TEICHMÜLLER THEORY FOR $\mathbb{C} \setminus \mathbb{Z}$

($\mathbb{C} \setminus \mathbb{Z}$に対するタイヒミュラー理論)

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Introduction

Let R be a Riemann surface. The Teichmüller space $T(R)$ of R is a space which describes all quasiconformal deformations of R. The Teichmüller distance, denoted by $d_{T(R)}$, is defined on $T(R)$ by using the maximal dilatations of quasiconformal mappings. Then $(T(R), d_{T(R)})$ becomes a complete metric space, and the Teichmüller modular group $\mathrm{Mod}(R)$ acts on $T(R)$ isometrically. By the so-called Bers embedding, $T(R)$ admits a finite or infinite dimensional complex structure on which $\mathrm{Mod}(R)$ acts biholomorphically. If R is of finite type, that is, R is obtained from a compact Riemann surface by removing at most a finite number of points, then $T(R)$ is a finite dimensional complex manifold. Further, $\mathrm{Mod}(R)$ acts on $T(R)$ properly discontinuously. On the other hand, if R is not of finite type, then $T(R)$ becomes an infinite dimensional, non separable, Banach analytic manifold, and $\mathrm{Mod}(R)$ does not act on $T(R)$ properly discontinuously in general. We say that such Riemann surfaces are of infinite type. For example, see [**11**, **15**, **19**].

Two Riemann surfaces are said to be quasiconformally equivalent to each other if there exists a quasiconformal mapping between them. For Riemann surfaces of finite type, quasiconformal equivalences are completely characterized by the genus and the number of punctures. On the other hand, in the case of infinite type Riemann surfaces, the situation is quite complicated. For example, $\mathbb{C} \setminus \mathbb{Z}$ and $\mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z})$ are homeomorphic to each other, however, there are no quasiconformal mappings between them. This fact follows from the invariance of porosity under quasiconformal mappings, see Väisälä [**25**] and Chapter 3. We will see that $\mathbb{C} \setminus \{\pm e^n\}_{n=0}^{\infty}$ and $\mathbb{C} \setminus \{e^n\}_{n=0}^{\infty}$ are quasiconforamlly equivalent, nevertheless $\mathbb{C}\setminus\mathbb{Z}$ and $\mathbb{C}\setminus\mathbb{N}$ are not quasiconformally equivalent, see Remark 2.3.5. Moreover, we can prove that any two of $\mathbb{C} \setminus \mathbb{Z}$, $\mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z})$, $\mathbb{C} \setminus \mathbb{N}$ and $\mathbb{C} \setminus \{e^n\}_{n=0}^{\infty}$ cannot be mapped to each other by quasiconformal mappings. Since $T(R)$ describes all quasiconformal deformations, it is important to determine Riemann surfaces that are quasiconformally equivalent to given R.

In this thesis, we try to find all Riemann surfaces that are quasiconformally equivalent to $\mathbb{C}\setminus\mathbb{Z}$. If a Riemann surface R is quasiconformally equivalent to $\mathbb{C}\setminus\mathbb{Z}$, then R is biholomorphically equivalent to $\mathbb{C}\setminus A$ for some closed discrete subsets $A\subset\mathbb{C}$. This is a direct consequence of the classical uniformization theorem and the removable isolated singularity theorem for quasiconformal mappings, see [**14**] or [**24**]. We will give some criteria for the complements of closed discrete subsets $A\subset\mathbb{C}$ to be quasiconformally equivalent to $\mathbb{C}\setminus\mathbb{Z}$.

This thesis is organized by three chapters. Chapter 1 is a preliminary for later chapters. We will give some definitions and basic properties. Here, we explain Chapter 2 and Chapter 3 respectively.

Chapter 2. Let $\mathscr{C}$ be the Cantor ternary set;

$$\mathscr{C}=[0,1]\setminus\bigcup_{m=1}^{\infty}\bigcup_{k=0}^{3^{m-1}-1}\left(\frac{3k+1}{3^m},\frac{3k+2}{3^m}\right).$$

In [**17**, Theorem 3, 1999], MacManus completely characterized subsets of $\mathbb{C}$ whose compliments are quasiconformally equivalent to $\mathbb{C}\setminus\mathscr{C}$ by some geometric conditions. As the first step in his proof, he gave a geometric characterization of subsets of $\mathbb{R}$ whose compliments are quasiconformally equivalent to $\mathbb{C}\setminus\mathscr{C}$. In Chapter 2, we first characterize subsets of $\mathbb{R}$ whose complements are quasiconformally equivalent to $\mathbb{C}\setminus\mathbb{Z}$.

Theorem A. *For a subset $A\subset\mathbb{R}$, the following conditions are quantitatively equivalent;*

1. *There exists an η-quasisymmetric bijection $f:\mathbb{Z}\to A$.*
2. *A can be written as a monotone increasing sequence $A=\{a_n\}_{n\in\mathbb{Z}}$ with $a_n\to\pm\infty$ as $n\to\pm\infty$, and there exists a constant $M\geq 1$ such that the following inequality holds for all $n\in\mathbb{Z}$ and $k\in\mathbb{N}$;*
$$\frac{1}{M}\leq\frac{a_{n+k}-a_n}{a_n-a_{n-k}}\leq M.$$
3. *There exists a K-quasiconformal mapping $F:\mathbb{C}\to\mathbb{C}$, such that $F(\mathbb{Z})=A$.*

Further, if A satisfies the second condition, then there exists a $K=K(M)$-quasiconformal mapping $F:\mathbb{C}\to\mathbb{C}$ such that $F(n)=a_n$ for all $n\in\mathbb{Z}$.

It immediately follows from the equivalence (2) $\Leftrightarrow$ (3) that $\mathbb{C}\setminus\mathbb{Z}$ and $\mathbb{C}\setminus\mathbb{N}$ are not quasiconformally equivalent.

Next, we have the following result concerning the study of $\mathrm{Mod}(\mathbb{C} \setminus \mathbb{Z})$ which consists of all homotopy classes of quasiconformal self-homeomorphisms of $\mathbb{C} \setminus \mathbb{Z}$:

THEOREM B. *For a bijection $f : \mathbb{Z} \to \mathbb{Z}$, the following conditions are quantitatively equivalent;*

1. *f is η-quasisymmetric.*
2. *f satisfies the λ-three point condition.*
3. *f is M-biLipschitz.*
4. *f admits an M-biLipschitz extension $F : \mathbb{C} \to \mathbb{C}$.*
5. *f admits a K-quasiconformal extension $F : \mathbb{C} \to \mathbb{C}$.*

As a corollary of these theorems, we immediately obtain the following:

THEOREM C. *Every η-quasisymmetric embedding $f : \mathbb{Z} \to \mathbb{R}$ admits a $K = K(\eta)$-quasiconformal extension $F : \mathbb{C} \to \mathbb{C}$ where $K(\eta)$ is a constant depending only on η.*

This result means that the integer set $\mathbb{Z}$ is an example of unbounded discrete subset for which the one dimensional Väisälä problem, stated below, can be solved affirmatively.

