mid \frac{1}{2} \leq |\xi| \leq 4,\ \psi^{\mp}(\xi) = \mp c|\xi| + \frac{c_0}{N_0} \right\}.$$

We recall that the unit normals on $\sigma_i \in S_i$ $(i = 1,2,3)$ are written as;

$$\mathfrak{n}_1(\sigma_1) = \frac{1}{\sqrt{2}} \left( -1,\ \frac{(\xi_1)_1}{|\xi_1|},\ \frac{(\xi_1)_2}{|\xi_1|} \right),$$
$$\mathfrak{n}_2(\sigma_2) = \frac{1}{\sqrt{2}} \left( 1,\ \frac{(\xi_2)_1}{|\xi_2|},\ \frac{(\xi_2)_2}{|\xi_2|} \right),$$
$$\mathfrak{n}_3(\sigma_3) = \frac{1}{\sqrt{c^2+1}} \left( \pm 1,\ c\frac{(\xi)_1}{|\xi|},\ c\frac{(\xi)_2}{|\xi|} \right).$$

where

$$\sigma_1 = \phi^+_{\tilde{c_1}}(\xi_1), \quad \sigma_2 = \phi^+_{\tilde{c_2}}(\xi_2), \quad \sigma_3 = (\psi(\xi), \xi).$$

We may assume that there exist $\xi_1', \xi_2', \xi' \in \mathbb{R}^2$ such that

$$\xi_1' + \xi_2' = \xi', \quad (\sigma_1' :=)\phi^+_{\tilde{c}_1}(\xi_1') \in S_1, \ (\sigma_2' :=)\phi^-_{\tilde{c}_2}(\xi_2') \in S_2, \ (\sigma_3' :=)(\psi^{\mp}(\xi'), \xi') \in S_3.$$

From $S_1 \subset \mathfrak{D}^{A_1}_{j_1'}$, $S_2 \subset \mathfrak{D}^{A_1}_{j_2'}$ and $S_3 \subset \mathfrak{D}^{A_1}_{j'}$, we easily observe

$$|\mathfrak{n}_1(\sigma_1) - \mathfrak{n}_1(\sigma_1')| \leq 2^{-10}(1-c)A^{-1}, \tag{3.49}$$

$$|\mathfrak{n}_2(\sigma_2) - \mathfrak{n}_2(\sigma_2')| \leq 2^{-10}(1-c)A^{-1}, \tag{3.50}$$

$$|\mathfrak{n}_3(\sigma_3) - \mathfrak{n}_3(\sigma_3')| \leq 2^{-10}(1-c)A^{-1}. \tag{3.51}$$

The above estimates (3.49)-(3.51) give

$$|(\sigma_1 - \sigma_1') \cdot \mathfrak{n}_1(\sigma_1')| = \left| \left( \sigma_1 - \frac{|\sigma_1|}{|\sigma_1'|}\sigma_1' \right) \cdot \mathfrak{n}_1(\sigma_1') \right| \leq 2^{-20}(1-c)^2 A^{-2},$$
$$|(\sigma_2 - \sigma_2') \cdot \mathfrak{n}_2(\sigma_2')| = \left| \left( \sigma_2 - \frac{|\sigma_2|}{|\sigma_2'|}\sigma_2' \right) \cdot \mathfrak{n}_2(\sigma_2') \right| \leq 2^{-20}(1-c)^2 A^{-2},$$
$$|(\sigma_3 - \sigma_3') \cdot \mathfrak{n}_3(\sigma_3')| = \left| \left( \sigma_3 - \frac{|\sigma_3|}{|\sigma_3'|}\sigma_3' \right) \cdot \mathfrak{n}_3(\sigma_3') \right| \leq 2^{-20}(1-c)^2 A^{-2}.$$

By the same argument as in the proof of Proposition 3.9, we can assume

$$|(\sigma_i - \sigma_i') \cdot \mathfrak{n}_j(\sigma_j')| \leq 2^{-20}(1-c)^2 A^{-2} \quad \text{for any } i, j = 1, 2, 3. \tag{3.52}$$

The remaining part is only to prove (I)-(IV) in Proposition 3.9 with

$$T = 2^{-10}(1-c)^2 A^{-2} (N^{\top})^{-1}, \qquad N = N(\sigma_1', \sigma_2', \sigma_3')$$

and $\tilde{S}_i := T^{-1} S_i$ $(i = 1, 2, 3)$. (I)-(IV) are verified from (3.49)-(3.52) as we proved in the proof of Proposition 3.9. To avoid redundancy, we omit the proof of them. □

Lastly, we consider the case of sufficiently small $\angle(\xi_1, \xi_2)$.

**Proposition 3.14.** *Let $f$, $g_1$, $g_2 \in L^2$ and $M_0$ is the minimal dyadic number which satisfies $N_0^{\frac{1}{2}} \leq M_0$. We assume that $f$, $g_1$, $g_2$ satisfy*

$$\operatorname{supp} f \subset K^{\pm, c}_{N_0, L_0}, \quad \operatorname{supp} g_1 \subset \mathfrak{D}^{M_0}_{j_1} \cap K^-_{N_1, L_1}, \quad \operatorname{supp} g_2 \subset \mathfrak{D}^{M_0}_{j_2} \cap K^+_{N_2, L_2},$$

*with $N_0 \sim N_1 \sim N_2$, $|j_1 - j_2| \leq 16$. Then the following estimate holds:*

$$\left| \int f(\tau, \xi) g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right|$$
$$\lesssim N_0^{\frac{1}{4}} (L_0 L_{\min}^{12})^{\frac{1}{2}} \|f\|_{L^2_{\xi,\tau}} \|g_1\|_{L^2_{\xi,\tau}} \|g_2\|_{L^2_{\xi,\tau}}. \tag{3.53}$$

*Proof.* From $\operatorname{supp} g_1 \subset \mathfrak{D}_{j_1}^{M_0}$ and $\operatorname{supp} g_2 \subset \mathfrak{D}_{j_2}^{M_0}$, we may assume $\operatorname{supp} f \subset \mathfrak{D}_{j}^{M_0}$. We also assume $L_1 \leq L_2$ by symmetry. By Hölder inequality, (3.53) is established if we show

$$\|P_{K^{+}_{N_2,L_2}}((P_{K^{\pm,c}_{N_0,L_0}} f)(P_{K^{-}_{N_1,L_1}} g_1))\|_{L^2_{x,t}} \lesssim N_0^{\frac{1}{4}}(L_0L_1)^{1/2}\|f\|_{L^2_{x,t}}\|g_1\|_{L^2_{x,t}}. \tag{3.54}$$

It is easily confirmed that (3.54) can be verified by the proof of Proposition 3.4 with minor modification. Indeed, same as in the proof of Proposition 3.4, we find that the desired estimate (3.54) is shown by

$$\sup_{\tau,\xi} |E(\tau,\xi)| \lesssim N_0^{\frac{1}{2}} L_0 L_1 \tag{3.55}$$

where

$$E(\tau,\xi) := \{(\tau_1,\xi_1) \in \mathfrak{D}_0^{M_0} \mid \langle \tau - \tau_1 \pm c|\xi - \xi_1|\rangle \sim L_0, \langle \tau_1 - |\xi_1|\rangle \sim L_1\}.$$

Applying the same proof as in Proposition 3.4, we immediately obtain (3.55) thanks to $N_1 M_0^{-1} \sim N_0^{\frac{1}{2}}$. □

We now prove the crucial estimates (3.16) and (3.17) with $\pm_2 = +$ and $N_0 \lesssim N_1 \sim N_2$.

**Theorem 3.15.** *For any $s \in (-3/4, 0)$, there exists $b \in (1/2, 1)$ such that for $f, g_1, g_2 \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$, the following estimates hold:*

$$\sum_{N_1} \sum_{1 \leq N_0 \lesssim N_1 \sim N_2} \sum_{L_j} I_1^{+} \lesssim \|f\|_{X^{s,\,b}_{\pm,\,c}} \|g_1\|_{X^{s,\,b}_{-}} \|g_2\|_{X^{-s,\,1-b}_{+}}, \tag{3.56}$$

$$\sum_{N_1} \sum_{1 \leq N_0 \lesssim N_1 \sim N_2} \sum_{L_j} I_2^{+} \lesssim \|f\|_{X^{-s,\,1-b}_{\pm,\,c}} \|g_1\|_{X^{s,\,b}_{-}} \|g_2\|_{X^{s,\,b}_{+}}, \tag{3.57}$$

*where*

$$I_1^{+} := \left| N_1^{-1} \int (P_{K^{\pm,c}_{N_0,L_0}} f)(P_{K^{-}_{N_1,L_1}} g_1)(P_{K^{+}_{N_2,L_2}} g_2) dtdx \right|,$$

$$I_2^{+} := \left| N_0 N_1^{-1} N_2^{-1} \int (P_{K^{\pm,c}_{N_0,L_0}} f)(P_{K^{-}_{N_1,L_1}} g_1)(P_{K^{+}_{N_2,L_2}} g_2) dtdx \right|.$$

*Proof.* We first note that if $N_1 \lesssim L_{\max}^{012}$ then (3.56) and (3.57) are obtained by the same proof as that of Theorem 3.6. Therefore we can assume $L_{\max}^{012} \lesssim N_1$. We can also assume that $1 \ll N_0$. Indeed, if $N_0 \sim 1$ (3.56) and (3.57) are immediately obtained by using Proposition 3.3 as $N_0 \sim 1$.

If $s \in (-3/4, -1/2)$, considering $N_0 \lesssim N_1 \sim N_2$, we observe that the latter estimate (3.57) is difficult to show compared with the former one. Clearly, the proof of (3.56) and (3.57) become easier as $s$ gets greater. Therefore, we here focus our attention on proving (3.57) for $s \in (-3/4, -1/2)$.

Considering (3.57) in Fourier side, it is easily confirmed that (3.57) is equivalent to

$$\sum_{N_1}\sum_{N_0\lesssim N_1\sim N_2}\sum_{L_j\lesssim N_1} N_0N_1^{-2}\left|\int f^{\pm,c}(\tau_1+\tau_2,\xi_1+\xi_2)g_1^-(\tau_1,\xi_1)g_2^+(\tau_2,\xi_2)d\tau_1d\tau_2d\xi_1d\xi_2\right| \lesssim \|f\|_{\widehat{X}^{-s,\,1-b}_{\pm,\,c}}\|g_1\|_{\widehat{X}^{s,\,b}_{-}}\|g_2\|_{\widehat{X}^{s,\,b}_{+}}, \qquad (3.58)$$

Here we utilized the redefined denotations $f^{\pm,c} := \chi_{K^{\pm,c}_{N_0,L_0}}f$, $g_1^- := \chi_{K^-_{N_1,L_1}}g$, $g_2^+ := \chi_{K^+_{N_2,L_2}}$, and the norms

$$\|\widehat{\cdot}\|_{\widehat{X}^{-s,\,1-b}_{\pm,\,c}} = \|\cdot\|_{X^{-s,\,1-b}_{\pm,\,c}},\quad \|\widehat{\cdot}\|_{\widehat{X}^{s,\,b}_{-}} = \|\cdot\|_{X^{s,\,b}_{-}},\quad \|\widehat{\cdot}\|_{\widehat{X}^{s,\,b}_{+}} = \|\cdot\|_{X^{s,\,b}_{+}}.$$

For simplicity, we use

$$I(f,g,h) := N_0N_1^{-2}\left|\int f(\tau,\xi)g(\tau_1,\xi_1)h(\tau_2,\xi_2)d\tau_1d\tau_2d\xi_1d\xi_2\right|$$

where $\tau=\tau_1+\tau_2$ and $\xi=\xi_1+\xi_2$. By the decomposition of $\mathbb{R}^3\times\mathbb{R}^3$

$$\mathbb{R}^3\times\mathbb{R}^3 = \bigcup_{\substack{-M_0\le j_1,j_2\le M_0-1\\ |j_1-j_2|\le 16}}\mathfrak{D}^{M_0}_{j_1}\times\mathfrak{D}^{M_0}_{j_2}\cup\bigcup_{64\le A\le M_0}\bigcup_{\substack{-A\le j_1,j_2\le A-1\\ 16\le|j_1-j_2|\le 32}}\mathfrak{D}^{A}_{j_1}\times\mathfrak{D}^{A}_{j_2}$$
$$\cup\bigcup_{\substack{-M_1\le j_1,j_2\le M_1-1\\ |j_1-j_2\pm M_1|\le 16}}\mathfrak{D}^{M_1}_{j_1}\times\mathfrak{D}^{M_1}_{j_2}\cup\bigcup_{64\le A\le M_1}\bigcup_{\substack{-A\le j_1,j_2\le A+1\\ 16\le|j_1-j_2\pm A|\le 32}}\mathfrak{D}^{A}_{j_1}\times\mathfrak{D}^{A}_{j_2}.$$

where $M_0$ and $M_1$ are the minimal dyadic number which satisfies respectively

$$N_0^{\frac{1}{2}}\le M_0,\qquad 2^7(1-c)^{-\frac{1}{2}}\frac{(N_1N_2)^{\frac{1}{2}}}{N_0}\le M_1,$$

we only need to show

$$\text{(I)}\ \sum_{N_0\sim N_1\sim N_2}\sum_{L_j\lesssim N_1}\sum_{\substack{-M_0\le j_1,j_2\le M_0-1\\ |j_1-j_2|\le 16}} I(f^{\pm,c},g_1^{-,M_0,j_1},g_2^{+,M_0,j_2}) \lesssim \|f\|_{\widehat{X}^{-s,\,1-b}_{\pm,\,c}}\|g_1\|_{\widehat{X}^{s,\,b}_{-}}\|g_2\|_{\widehat{X}^{s,\,b}_{+}},$$

$$\text{(II)}\ \sum_{N_0\sim N_1\sim N_2}\sum_{L_j\lesssim N_1}\sum_{64\le A\le M_0}\sum_{\substack{-M_0\le j_1,j_2\le M_0-1\\ |j_1-j_2|\le 16}} I(f^{\pm,c},g_1^{-,A,j_1},g_2^{+,A,j_2}) \lesssim \|f\|_{\widehat{X}^{-s,\,1-b}_{\pm,\,c}}\|g_1\|_{\widehat{X}^{s,\,b}_{-}}\|g_2\|_{\widehat{X}^{s,\,b}_{+}},$$

$$\text{(III)}\ \sum_{N_1}\sum_{1\ll N_0\lesssim N_1\sim N_2}\sum_{L_j\lesssim N_1}\sum_{\substack{-M_1\le j_1,j_2\le M_1-1\\ |j_1-j_2\pm M_1|\le 16}} I(f^{\pm,c},g_1^{-,M_1,j_1},g_2^{+,M_1,j_2}) \lesssim \|f\|_{\widehat{X}^{-s,\,1-b}_{\pm,\,c}}\|g_1\|_{\widehat{X}^{s,\,b}_{-}}\|g_2\|_{\widehat{X}^{s,\,b}_{+}},$$

$$\text{(IV)}\ \sum_{N_1}\sum_{1\ll N_0\lesssim N_1\sim N_2}\sum_{L_j\lesssim N_1}\sum_{64\le A\le M_1}\sum_{\substack{-A\le j_1,j_2\le A-1\\ |j_1-j_2\pm A|\le 16}} I(f^{\pm,c},g_1^{-,A,j_1},g_2^{+,A,j_2}) \lesssim \|f\|_{\widehat{X}^{-s,\,1-b}_{\pm,\,c}}\|g_1\|_{\widehat{X}^{s,\,b}_{-}}\|g_2\|_{\widehat{X}^{s,\,b}_{+}},$$

where $g_1^{-,A,j_1} := g_1^-|_{\mathfrak{D}_{j_1}^A}$ and $g_2^{+,A,j_2} := g_2^+|_{\mathfrak{D}_{j_2}^A}$. We further simplify (I)-(IV). From $\ell^2$ Cauchy-Schwarz inequality and $L_j \lesssim N_1$, it suffices to show that there exists $0 < \varepsilon < 1$ such that the following estimates hold;

$$\text{(I)}' \quad \sum_{\substack{-M_0 \le j_1, j_2 \le M_0 - 1 \\ |j_1 - j_2| \le 16}} I(f^{\pm,c}, g_1^{-,M_0,j_1}, g_2^{+,M_0,j_2}) \lesssim N_0^{s-\varepsilon}(L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \|g_1^-\|_{L^2_{\xi,\tau}} \|g_2^+\|_{L^2_{\xi,\tau}},$$

$$\text{(II)}' \quad \sum_{64 \le A \le M_0} \sum_{\substack{-A \le j_1, j_2 \le A-1 \\ |j_1 - j_2| \le 16}} I(f^{\pm,c}, g_1^{-,A,j_1}, g_2^{+,A,j_2}) \lesssim N_0^{s-\varepsilon}(L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \|g_1^-\|_{L^2_{\xi,\tau}} \|g_2^+\|_{L^2_{\xi,\tau}},$$

$$\text{(III)}' \quad \sum_{\substack{-M_1 \le j_1, j_2 \le M_1 - 1 \\ |j_1 - j_2 \pm M_1| \le 16}} I(f^{\pm,c}, g_1^{-,M_1,j_1}, g_2^{+,M_1,j_2}) \lesssim N_0^{-s} N_1^{2s-\varepsilon}(L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \|g_1^-\|_{L^2_{\xi,\tau}} \|g_2^+\|_{L^2_{\xi,\tau}},$$

$$\text{(IV)}' \quad \sum_{64 \le A \le M_1} \sum_{\substack{-A \le j_1, j_2 \le A-1 \\ |j_1 - j_2 \pm A| \le 16}} I(f^{\pm,c}, g_1^{-,A,j_1}, g_2^{+,A,j_2}) \lesssim N_0^{-s} N_1^{2s-\varepsilon}(L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \|g_1^-\|_{L^2_{\xi,\tau}} \|g_2^+\|_{L^2_{\xi,\tau}}.$$

If $-3/4 < s$, (I)$'$ is immediately established by using Proposition 3.13.

$$\begin{aligned}
&\sum_{\substack{-M_0 \le j_1, j_2 \le M_0 - 1 \\ |j_1 - j_2| \le 16}} I(f^{\pm,c}, g_1^{-,M_0,j_1}, g_2^{+,M_0,j_2}) \\
&\sim \sum_{\substack{-M_0 \le j_1, j_2 \le M_0 - 1 \\ |j_1 - j_2| \le 16}} N_0^{-1} \left| \int f^{\pm,c}(\tau,\xi) g_1^{-,M_0,j_1}(\tau_1,\xi_1) g_2^{+,M_0,j_2}(\tau_2,\xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right| \\
&\lesssim N_0^{-\frac{3}{4}} (L_0L_1)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \sum_{\substack{-M_0 \le j_1, j_2 \le M_0 - 1 \\ |j_1 - j_2| \le 16}} \|g_1^{-,M_0,j_1}\|_{L^2_{\xi,\tau}} \|g_2^{+,M_0,j_2}\|_{L^2_{\xi,\tau}}, \\
&\lesssim N_0^{-\frac{3}{4}} (L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \|g_1^-\|_{L^2_{\xi,\tau}} \|g_2^+\|_{L^2_{\xi,\tau}}.
\end{aligned}$$

Next we prove (II)$'$. It follows from Proposition 3.13 that

$$\begin{aligned}
&\sum_{64 \le A \le M_0} \sum_{\substack{-A \le j_1, j_2 \le A-1 \\ |j_1 - j_2| \le 16}} I(f^{\pm,c}, g_1^{-,A,j_1}, g_2^{+,A,j_2}) \\
&\sim \sum_{64 \le A \le M_0} \sum_{\substack{-A \le j_1, j_2 \le A-1 \\ |j_1 - j_2| \le 16}} N_0^{-1} \left| \int f^{\pm,c}(\tau,\xi) g_1^{-,A,j_1}(\tau_1,\xi_1) g_2^{+,A,j_2}(\tau_2,\xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right|
\end{aligned}$$

$$\lesssim \sum_{64\le A\le M_0} N_0^{-1} A^{\frac12}(L_0L_1L_2)^{\frac12}\|f^{\pm,c}\|_{L^2_{\xi,\tau}} \sum_{\substack{-A\le j_1,j_2\le A-1\\ |j_1-j_2|\le 16}} \|g_1^{-,A,j_1}\|_{L^2_{\xi,\tau}}\|g_2^{+,A,j_2}\|_{L^2_{\xi,\tau}}$$

$$\lesssim N_0^{-\frac34}(L_0L_1L_2)^{\frac12}\|f^{\pm,c}\|_{L^2_{\xi,\tau}}\|g_1^{-}\|_{L^2_{\xi,\tau}}\|g_2^{+}\|_{L^2_{\xi,\tau}}.$$

(III)$'$ is verified as follows. By Lemma 3.8, we have $N_0 \lesssim L_{\max}^{012}$. For the sake of simplicity, we here consider the case of $N_0 \lesssim L_0$. The other cases can be proved similarly. We deduce from Proposition 3.3 and Hölder inequality that

$$\begin{aligned}
&\sum_{\substack{-M_1\le j_1,j_2\le M_1-1\\ |j_1-j_2\pm M_1|\le 16}} I(f^{\pm,c}, g_1^{-,M_1,j_1}, g_2^{+,M_1,j_2})\\
\sim& N_0N_1^{-2}\sum_{\substack{-M_1\le j_1,j_2\le M_1-1\\ |j_1-j_2\pm M_1|\le 16}} \left|\int f^{\pm,c}(\tau,\xi)g_1^{-,M_1,j_1}(\tau_1,\xi_1)g_2^{+,M_1,j_2}(\tau_2,\xi_2)d\tau_1d\tau_2d\xi_1d\xi_2\right|\\
\lesssim& N_0N_1^{-2}\sum_{\substack{-M_1\le j_1,j_2\le M_1-1\\ |j_1-j_2\pm M_1|\le 16}} \|\chi_{K^{\pm,c}_{N_0,L_0}}(g_1^{-,M_1,j_1}*g_2^{+,M_1,j_2})\|_{L^2_{\xi,\tau}}\|f^{\pm,c}\|_{L^2_{\xi,\tau}}\\
\lesssim& N_0N_1^{-2}N_0^{\frac12}N_1^{\frac14}L_1^{\frac12}L_2^{\frac14}N_0^{-\frac12}L_0^{\frac12}\|f^{\pm,c}\|_{L^2_{\xi,\tau}}\sum_{\substack{-M_1\le j_1,j_2\le M_1-1\\ |j_1-j_2\pm M_1|\le 16}}\|g_1^{-,M_0,j_1}\|_{L^2_{\xi,\tau}}\|g_2^{+,M_0,j_2}\|_{L^2_{\xi,\tau}}.\\
\lesssim& N_0^{-s}N_1^{2s-\varepsilon}(L_0L_1L_2)^{\frac12}\|f^{\pm,c}\|_{L^2_{\xi,\tau}}\|g_1^{-}\|_{L^2_{\xi,\tau}}\|g_2^{+}\|_{L^2_{\xi,\tau}}.
\end{aligned}$$

Lastly, we prove (IV)$'$. We use the two estimations depending on $N_0$ and $N_1$. Precisely, we utilize Proposition 3.4 if $N_0^3 \lesssim N_1^2$, and if not so, we employ Proposition 3.9. We first assume $N_0^3 \lesssim N_1^2$.

$$\begin{aligned}
&\sum_{64\le A\le M_1}\sum_{\substack{-A\le j_1,j_2\le A-1\\ |j_1-j_2\pm A|\le 16}} I(f^{\pm,c}, g_1^{-,A,j_1}, g_2^{+,A,j_2})\\
\sim& N_0N_1^{-2}\sum_{64\le A\le M_1}\sum_{\substack{-A\le j_1,j_2\le A-1\\ |j_1-j_2\pm A|\le 16}} \left|\int f^{\pm,c}(\tau,\xi)g_1^{-,A,j_1}(\tau_1,\xi_1)g_2^{+,A,j_2}(\tau_2,\xi_2)d\tau_1d\tau_2d\xi_1d\xi_2\right|\\
\lesssim& N_0N_1^{-2}\sum_{64\le A\le M_1}\sum_{\substack{-A\le j_1,j_2\le A-1\\ |j_1-j_2\pm A|\le 16}} \|\chi_{K^{-}_{N_1,L_1}}(f^{\pm,c}*g_{2,-}^{+,A,j_2})\|_{L^2_{\xi,\tau}}\|g_1^{-,A,j_1}\|_{L^2_{\xi,\tau}}\\
\lesssim& N_0N_1^{-2}N_0^{\frac12}(L_0L_2)^{\frac12}\sum_{64\le A\le M_1}\|f^{\pm,c}\|_{L^2_{\xi,\tau}}\sum_{\substack{-A\le j_1,j_2\le A-1\\ |j_1-j_2\pm A|\le 16}}\|g_1^{-,A,j_1}\|_{L^2_{\xi,\tau}}\|g_2^{+,A,j_2}\|_{L^2_{\xi,\tau}}.\\
\lesssim& N_0^{\frac32}N_1^{-2+\varepsilon}(L_0L_1L_2)^{\frac12}\|f^{\pm,c}\|_{L^2_{\xi,\tau}}\|g_1^{-}\|_{L^2_{\xi,\tau}}\|g_2^{+}\|_{L^2_{\xi,\tau}}\\
\lesssim& N_0^{\frac32}N_1^{-2-2s+2\varepsilon}N_1^{2s-\varepsilon}(L_0L_1L_2)^{\frac12}\|f^{\pm,c}\|_{L^2_{\xi,\tau}}\|g_1^{-}\|_{L^2_{\xi,\tau}}\|g_2^{+}\|_{L^2_{\xi,\tau}}\\
\lesssim& N_0^{-s+3\left(\varepsilon-\frac23\left(s+\frac34\right)\right)}N_1^{2s-\varepsilon}(L_0L_1L_2)^{\frac12}\|f^{\pm,c}\|_{L^2_{\xi,\tau}}\|g_1^{-}\|_{L^2_{\xi,\tau}}\|g_2^{+}\|_{L^2_{\xi,\tau}}.
\end{aligned}$$

If $0 < \varepsilon \leq \frac{2}{3}\left(s+\frac{3}{4}\right)$, this completes (IV)$'$. We next assume $N_0^3 \gtrsim N_1^2$. From Proposition 3.9 and $M_1 \sim N_1/N_0$, we observe that

$$\begin{aligned}
&\sum_{64\leq A\leq M_1} \sum_{\substack{-A\leq j_1,j_2\leq A-1\\ |j_1-j_2\pm A|\leq 16}} I(f^{\pm,c}, g_1^{-,A,j_1}, g_2^{+,A,j_2})\\
&\sim N_0 N_1^{-2} \sum_{64\leq A\leq M_1} \sum_{\substack{-A\leq j_1,j_2\leq A-1\\ |j_1-j_2\pm A|\leq 16}} \left| \int f^{\pm,c}(\tau,\xi) g_1^{-,A,j_1}(\tau_1,\xi_1) g_2^{+,A,j_2}(\tau_2,\xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \right|\\
&\lesssim N_0 N_1^{-2} \sum_{64\leq A\leq M_1} A^{\frac{7}{8}} (L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \sum_{\substack{-A\leq j_1,j_2\leq A-1\\ |j_1-j_2\pm A|\leq 16}} \|g_1^{-,A,j_1}\|_{L^2_{\xi,\tau}} \|g_2^{+,A,j_2}\|_{L^2_{\xi,\tau}}\\
&\lesssim N_0 N_1^{-2} N_1^{\frac{7}{8}} N_0^{-\frac{7}{8}} (L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \|g_1^{-}\|_{L^2_{\xi,\tau}} \|g_2^{+}\|_{L^2_{\xi,\tau}}\\
&\lesssim N_0^{\frac{1}{8}} N_1^{-\frac{9}{8}} (L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \|g_1^{-}\|_{L^2_{\xi,\tau}} \|g_2^{+}\|_{L^2_{\xi,\tau}}\\
&\lesssim N_0^{-s} N_0^{s+\frac{1}{8}} N_1^{-\frac{9}{8}} (L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \|g_1^{-}\|_{L^2_{\xi,\tau}} \|g_2^{+}\|_{L^2_{\xi,\tau}}\\
&\lesssim N_0^{-s} N_1^{2s-\left(\frac{4}{3}s+\frac{25}{24}\right)} (L_0L_1L_2)^{\frac{1}{2}} \|f^{\pm,c}\|_{L^2_{\xi,\tau}} \|g_1^{-}\|_{L^2_{\xi,\tau}} \|g_2^{+}\|_{L^2_{\xi,\tau}}.
\end{aligned}$$

This completes the proof of (IV)$'$. □

## 4. Well-posedness of the Klein-Gordon-Zakharov system in $d \geq 5$

4.1. **Introduction.** We continue the study of the Cauchy problem of the Klein-Gordon-Zakharov system:

$$\begin{cases} (\partial_t^2 - \Delta + 1)u = -nu, & (t,x) \in [-T,T] \times \mathbb{R}^d, \\ (\partial_t^2 - c^2\Delta)n = \Delta|u|^2, & (t,x) \in [-T,T] \times \mathbb{R}^d, \\ (u, \partial_t u, n, \partial_t n)|_{t=0} = (u_0, u_1, n_0, n_1) & \\ \qquad\qquad \in H^{s+1}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d), & \end{cases} \tag{4.1}$$

where $u, n$ are real valued functions, $d \geq 5, c > 0$ and $c \neq 1$. As opposed to $d = 2$, the proof for $c > 1$ is quite similar to that of the case $0 < c < 1$. Therefore we also consider the case $c > 1$. Similarly to 2D, (4.1) is equivalent to the following.

$$\begin{cases} (i\partial_t \mp \omega_1)u_\pm = \pm(1/4)(n_+ + n_-)(\omega_1^{-1}u_+ + \omega_1^{-1}u_-), & (t,x) \in [-T,T] \times \mathbb{R}^d, \\ (i\partial_t \mp c\omega)n_\pm = \pm(4c)^{-1}\omega|\omega_1^{-1}u_+ + \omega_1^{-1}u_-|^2, & (t,x) \in [-T,T] \times \mathbb{R}^d, \\ (u_\pm, n_\pm)|_{t=0} = (u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d). & \end{cases} \tag{4.2}$$

Our main result is as follows.