QUESTION (The Väisälä 8th problem [**23**, 1995]). *Let $X \subset \mathbb{R}^n$. Can η-quasisymmetric mapping $f : X \to \mathbb{R}^n$ be extended to a K-quasiconformal mapping $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, where $K \geq 1$ is a constant depending only on η and n?*

The Väisälä problem has been studied mainly in the case that $X \subset \mathbb{R}^n$ is connected or bounded, see [**2**,**5**,**6**,**22**,**26**]. In [**20**, Theorem 6.21, 1999], Trotsenko–Väisälä proved that; if $X \subset \mathbb{R}^n$ is not relatively connected, then there is a quasisymmetric embedding $f : X \to \mathbb{R}^n$ which admits no quasiconformal extensions $F : H \to H$ to any Hilbert spaces H. Therefore, the Väisälä problem cannot be solved affirmatively for general subsets even if $n = 1$.

This chapter is based on the papers [**9**, Section 3] and [**10**].

Chapter 3. Let $[f], [g] \in T(R)$, where $[f]$ denotes the Teichmüller equivalence class of the quasiconformal mapping $f : R \to f(R)$ from R to another Riemann surface $f(R)$. Then the Teichmüller distance between $[f]$ and $[g]$ is defined by

$$d_{T(R)}([f], [g]) = \frac{1}{2} \inf \log K(h),$$

where the infimum is taken over all quasiconformal mappings $h : f(R) \to g(R)$ which are homotopic to $g \circ f^{-1}$ relative to the ideal boundary of $f(R)$, and $K(h)$ denotes the maximal dilatation of h.

Let $p : S \to R$ be a covering. Then every quasiconformal deformation of R lifts to a quasiconformal deformation of S. Thus the covering mapping $p : S \to R$ naturally induces the embedding $p^* : T(R) \to T(S)$. By the definitions, the mapping p^* is 1-Lipschitz, that is,

$$d_{T(S)}(p^*[f], p^*[g]) \leq d_{T(R)}([f], [g])$$

for all $[f], [g] \in T(R)$. Suppose R is of finite type and is hyperbolic, that is, R is covered by the unit disk $\mathbb{D}$. Then, in [**18**, Corollary 1.2, 1989], McMullen showed that for any covering $p : S \to R$, the induced mapping p^* is either a contraction ($d_{T(S)}(p^*{*}, p^*{*}) < d_{T(R)}(*, *)$), or a global isometry ($d_{T(S)}(p^*{*}, p^*{*}) = d_{T(R)}(*, *)$). Furthermore, he characterized coverings which induce global isometries. This result completely solved the Kra conjecture [**13**, 1981]: For any universal covering $p : \mathbb{D} \to R$ of a hyperbolic Riemann surface R of finite type, the induced embedding $p^* : T(R) \to T(\mathbb{D})$ is a contraction. Notice that the Teichmüller space $T(\mathbb{D})$ of the unit disk contains all Teichmüller spaces of hyperbolic Riemann surfaces. Therefore $T(\mathbb{D})$ is called the universal Techmüller space.

Let $R_n = (\mathbb{C} \setminus \mathbb{Z})/\langle z + n \rangle$ for $n \in \mathbb{Z}$, and let $p_n : R \to R_n$ be the projection. Then R_n is an $(n + 2)$-punctured Riemann sphere. Since the covering transformation group $\mathrm{Deck}(p_n) = \langle z + n \rangle$, every quasiconformal deformation $f : R_n \to f(R_n)$ lifts to a periodic quasiconformal deformation of $\mathbb{C} \setminus \mathbb{Z}$ with period n. In Chapter 3, we discuss periodic quasiconformal deformations of $\mathbb{C} \setminus \mathbb{Z}$. First, we prove the following:

THEOREM D. *Let $A \subset \mathbb{C}$ be a closed discrete subset which has the following form;*

$$A = \mathbb{Z} + \{a_n\}_{n=1}^{k}$$

where $k \leq \infty$ and each $a_n \in \mathbb{C}$ satisfies $\mathrm{Re}(a_n) \in [0, 1)$. Then, $\mathbb{C} \setminus A$ is quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$ if and only if $k < \infty$.

By the McMullen theorem, it turns out that $p_n^* : T(R_n) \to T(\mathbb{C} \setminus \mathbb{Z})$ is globally isometric. Further, any two points of $T(R_n)$ can be joined by a geodesic. Thus $T_n = p_n^*(T(R_n))$ is a geodesically convex subspace of $T(\mathbb{C} \setminus \mathbb{Z})$. Note that T_n is a subspace which describes periodic quasiconformal deformations of $\mathbb{C} \setminus \mathbb{Z}$ with period n.

Let T_∞ be the set of all $[f] \in T(\mathbb{C} \setminus \mathbb{Z})$ such that $f(\mathbb{C} \setminus \mathbb{Z})$ has a biholomorphic automorphism of infinite order. Then the Teichmüller modular group $\mathrm{Mod}(\mathbb{C} \setminus \mathbb{Z})$ does not act properly discontinuously on $\overline{T_\infty} \subset T(\mathbb{C} \setminus \mathbb{Z})$, since the stabilizer $\mathrm{Stab}_{\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})}([f])$ is isomorphic to the group $\mathrm{Aut}(f(\mathbb{C}\setminus\mathbb{Z}))$ of biholomorphic automorphisms of $f(\mathbb{C}\setminus\mathbb{Z})$, see [**7**, Lemma 2]. Theorem D implies that if $[f] \in T_\infty$ then there is a periodic quasiconformal deformation $g : \mathbb{C}\setminus\mathbb{Z} \to g(\mathbb{C}\setminus\mathbb{Z})$ such that $g(\mathbb{C}\setminus\mathbb{Z})$ is biholomorphically equivalent to $f(\mathbb{C}\setminus\mathbb{Z})$. Therfore, as a corollary of Theorem D, we have the next theorem:

THEOREM E. $T_0 = \bigcup_{n=1}^{\infty} T_n$ *is separable and geodesically convex. Further, the following equality holds;*

$$T_\infty = \bigcup_{[f]\in\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})} [f]_*(T_0).$$

Here, $[f]_* : T(\mathbb{C}\setminus\mathbb{Z}) \to T(\mathbb{C}\setminus\mathbb{Z})$, $[g] \mapsto [g\circ f^{-1}]$ denotes the action of $[f] \in \mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})$ on $T(\mathbb{C}\setminus\mathbb{Z})$. Further, we can show the following:

THEOREM F. *Let* $[f] \in \mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})$. *If* $[f]_*(T_0) \cap T_0 \neq \emptyset$, *then* $[f]_*(T_0) = T_0$.