**Theorem 4.1.** *Let $d \geq 5, s = s_c = d/2 - 2$ and assume the initial data $(u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$ is small. Then, (4.2) is globally well-posed in $H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$.*

**Corollary 4.2.** *The solution obtained in Theorem 4.1 scatters as $t \to \pm\infty$.*

For more precise statement of Theorem 4.1 and Corollary 4.2, see Propositions 4.25, 4.26. [25] considered (4.2) for $d \geq 4, 0 < c$ and $c \neq 1$. [25] applied $U^2, V^2$ type spaces and established that (4.2) is globally well-posed in $H^{s_c}(\mathbb{R}^d) \times \dot{H}^{s_c}(\mathbb{R}^d)$ if the initial data is small and radial. $U^2, V^2$ type spaces were introduced by Koch and Tataru [31]. As we mentioned in Introduction, these spaces work well when we consider well-posedness at the critical space [18], [22], [23], [26]. Theorem 4.1 is proved by the Banach fixed point theorem. The key is the bilinear estimate (Proposition 4.24). For $d \geq 5$, it appears to be difficult to prove Proposition 4.24 only by applying $U^2, V^2$ type spaces, the modulation estimate (Proposition 4.14, Lemma 4.15) and the Strichartz type estimates (Proposition 4.10) for a nonlinear interaction [25]. In this thesis, to overcome the difficulty, we derive the bilinear Strichartz estimate for the nonlinear interaction and then we are able to prove Proposition 4.24. See Proposition 4.23 for the bilinear Strichartz estimate. $c \neq 1$

plays an important role in the proof of the bilinear Strichartz estimate as well as in the proof of Lemma 4.15.

In Section 4.2, we prepare some notations and lemmas with respect to $U^p, V^p$, in Section 4.3, we prove the bilinear estimates and in Section 4.4, we prove the main result.

4.2. **Notations and Preliminary Lemmas.** In this section, we define $U^p$ $V^p$ spaces and prepare some lemmas, propositions and notations to prove the main theorem. Let $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < ... < t_K = \infty$ and let $\mathcal{Z}_0$ be the set of finite partitions $-\infty < t_0 < t_1 < ... < t_K \leq \infty$.

*Definition* 1. Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L_x^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{L_x^2}^p = 1$, we call the function $a : \mathbb{R} \to L_x^2$ given by

$$a = \sum_{k=1}^{K} \chi_{[t_{k-1}, t_k)} \phi_{k-1}$$

a $U^p$-atom. Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \,\middle|\, a_j : U^p\text{-atom}, \lambda_j \in \mathbb{C} \textit{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with norm

$$\|u\|_{U^p} := \inf\left\{ \sum_{j=1}^{\infty} |\lambda_j| \,\middle|\, u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j : U^p\text{-atom} \right\}.$$

**Proposition 4.3.** *Let* $1 \leq p < q < \infty$.
*(i)* $U^p$ *is a Banach space.*
*(ii) The embeddings* $U^p \subset U^q \subset L_t^\infty(\mathbb{R}; L_x^2)$ *are continuous.*
*(iii) For* $u \in U^p$*, it holds that* $\lim_{t \to t_0+} \|u(t) - u(t_0)\|_{L_x^2} = 0$*, i.e. every* $u \in U^p$ *is right-continuous.*
*(iv) The closed subspace* $U_c^p$ *of all continuous functions in* $U^p$ *is a Banach space.*

The above proposition is in [18] (Proposition 2.2).

*Definition* 2. Let $1 \leq p < \infty$. We define $V^p$ as the normed space of all functions $v : \mathbb{R} \to L_x^2$ such that $\lim_{t \to \pm\infty} v(t)$ exist and for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left( \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L_x^2}^p \right)^{1/p}$$

is finite, where we use the convention that $v(-\infty) := \lim_{t \to -\infty} v(t)$ and $v(\infty) := 0$. Likewise, let $V_-^p$ denote the closed subspace of all $v \in V^p$ with $\lim_{t \to -\infty} v(t) = 0$.

The definitions of $V^p$ and $V^p_-$, see also [19].

**Proposition 4.4.** *Let $1 \le p < q < \infty$.*
*(i) Let $v : \mathbb{R} \to L^2_x$ be such that*

$$\|v\|_{V^p_0} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}_0} \Big( \sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|^p_{L^2_x} \Big)^{1/p}$$

*is finite. Then, it follows that $v(t_0^+) := \lim_{t\to t_0+} v(t)$ exists for all $t_0 \in [-\infty, \infty)$ and $v(t_0^-) := \lim_{t \to t_0-} v(t)$ exists for all $t_0 \in (-\infty, \infty]$ and moreover,*

$$\|v\|_{V^p} = \|v\|_{V^p_0}.$$

*(ii) We define the closed subspace $V^p_{rc}$ ($V^p_{-,rc}$) of all right-continuous $V^p$ functions ($V^p_-$ functions). The spaces $V^p$, $V^p_{rc}$, $V^p_-$ and $V^p_{-,rc}$ are Banach spaces.*
*(iii) The embeddings $U^p \subset V^p_{-,rc} \subset U^q$ are continuous.*
*(iv) The embeddings $V^p \subset V^q$ and $V^p_- \subset V^q_-$ are continuous.*

The proof of Proposition 4.4 is in [18] (Proposition 2.4 and Corollary 2.6). Let $\{\mathcal{F}^{-1}_\xi[\varphi_n](x)\}_{n\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^d)$ be the Littlewood-Paley decomposition with respect to $x$, that is to say

$$\begin{cases} \varphi(\xi) \ge 0, \\ \operatorname{supp} \varphi(\xi) = \{\xi \,|\, 2^{-1} \le |\xi| \le 2\}, \end{cases}$$

$$\varphi_n(\xi) := \varphi(2^{-n}\xi), \;\; \sum_{n=-\infty}^{\infty} \varphi_n(\xi) = 1 \;\, (\,\xi \neq 0), \;\; \psi(\xi) := 1 - \sum_{n=0}^{\infty} \varphi_n(\xi).$$

Let $N = 2^n$ ($n \in \mathbb{Z}$) be dyadic number. $P_N$ and $P_{<1}$ denote

$$\mathcal{F}_x[P_N f](\xi) := \varphi(\xi/N)\mathcal{F}_x[f](\xi) = \varphi_n(\xi)\mathcal{F}_x[f](\xi),$$
$$\mathcal{F}_x[P_{<1} f](\xi) := \psi(\xi)\mathcal{F}_x[f](\xi).$$

Similarly, let $\tilde{Q}_N$ be

$$\mathcal{F}_t[\tilde{Q}_N g](\tau) := \phi(\tau/N)\mathcal{F}_t[g](\tau) = \phi_n(\tau)\mathcal{F}_t[g](\tau),$$

where $\{\mathcal{F}^{-1}_\tau[\phi_n](t)\}_{n\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R})$ be the Littlewood-Paley decomposition with respect to $t$. Let $K_\pm(t) = \exp\{\mp it(1-\Delta)^{1/2}\} : L^2_x \to L^2_x$ be the Klein-Gordon unitary operator such that $\mathcal{F}_x[K_\pm(t)u_0](\xi) = \exp\{\mp it\langle\xi\rangle\}\,\mathcal{F}_x[u_0](\xi)$. Similarly, we define the wave unitary operator $W_{\pm c}(t) = \exp\{\mp ict(-\Delta)^{1/2}\} : L^2_x \to L^2_x$ such that $\mathcal{F}_x[W_{\pm c}(t)n_0](\xi) = \exp\{\mp ict|\xi|\}\,\mathcal{F}_x[n_0](\xi)$. We set

$$W_L^{\pm c} := \big\{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^d \,|\, L/2 \le \big|\tau \pm c|\xi|\big| \le 2L\big\},$$
$$KG_L^{\pm} := \big\{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^d \,|\, L/2 \le \big|\tau \pm \langle\xi\rangle\big| \le 2L\big\}.$$

*Definition* 3. We define
(*i*) $U^p_{K_\pm} = K_\pm(\cdot)U^p$ with norm $\|u\|_{U^p_{K_\pm}} = \|K_\pm(-\cdot)u\|_{U^p}$,
(*ii*) $V^p_{K_\pm} = K_\pm(\cdot)V^p$ with norm $\|u\|_{V^p_{K_\pm}} = \|K_\pm(-\cdot)u\|_{V^p}$.
For dyadic numbers $N, M$,

$$Q_N^{K_\pm} := K_\pm(\cdot)\tilde{Q}_N K_\pm(-\cdot), \quad Q_{\ge M}^{K_\pm} := \sum_{N\ge M} Q_N, \quad Q_{<M}^{K_\pm} := Id - Q_{\ge M}^{K_\pm}.$$

Here summation over $N$ means summation over $n\in\mathbb{Z}$. Similarly, we define $U^p_{W_{\pm c}}, V^p_{W_{\pm c}}$.

*Remark* 4.1. For $L^2_x$ unitary operator $A = K_\pm$ or $W_{\pm c}$,

$$U^2_A \subset V^2_{-,rc,A} \subset L^\infty(\mathbb{R};L^2_x)$$

*Definition* 4. For the Klein-Gordon equation, we define $Y^s_{K_\pm}$ (resp. $Z^s_{K_\pm}$) as the closure of all $u\in C(\mathbb{R};H^s_x(\mathbb{R}^d))\cap\langle\nabla_x\rangle^{-s}V^2_{-,rc,K_\pm}$ (resp. $u\in C(\mathbb{R};H^s_x(\mathbb{R}^d))\cap\langle\nabla_x\rangle^{-s}U^2_{K_\pm}$) with $Y^s_{K_\pm}$ (resp. $Z^s_{K_\pm}$) norm, where

$$\|u\|_{Y^s_{K_\pm}} := \|P_{<1}u\|_{V^2_{K_\pm}} + \Big(\sum_{N\ge 1} N^{2s}\|P_N u\|^2_{V^2_{K_\pm}}\Big)^{1/2},$$
$$\|u\|_{Z^s_{K_\pm}} := \|P_{<1}u\|_{U^2_{K_\pm}} + \Big(\sum_{N\ge 1} N^{2s}\|P_N u\|^2_{U^2_{K_\pm}}\Big)^{1/2}.$$

For the wave equation, we define $\dot{Y}^s_{W_{\pm c}}, \dot{Z}^s_{W_{\pm c}}$ as the closure of all $n\in C(\mathbb{R};H^s_x(\mathbb{R}^d))\cap |\nabla_x|^{-s}V^2_{-,rc,W_{\pm c}}$ (resp. $n\in C(\mathbb{R};H^s_x(\mathbb{R}^d))\cap|\nabla_x|^{-s}U^2_{W_{\pm c}}$) with $\dot{Y}^s_{W_{\pm c}}$ (resp. $\dot{Z}^s_{W_{\pm c}}$) norm, where

$$\|n\|_{\dot{Y}^s_{W_{\pm c}}} := \Big(\sum_N N^{2s}\|P_N n\|^2_{V^2_{W_{\pm c}}}\Big)^{1/2}, \qquad \|n\|_{\dot{Z}_{W_{\pm c}}} := \Big(\sum_N N^{2s}\|P_N n\|^2_{U^2_{W_{\pm c}}}\Big)^{1/2}.$$

*Definition* 5. For a Hilbert space $H$ and a Banach space $X\subset C(\mathbb{R};H)$, we define

$$B_r(H) := \{f\in H \,|\, \|f\|_H \le r\},$$
$$X([0,T)) := \{u\in C([0,T);H) \,|\, \exists\tilde{u}\in X, \tilde{u}(t) = u(t), t\in[0,T)\}$$

endowed with the norm $\|u\|_{X([0,T))} = \inf\{\|\tilde{u}\|_X \,|\, \tilde{u}(t) = u(t), t\in[0,T)\}$.

We denote the Duhamel term

$$I_{T,K_\pm}(n,v) := \pm\int_0^t \chi_{[0,T]}(t')K_\pm(t-t')n(t')(\omega_1^{-1}v(t'))dt',$$
$$I_{T,W_{\pm c}}(u,v) := \pm\int_0^t \chi_{[0,T]}(t')W_{\pm c}(t-t')\omega\big((\omega_1^{-1}u(t'))(\overline{\omega_1^{-1}v(t')})\big)dt'$$

for the Klein-Gordon equation and the wave equation respectively. The following proposition is in [18] (Theorem 2.8 and Proposition 2.10).

**Proposition 4.5.** *Let $u\in V^1_{-,rc}\subset U^2$ be absolutely continuous on compact intervals. Then,* $\|u\|_{U^2} = \sup_{v\in V^2,\,\|v\|_{V^2}=1}\Big|\int_{-\infty}^{\infty}\langle u'(t),v(t)\rangle_{L^2_x}dt\Big|.$

**Corollary 4.6.** *Let $A=K_\pm$ or $W_{\pm c}$ and $u\in V^1_{-,rc,A}\subset U^2_A$ be absolutely continuous on compact intervals. Then,*

$$\|u\|_{U^2_A} = \sup_{v\in V^2_A,\,\|v\|_{V^2_A}=1}\Big|\int_{-\infty}^{\infty}\langle A(t)(A(-\cdot)u)'(t),v(t)\rangle_{L^2_x}dt\Big|.$$

**Proposition 4.7.** *Let $T_0: L^2_x\times\ldots\times L^2_x\to L^1_{loc}(\mathbb{R}^d;\mathbb{C})$ be a n-linear operator. Assume that for some $1\le p<\infty$ and $1\le q\le\infty$, it holds that*

$$\|T_0(K_\pm(\cdot)\phi_1,\ldots,K_\pm(\cdot)\phi_n)\|_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))}\lesssim\prod_{i=1}^n\|\phi_i\|_{L^2_x}.$$

*Then, there exists $T: U^p_{K_\pm}\times\ldots\times U^p_{K_\pm}\to L^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))$ satisfying*

$$\|T(u_1,\ldots,u_n)\|_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))}\lesssim\prod_{i=1}^n\|u_i\|_{U^p_{K_\pm}},$$

*such that $T(u_1,\ldots,u_n)(t)(x)=T_0(u_1(t),\ldots,u_n(t))(x)$ a.e.*

See Proposition 2.19 in [18] for the proof of the above proposition.

**Proposition 4.8.** *Let $d\ge3, 2\le r<\infty, 2/q=(d-1)(1/2-1/r), (q,r)\ne(2,2(d-1)/(d-3))$ and $s=1/q-1/r+1/2$. Then it holds that*

$$\|W_{\pm c}(t)f\|_{L^q_t\dot{W}^{-s,r}_x(\mathbb{R}^{1+d})}\lesssim\|f\|_{L^2_x(\mathbb{R}^d)}.$$

For the proof of Proposition 4.8, see [28], [16].

**Proposition 4.9.** *Let $d\ge3, 2\le r<\infty, 2/q=(d-1)(1/2-1/r), (q,r)\ne(2,2(d-1)/(d-3))$ and $s=1/q-1/r+1/2$. Then, it holds that*

$$\|K_\pm(t)f\|_{L^q_tW^{-s,r}_x(\mathbb{R}^{1+d})}\lesssim\|f\|_{L^2_x(\mathbb{R}^d)}.$$

For the proof of Proposition 4.9, see [34]. Combining Proposition 4.4, Proposition 4.7, Proposition 4.8 and Proposition 4.9, we have the following.