Therefore, the subset T_∞ contained in the limit set of $\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})$ is the disjoint union of isometric copies of the separable geodesically convex subspace T_0, which describes all periodic quasiconformal deformations of $\mathbb{C}\setminus\mathbb{Z}$. For general results on the limit sets for infinite type Riemann surfaces, see Fujikawa [**8**].

This chapter is based on the paper [**9**, Section 4].

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CHAPTER 1

Some classes of mappings

In this chapter, we introduce some classes of mappings and give relations between each pair of these classes, as a preliminary to later chapters. We only consider mappings from a planar subset into the plane. However, we remark that all of these classes are generalized to any metric spaces, see [**12**] and [**24**].

1.1. Quasiconformal mappings

Let $\Omega \subset \mathbb{C}$ be a domain, and $f : \Omega \to \mathbb{C}$ be a homeomorphism into $\mathbb{C}$. For $z \in \Omega$ and $r > 0$ such that $S(z,r) = \{w \in \mathbb{C} \mid |z-w| = r\} \subset \Omega$, we set

$$\begin{aligned} L(z,f,r) &= \max_{w \in S(z,r)} |f(z) - f(w)|, \\ \ell(z,f,r) &= \min_{w \in S(z,r)} |f(z) - f(w)|. \end{aligned}$$

Then the circular dilatation $H_f(z) \in [1,\infty]$ of f at z is defined by

$$H_f(z) = \limsup_{r \to 0} \frac{L(z,f,r)}{\ell(z,f,r)}.$$

Let $K \geq 1$. Then f is said to be K-*quasiconformal* if $H_f(z) < \infty$ for all points $z \in \Omega$ and satisfies $H_f(z) \leq K$ almost everywhere in Ω. For a quasiconformal mapping $f : \Omega \to \mathbb{C}$, the quantity $K(f) = \mathrm{ess.sup}_{\Omega} H_f(z)$ is called the maximal dilatation of f.

Suppose f is differentiable at z. Its derivative df maps infinitesimal circle to (non degenerate or degenerate) infinitesimal ellipse. Then $H_f(z)$ coincides with the ratio of long axis and short axis of ellipse. In particular, if $H_f(z) = 1$, then df maps infinitesimal circle to infinitesimal circle. Therefore, smooth orientation preserving 1-quasiconformal mappings are conformal. More generally, this fact is true even if the mapping is not smooth, that is, for an orientation preserving homeomorphism f, f is 1-quasiconformal if and only if f is conformal.

1.2. Quasisymmetric mappings

Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism and $X \subset \mathbb{C}$ be a subset. Then an injection $f : X \to \mathbb{C}$ is said to be *η-quasisymmetric* if the following inequality holds for any three points $x, y, z \in X$ $(x \neq z)$;

$$\left|\frac{f(x) - f(y)}{f(x) - f(z)}\right| \leq \eta\left(\left|\frac{x - y}{x - z}\right|\right). \tag{QS}$$

If $x \neq y$, replacing y and z, the following lower estimate also holds;

$$\left|\frac{f(x) - f(y)}{f(x) - f(z)}\right| \geq \eta\left(\left|\frac{x - y}{x - z}\right|^{-1}\right)^{-1}.$$

Therefore, if $\eta(t) = t$, then η-quasisymmetric mapping f satisfies

$$\left|\frac{f(x) - f(y)}{f(x) - f(z)}\right| = \left|\frac{x - y}{x - z}\right|,$$

for any distinct points $x, y, z \in X$. This implies that f is the restriction of an Affine transformation of $\mathbb{C}$ or its complex conjugation.

Notice that if X contains at least two elements and there exists at least one η-quasisymmetric mapping, applying (QS) to $y = z$, it turns out that η must satisfy $\eta(1) \geq 1$.

1.3. Egg-yolk principle

For a homeomorphism between planar domains, the quasiconformality and the quasisymmetry are closely related by the so-called egg-yolk principle as follows, see [**12**, Theorem11.14]:

Let $\Omega, \Omega' \subset \mathbb{C}$ be domains, and let $f : \Omega \to \Omega'$ be a homeomorphism. Then f is K-quasiconformal if and only if there exists an η such that f is η-quasisymmetric on $D(z, \mathrm{dist}(z, \partial\Omega)/2)$ for all $z \in \Omega$. Here $D(z, r)$ denotes the open disk with radius $r > 0$ centered at z. Further, in this statement, K and η are related quantitatively.

Therefore, every η-quasisymmetric mapping defined on a domain is a $K = K(\eta)$-quasiconformal mapping. Conversely, every K-quasiconformal mapping is locally $\eta = \eta_K$-quasisymmetric. In particular, for a self-homeomorphism of $\mathbb{C}$, the quasiconformality and the quasisymmetry are quantitatively equivalent.

1.4. biLipschitz mappings

Let $X \subset \mathbb{C}$ be a subset and $M \geq 1$. Then a mapping $f : X \to \mathbb{C}$ is said to be M-*biLipschitz* if

$$\frac{1}{M}|x - y| \leq |f(x) - f(y)| \leq M|x - y|$$

for any $x, y \in X$. Clearly, biLipschitz mappings are homeomorphisms into $\mathbb{C}$, and 1-biLipschitz mappings are isometries.

If $f : X \to \mathbb{C}$ is M-biLipschitz, then

$$\left|\frac{f(x) - f(y)}{f(x) - f(z)}\right| \leq M^2 \left|\frac{x - y}{x - z}\right|$$

for all $x, y, z \in X$ $(x \neq z)$. Thus every M-biLipschitz mappping is $\eta(t) = M^2 t$-quasisymmetric. Furthermore, if X is a domain, then it immediately follows from the definition of quasiconformality that f is M^2-quasiconformal.

The biLipschitz extendability of biLipschitz mapping has been studied by many authors, and many detailed results were obtained so far compared with the case of quasisymmteric or quasiconformal mappings. For example, Alestalo–Väisälä [**3**, Theorem 5.5] showed that every M-biLipschitz mapping f from $X \subset \mathbb{R}^k$ into $\mathbb{R}^n$ $(n \geq k)$ admits a $\sqrt{7}M^2$-biLipschitz extension $F : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$. In particular, every M-biLipschitz embedding $f : X \to \mathbb{R}$ $(X \subset \mathbb{R})$ can be extended to a $\sqrt{7}M^2$-biLipschitz self-homeomorphism of $\mathbb{C}$ (this fact will be used in a later section). Moreover, in the case of $X \subset \mathbb{R}$, more detailed result was obtained by MacManus [**16**, Theorem 1]; if $X \subset \mathbb{R}$, then any M-biLipschitz mapping $f : X \to \mathbb{C}$ admits a M'-biLipschitz extension $F : \mathbb{C} \to \mathbb{C}$, where M' is a constant depending only on M. The same statements cannot hold for quasisymmetric mappings since there is a subset $X \subset \mathbb{R}$ and a quasisymmetric mapping $f : X \to \mathbb{R}$ which cannot be extended to quasisymmetric self-homeomorphism of any Hilbert space, see Introduction, [**20**, Theorem 6.21] or [**12**, p.89].

CHAPTER 2

Extendability of quasisymmetric embedding of $\mathbb{Z}$

2.1. M-condition and λ-three point condition

Let $M \geq 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. We say that f satisfies the *M-condition* if the following inequality holds for any $x \in \mathbb{R}$ and $t > 0$;

$$\frac{1}{M} \leq \left| \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \right| \leq M.$$

The M-condition was introduced by Beurling–Ahlfors in [**5**] as the quasisymmetry on the real line, and later, the M-condition was generalized to the quasisymmetry defined in Chapter 1. In fact, for a self-homeomorphism of $\mathbb{R}$, the η-quasisymmetry and the M-condition are quantitatively equivalent, see [**21**]. Further, it was shown in [**5**] that for a self-homeomorphism f of $\mathbb{R}$, f can be extended to a K-quasiconformal self-homeomorphism of $\mathbb{C}$ if and only if f satisfies the M-condition. Further K and M are quantitatively related to each other, see also [**1**].

The aim of this chapter is to show that every quasisymmetric embedding from $\mathbb{Z}$ to $\mathbb{R}$ admits a quasiconformal extension to $\mathbb{C}$.

Every self-homeomorphism of $\mathbb{R}$ is monotone. However, quasisymmetric embeddings from $\mathbb{Z}$ to $\mathbb{R}$ need not to be monotone. This is one of the difficulty compared with the above Beurling–Ahlfors theorem. Here, we introduce the *λ-three point condition* which will play a "non-monotone" part of a quasisymmetric embedding from $\mathbb{Z}$ to $\mathbb{R}$, see Remark 2.5.1:

Definition 2.1.1. *Let $A \subset \mathbb{R}$ be a subset and let $\lambda \geq 1$. Then we say that an injection $f : A \to \mathbb{R}$ satisfies the λ-three point condition if the following inequality holds for any $x, y, z \in A$ with $x < y \leq z$;*

$$\left| \frac{f(x) - f(y)}{f(x) - f(z)} \right| \leq \lambda.$$

Note that the three point condition does not depend on the distances of $x, y, z \in A$. In particular, if $f : A \to \mathbb{R}$ satisfies the λ-three point condition, and if $h : A \to \mathbb{R}$ is strictly monotone increasing, then $f \circ h^{-1} : h(A) \to \mathbb{R}$ also satisfies the λ-three point condition.

REMARK 2.1.2. *Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be an η-quasisymmetric mapping. Then for any points $x, y, z \in A$ with $x < y \leq z$, we have*

$$\left|\frac{f(x)-f(y)}{f(x)-f(z)}\right| \leq \eta\left(\left|\frac{x-y}{x-z}\right|\right) < \eta(1).$$

Thus, every η-quasisymmetric mapping satisfies the $\eta(1)$-three point condition.

2.2. Lemma

The following lemma will play an important role in later sections.

LEMMA 2.2.1. *Let $A = \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly monotone increasing sequence. If a bijection $g : E \to \mathbb{Z}$ satisfies the μ-three point condition ($\mu \geq 1$), then for any $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, the following inequality holds;*

$$\frac{k}{2\mu} < |g(a_n) - g(a_{n+k})| < 2\mu k.$$

Proof. We first prove the following estimation;

CLAIM 1. $|g(a_n) - g(a_{n+1})| < 2\mu$ *for any* $n \in \mathbb{Z}$.

PROOF. Since $\mu \geq 1$, it suffices to consider the case where $|g(a_n) - g(a_{n+1})| \geq 2$. Then we may assume $g(a_{n+1}) > g(a_n)$ since the same argument mentioned below can be applied to the case where $g(a_n) > g(a_{n+1})$.

Letting $m \leq n$ satisfy

$$g(a_m) = \max\{g(a_j) \mid j \leq n \text{ and } g(a_n) \leq g(a_j) < g(a_{n+1})\}$$

and $\ell \in \mathbb{Z}$ satisfy $g(a_\ell) = g(a_m)+1$ (then $\ell \geq n+1$ by the construction), we can construct $m, \ell \in \mathbb{Z}$ which satisfy the following conditions, see Figure 1;

1. $m \leq n$ and $n+1 \leq \ell$,
2. $g(a_n) \leq g(a_m) < g(a_\ell) = g(a_m) + 1 \leq g(a_{n+1})$.

First, suppose $g(a_m) - g(a_n) \geq (g(a_{n+1}) - g(a_n))/2 \geq 1$. By the three point condition,

$$\begin{aligned} \mu &\geq \left|\frac{g(a_m)-g(a_n)}{g(a_m)-g(a_\ell)}\right| \\ &= g(a_m) - g(a_n) \geq \frac{g(a_{n+1})-g(a_n)}{2}. \end{aligned}$$

Thus we have $g(a_{n+1}) - g(a_n) < 2\mu$.

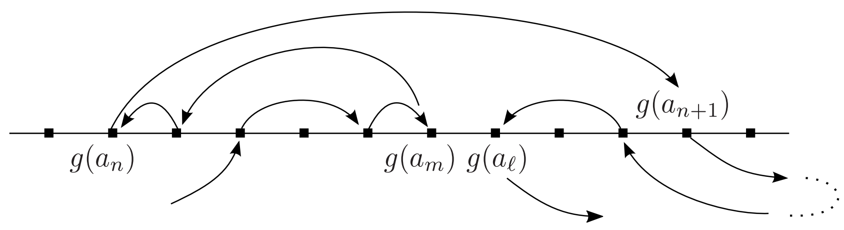

Figure 1 : a part of the orbit of the sequence $\big(g(a_j)\big)_{j\in\mathbb{Z}}$.

Next, suppose $g(a_m)-g(a_n) < (g(a_{n+1}) - g(a_n))/2$. Then $g(a_{n+1})-g(a_m) > (g(a_{n+1}) - g(a_n))/2$. Similarly we have $g(a_{n+1}) - g(a_n) < 2\mu$. □

CLAIM 2. *Lemma 2.2.1 holds.*

PROOF. (*Upper bound*) By the triangle inequality, it immediately follows from Claim 1 that $|g(a_n) - g(a_{n+k})| < 2\mu k$.