**Proposition 4.10.** *Let $d \geq 3, 2 \leq r < \infty, 2/q = (d-1)(1/2 - 1/r), (q,r) \neq (2, 2(d-1)/(d-3))$ and $s = 1/q - 1/r + 1/2$. If $p < q$, then it holds that*

$$\|f\|_{L_t^q W_x^{-s,r}(\mathbb{R}^{1+d})} \lesssim \|f\|_{V^p_{K_\pm}}, \qquad \|f\|_{L_t^q \dot{W}_x^{-s,r}(\mathbb{R}^{1+d})} \lesssim \|f\|_{V^p_{W_{\pm c}}}.$$

**Proposition 4.11.** *(i) Let $T > 0$ and $u \in Y^s_{K_\pm}([0,T]), u(0) = 0$. Then, there exists $0 \leq T' \leq T$ such that $\|u\|_{Y^s_{K_\pm}([0,T'])} < \varepsilon$.*
*(ii) Let $T > 0$ and $n \in \dot{Y}^s_{W_{\pm c}}([0,T]), n(0) = 0$. Then, there exists $0 \leq T' \leq T$ such that $\|n\|_{\dot{Y}^s_{W_{\pm c}}([0,T'])} < \varepsilon$.*

For the proofs of $(i)$ and $(ii)$, see Proposition 2.24 in [18].

**Lemma 4.12.** *Let $a \geq 0$. Then for $A = K_\pm$ or $W_{\pm c}$, it holds that*

$$\|\langle \nabla_x \rangle^a f\|_{V^2_A} \lesssim \|f\|_{Y^a_A}.$$

*Proof.* We only prove it for $A = K_\pm$ since we can prove it similarly for $A = W_{\pm c}$. By $L^2_x$ orthogonality, we have

$$\begin{aligned}
\|\langle \nabla_x \rangle^a f\|^2_{V^2_{K_\pm}} \lesssim & \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I (\|P_{<1}(K_\pm(-t_i)f(t_i) - K_\pm(-t_{i-1})f(t_{i-1}))\|^2_{L^2_x} \\
& + \sum_{N \geq 1} N^{2a} \|P_N(K_\pm(-t_i)f(t_i) - K_\pm(-t_{i-1})f(t_{i-1}))\|^2_{L^2_x}) \\
\lesssim & \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I \|K_\pm(-t_i)P_{<1}f(t_i) - K_\pm(-t_{i-1})P_{<1}f(t_{i-1})\|^2_{L^2_x} \\
& + \sum_{N \geq 1} N^{2a} \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I \|K_\pm(-t_i)P_N f(t_i) - K_\pm(-t_{i-1})P_N f(t_{i-1})\|^2_{L^2_x} \\
\lesssim & \|f\|^2_{Y^a_{K_\pm}}.
\end{aligned}$$

□

*Remark* 4.2. Similarly, we see

$$\||\nabla_x|^a f\|_{V^2_A} \lesssim \|f\|_{\dot{Y}^a_A}.$$

**Lemma 4.13.** *If $f, g$ are measurable functions, then for $Q = Q^A_{<M}$ or $Q^A_{\geq M}, A = K_\pm$ or $W_{\pm c}$, it holds that*

$$\int_{\mathbb{R}^{1+d}} f(t,x)\overline{Qg(t,x)}dxdt = \int_{\mathbb{R}^{1+d}} \big(Qf(t,x)\big)\overline{g(t,x)}dxdt.$$

For the proof of Lemma 4.13, see [26], Lemma 2.17. Since $Q^A_{<M} = Id - Q^A_{\geq M}$, we also obtain the result for $Q = Q^A_{<M}$.

**Proposition 4.14.** *It holds that*

$$\|Q_M^{K_\pm}u\|_{L^2_{t,x}(\mathbb{R}^{1+d})} \lesssim M^{-1/2}\|u\|_{V^2_{K_\pm}}, \qquad \|Q_{\geq M}^{K_\pm}u\|_{L^2_{t,x}(\mathbb{R}^{1+d})} \lesssim M^{-1/2}\|u\|_{V^2_{K_\pm}}, \tag{4.3}$$

$$\|Q_{<M}^{K_\pm}u\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}, \qquad \|Q_{\geq M}^{K_\pm}u\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}},$$

$$\|Q_{<M}^{K_\pm}u\|_{U^2_{K_\pm}} \lesssim \|u\|_{U^2_{K_\pm}}, \qquad \|Q_{\geq M}^{K_\pm}u\|_{U^2_{K_\pm}} \lesssim \|u\|_{U^2_{K_\pm}}.$$

*The same estimates hold by replacing the Klein-Gordon operator $K_\pm$ by the wave operator $W_{\pm c}$.*

**Lemma 4.15.** *Let $c > 0, c \neq 1$ and $\tau_3 = \tau_1 - \tau_2,\ \xi_3 = \xi_1 - \xi_2$. If $|\xi_1| \gg \langle \xi_2 \rangle$ or $\langle \xi_1 \rangle \ll |\xi_2|$, then it holds that*

$$\max\left\{\left|\tau_1 \pm \langle \xi_1 \rangle\right|, \left|\tau_2 \pm \langle \xi_2 \rangle\right|, \left|\tau_3 \pm c|\xi_3|\right|\right\} \gtrsim \max\{|\xi_1|, |\xi_2|\}. \tag{4.4}$$

*Proof.* We only prove the case $|\xi_1| \gg \langle \xi_2 \rangle$ since the case $\langle \xi_1 \rangle \ll |\xi_2|$ is proved by the same manner.

$$\text{(l.h.s.)} \gtrsim \left|\left(\tau_1 \pm \left(1 + |\xi_1|\right)\right) - \left(\tau_2 \pm (1 + |\xi_2|)\right) - (\tau_3 \pm c|\xi_3|)\right| \tag{4.5}$$

If $0 < c < 1$, then we take $\varepsilon_c$ such that $0 < \varepsilon_c < (1-c)/(1+c), |\xi_2| \leq \varepsilon_c|\xi_1|$. Then, the right hand side of (4.5) is bounded by

$$(1 + |\xi_1|) - (1 + |\xi_2|) - c|\xi_1 - \xi_2| \geq |\xi_1| - \varepsilon_c|\xi_1| - c(1 + \varepsilon_c)|\xi_1| \gtrsim |\xi_1|.$$

If $c > 1$, then we take $\tilde{\varepsilon}_c$ such that $0 < \tilde{\varepsilon}_c < (c-1)/(c+3), |\xi_2| \leq \tilde{\varepsilon}_c|\xi_1|, |\xi_1| \geq 1/\tilde{\varepsilon}_c$. Then, the right hand side of (4.5) is bounded by

$$c|\xi_1 - \xi_2| - (1 + |\xi_1|) - (1 + |\xi_2|) \geq c(1 - \tilde{\varepsilon}_c)|\xi_1| - (1 + \tilde{\varepsilon}_c)|\xi_1| - 2\tilde{\varepsilon}_c|\xi_1| \gtrsim |\xi_1|.$$

□

*Remark* 4.3. From (4.3) and (4.4), we can obtain a half derivative.

**Lemma 4.16.** *Let $\tilde{u}_{N_1} := \chi_{[0,T)}P_{N_1}u, \tilde{v}_{N_2} := \chi_{[0,T)}P_{N_2}v, \tilde{n}_{N_3} := \chi_{[0,T)}P_{N_3}n, Q_1, Q_2 \in \{Q_{<M}^{K_\pm}, Q_{\geq M}^{K_\pm}\}, Q_3 \in \{Q_{<M}^{W_{\pm c}}, Q_{\geq M}^{W_{\pm c}}\}$. Let $d \geq 5,\ s = s_c = d/2 - 2$. Then the following estimates hold for all $0 < T < \infty$ :*

*(i) If $N_3 \lesssim N_2 \sim N_1$, then*

$$|I_1| := \left|\int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}})dxdt\right| \lesssim N_3^s\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}.$$

*(ii) It holds that*

$$|I_2| := \left|\int_{\mathbb{R}^{1+d}} \tilde{n}(\omega_1^{-1}\tilde{v})(\overline{P_{<1}\tilde{u}})dxdt\right| \lesssim \|n\|_{\dot{Y}^s_{W_{\pm c}}}\|v\|_{Y^s_{K_\pm}}\|P_{<1}u\|_{V^2_{K_\pm}}.$$

*(iii) If* $N_1 \sim N_2$*, then*

$$|I_3| := \Big|\int_{\mathbb{R}^{1+d}} \Big(\sum_{N_3 \lesssim N_2} \tilde{n}_{N_3}\Big)(\omega_1^{-1}\tilde{v}_{N_2})\overline{\tilde{u}_{N_1}}dxdt\Big| \lesssim \|n\|_{\dot{Y}^s_{W_{\pm c}}}\|v_{N_2}\|_{V^2_{K_\pm}}\|u_{N_1}\|_{V^2_{K_\pm}}.$$

*(iv) If* $N_1 \sim N_3, N_1 \gg 1, M = \varepsilon N_1$ *and* $\varepsilon > 0$ *is sufficiently small, then*

$$|I_i| \lesssim \|n_{N_3}\|_{V^2_{W_{\pm c}}}\|v\|_{Y^s_{K_\pm}}\|u_{N_1}\|_{V^2_{K_\pm}}, \quad (i = 4, 5)$$

*where*

$$I_4 := \int_{\mathbb{R}^{1+d}} (Q^{W_{\pm c}}_{\geq M}\tilde{n}_{N_3})\Big(\sum_{N_2 \ll N_1} Q_2\omega_1^{-1}\tilde{v}_{N_2}\Big)(\overline{Q_1\tilde{u}_{N_1}})dxdt,$$

$$I_5 := \int_{\mathbb{R}^{1+d}} (Q_3\tilde{n}_{N_3})\Big(\sum_{N_2 \ll N_1} Q_2\omega_1^{-1}\tilde{v}_{N_2}\Big)(\overline{Q^{K_\pm}_{\geq M}\tilde{u}_{N_1}})dxdt.$$

*Proof.* We show $(i)$ first. For $f \in V^2_A,\ A \in \{K_\pm, W_{\pm c}\}$, we see

$$\|\chi_{[0,T)}f\|_{V^2_A} \lesssim \|f\|_{V^2_A}. \tag{4.6}$$

For $d \geq 5$, we apply the Hölder inequality to have

$$|I_1| \lesssim \|\omega_1^{-1}\tilde{u}_{N_1}\|_{L^{2(d+1)/(d-1)}_{t,x}}\|\omega_1^{-1}\tilde{v}_{N_2}\|_{L^{2(d+1)/(d-1)}_{t,x}}\|\omega\tilde{n}_{N_3}\|_{L^{(d+1)/2}_{t,x}}. \tag{4.7}$$

We apply Proposition 4.10, (4.6) and the Sobolev inequality, then we have

$$\|\omega_1^{-1}\tilde{f}_N\|_{L^{2(d+1)/(d-1)}_{t,x}} \lesssim \langle N\rangle^{1/2-1}\|f_N\|_{V^2_{K_\pm}} = \langle N\rangle^{-1/2}\|f_N\|_{V^2_{K_\pm}}, \tag{4.8}$$

$$\begin{aligned}\|\omega\tilde{n}_{N_3}\|_{L^{(d+1)/2}_{t,x}} &\lesssim \||\nabla_x|^{d(d-5)/2(d-1)}\omega\tilde{n}_{N_3}\|_{L^{(d+1)/2}_t L^{2(d^2-1)/(d^2-9)}_x} \\ &\lesssim \||\nabla_x|^{d/2-2}\omega\tilde{n}_{N_3}\|_{V^2_{W_{\pm c}}} \qquad (4.9)\\ &\lesssim N_3^{s_c+1}\|n_{N_3}\|_{V^2_{W_{\pm c}}} \qquad (4.10)\end{aligned}$$

Collecting (4.7), (4.8), (4.10) and $N_3 \lesssim N_1 \sim N_2$, we obtain

$$|I_1| \lesssim N_3^{s_c}\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}.$$

Next, we prove $(ii)$. For $d \geq 5$, by the Hölder inequality to have

$$|I_2| \lesssim \|\tilde{n}\|_{L^{(d+1)/2}_{t,x}}\|\omega_1^{-1}\tilde{v}\|_{L^{2(d+1)/(d-1)}_{t,x}}\|P_{<1}\tilde{u}\|_{L^{2(d+1)/(d-1)}_{t,x}}. \tag{4.11}$$

From Proposition 4.10, (4.9), Remark 4.2 and Lemma 4.12, we obtain

$$\|\tilde{n}\|_{L^{(d+1)/2}_{t,x}} \lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}}, \tag{4.12}$$

$$\|\omega_1^{-1}\tilde{v}\|_{L^{2(d+1)/(d-1)}_{t,x}} \lesssim \|\langle\nabla_x\rangle^{-1/2}v\|_{V^2_{K_\pm}} \lesssim \|\langle\nabla_x\rangle^{s_c}v\|_{V^2_{K_\pm}} \lesssim \|v\|_{Y^{s_c}_{K_\pm}}, \tag{4.13}$$

$$\|P_{<1}\tilde{u}\|_{L^{2(d+1)/(d-1)}_{t,x}} \lesssim \|\langle\nabla_x\rangle^{1/2}P_{<1}u\|_{V^2_{K_\pm}} \lesssim \|P_{<1}u\|_{V^2_{K_\pm}}. \tag{4.14}$$

Collecting (4.11)–(4.14), we obtain

$$|I_2| \lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|_{Y^{s_c}_{K_\pm}} \|P_{<1}u\|_{V^2_{K_\pm}}.$$

We prove $(iii)$ for $d \geq 5$. We apply the Hölder inequality to have

$$|I_3| \lesssim \Big\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{L^{(d+1)/2}_{t,x}} \|\omega_1^{-1}\tilde{v}_{N_2}\|_{L^{2(d+1)/(d-1)}_{t,x}} \|\tilde{u}_{N_1}\|_{L^{2(d+1)/(d-1)}_{t,x}}. \tag{4.15}$$

Similar to (4.9), the Sobolev inequality and Proposition 4.10, we have

$$\Big\| \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{L^{(d+1)/2}_{t,x}} \lesssim \Big\| |\nabla_x|^{s_c} \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|_{V^2_{W_{\pm c}}}. \tag{4.16}$$

By the $L^2_x$ orthogonality, we obtain

$$\begin{aligned} \Big\| |\nabla_x|^{s_c} \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3} \Big\|^2_{V^2_{W_{\pm c}}} \lesssim \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I \sum_N N^{2s_c} \Big\| P_N \Big\{ W_{\pm c}(-t_i) \Big( \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3}(t_i) \Big) \\ - W_{\pm c}(-t_{i-1}) \Big( \sum_{N_3 \lesssim N_2} \tilde{n}_{N_3}(t_{i-1}) \Big) \Big\} \Big\|^2_{L^2_x}. \end{aligned} \tag{4.17}$$

Since $P_N \tilde{n}_{N_3} = 0$ if $N_3 > 2N$ or $N_3 < N/2$ and $P_N$ is projection, the right-hand side is bounded by

$$\begin{aligned} &\sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \sum_{i=1}^I \sum_N N^{2s_c} \|W_{\pm c}(-t_i) P_N \tilde{n}(t_i) - W_{\pm c}(-t_{i-1}) P_N \tilde{n}(t_{i-1})\|^2_{L^2_x} \\ &\lesssim \sum_N N^{2s_c} \sup_{\{t_i\}_{i=0}^I \in \mathcal{Z}} \|W_{\pm c}(-t_i) P_N \tilde{n}(t_i) - W_{\pm c}(-t_{i-1}) P_N \tilde{n}(t_{i-1})\|^2_{L^2_x} \\ &\lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}}. \end{aligned} \tag{4.18}$$