(*Lower bound*) Since the open interval

$$\left(g(a_n) - \frac{k}{2},\ g(a_n) + \frac{k}{2}\right)$$

contains at most $k-1$ integers except $g(a_n)$, there exists an integer $m \in \mathbb{Z}$ $(n < m \le n+k)$ such that

$$|g(a_n) - g(a_m)| \ge \frac{k}{2}.$$

By the three point condition, we obtain

$$\mu \ \ge\ \left|\frac{g(a_n) - g(a_m)}{g(a_n) - g(a_{n+k})}\right| \ge \frac{k}{2|g(a_n) - g(a_{n+k})|},$$

that is, $|g(a_n) - g(a_{n+k})| > k/2\mu$. □

2.3. Images of $\mathbb{Z}$ under global quasiconformal mappings

In this section, we characterize subsets $A \subset \mathbb{R}$ whose compliments are quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$. First we prove the following:

PROPOSITION 2.3.1. *Let $A = \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly monotone increasing sequence with $a_n \to \pm\infty$ as $n \to \pm\infty$. If there exists a* *constant $M \geq 1$ such that the following inequality holds for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$,*

$$\frac{1}{M} \leq \frac{a_{n+k} - a_n}{a_n - a_{n-k}} \leq M,$$

then there exists a $K = K(M)$-quasiconformal mapping $F : \mathbb{C} \to \mathbb{C}$ such that $F(n) = a_n$ for all $n \in \mathbb{Z}$.

Proof. Set $\phi(x) := (a_{n+1} - a_n)(x - n) + a_n$ for $x \in [n, n+1)$. Then ϕ defines an orientation preserving self-homeomorphism of $\mathbb{R}$ with $\phi(n) = a_n$. Further we can show that ϕ satisfies $C(M)$-condition, where $C(M) = M^4 + M^3 + M^2 + M$. Therefore we obtain a $K = K(M)$-quasicnoformal extension $F : \mathbb{C} \to \mathbb{C}$ of ϕ by the Beurling–Ahlfors extension theorem.

Let $x = n + t_1,\ t = m + t_2$ $(n \in \mathbb{Z},\ m \in \mathbb{Z}_{\geq 0},\ t_1, t_2 \in [0, 1))$. To prove that ϕ satisfies $C(M)$-condition, we have to show the following inequality,

$$\frac{1}{C(M)} \leq I := \frac{\phi(x + t) - \phi(x)}{\phi(x) - \phi(x - t)} \leq C(M).$$

We divide the calculations into the following four cases.

1. $t_1 + t_2 \in [0, 1)$ and $t_1 - t_2 \in (-1, 0)$.
2. $t_1 + t_2 \in [0, 1)$ and $t_1 - t_2 \in [0, 1)$.
3. $t_1 + t_2 \in [1, 2)$ and $t_1 - t_2 \in (-1, 0)$.
4. $t_1 + t_2 \in [1, 2)$ and $t_1 - t_2 \in [0, 1)$.

However we only check the first case here as the calculations are almost the same and easy. To simplify the calculation, we use the next inequality.

Lemma 2.3.2. *Under the above assumptions, the following inequalities hold.*

(I) *For $n, m \in \mathbb{Z}$ $(n < m)$, and $k \in \mathbb{Z}_{\geq 0}$,*

$$\begin{aligned} \frac{a_{m+k} - a_n}{a_m - a_n} &\leq M^k + M^{k-1} + \cdots + M + 1, \\ \frac{a_m - a_{n-k}}{a_m - a_n} &\leq M^k + M^{k-1} + \cdots + M + 1. \end{aligned}$$

(II) *For $p, q \in \mathbb{R}$ and $k \in \mathbb{Z}$, if $k - 1 \leq p \leq k \leq q \leq k + 1$, then*

$$\frac{1}{M}(a_{k+1} - a_k)(q - p) \leq \phi(q) - \phi(p) \leq M(a_{k+1} - a_k)(q - p).$$

The inequality (II) also holds for $(a_k - a_{k-1})$ instead of $(a_{k+1} - a_k)$.

PROOF.

$$\text{(I)} \quad \frac{a_{m+k}-a_n}{a_m-a_n} \le \frac{a_{m+k}-a_{m-1}}{a_m-a_{m-1}} = \sum_{j=0}^{k} \frac{a_{m+j}-a_{m+j-1}}{a_m-a_{m-1}} \le M^k + M^{k-1} + \cdots + M + 1.$$

The second inequality also follows from the same calculation.

$$\begin{aligned}\text{(II)} \quad \phi(q)-\phi(p) &= (a_{k+1}-a_k)(q-k)+a_k-(a_k-a_{k-1})(p-k+1)-a_{k-1} \\ &= (a_{k+1}-a_k)(q-k)+(a_k-a_{k-1})(k-p)\cdots(*)\end{aligned}$$

$$\begin{aligned}(*) &\le (a_{k+1}-a_k)(q-k)+M(a_{k+1}-a_k)(k-p) \\ &= M(a_{k+1}-a_k)(q-p)+(1-M)(a_{k+1}-a_k)(q-k) \le M(a_{k+1}-a_k)(q-p) \\ (*) &\le M(a_k-a_{k-1})(q-k)+(a_k-a_{k-1})(k-p) \\ &= M(a_k-a_{k-1})(q-p)+(1-M)(a_k-a_{k-1})(q-k) \le M(a_k-a_{k-1})(q-p)\end{aligned}$$

We can prove the lower bounds in the same way. □

Continuation of Proof of Proposition 2.3.1. Suppose $t_1+t_2 \in [0,1)$ and $t_1-t_2 \in (-1,0)$.

(*Upper bound*). First if $m \neq 0$, since ϕ is monotone increasing,

$$\begin{aligned} I &\le \frac{\phi(n+m+1)-\phi(n)}{\phi(n)-\phi(n-m)} \\ &= \frac{a_{n+m+1}-a_n}{a_n-a_{n-m}} \le M\frac{a_{n+m+1}-a_n}{a_{n+m}-a_n} \le M(M+1) < C(M).\end{aligned}$$

Next if $m=0$, since $n-1 \le n+t_1+t_2 \le n \le n+t_1 \le n+1$,

$$I \le \frac{(a_{n+1}-a_n)(t_1+t_2)-(a_{n+1}-a_n)t_1}{\frac{1}{M}(a_{n+1}-a_n)t_2} = M < C(M).$$

(*Lower bound*). First if $m \neq 0, 1$, by the monotonicity of ϕ

$$\begin{aligned} I &> \frac{\phi(n+m+1)-\phi(n)}{\phi(n)-\phi(n-m)} \\ &\ge \frac{1}{M}\frac{a_{n+m}-a_{n+1}}{a_{n+m+3}-a_{n+1}} \ge \frac{1}{M(M^3+M^2+M+1)} = \frac{1}{C(M)}.\end{aligned}$$

Next if $m = 0$, for the same reason as in the case of upper bound,

$$I \geq \frac{(a_{n+1} - a_n)(t_1 + t_2) - (a_{n+1} - a_n)t_1}{M(a_{n+1} - a_n)t_2} = \frac{1}{M} > \frac{1}{C(M)}.$$

Finally if $m = 1$, since $n \leq n + t_1 \leq n + 1 \leq n + t_1 + 1 \leq n + 2$ we have $\phi(x+t) - \phi(x) \geq \phi(x+1) - \phi(x) \geq \frac{1}{M}(a_{n+1} - a_n)$. On the other hand, $\phi(x) - \phi(x-t) \leq \phi(n+1) - \phi(n-2) = a_{n+1} - a_{n-2}$. Thus we have

$$I \geq \frac{1}{M} \frac{a_{n+1} - a_n}{a_{n+1} - a_{n-2}} \geq \frac{1}{M(M^2 + M + 1)} \geq \frac{1}{C(M)}.$$

□

Next, we show that if $f : \mathbb{Z} \to \mathbb{R}$ is quasisymmetric, then $A = f(\mathbb{Z})$ satisfies the assumption in Proposition 2.3.1.

REMARK 2.3.3. *Quasisymmetric mappings take Cauchy sequences to Cauchy sequences. Therefore, if $A \subset \mathbb{R}$ is an image of a quasisymmetric mapping $f : \mathbb{Z} \to \mathbb{R}$, then A must be closed and discrete in $\mathbb{R}$.*

LEMMA 2.3.4. *Let $f : \mathbb{Z} \to \mathbb{R}$ be an η-quasisymmetric mapping, and let $A := f(\mathbb{Z})$. Then* $\sup A = \infty$ *and* $\inf A = -\infty$.

PROOF. To obtain a contradiction, we assume $\inf A > -\infty$. Since A is closed and discrete, we have $\sup A = \infty$. Thus A can be written as a monotone increasing sequence $A = \{a_n\}_{n \in \mathbb{N}}$ with $a_n \to \infty$ as $n \to \infty$.

Let $g := f^{-1} : A \to \mathbb{Z}$. By translation, we may assume $g(a_1) = 0$. Further, note that g is η'-quasisymmetric where $\eta'(t) = 1/\eta^{-1}(1/t)$, see [**21**, Theorem 2.2]. Let $\mu := \eta'(1)$ and consider the set

$$S := \left\{ k \in \mathbb{N} \;\middle|\; g(a_k) = \max_{j=1,2,\cdots,k} g(a_j) \geq \mu \right\}.$$

Since $g : A \to \mathbb{Z}$ is bijective, S consists of infinitely many elements. We number $S = \{k_j\}_{j \in \mathbb{N}}$ in ascending order. Then the sequence $\{g(a_{k_j})\}_{j \in \mathbb{N}} \subset \mathbb{Z}$ is monotone increasing. On the other hand, there exist infinitely many $n \in \mathbb{N}$ with $g(a_n) < 0$. Thus we can find $j, \ell \in \mathbb{N}$ such that $k_j < \ell < k_{j+1}$ and $g(a_\ell) < 0$. Moreover since $g(a_n) \leq g(a_{k_j})$ for all $n = 1, 2, \ldots, k_{j+1} - 1$, if $g(a_m) = g(a_{k_j}) + 1$ then $m \geq k_{j+1}$. Consequently we confirmed that there exists $k \in S$ and exist $\ell, m \in \mathbb{N}$ such that

- $k < \ell < m$,
- $g(a_\ell) < 0$ and $g(a_m) = g(a_k) + 1$, see Figure 2.

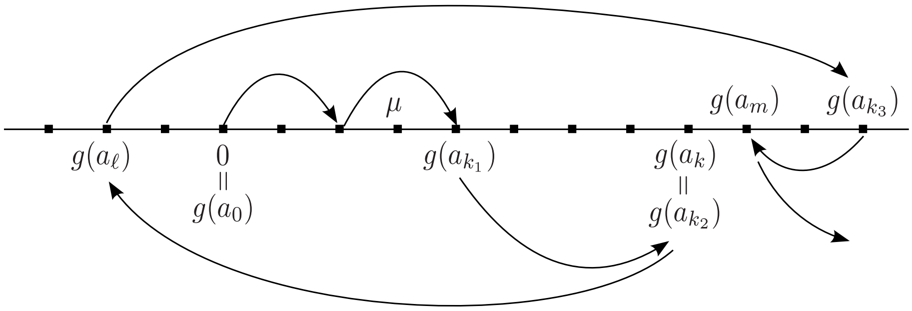

Figure 2

Therefore, we have a contradiction;

$$\begin{aligned} \mu > \eta'\left(\left|\frac{a_k - a_\ell}{a_k - a_m}\right|\right) &\geq \left|\frac{g(a_k) - g(a_\ell)}{g(a_k) - g(a_m)}\right| \\ &= g(a_k) - g(a_\ell) > g(a_k) \geq \mu. \end{aligned}$$

□

REMARK 2.3.5. *Preceding Lemma 2.3.4 extremely depends on the particularity of* $\mathbb{Z}$. *For example, Lemma 2.3.4 no longer holds for* $X = \{\pm e^n\}_{n=0}^\infty$. *In fact, there exists a quasisymmetric mapping which maps* X *to* $Y = \{e^{n/2}\}_{n=0}^\infty$*: Define* $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ *by* $f_1(re^{i\theta}) = re^{i(\theta+2\pi \log r)}$, $f_2(x+iy) = x+iy$ $(x \geq 0)$, *and* $f_2(x+iy) = x/\sqrt{e}+iy$ $(x < 0)$. *Then* $f = f_2 \circ f_1 : \mathbb{C} \to \mathbb{C}$ *is quasiconformal and maps* Y *to* X. *Thus* $f^{-1}|_X$ *is a quasisymmetric mapping whose image is* Y.

LEMMA 2.3.6. *Let* $A = \{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ *be a monotone increasing sequence with* $a_n \to \pm\infty$ *as* $n \to \pm\infty$. *If* $g : A \to \mathbb{Z}$ *is an* η'-*quasisymmetric bijection, then there exists a constant* $L \geq 1$ *depending only on* $\mu := \eta'(1)$ *which satisfies the following inequality for all* $n \in \mathbb{Z}$ *and* $k \in \mathbb{N}$*;*

$$\frac{1}{L} < \left|\frac{g(a_{n+k}) - g(a_n)}{g(a_n) - g(a_{n-k})}\right| < L.$$

PROOF. Since g satisfies the $\eta'(1)$-three point condition, by Lemma 2.2.1 we have

$$\frac{1}{4\mu^2} < \left|\frac{g(a_{n+k}) - g(a_n)}{g(a_n) - g(a_{n-k})}\right| < 4\mu^2,$$

where $\mu = \eta'(1)$. □

THEOREM A. *For a subset* $A \subset \mathbb{R}$, *the following conditions are quantitatively equivalent;*

1. *There exists an η-quasisymmetric bijection $f : \mathbb{Z} \to A$.*
2. *A can be written as a monotone increasing sequence $A = \{a_n\}_{n\in\mathbb{Z}}$ with $a_n \to \pm\infty$ as $n \to \pm\infty$, and there exists a constant $M \geq 1$ such that the following inequality holds for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$;*
$$\frac{1}{M} \leq \frac{a_{n+k} - a_n}{a_n - a_{n-k}} \leq M.$$
3. *There exists a K-quasiconformal mapping $F : \mathbb{C} \to \mathbb{C}$, such that $F(\mathbb{Z}) = A$.*

Further, if A satisfies the second condition, then there exists a $K = K(M)$-quasiconformal mapping $F : \mathbb{C} \to \mathbb{C}$ such that $F(n) = a_n$ for all $n \in \mathbb{Z}$.