Hence, from (4.15)–(4.18), (4.8) and $N_1 \sim N_2$, we have

$$\begin{aligned} |I_3| &\lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \langle N_2 \rangle^{-1/2} \|v_{N_2}\|_{V^2_{K_\pm}} \langle N_1 \rangle^{1/2} \|u_{N_1}\|_{V^2_{K_\pm}} \\ &\lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}. \end{aligned}$$

We prove $(iv)$. The estimate for $I_5$ is obtained by the same manner as the estimate for $I_4$, so we only estimate $I_4$. We apply the Hölder inequality to have

$$|I_4| \lesssim \|Q^{W_{\pm c}}_{\geq M} \tilde{n}_{N_3}\|_{L^2_{t,x}} \Big\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^{d+1}_{t,x}} \|Q_1 \tilde{u}_{N_1}\|_{L^{2(d+1)/(d-1)}_{t,x}}. \tag{4.19}$$

By Proposition 4.14, (4.8) and (4.6), we have

$$\|Q^{W_{\pm c}}_{\geq M} \tilde{n}_{N_3}\|_{L^2_{t,x}} \lesssim N_1^{-1/2} \|n_{N_3}\|_{V^2_{W_{\pm c}}}, \tag{4.20}$$

$$\|Q_1 \tilde{u}_{N_1}\|_{L^{2(d+1)/(d-1)}_{t,x}} \lesssim \langle N_1 \rangle^{1/2} \|u_{N_1}\|_{V^2_{K_\pm}}. \tag{4.21}$$

We apply the Sobolev inequality, Proposition 4.10, Proposition 4.14 and (4.6), we have

$$\begin{aligned}\Big\| \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L^{d+1}_{t,x}} &\lesssim \Big\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)} \sum_{N_2 \ll N_1} Q_2 \omega_1^{-1} \tilde{v}_{N_2} \Big\|_{L_t^{d+1} L_x^{2(d^2-1)/(d^2-5)}} \\ &\lesssim \Big\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)+1/(d-1)-1} \sum_{N_2 \ll N_1} \tilde{v}_{N_2} \Big\|_{V^2_{K_\pm}}. \end{aligned} \tag{4.22}$$

Similar to (4.17) and (4.18), we have

$$\Big\| \langle \nabla_x \rangle^{d(d-3)/2(d-1)+1/(d-1)-1} \sum_{N_2 \ll N_1} \tilde{v}_{N_2} \Big\|_{V^2_{K_\pm}} \lesssim \|v\|_{Y^{s_c}_{K_\pm}}. \tag{4.23}$$

Collecting (4.19)–(4.23) and $N_1 \gg 1$, we obtain

$$|I_4| \lesssim \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v\|_{Y^{s_c}_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}.$$

□

The following proposition is in [43], Proposition 10.

**Proposition 4.17.** *($L^4$ Strichartz estimate) For all dyadic numbers $H \geq 1$ and $N$, it holds that*

$$\|W_{\pm c}(t) P_N \phi\|_{L^4_{t,x}} \lesssim N^{(d-1)/4} \|P_N \phi\|_{L^2_x}, \qquad \|K_\pm(t) P_H \varphi\|_{L^4_{t,x}} \lesssim H^{(d-1)/4} \|P_H \varphi\|_{L^2_x}.$$

From Proposition 4.7 and the above proposition, we obtain the following.

**Proposition 4.18.** *For dyadic numbers $H \geq 1$ and $N$, it holds that*

$$\|u_N\|_{L^4_{t,x}} \lesssim N^{(d-1)/4} \|u_N\|_{U^4_{W_{\pm c}}}, \qquad \|v_H\|_{L^4_{t,x}} \lesssim H^{(d-1)/4} \|v_H\|_{U^4_{K_\pm}}.$$

**Proposition 4.19.** *Let $u_M, v_N \in L^2(\mathbb{R}^{1+d})$ be such that*

$$\operatorname{supp} \mathcal{F} u_M \subset W^{\pm c}_{L_1} \cap \big(\mathbb{R} \times (C \cap P_M)\big), \qquad \operatorname{supp} \mathcal{F} v_N \subset KG^{\pm}_{L_2} \cap (\mathbb{R} \times P_N)$$

*for dyadic numbers $L_1, L_2, M, N$ and a cube $C \subset \mathbb{R}^d$ of side length $L$. If $L \ll M \sim N, c > 0$ and $c \neq 1$, it holds that*

$$\|P_L(u_M v_N)\|_{L^2_{t,x}} \lesssim L^{(d-1)/2} (L_1 L_2)^{1/2} \|u_M\|_{L^2_{t,x}} \|v_N\|_{L^2_{t,x}}.$$

*Proof.* Let $f := \mathcal{F} u_M, g := \mathcal{F} v_N$. By the Cauchy-Schwarz inequality, we have

$$\Big\| \int_{|\xi| \sim L} f(\tau_1, \xi_1) g(\tau - \tau_1, \xi - \xi_1) d\tau_1 d\xi_1 \Big\|_{L^2_{\tau,\xi}} \lesssim \sup_{\tau, \xi} |E(\tau, \xi)|^{1/2} \|f\|_{L^2} \|g\|_{L^2}$$

where

$$E(\tau, \xi) = \{(\tau_1, \xi_1) \in \operatorname{supp} f; (\tau - \tau_1, \xi - \xi_1) \in \operatorname{supp} g, |\xi| \sim L\} \subset \mathbb{R}^{1+d}.$$

Put $\underline{l} := \min\{L_1, L_2\}, \bar{l} := \max\{L_1, L_2\}$. By the Fubini theorem,

$$|E(\tau,\xi)| \le \underline{l} \left|\{\xi_1; |\tau \pm c|\xi_1| \pm |\xi-\xi_1|| \lesssim \bar{l}, \xi_1 \in C, |\xi_1| \sim M, |\xi-\xi_1| \sim N, |\xi| \sim L\}\right|.$$

For some $i \in \{1, ..., d\}$, we set $|(\xi-\xi_1)_i| \gtrsim N$, where $(\xi-\xi_1)_i$ denotes the $i$-th component of $\xi - \xi_1$. We compute

$$|\partial_{\xi_{1,i}}(\tau \pm c|\xi_1| \pm (1+|\xi-\xi_1|))| = \left|\frac{(\xi-\xi_1)_i}{|\xi-\xi_1|} \pm c\frac{\xi_{1,i}}{|\xi_1|}\right|, \tag{4.24}$$

where $\xi_{1,i}$ be the $i$-th component of $\xi_1$. Since $|(\xi-\xi_1)_i| \gtrsim N$ and $|\xi| \sim L$, it suffices to consider the case $|\xi_{0,i}| \ll |\xi_{1,i}|$, where $\xi_{0,i}$ be the $i$-th component of $\xi$. Firstly, we consider the case $0 < c \ll 1$. We have

$$r.h.s.\ of\ (4.24) \ge \frac{|(\xi-\xi_1)_i|}{|\xi-\xi_1|} - c\frac{|\xi_{1,i}|}{|\xi_1|} \gtrsim 1-c$$

from $|(\xi-\xi_1)_i| \gtrsim N \sim |\xi-\xi_1|$ and $|\xi_1| \ge |\xi_{1,i}|$. Secondly, we consider the case $c \sim 1, c \ne 1$. The assumption $L \ll N \sim M$ implies

$$\max\{(1-(1-c)^2),\ 1/2\}\frac{|\xi_{1,i}|}{|\xi_1|} \le \frac{|(\xi-\xi_1)_i|}{|\xi-\xi_1|} \le \min\{(1+(1-c)^2),\ 3/2\}\frac{|\xi_{1,i}|}{|\xi_1|}.$$

From the above inequality, we obtain

$$r.h.s.\ of\ (4.24) \gtrsim \left|c\frac{|\xi_{1,i}|}{|\xi_1|} - \frac{|(\xi-\xi_1)_i|}{|\xi-\xi_1|}\right| \gtrsim |c-1|.$$

Finally, we consider the case $c \gg 1$. We have

$$r.h.s.\ of\ (4.24) \gtrsim c\frac{|\xi_{1,i}|}{|\xi_1|} - \frac{|(\xi-\xi_1)_i|}{|\xi-\xi_1|} \gtrsim c-1$$

since $|(\xi-\xi_1)_i| \gtrsim N$ and $|\xi_{0,i}| \ll |\xi_{1,i}|$. Therefore,

$$|\partial_{\xi_{1,i}}(\tau \pm c|\xi_1| \pm (1+|\xi-\xi_1|))| \gtrsim |c-1|. \tag{4.25}$$

Hence by (4.25) and the mean value theorem, we have

$$\begin{aligned}&\left|\{\xi_1; |\tau \pm c|\xi_1| \pm |\xi-\xi_1|| \lesssim \bar{l}, \xi_1 \in C, |\xi_1| \sim M, |\xi-\xi_1| \sim N, |\xi| \sim L\}\right| \\ &\lesssim |c-1|^{-1} m^{d-1}\bar{l}.\end{aligned}$$

From $m \sim L$, we have

$$|E(\xi,\tau)|^{1/2} \lesssim (\underline{l}|c-1|^{-1}m^{d-1}\bar{l})^{1/2} \sim |c-1|^{-1/2}(L_1L_2)^{1/2}L^{(d-1)/2}.$$

Thus, we obtain the result. □

Proposition 4.19 implies the following.

**Proposition 4.20.** *Let $L \ll M \sim N, c > 0$ and $c \neq 1$. For $u_M = W_{\pm c}(t)P_M\phi, v_N = K_{\pm}(t)P_N\varphi$, it holds that*

$$\|P_L(u_M v_N)\|_{L^2_{t,x}} \lesssim L^{(d-1)/2}\|P_M\phi\|_{L^2_x}\|P_N\varphi\|_{L^2_x}.$$

From Proposition 4.7 and the above proposition, we have the following.

**Proposition 4.21.** *Let $L \ll M \sim N, c > 0$ and $c \neq 1$. It holds that*

$$\|P_L(u_M v_N)\|_{L^2_{t,x}} \lesssim L^{(d-1)/2}\|u_M\|_{U^2_{W_{\pm c}}}\|v_N\|_{U^2_{K_{\pm}}}.$$

The following proposion is in [18], Proposition 2.20.

**Proposition 4.22.** *Let $q > 1$, $E$ be a Banach space, $A = K_{\pm}$ or $W_{\pm c}$ and $T : U^q_A \to E$ be a bounded, linear operator with $\|Tu\|_E \leq C_q\|u\|_{U^q_A}$ for all $u \in U^q_A$. In addition, assume that for some $1 \leq p < q$ there exists $C_p \in (0, C_q]$ such that the estimate $\|Tu\|_E \leq C_p\|u\|_{U^p_A}$ holds true for all $u \in U^p_A$. Then, $T$ satisfies the estimate*

$$\|Tu\|_E \leq C_p\big(1 + \ln(C_q/C_p)\big)\|u\|_{V^p_A}, \quad u \in V^p_A.$$

**Proposition 4.23.** *Let $L \ll M \sim N, N \geq 1, c > 0$ and $c \neq 1$. For sufficiently small $\varepsilon > 0$, it holds that*

$$\|P_L(u_M v_N)\|_{L^2_{t,x}} \lesssim L^{(d-1)/2}\,(M/L)^{\varepsilon}\,\|u_M\|_{V^2_{W_{\pm c}}}\|v_N\|_{V^2_{K_{\pm}}}.$$

*Proof.* By the Hölder inequality, $M \sim N, N \geq 1$ and Proposition 4.18, we obtain

$$\|P_L(u_M v_N)\|_{L^2_{t,x}} \lesssim \|u_M\|_{L^4_{t,x}}\|v_N\|_{L^4_{t,x}} \lesssim M^{(d-1)/2}\|u_M\|_{U^4_{W_{\pm c}}}\|v_N\|_{U^4_{K_{\pm}}}. \tag{4.26}$$

Let $Sv := P_L(\tilde{P}_M u\tilde{P}_N v)$, where $\tilde{P}_M = P_{M/2} + P_M + P_{2M}$, such that $\tilde{P}_M P_M = P_M$. $\tilde{P}_N$ is defined by the same manner as $\tilde{P}_M$. From (4.26) and $U^2_{W_{\pm c}} \subset U^4_{W_{\pm c}}$, we have

$$\|S\|_{U^4_{K_{\pm}} \to L^2} \lesssim M^{(d-1)/2}\|u\|_{U^4_{W_{\pm c}}} \lesssim M^{(d-1)/2}\|u\|_{U^2_{W_{\pm c}}}. \tag{4.27}$$

From Proposition 4.21, we have

$$\|S\|_{U^2_{K_{\pm}} \to L^2} \lesssim L^{(d-1)/2}\|u\|_{U^2_{W_{\pm c}}}. \tag{4.28}$$

From (4.27), (4.28) and Proposition 4.22, for sufficiently small $\varepsilon' > 0$, we have

$$\|S\|_{V^2_{K_{\pm}} \to L^2} \lesssim L^{(d-1)/2}\,(M/L)^{\varepsilon'}\,\|u\|_{U^2_{W_{\pm c}}}. \tag{4.29}$$

Let $Tu := P_L(\tilde{P}_M u\tilde{P}_N v)$. From Proposition 4.18, $M \sim N$ and $V^2_{K_{\pm}} \subset U^4_{K_{\pm}}$, we have

$$\|T\|_{U^4_{W_{\pm c}} \to L^2} \lesssim N^{(d-1)/2}\|v_N\|_{U^4_{K_{\pm}}} \lesssim N^{(d-1)/2}\|v_N\|_{V^2_{K_{\pm}}} \lesssim N^{(d-1)/2}\|v\|_{V^2_{K_{\pm}}}. \tag{4.30}$$

By (4.29), we have

$$\|T\|_{U^2_{W_{\pm c}} \to L^2} \lesssim L^{(d-1)/2}\,(M/L)^{\varepsilon'}\,\|v\|_{V^2_{K_{\pm}}}. \tag{4.31}$$

Collecting (4.30), (4.31), $M \sim N$ and Proposition 4.22, we obtain

$$\|T\|_{V^2_{W_{\pm c}} \to L^2} \lesssim L^{(d-1)/2} (M/L)^{2\varepsilon'} \|v\|_{V^2_{K_\pm}}.$$

Taking $\varepsilon = 2\varepsilon'$, the claim follows. □

4.3. **Bilinear estimates.**

**Proposition 4.24.** *Let $d \geq 5, s = s_c = d/2 - 2$ and $c > 0, c \neq 1$. Then for all $0 < T < \infty$, it holds that*

$$\|I_{T,K_\pm}(n,v)\|_{Z^s_{K_\pm}} \lesssim \|n\|_{\dot{Y}^s_{W_{\pm c}}} \|v\|_{Y^s_{K_\pm}}, \tag{4.32}$$

$$\|I_{T,W_{\pm c}}(u,v)\|_{\dot{Z}^s_{W_{\pm c}}} \lesssim \|u\|_{Y^s_{K_\pm}} \|v\|_{Y^s_{K_\pm}}. \tag{4.33}$$

*Remark* 4.4. In (4.32) and (4.33), the implicit constant does not depend on $T$.