Proof. First, (2) $\Rightarrow$ (3) and the last statement follow from Proposition 2.3.4. Next, since global K-quasiconformal mappings are quantitatively η-quasisymmetric, see Chapter 1, (3) $\Rightarrow$ (1) follows. Thus it suffices to show (1) $\Rightarrow$ (2).

Let us assume that there exists an η-quasisymmetric bijection $f : \mathbb{Z} \to A$. By Lemma 2.3.4, A can be written as a monotone increasing sequence $A = \{a_n\}_{n\in\mathbb{Z}}$ with $a_n \to \pm\infty$ as $n \to \pm\infty$ (recall that A must be closed and discrete in $\mathbb{R}$). Let $g := f^{-1}$. Then g is η'-quasisymmetric where $\eta'(t) = 1/\eta^{-1}(1/t)$. By Lemma 2.3.6, there exists a constant $L \geq 1$ depending only on $\eta'(1) = 1/\eta^{-1}(1)$ which satisfies the following inequality for any $n \in \mathbb{Z}$ and $k \in \mathbb{N}$;
$$\frac{1}{L} < \left|\frac{g(a_{n+k}) - g(a_n)}{g(a_n) - g(a_{n-k})}\right| < L.$$
Therefore we obtain
$$\left|\frac{a_{n+k} - a_n}{a_n - a_{n-k}}\right| \leq \eta\left(\left|\frac{g(a_{n+k}) - g(a_n)}{g(a_n) - g(a_{n-k})}\right|\right) < \eta(L).$$
and
$$\left|\frac{a_{n+k} - a_n}{a_n - a_{n-k}}\right| \geq \eta\left(\left|\frac{g(a_{n+k}) - g(a_n)}{g(a_n) - g(a_{n-k})}\right|^{-1}\right)^{-1} > \frac{1}{\eta(L)}.$$
$\square$

2.4. Quasisymmetric automorphism of $\mathbb{Z}$

From Lemma 2.2.1, the next theorem immediately follows:

Theorem B. *For a bijection $f : \mathbb{Z} \to \mathbb{Z}$, the following conditions are quantitatively equivalent;*

1. *f is η-quasisymmetric.*
2. *f satisfies the λ-three point condition.*
3. *f is M-biLipschitz.*
4. *f admits an M-biLipschitz extension $F : \mathbb{C} \to \mathbb{C}$.*
5. *f admits a K-quasiconformal extension $F : \mathbb{C} \to \mathbb{C}$.*

PROOF. First, (1) $\Rightarrow$ (2) is clear, see Section 2.1.

Next, we suppose f satisfies the λ-three point condition. By applying Lemma 2.2.1 to f and $A = \mathbb{Z}$ (that is, $a_n = n$), we have

$$\frac{k}{2\lambda} < |f(n+k) - f(n)| < 2\lambda k,$$

for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Thus we obtain (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4) is follows from Alestalo–Väisälä [**3**, Theorem 5.5].

Generally, M-biLipschitz mappings are M^2-quasiconformal, see Chapter 1. Thus we have (4) $\Rightarrow$ (5).

Finally, (5) $\Rightarrow$ (1) is also clear, since K-quasiconformal self-homeomorphisms of $\mathbb{C}$ are η-quasisymmetric with an η depending only on K (thus the restrictions to $\mathbb{Z}$ are also η-quasisymmetric with the same η).

□

REMARK 2.4.1. *An analogous theorem holds for the set $E = \{e^n\}_{n\in\mathbb{Z}}$, see Appendix 2.6. Also in this case, for a bijection of E, biLipschitzness and quasisymmetry are equivalent, however, this equivalence cannot be quantitative.*

2.5. The Väisälä problem

Now, we can prove that the integer set $\mathbb{Z}$ is an example of closed discrete subset for which the one dimensional Väisälä problem can be solved affirmatively, that is:

THEOREM C. *Every η-quasisymmetric embedding $f : \mathbb{Z} \to \mathbb{R}$ admits a $K = K(\eta)$-quasiconformal extension $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ where $K(\eta)$ is a constant depending only on η.*

PROOF. Let $f : \mathbb{Z} \to \mathbb{R}$ be an η-quasisymmetric embedding, and let $A := f(\mathbb{Z})$. Then, by Theorem A, there exists a K'-quasiconformal mapping $F : \mathbb{C} \to \mathbb{C}$ such that $F(\mathbb{Z}) = A$, where K' depends only on η. Since compositions of quasisymmetric mappings are also quasisymmetric, $F^{-1} \circ f : \mathbb{Z} \to \mathbb{Z}$ becomes an η'-quasisymmetric automorphism where η' depends only on η. By Theorem B, $F^{-1} \circ f$ admits a K''-quasiconformal extension $G : \mathbb{C} \to \mathbb{C}$, where K'' depends only on η. Therefore, we obtain a $K = K'K''$-quasiconformal extension $\widetilde{f} = F \circ G : \mathbb{C} \to \mathbb{C}$ of f. The proof is completed. □

REMARK 2.5.1. *Applying the last statement of Theorem A to the preceding proof, it turns out that every quasisymmetric embedding $f : \mathbb{Z} \to \mathbb{R}$ admits the decomposition $f = F \circ G$ where $F : \mathbb{Z} \to f(\mathbb{Z})$ is strictly monotone increasing and $G : \mathbb{Z} \to \mathbb{Z}$ satisfies the three point condition.*

2.6. Appendix: Quasisymmetric automorphisms of $\{e^n\}_{n\in\mathbb{Z}}$

Let $E = \{e^n\}_{n\in\mathbb{Z}}$. We first prove the following.