*Proof.* We denote $\tilde{u}_{N_1} := \chi_{[0,T)} P_{N_1} u, \tilde{v}_{N_2} := \chi_{[0,T)} P_{N_2} v, \tilde{n}_{N_3} := \chi_{[0,T)} P_{N_3} n$. To prove (4.32), we need to estimate the following.

$$\|I_{T,K_\pm}(n,v)\|^2_{Z^{s_c}_{K_\pm}} \lesssim \sum_{i=0}^{3} J_i$$

where

$$J_0 := \left\| \int_0^t \chi_{[0,T)}(t') K_\pm(t-t') P_{<1}(\tilde{n}(\omega_1^{-1}\tilde{v}))(t') dt' \right\|^2_{U^2_{K_\pm}},$$

$$J_1 := \sum_{N_1 \geq 1} N_1^{2s_c} \left\| \int_0^t \chi_{[0,T)}(t') K_\pm(t-t') \sum_{N_2 \sim N_1} \sum_{N_3 \lesssim N_2} P_{N_1}(\tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2}))(t') dt' \right\|^2_{U^2_{K_\pm}},$$

$$J_2 := \sum_{N_1 \geq 1} N_1^{2s_c} \left\| \int_0^t \chi_{[0,T)}(t') K_\pm(t-t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1}(\tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2}))(t') dt' \right\|^2_{U^2_{K_\pm}},$$

$$J_3 := \sum_{N_1 \geq 1} N_1^{2s_c} \left\| \int_0^t \chi_{[0,T)}(t') K_\pm(t-t') \sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} P_{N_1}(\tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2}))(t') dt' \right\|^2_{U^2_{K_\pm}}.$$

By Corollary 4.6 and Lemma 4.16 $(ii)$, we have

$$\begin{aligned} J_0^{1/2} &\lesssim \sup_{\|u\|_{V^2_{K_\pm}}=1} \left| \int_{\mathbb{R}^{1+d}} \tilde{n}(\omega_1^{-1}\tilde{v})(\overline{P_{<1}\tilde{u}}) dx dt \right| \\ &\lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|_{Y^{s_c}_{K_\pm}}. \end{aligned} \tag{4.34}$$

We apply Corollary 4.6, $N_1 \sim N_2$, Lemma 4.16 $(iii)$ and $\|\tilde{u}_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, then

$$\begin{aligned} J_1 &\lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \sup_{\|u\|_{V^2_{K_\pm}}=1} \Big| \sum_{N_2 \sim N_1} \sum_{N_3 \lesssim N_2} \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \overline{\tilde{u}_{N_1}} dxdt \Big|^2 \\ &\lesssim \sum_{N_2 \gtrsim 1} N_2^{2s_c} \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v_{N_2}\|^2_{V^2_{K_\pm}} \\ &\lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|^2_{Y^{s_c}_{K_\pm}}. \end{aligned} \tag{4.35}$$

For the estimate of $J_2$, we take $M = \varepsilon N_1$ for sufficiently small $\varepsilon > 0$. Then, from Lemma 4.15, we have

$$\begin{aligned} &P_{N_1} Q^{K_\pm}_{<M} \big( (Q^{W_{\pm c}}_{<M} \tilde{n}_{N_3})(Q^{K_\pm}_{<M} \omega_1^{-1} \tilde{v}_{N_2}) \big) \\ &= P_{N_1} Q^{K_\pm}_{<M} \Big[ \mathcal{F}^{-1} \Big( \int_{\tau_1 = \tau_2 + \tau_3,\, \xi_1 = \xi_2 + \xi_3} (\widehat{Q^{W_{\pm c}}_{<M} \tilde{n}_{N_3}})(\tau_3, \xi_3) (\widehat{Q^{K_\pm}_{<M} \omega_1^{-1}} \tilde{v}_{N_2})(\tau_2, \xi_2) \Big) \Big] = 0 \end{aligned}$$

when $N_1 \gg \langle N_2 \rangle$. Therefore,

$$P_{N_1} \big( \tilde{n}_{N_3} (\omega_1^{-1} \tilde{v}_{N_2}) \big) = \sum_{i=1}^{3} P_{N_1} F_i,$$

where

$$\begin{aligned} &F_1 := Q_1 \big( (Q^{W_{\pm c}}_{\geq M} \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2}) \big), \qquad F_2 := Q_1 \big( (Q_3 \tilde{n}_{N_3})(Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2}) \big), \\ &F_3 := Q^{K_\pm}_{\geq M} \big( (Q_3 \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2}) \big). \end{aligned}$$

Here, $Q_1, Q_2 \in \{Q^{K_\pm}_{<M}, Q^{K_\pm}_{\geq M}\}$ and $Q_3 \in \{Q^{W_{\pm c}}_{<M}, Q^{W_{\pm c}}_{\geq M}\}$. For the estimate of $F_1$, we apply Corollary 4.6, Lemma 4.13, Lemma 4.16 $(iv)$, $N_3 \sim N_1 \geq 1$ and $\|\tilde{u}_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, then we have

$$\begin{aligned} &\sum_{N_1 \geq 1} N_1^{2s_c} \Big\| \int_0^t \chi_{[0,T)}(t') K_\pm(t-t') \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} P_{N_1} F_1(t') dt' \Big\|^2_{U^2_{K_\pm}} \\ &\lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \sup_{\|u\|_{V^2_{K_\pm}}=1} \Big| \sum_{N_2 \ll N_1} \sum_{N_3 \sim N_1} \int_{\mathbb{R}^{1+d}} (Q^{W_{\pm c}}_{\geq M} \tilde{n}_{N_3})(Q_2 \omega_1^{-1} \tilde{v}_{N_2})(\overline{Q_1 \tilde{u}_{N_1}}) dxdt \Big|^2 \\ &\lesssim \sum_{N_3 \gtrsim 1} N_3^{2s_c} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v\|^2_{Y^{s_c}_{K_\pm}} \\ &\lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|^2_{Y^{s_c}_{K_\pm}}. \end{aligned} \tag{4.36}$$

For the estimate of $F_2$, we apply Corollary 4.6, Lemma 4.13 and the triangle inequality, we have

$$\begin{aligned}
&\sum_{N_1\geq 1} N_1^{2s_c}\Big\|\int_0^t \chi_{[0,T)}(t')K_\pm(t-t')\sum_{N_2\ll N_1}\sum_{N_3\sim N_1} P_{N_1}F_2(t')dt'\Big\|^2_{U^2_{K_\pm}}\\
&\lesssim \sum_{N_1\geq 1} N_1^{2s_c}\sup_{\|u\|_{V^2_{K_\pm}}=1}\Big|\sum_{N_2\ll N_1}\sum_{N_3\sim N_1}\int_{\mathbb{R}^{1+d}}(Q_3\tilde{n}_{N_3})(Q^{K_\pm}_{\geq M}\omega_1^{-1}\tilde{v}_{N_2})(\overline{Q_1\tilde{u}_{N_1}})dxdt\Big|^2\\
&\lesssim \sum_{N_1\geq 1} N_1^{2s_c}\sup_{\|u\|_{V^2_{K_\pm}}=1}\sum_{N_2\ll N_1}\sum_{N_3\sim N_1}\Big|\int_{\mathbb{R}^{1+d}}(Q_3\tilde{n}_{N_3})(Q^{K_\pm}_{\geq M}\omega_1^{-1}\tilde{v}_{N_2})(\overline{Q_1\tilde{u}_{N_1}})dxdt\Big|^2.
\end{aligned}\tag{4.37}$$

By Proposition 4.23, $N_2 \ll N_1 \sim N_3, N_1 \geq 1$ and Proposition 4.14, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^{1+d}}(Q_3\tilde{n}_{N_3})(Q^{K_\pm}_{\geq M}\omega_1^{-1}\tilde{v}_{N_2})(\overline{Q_1\tilde{u}_{N_1}})dxdt\Big|\\
&\lesssim \|Q^{K_\pm}_{\geq M}\omega_1^{-1}\tilde{v}_{N_2}\|_{L^2_{t,x}}\|P_{N_2}\big((Q_3\tilde{n}_{N_3})(\overline{Q_1\tilde{u}_{N_1}})\big)\|_{L^2_{t,x}}\\
&\lesssim N_3^{-1/2}\langle N_2\rangle^{-1}\|v_{N_2}\|_{V^2_{K_\pm}}N_2^{(d-1)/2}(N_3/N_2)^{\varepsilon}\|n_{N_3}\|_{V^2_{\pm c}}\|u_{N_1}\|_{V^2_{K_\pm}}\\
&\lesssim N_2^{s_c}(N_2/N_3)^{1/2-\varepsilon}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{\pm c}}\|u_{N_1}\|_{V^2_{K_\pm}}.
\end{aligned}\tag{4.38}$$

By (4.38) and the Cauchy-Schwarz inequality, the right-hand side of (4.37) is bounded by

$$\begin{aligned}
&\sum_{N_3\ \gtrsim\ 1} N_3^{2s_c}\|n_{N_3}\|^2_{V^2_{W_{\pm c}}}\Big(\sum_{N_2\ll N_3}\big(N_2/N_3\big)^{1/2-\varepsilon}N_2^{s_c}\|v_{N_2}\|_{V^2_{K_\pm}}\Big)^2\\
&\lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}}\|v\|^2_{Y^{s_c}_{K_\pm}}.
\end{aligned}\tag{4.39}$$

For the estimate for $F_3$, we apply Corollary 4.6, Lemma 4.13, Lemma 4.16 $(iv)$, $N_3 \sim N_1 \geq 1$ and $\|\tilde{u}_{N_1}\|_{V^2_{K_\pm}} \lesssim \|u\|_{V^2_{K_\pm}}$, then we obtain

$$\begin{aligned}
&\sum_{N_1\geq 1} N_1^{2s_c}\Big\|\int_0^t \chi_{[0,T)}(t')K_\pm(t-t')\sum_{N_2\ll N_1}\sum_{N_3\sim N_1} P_{N_1}F_3(t')dt'\Big\|^2_{U^2_{K_\pm}}\\
&\lesssim \sum_{N_1\geq 1} N_1^{2s_c}\sup_{\|u\|_{V^2_{K_\pm}}=1}\Big|\sum_{N_2\ll N_1}\sum_{N_3\sim N_1}\int_{\mathbb{R}^{1+d}}(Q_3\tilde{n}_{N_3})(Q_2\omega_1^{-1}\tilde{v}_{N_2})(\overline{Q^{K_\pm}_{\geq M}\tilde{u}_{N_1}})dxdt\Big|^2\\
&\lesssim \sum_{N_3\ \gtrsim\ 1} N_3^{2s_c}\|n_{N_3}\|^2_{V^2_{W_{\pm c}}}\|v\|^2_{Y^{s_c}_{K_\pm}}\\
&\lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}}\|v\|^2_{Y^{s_c}_{K_\pm}}.
\end{aligned}\tag{4.40}$$

Collecting (4.36), (4.39) and (4.40), we have

$$J_2 \lesssim \|n\|^2_{\dot{Y}^{s_c}_{W_{\pm c}}}\|v\|^2_{Y^{s_c}_{K_\pm}}.\tag{4.41}$$

By Corollary 4.6 and the triangle inequality to have

$$
\begin{aligned}
J_3 &\lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \sup_{\|u\|_{V^2_{K_\pm}}=1} \Big| \sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2})\overline{\tilde{u}_{N_1}} dxdt \Big|^2 \\
&\lesssim \sum_{N_1 \geq 1} N_1^{2s_c} \Big( \sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} \sup_{\|u\|_{V^2_{K_\pm}}=1} \Big| \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2})\overline{\tilde{u}_{N_1}} dxdt \Big| \Big)^2. \qquad (4.42)
\end{aligned}
$$

By the same manner as the estimate for Lemma 4.16 $(iii)$, we obtain

$$
\Big| \int_{\mathbb{R}^{1+d}} \tilde{n}_{N_3}(\omega_1^{-1}\tilde{v}_{N_2})\overline{\tilde{u}_{N_1}} dxdt \Big| \lesssim N_3^{s_c} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \|u_{N_1}\|_{V^2_{K_\pm}}. \qquad (4.43)
$$

From (4.43), the right-hand side of (4.42) is bounded by

$$
\sum_{N_1 \geq 1} \Big( \sum_{N_2 \gg N_1} \sum_{N_3 \sim N_2} N_1^{s_c} N_3^{s_c} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \Big)^2.
$$

From $s_c > 0, \|\cdot\|_{l^2 l^1} \lesssim \|\cdot\|_{l^1 l^2}$ and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
J_3^{1/2} &\lesssim \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} \Big( \sum_{N_1 \ll N_2} N_1^{2s_c} N_3^{2s_c} \|n_{N_3}\|^2_{V^2_{W_{\pm c}}} \|v_{N_2}\|^2_{V^2_{K_\pm}} \Big)^{1/2} \\
&\lesssim \sum_{N_2 \gtrsim 1} \sum_{N_3 \sim N_2} N_2^{s_c} N_3^{s_c} \|n_{N_3}\|_{V^2_{W_{\pm c}}} \|v_{N_2}\|_{V^2_{K_\pm}} \\
&\lesssim \|n\|_{\dot{Y}^{s_c}_{W_{\pm c}}} \|v\|_{Y^{s_c}_{K_\pm}}. \qquad (4.44)
\end{aligned}
$$

Collecting (4.34), (4.35), (4.41) and (4.44), we obtain (4.32). We prove (4.33) below. By Corollary 4.6, we only need to estimate $K_i$ $(i = 1, 2, 3)$:

$$
\begin{aligned}
K_1 &:= \sum_{N_3} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}}) dxdt \Big|^2, \\
K_2 &:= \sum_{N_3} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \ll N_3} \sum_{N_1 \sim N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}}) dxdt \Big|^2, \\
K_3 &:= \sum_{N_3} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \gtrsim N_3} \sum_{N_1 \sim N_2} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}}) dxdt \Big|^2.
\end{aligned}
$$

First, we estimate $K_1$. Put $K_1 = K_{1,1} + K_{1,2}$ where

$$
\begin{aligned}
K_{1,1} &:= \sum_{N_3 \lesssim 1} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}}) \\
&\qquad\qquad \times (\overline{\omega\tilde{n}_{N_3}}) dxdt \Big|^2, \qquad (4.45) \\
K_{1,2} &:= \sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (\omega_1^{-1}\tilde{u}_{N_1})(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}}) dxdt \Big|^2.
\end{aligned}
$$

By the same manner as the proof for Lemma (4.16) ($i$), we see

$$\Big|\int_{\mathbb{R}^{1+d}}\Big(\sum_{N_1\ll N_3}\omega_1^{-1}\tilde{u}_{N_1}\Big)(\overline{\omega_1^{-1}\tilde{v}_{N_2}})(\overline{\omega\tilde{n}_{N_3}})dxdt\Big| \lesssim \langle N_2\rangle^{-1/2}\langle N_3\rangle^{3/2}\|u\|_{Y^{s_c}_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}. \tag{4.46}$$

Collecting (4.45), (4.46) and $N_2\sim N_3\lesssim 1$, we obtain

$$\begin{aligned}K_{1,1} &\lesssim \sum_{N_2\lesssim 1} N_2^{2s_c}(\|u\|_{Y^{s_c}_{K_\pm}}\langle N_2\rangle^{-1/2+3/2}\|v_{N_2}\|_{V^2_{K_\pm}})^2\\ &\lesssim \|u\|^2_{Y^{s_c}_{K_\pm}}\sum_{N_2\lesssim 1}N_2^{2s_c}\|v_{N_2}\|^2_{V^2_{K_\pm}}\\ &\lesssim \|u\|^2_{Y^{s_c}_{K_\pm}}\|v\|^2_{Y^{s_c}_{K_\pm}}.\end{aligned}$$

For the estimate for $K_{1,2}$, we take $M=\varepsilon N_2$ for sufficiently small $\varepsilon>0$. Then, from Lemma 4.15, we have

$$\begin{aligned}&P_{N_1}Q^{K_\pm}_{<M}\omega_1^{-1}\big((Q^{K_\pm}_{<M}\omega_1^{-1}\tilde{v}_{N_2})(Q^{W_{\pm c}}_{<M}\omega\,\tilde{n}_{N_3})\big)\\ &=P_{N_1}Q^{K_\pm}_{<M}\omega_1^{-1}\Big[\mathcal{F}^{-1}\Big(\int_{\tau_1=\tau_2+\tau_3,\,\xi_1=\xi_2+\xi_3}(\widehat{Q^{K_\pm}_{<M}\omega_1^{-1}}\tilde{v}_{N_2})(\tau_2,\xi_2)(\widehat{Q^{W_{\pm c}}_{<M}\omega}\,\tilde{n}_{N_3})(\tau_3,\xi_3)\Big)\Big]\\ &=0\end{aligned}$$

when $N_2\gg\langle N_1\rangle$. Therefore,

$$P_{N_1}\big((\omega_1^{-1}\tilde{v}_{N_2})(\omega\,\tilde{n}_{N_3})\big)=\sum_{i=1}^{3}P_{N_1}G_i,$$

where

$$\begin{aligned}&G_1:=Q^{K_\pm}_{\ge M}\big((Q_2\omega_1^{-1}\tilde{v}_{N_2})(Q_3\omega\,\tilde{n}_{N_3})\big),\qquad G_2:=Q_1\big((Q^{K_\pm}_{\ge M}\omega_1^{-1}\tilde{v}_{N_2})(Q_3\omega\,\tilde{n}_{N_3})\big),\\ &G_3:=Q_1\big((Q_2\omega_1^{-1}\tilde{v}_{N_2})(Q^{W_{\pm c}}_{\ge M}\omega\,\tilde{n}_{N_3})\big).\end{aligned}$$

Here, $Q_1,Q_2\in\{Q^{K_\pm}_{<M},Q^{K_\pm}_{\ge M}\}$ and $Q_3\in\{Q^{W_{\pm c}}_{<M},Q^{W_{\pm c}}_{\ge M}\}$. Hence, it follows that

$$K_{1,2}\le\sum_{i=1}^{3}K_{1,2,i}$$

where

$$K_{1,2,i}:=\sum_{N_3\gg 1}N_3^{2s_c}\sup_{\|n\|_{V^2_{W_{\pm c}}}=1}\Big|\sum_{N_2\sim N_3}\sum_{N_1\ll N_3}\int_{\mathbb{R}^{1+d}}(\omega_1^{-1}\tilde{u}_{N_1})\overline{G_i}dxdt\Big|^2,\qquad i=1,2,3.$$

By Lemma 4.13, we have

$$K_{1,2,1} \lesssim \sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) \times (\overline{Q_3 \omega \tilde{n}_{N_3}}) dxdt \Big|^2, \tag{4.47}$$

$$K_{1,2,2} \lesssim \sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2}}) \times (\overline{Q_3 \omega \tilde{n}_{N_3}}) dxdt \Big|^2, \tag{4.48}$$

$$K_{1,2,3} \lesssim \sum_{N_3 \gg 1} N_3^{2s_c} \sup_{\|n\|_{V^2_{W_{\pm c}}}=1} \Big| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_3} \int_{\mathbb{R}^{1+d}} (Q_1 \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) \times (\overline{Q^{W_{\pm c}}_{\geq M} \omega \tilde{n}_{N_3}}) dxdt \Big|^2. \tag{4.49}$$

By the same manner as the estimate for $F_2$, we apply Proposition 4.23, $N_1 \ll N_2 \sim N_3, N_3 \gg 1$ and Proposition 4.14, then we obtain

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^{1+d}} (Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{u}_{N_1}) (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q_3 \omega \tilde{n}_{N_3}}) dxdt \Big| \\
&\lesssim \|Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{u}_{N_1}\|_{L^2_{t,x}} \|P_{N_1}\big( (\overline{Q_2 \omega_1^{-1} \tilde{v}_{N_2}})(\overline{Q_3 \omega \tilde{n}_{N_3}}) \big)\|_{L^2_{t,x}} \\
&\lesssim N_3^{-1/2} \langle N_1 \rangle^{-1} \|u_{N_1}\|_{V^2_{K_\pm}} N_1^{(d-1)/2} (N_3/N_1)^{\varepsilon} \langle N_2 \rangle^{-1} \|v_{N_2}\|_{V^2_{K_\pm}} N_3 \|n_{N_3}\|_{V^2_{W_{\pm c}}} \\
&\lesssim N_1^{s_c} (N_1/N_3)^{1/2-\varepsilon} \langle N_2 \rangle^{-1} N_3 \|u_{N_1}\|_{V^2_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}} \|n_{N_3}\|_{V^2_{W_{\pm c}}}.
\end{aligned} \tag{4.50}$$

From (4.47), (4.50), $N_3 \gg 1, N_2 \sim N_3$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
K_{1,2,1} &\lesssim \sum_{N_2 \gg 1} N_2^{2s_c} \Big( \sum_{N_1 \ll N_2} N_1^{s_c} \|u_{N_1}\|_{V^2_{K_\pm}} (N_1/N_2)^{1/2-\varepsilon} \langle N_2 \rangle^{-1} N_2 \|v_{N_2}\|_{V^2_{K_\pm}} \Big)^2 \\
&\lesssim \|u\|^2_{Y^{s_c}_{K_\pm}} \|v\|^2_{Y^{s_c}_{K_\pm}}.
\end{aligned}$$

By Lemma 4.16 $(iv), i = 5$, we obtain

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^{1+d}} \Big( \sum_{N_1 \ll N_3} Q_1 \omega_1^{-1} \tilde{u}_{N_1} \Big) (\overline{Q^{K_\pm}_{\geq M} \omega_1^{-1} \tilde{v}_{N_2}}) (\overline{Q_3 \omega \tilde{n}_{N_3}}) dxdt \Big| \\
&\lesssim \langle N_2 \rangle^{-1} N_3 \|u\|_{Y^{s_c}_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}} \|n_{N_3}\|_{V^2_{W_{\pm c}}}.
\end{aligned} \tag{4.51}$$

From (4.48), (4.51), $N_3 \gg 1$ and $N_2 \sim N_3$, we have

$$K_{1,2,2} \lesssim \sum_{N_2 \gg 1} N_2^{2s_c} (\|u\|_{Y^{s_c}_{K_\pm}} \|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim \|u\|^2_{Y^{s_c}_{K_\pm}} \|v\|^2_{Y^{s_c}_{K_\pm}}.$$

By Lemma 4.16 $(iv), i = 4$, we obtain

$$\begin{aligned} &\Big|\int_{\mathbb{R}^{1+d}}\Big(\sum_{N_1\ll N_3} Q_1\omega_1^{-1}\tilde{u}_{N_1}\Big)\overline{(Q_2\omega_1^{-1}\tilde{v}_{N_2})}\overline{(Q_{\geq M}^{W_{\pm c}}\omega\tilde{n}_{N_3})}dxdt\Big| \\ &\lesssim \langle N_2\rangle^{-1}N_3\|u\|_{Y^{s_c}_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}}\|n_{N_3}\|_{V^2_{W_{\pm c}}}. \end{aligned} \tag{4.52}$$

From (4.49), (4.52), $N_3 \gg 1$ and $N_2 \sim N_3$, we have

$$K_{1,2,3} \lesssim \sum_{N_2\gg 1} N_2^{2s_c}(\|u\|_{Y^{s_c}_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}})^2 \lesssim \|u\|^2_{Y^{s_c}_{K_\pm}}\|v\|^2_{Y^{s_c}_{K_\pm}}.$$

By symmetry, the estimate for $K_2$ is obtained by the same manner as the estimate for $K_1$. Hence, we omit the estimate for $K_2$. By the triangle inequality, Lemma 4.16 $(i)$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} K_3^{1/2} &\lesssim \sum_{N_2}\sum_{N_1\sim N_2}\Big\{\sum_{N_3\lesssim N_2} N_3^{2s_c}\sup_{\|n\|_{V^2_{W_{\pm c}}}=1}\Big|\int_{\mathbb{R}^{1+d}}(\omega_1^{-1}\tilde{u}_{N_1})\overline{(\omega_1^{-1}\tilde{v}_{N_2})}\overline{(\omega\tilde{n}_{N_3})}dxdt\Big|^2\Big\}^{1/2} \\ &\lesssim \sum_{N_2}\sum_{N_1\sim N_2}\Big\{\sum_{N_3\lesssim N_2} N_3^{2s_c}(N_3^{s_c}\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}})^2\Big\}^{1/2} \\ &\lesssim \sum_{N_2}\sum_{N_1\sim N_2} N_1^{s_c}N_2^{s_c}\|u_{N_1}\|_{V^2_{K_\pm}}\|v_{N_2}\|_{V^2_{K_\pm}} \\ &\lesssim \|u\|_{Y^{s_c}_{K_\pm}}\|v\|_{Y^{s_c}_{K_\pm}}. \end{aligned}$$

Therefore, we obtain (4.33). □

4.4. **The proof of the main theorem.** We define

$$u_\pm := \omega_1 u \pm i\partial_t u, \quad n_\pm := n \pm i(c\omega)^{-1}\partial_t n$$

where $\omega_1 := (1-\Delta)^{1/2}, \omega := (-\Delta)^{1/2}$. Then the wave equation in (4.1) is rewritten into

$$\begin{cases} i\partial_t u_\pm \mp \omega_1 u_\pm = \pm(1/4)(n_+ + n_-)(\omega_1^{-1}u_+ + \omega_1^{-1}u_-), & (t,x)\in[-T,T]\times\mathbb{R}^d, \\ i\partial_t n_\pm \mp c\omega n_\pm = \pm(4c)^{-1}\omega|\omega_1^{-1}u_+ + \omega_1^{-1}u_-|^2, & (t,x)\in[-T,T]\times\mathbb{R}^d, \\ (u_\pm, n_\pm)|_{t=0} = (u_{\pm 0}, n_{\pm 0}) \in H^s(\mathbb{R}^d)\times \dot{H}^s(\mathbb{R}^d). \end{cases} \tag{4.53}$$

Hence by the Duhamel principle, we consider the following integral equation corresponding to (4.53) on the time interval $[0,T)$ with $0 < T \leq \infty$ :

$$u_\pm = \Phi_1(u_\pm, n_+, n_-), \quad n_\pm = \Phi_2(n_\pm, u_+, u_-), \tag{4.54}$$

where

$$\Phi_1(u_\pm, n_+, n_-) := K_\pm(t)u_{\pm 0} \pm (1/4)\{I_{T,K_\pm}(n_+, u_+)(t) + I_{T,K_\pm}(n_+, u_-)(t) + I_{T,K_\pm}(n_-, u_+)(t) + I_{T,K_\pm}(n_-, u_-)(t)\},$$

$$\Phi_2(n_\pm, u_+, u_-) := W_{\pm c}(t)n_{\pm 0} \pm (4c)^{-1}\{I_{T,W_{\pm c}}(u_+, u_+)(t) + I_{T,W_{\pm c}}(u_+, u_-)(t) + I_{T,W_{\pm c}}(u_-, u_+)(t) + I_{T,W_{\pm c}}(u_-, u_-)(t)\}.$$

**Proposition 4.25.** *(i) Let* $d \geq 5, s = s_c = d/2 - 2$ *and* $\delta > 0$ *be sufficiently small. For all* $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$ *and for all* $0 < T < \infty$*, there exists a unique solution of* (4.54) *on* $[0, T]$ *such that*

$$(u_\pm, n_\pm) \in Y^s_{K_\pm}([0,T]) \times \dot{Y}^s_{W_{\pm c}}([0,T]) \subset C([0,T]; H^s(\mathbb{R}^d)) \times C([0,T]; \dot{H}^s(\mathbb{R}^d)).$$

*(ii) The flow map obtained by (i):*
$B_\delta(H^s(\mathbb{R}^d)) \times B_\delta(\dot{H}^s(\mathbb{R}^d)) \ni (u_{\pm 0}, n_{\pm 0}) \mapsto (u_\pm, n_\pm) \in Y^s_{K_\pm}([0,T]) \times \dot{Y}^s_{W_{\pm c}}([0,T])$ *is Lipschitz continuous.*

*Remark* 4.5. Due to the time reversibility of the Klein-Gordon-Zakharov equation, Porpositions 4.25 also holds in corresponding time interval $[-T, 0]$

*Remark* 4.6. By $(i)$ in Proposition 4.25 and Remark 4.5, for any $T > 0$, we have solutions to (4.54) $(u_\pm(t), n_\pm(t))$ on $[0, T]$ and $[-T, 0]$. If initial data $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$, then we can take $T$ arbitrary large and by uniqueness, $(u_\pm(t), n_\pm(t)) \in C((-\infty, \infty); H^s(\mathbb{R}^d)) \times C((-\infty, \infty); \dot{H}^s(\mathbb{R}^d))$ can be defined uniquely.

**Proposition 4.26.** *Let the solution* $(u_\pm(t), n_\pm(t))$ *to* (4.54) *on* $(-\infty, \infty)$ *obtained by Proposirion 4.25, Remark 4.5 and Remark 4.6 with initial data* $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$*. Then, there exist* $(u_{\pm,+\infty}, n_{\pm,+\infty})$ *and* $(u_{\pm,-\infty}, n_{\pm,-\infty})$ *in* $H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d)$ *such that*

$$\lim_{t \to +\infty} (\|u_\pm(t) - K_\pm(t)u_{\pm,+\infty}\|_{H^s_x(\mathbb{R}^d)} + \|n_\pm(t) - W_{\pm c}(t)n_{\pm,+\infty}\|_{\dot{H}^s_x(\mathbb{R}^d)}) = 0,$$

$$\lim_{t \to -\infty} (\|u_\pm(t) - K_\pm(t)u_{\pm,-\infty}\|_{H^s_x(\mathbb{R}^d)} + \|n_\pm(t) - W_{\pm c}(t)n_{\pm,-\infty}\|_{\dot{H}^s_x(\mathbb{R}^d)}) = 0.$$

*proof of Proposition 4.25.* First, we prove $(i)$. By Proposition 4.10, there exists $C > 0$ such that

$$\|K_\pm(t)u_{\pm 0}\|_{Y^s_{K_\pm}} \leq C\|u_{\pm 0}\|_{H^s}, \qquad \|W_{\pm c}(t)n_{\pm 0}\|_{\dot{Y}^s_{W_{\pm c}}} \leq C\|n_{\pm 0}\|_{\dot{H}^s}.$$

We denote time interval $I := [0, T]$. If $(u_{\pm 0}, n_{\pm 0}) \in B_\delta(H^s(\mathbb{R}^d) \times \dot{H}^s(\mathbb{R}^d))$ is small and $(u_\pm, n_\pm) \in B_r(Y^s_{K_\pm}(I) \times \dot{Y}^s_{W_{\pm c}}(I)), s = d/2 - 2$, then by Proposition 4.24 and

Remark 4.4, we have

$$\begin{aligned}
&\|\Phi_1(u_\pm, n_+, n_-)\|_{Y^s_{K_\pm}(I)} \\
&\le C\delta + (C/4)(\|n_+\|_{\dot{Y}^s_{W_{+c}}(I)}\|u_+\|_{Y^s_{K_+}(I)} + \|n_+\|_{\dot{Y}^s_{W_{+c}}(I)}\|u_-\|_{Y^s_{K_-}(I)} \\
&\qquad + \|n_-\|_{\dot{Y}^s_{W_{-c}}(I)}\|u_+\|_{Y^s_{K_+}(I)} + \|n_-\|_{\dot{Y}^s_{W_{-c}}(I)}\|u_-\|_{Y^s_{K_-}(I)}), \\
&\|\Phi_2(n_\pm, u_+, u_-)\|_{\dot{Y}^s_{W_{\pm c}}(I)} \\
&\le C\delta + (C/4c)(\|u_+\|^2_{Y^s_{K_+}(I)} + 2\|u_+\|_{Y^s_{K_+}(I)}\|u_-\|_{Y^s_{K_-}(I)} + \|u_-\|^2_{Y^s_{K_-}(I)}).
\end{aligned}$$

Taking $\delta = r^2$ and $r = \min\{1, c\}/(4C)$, then we have

$$\|\Phi_1(u_\pm, n_+, n_-)\|_{Y^s_{K_\pm}(I)} \le r, \qquad \|\Phi_2(n_\pm, u_+, u_-)\|_{\dot{Y}^s_{W_{\pm c}}(I)} \le r.$$

Hence, $(\Phi_1, \Phi_2)$ is a map from $B_r(Y^s_{K_\pm}([0,T]) \times \dot{Y}^s_{W_{\pm c}}([0,T]))$ into itself. If we also assume $(v_\pm, m_\pm) \in B_r(Y^s_{K_\pm}(I) \times \dot{Y}^s_{W_{\pm c}}(I))$, then we have

$$\begin{aligned}
&\|\Phi_1(u_\pm, n_+, n_-) - \Phi_1(v_\pm, m_+, m_-)\|_{Y^s_{K_\pm}(I)} \\
&\le (1/8)(\|u_+ - v_+\|_{Y^s_{K_+}(I)} + \|u_- - v_-\|_{Y^s_{K_-}(I)} \\
&\qquad + \|n_+ - m_+\|_{\dot{Y}^s_{W_{+c}}(I)} + \|n_- - m_-\|_{\dot{Y}^s_{W_{-c}}(I)}), \qquad (4.55) \\
&\|\Phi_2(n_\pm, u_+, u-) - \Phi_2(m_\pm, v_+, v_-)\|_{\dot{Y}^s_{W_{\pm c}}(I)} \\
&\le (1/4)(\|u_+ - v_+\|_{Y^s_{K_+}(I)} + \|u_- - v_-\|_{Y^s_{K_-}(I)}). \qquad (4.56)
\end{aligned}$$

Thus, $(\Phi_1, \Phi_2)$ is a contraction mapping on $B_r(Y^s_{K_\pm}([0,T]) \times \dot{Y}^s_{W_{\pm c}}([0,T]))$. Hence, by the Banach fixed point theorem, we have a solution to (4.54) in it. We assume that $(u_\pm(0), n_\pm(0)), (v_\pm(0), m_\pm(0))$ are both small and $s = d/2 - 2$ for $d \ge 5$. Let $(u_\pm, n_\pm), (v_\pm, m_\pm) \in Y^s_{K_\pm}([0,T]) \times \dot{Y}^s_{W_{\pm c}}([0,T])$ are two solutions satisfying $(u_\pm(0), n_\pm(0)) = (v_\pm(0), m_\pm(0))$. Moreover,

$$T' := \sup\{0 \le t \le T\,; u_\pm(t) = v_\pm(t), n_\pm(t) = m_\pm(t)\} < T.$$

By a translation in $t$, it suffices to consider $T' = 0$. Let $0 < \tau \le T$ be fixed later. From (4.55)–(4.56) and Proposition 4.11, we obtain

$$\begin{aligned}
&\|u_\pm - v_\pm\|_{Y^s_{K_\pm}([0,\tau])} \\
&\le (1/7)(\|n_+ - m_+\|_{\dot{Y}^s_{W_{+c}}([0,\tau])} + \|n_- - m_-\|_{\dot{Y}^s_{W_{-c}}([0,\tau])} + \|u_\mp - v_\mp\|_{Y^s_{K_\mp}([0,\tau])}), \qquad (4.57)
\end{aligned}$$

$$\|n_\pm - m_\pm\|_{\dot{Y}^s_{W_{\pm c}}([0,\tau])} \le (1/4)(\|u_+ - v_+\|_{Y^s_{K_+}([0,\tau])} + \|u_- - v_-\|_{Y^s_{K_-}([0,\tau])}). \qquad (4.58)$$

From (4.57) and (4.58), we obtain

$$u_\pm = v_\pm, \quad n_\pm = m_\pm$$

on $[0, \tau]$ if $0 < \tau \le T$ be sufficiently small. This contradicts the definition of $T'$. Therefore, the uniqueness of the solution $(u_\pm, n_\pm)$ is showed. $(ii)$ follows from the standard argument, so we omit the proof. □

Finally, we prove Proposition 4.26. The proof is the same manner as the proof for Proposition 4.2 in [26].

*Proof.* There exists $M > 0$ such that for all $0 < T < \infty$,

$$\|u_\pm\|_{Y^s_{K_\pm}([0,T])} + \|n_\pm\|_{\dot{Y}^s_{W_{\pm c}}([0,T])} < M,$$
$$\|u_\pm\|_{Y^s_{K_\pm}([-T,0])} + \|n_\pm\|_{\dot{Y}^s_{W_{\pm c}}([-T,0])} < M$$

holds since $r$ in the proof of Proposition 4.25 does not depend on $T$. Take $\{t_k\}_{k=0}^K \in \mathcal{Z}_0$ and $0 < T < \infty$ such that $-T < t_0, t_K < T$. By $L^2_x$ orthogonality,

$$\begin{aligned}
&\Big(\sum_{k=1}^{K} \|\langle \nabla_x \rangle^s \big(K_\pm(-t_k)u_\pm(t_k) - K_\pm(-t_{k-1})u_\pm(t_{k-1})\big)\|_{L^2_x}^2\Big)^{1/2} \\
&\lesssim \|\langle \nabla_x \rangle^s u_\pm\|_{V^2_{K_\pm}([0,T])} + \|\langle \nabla_x \rangle^s u_\pm\|_{V^2_{K_\pm}([-T,0])} \\
&\lesssim \|u_\pm\|_{Y^s_{K_\pm}([0,T])} + \|u_\pm\|_{Y^s_{K_\pm}([-T,0])} \\
&< 2M.
\end{aligned}$$

Thus,

$$\sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}_0} \Big(\sum_{k=1}^{K} \|\langle \nabla_x \rangle^s K_\pm(-t_k)u_\pm(t_k) - \langle \nabla_x \rangle^s K_\pm(-t_{k-1})u_\pm(t_{k-1})\|_{L^2_x}^2\Big)^{1/2} \lesssim M.$$

Hence, there exists $f_\pm := \lim_{t\to\pm\infty} \langle \nabla_x \rangle^s K_\pm(-t)u_\pm(t)$ in $L^2_x(\mathbb{R}^d)$. Then put $u_{\pm\infty} := \langle \nabla_x \rangle^{-s} f_\pm$, we obtain

$$\|\langle \nabla_x \rangle^s K_\pm(-t)u_\pm(t) - f_\pm\|_{L^2_x} = \|u_\pm(t) - K_\pm(t)u_{\pm\infty}\|_{H^s_x} \to 0$$

as $t \to \pm\infty$. The scattering result for the wave equation is obtained similarly.

□

## References

[1] I. Bejenaru, S. Herr, J. Holmer, and D. Tataru, *On the 2D Zakharov system with $L^2$ Schrödinger data*, Nonlinearity **22** (2009), 1063–1089.

[2] I. Bejenaru, S. Herr, *Convolutions of singular measures and applications to the Zakharov system*, J. Funct. Anal. **261** (2011), 478–506.

[3] I. Bejenaru, S. Herr, and D. Tataru, *A convolution estimate for two-dimensional hypersurfaces*, Rev. Mat. Iberoam. **26** (2010), 707–728.

[4] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, Berlin, 1976.

[5] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations I. Schrödinger equations*, GAFA **3** (1993), 107–156.

[6] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, 10. American Mathematical Society, 2003.

[7] T. Cazenave, F.B. Weissler. *The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$*, Nonlinear Anal. **14** (1990), no. 10, 807–836.

[8] Y. Cho, G. Hwang and T. Ozawa, *Global well-posedness of critical nonlinear Schrödinger equations below $L^2$*, Discrete Contin. Dyn. Syst. **33** (2013), no. 4, 1389–1405.

[9] Y. Cho, S. Lee, *Strichartz estimates in spherical coordinates*, Indi. Univ. Math. J. **62** (2013), 991–1020.

[10] Y. Cho and K. Nakanishi, *On the global existence of semirelativistic Hartee equations*, RIMS Kôkyûroku Bessatsu, **B22** (2010), 145–166.

[11] M. Christ and A. Kiselev, *Maximal functions associated to filtrations*, J. Funct. Anal. **179** (2001), no. 2, 409–425.

[12] F. M. Christ and M. I. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. **100** (1991), 87–109.

[13] D. Fang and C. Wang, *Weighted Strichartz estimates with angular regularity and their applications*, Forum Math. **23** (2011), no. 1, 181–205.

[14] J. Ginibre, T. Ozawa and G. Velo, *On the existence of the wave operators for a class of nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. **60** (1994), no. 2, 211–239.

[15] J. Ginibre, Y. Tsutsumi, and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151** (1997), no. 2, 384–436.

[16] J. Ginibre, G. Velo, *Generalized Strichartz Inequalities for the Wave Equation*, J. Funct. Anal. **133** (1995), 50–68.

[17] Z. Guo, K. Nakanishi, and S. Wang, *Small energy scattering for the Klein-Gordon-Zakharov system with radial symmetry*, Math. Res. Nett. **21** (2014), no. 4, 733–755.

[18] M. Hadac, S. Herr, and H. Koch, *Well-posedness and scattering for the KP-II equation in a critical space*, Ann. I. H. Poincaré AN **26** (2009), 917–941.

[19] M. Hadac, S. Herr, and H. Koch, *Erratum to "Well-posedness and scattering for the KP-II equation in a critical space"[Ann. I. H. Poincaré AN **26** (2009), 917–941]*, Ann. I. H. Poincaré AN **27** (2010), no. 3, 971–972.

[20] N. Hayashi and T. Ozawa, *Smoothing effect for Schrödinger equations*, J. Funct. Anal. **85** (1989), no. 2, 307–348.

[21] K. Hidano, *Nonlinear Schrödinger equations with radially symmetric data of critical regularity*, Funkcial. Ekvac. **51** (2008), no. 1, 135–147.

[22] H. Hirayama, *Well-posedness and scattering for nonlinear Schrödinger equations with a derivative nonlinearity at the scaling critical regularity*, Funkcialaj Ekvacioj **58** (2015), 431–450.

[23] M. Ikeda, N. Kishimoto, and M. Okamoto, *Well-posedness for a quadratic nonlinear Schrödinger system at the critical regularity*, J. Funct. Anal. **271** (2016), no. 4, 747–798.

[24] J. -C. Jiang, C. Wang, and X. Yu, *Generalized and weighted Strichartz estimates*, Comm. Pure Appl. Anal. **11** (2012), no. 5, 1723–1752.

[25] I. Kato, *Well-posedness for the Cauchy problem of the Klein-Gordon-Zakharov system in four and more spatial dimensions*, Comm. Pure. Appl. Anal. **15** (2016), no. 6, 2247–2280.

[26] I. Kato, K. Tsugawa, *Scattering and well-posedness for the Zakharov system at a critical space in four and more spatial dimensions*, arXiv:1512.00551v1.

[27] J. Kato, M. Nakamura and T. Ozawa, *A generalization of the weighted Strichartz estimates for wave equations and an application to self-similar solutions*, Comm. Pure Appl. Math. **60** (2007), no. 2, 164–186.

[28] T. Kato, *An $L^{q,r}$-theory for nonlinear Schrödinger equations, Spectral and scattering theory and applications*, Adv. Stud. Pure. Math **23** Math. Soc. Japan, Tokyo, (1994), 223–238.

[29] T. Kato, *On nonlinear Schrödinger equations II. $H^s$-solutions and unconditional well-posedness*. J. Anal. Math. **67** (1995), 281–306.

[30] C. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Soc. **9** (1996), 573–603.

[31] H. Koch, D. Tataru, *Dispersive estimates for principally normal pseudodifferential operators*, Comm. Pure. Appl. Math. **58** (2) (2005), 217–284.

[32] H. Lindblad, *A sharp counterexample to the local existence of low-regularity solutions to non-linear wave equations* , Duke Math.J. **72** (1993), 503–539.

[33] H. Lindblad, *Counterexamples to local existence for semi-linear wave equations*, Amer. J. Math. **118** (1996), 1–16.

[34] S. Machihara, K. Nakanishi, and T. Ozawa, *Small global solutions and the nonrelativistic limit for the Dirac equation*, Rev. Mat. Iberoamericana. **19** (2003), 179–194.

[35] C. Miao, G. Xu and L. Zhao, *Global well-posedness and scattering for the energy-critical, defocusing Hartree equation for radial data*, J. Funct. Anal. **253** (2007), no. 2, 605–627.

[36] C. Miao, G. Xu and L. Zhao, *The Cauchy problem of the hartree equations*, J. Partial Differential Equations **21** (2008), no. 1, 22–44.

[37] C. Miao, G. Xu and L. Zhao, *Global well-posedness and scattering for the energy-critical, defocusing Hartree equation in $\mathbb{R}^{1+n}$*, Comm. Partial Differential Equations **36** (2011), no. 5, 729–776.

[38] M. Nakamura and T. Ozawa, *Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces*, Rev. Math. Phys. **9** (1997), no. 3, 397–410.

[39] K. Nakanishi and T. Ozawa, *Remarks on scattering for nonlinear Schrödinger equations*, NoDEA Nonlinear Differential Equations Appl. **9** (2002), no. 1, 45–68.

[40] T. Ozawa, K. Tsutaya, and Y. Tsutsumi, *Well-posedness in energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions*, Math. Ann. **313** (1999), no. 1, 127–140.

[41] H. Pecher, *Solutions of semilinear Schrödinger equations in $H^{s+1}$*, Ann. Inst. H. Poincaré Phys. Théor. **67** (1997), no. 3, 259–296.

[42] S. Selberg, *Bilinear Fourier restriction estimates related to the 2D wave equation*, Adv. Diff. Eq. **16** (2011), no. 7–8, 667–690.

[43] T. Schottdorf, *Global existence without decay for quadratic Klein-Gordon equations*, arXiv:1209.1518v2.

[44] T. Tao, *Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation*, Comm. Partial Differential Equations **25** (2000), no. 7-8, 1471–1485.

[45] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, AMS (2006).

[46] H. Triebel, *Spaces of Besov-Hardy-Sobolev type on complete Riemannian manifolds*, Ark. Mat. **24** (1986), no. 2, 299–337.

[47] Y. Tsutsumi, $L^2$*-solutions for nonlinear Schrödinger equations and nonolinear groups*, Funkcial. Ekvac. **30** (1987), no. 1, 115–125.

(Shinya Kinoshita) Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya, 464-8602, Japan

*E-mail address*, Shinya Kinoshita: `m12018b@math.nagoya-u.ac.jp`