PROPOSITION 2.6.1. *If a bijection $f : E \to E$ satisfies λ-three point condition ($\lambda \geq 1$), $f(1) = 1$, and $f(e^n) \to 0$ as $n \to -\infty$, then f is M_λ-biLipschitz, where $M_\lambda = \dfrac{e^2}{e-1}\lambda(\lambda+1)^2$.*

Proof. Let $g := \log \circ f \circ \exp$, that is, $f(e^n) = e^{g(n)}$ for all $n \in \mathbb{Z}$. Then, by the assumptions, $g(0) = 0$ and $g(n) \to -\infty$ as $n \to -\infty$. Let $C_\lambda = \log(\lambda + 1)$.

CLAIM 1. *$g(k) - g(\ell) \leq C_\lambda$ if $k < \ell$.*

PROOF. Since $C_\lambda > 0$, it suffices to consider the case where $g(k) > g(\ell)$ and $k < \ell$. Since $g(n) \to -\infty$ as $n \to -\infty$, there exists an integer $j < k$ such that $g(j) < g(\ell)$. Thus, by the three point condition,

$$\lambda \geq \left| \frac{e^{g(j)} - e^{g(k)}}{e^{g(j)} - e^{g(\ell)}} \right| = \frac{e^{g(k)-g(\ell)} - e^{-(g(\ell)-g(j))}}{1 - e^{-(g(\ell)-g(j))}}.$$

Since $0 < e^{-(g(\ell)-g(j))} < 1$, we have $\lambda \geq e^{g(k)-g(\ell)} - 1$. Therefore $g(k) - g(\ell) \leq \log(\lambda + 1) = C_\lambda$. □

CLAIM 2. *$-(2C_\lambda + 1) \leq g(n) - n \leq 2C_\lambda + 1$ for all $n \in \mathbb{Z}$.*

PROOF. Let $n > 0$. By Claim 1 and $g(0) = 0$, we have $g(m) \geq -C_\lambda$ for all $m > 0$. Thus there exists an integer j ($0 \leq j < n$) such that $g(j) \geq -C_\lambda + n - 1$. Using Claim 1 again, we have

$$C_\lambda \geq g(j) - g(n) \geq -C_\lambda + n - 1 - g(n).$$

Thus $g(n) - n \geq -(2C_\lambda + 1)$. Further, to obtain a contradiction, we suppose $g(n) - n > 2C_\lambda + 1$. Note that $g(n) > 0$. Let

$$\begin{aligned} G &= \{1, 2, \cdots, g(n) - 1, g(n)\}, \\ H &= \{g(1), g(2), \cdots, g(n-1), g(n)\}, \\ I &= \{g(n+1), g(n+2), g(n+3), \cdots\}. \end{aligned}$$

By Claim 1, $g(n)-C_\lambda \leq g(m)$ for any $m > n$. This implies $\#(G\cap I) \leq C_\lambda$. Since $g : \mathbb{Z} \to \mathbb{Z}$ is bijective, we have

$$\begin{aligned} \#\left(G \setminus (H \cup I)\right) &= \#G - \#(G\cap H) - \#(G\cap I) \\ &\geq g(n) - n - C_\lambda > C_\lambda + 1. \end{aligned}$$

Thus there exists an integer $j < 0$ such that $g(j) > C_\lambda + 1$. By Claim 1, we have a contradiction;

$$C_\lambda \geq g(j) - g(0) > C_\lambda + 1.$$

Therefore $g(n) - n \leq 2C_\lambda + 1$.

The same argument can be applied to $n < 0$. We have the claim. □

CLAIM 3. *Proposition 2.6.1 holds.*

PROOF. Let $n, m \in \mathbb{Z}$ $(n > m)$ and let

$$A = \left|\frac{f(e^n) - f(e^m)}{e^n - e^m}\right| = e^{g(n)-n}\left|\frac{1 - e^{g(m)-g(n)}}{1 - e^{m-n}}\right|.$$

First, suppose $g(n) > g(m)$. Since $0 < e^{g(m)-g(n)}$, $e^{m-n} \leq e^{-1}$ and by Claim 2, we have

$$\begin{aligned} A &\leq e^{2C_\lambda+1}\frac{1}{1-e^{-1}} = \frac{e^2}{e-1}(\lambda+1)^2, \\ A &\geq e^{-(2C_\lambda+1)}\left(1 - \frac{1}{e}\right) = \left(\frac{e^2}{e-1}(\lambda+1)^2\right)^{-1}. \end{aligned}$$

Next, we suppose $g(n) < g(m)$. Note that, $1 \leq g(m) - g(n) \leq C_\lambda$ by Claim 1. Similarly we have

$$\begin{aligned} A &\leq e^{2C_\lambda+1}\frac{e^{C_\lambda}-1}{1-e^{-1}} = \frac{e^2}{e-1}\lambda(\lambda+1)^2, \\ A &\geq e^{-(2C_\lambda+1)}(e-1) = \left(\frac{e}{e-1}(\lambda+1)^2\right)^{-1}. \end{aligned}$$

Since $\lambda \geq 1$, we have $1/M_\lambda \leq A \leq M_\lambda$. □

COROLLARY 2.6.2. *For a bijection $f : E \to E$, f is quasisymmetric if and only if f is biLipschitz.*

PROOF. Generally, M-biLipschitz mappings are $\eta(t) = M^2t$-quasisymmetric. Thus "if" part follows.

Suppose f is η-quasisymmetric, and let $F(z) = f(z)/f(1)$. Then F is also η-quasisymmetric with the same η. Since quasisymmetric mappings take a Cauchy sequence to a Cauchy sequence, we have $F(e^n) \to 0$ as $n \to -\infty$. Further $F(1) = 1$ and F satisfies the $\lambda := \eta(1)$-three point condition, see Remark 2.1.2. Thus, by Proposition 2.6.1, F is M_λ-biLipschitz. Therefore $f(z) = f(1)F(z)$ is $\max\{f(1)M_\lambda, M_\lambda/f(1)\}$-biLipschitz. □

REMARK 2.6.3. *Contrary to Theorem B, the equivalence in Corollary 2.6.2 cannot be quantitative: Let $f_n : E \to E, z \mapsto e^n z$ for $n \in \mathbb{N}$. Then f_n is $\eta(t) = t$-quasisymmetric for any $n \in \mathbb{N}$. However f_n is e^n-biLipschitz, and this biLipschitz constant is sharp.*

Finally, we analogously obtain the following theorem.

THEOREM 2.6.4. *For a bijection $f : E = \{e^n\}_{n\in\mathbb{Z}} \to E$, the following conditions are quantitatively equivalent;*

1. *f is η-quasisymmetric.*
2. *f satisfies the λ-three point condition, and $f(e^n) \to 0$ as $n \to -\infty$.*
3. *f admits a K-quasiconformal extension $F : \mathbb{C} \to \mathbb{C}$.*

PROOF. (1) ⇒ (2) follows for the same reason as in the preceding proof.

Suppose f satisfies the λ-three point condition and $f(e^n) \to 0$ as $n \to -\infty$. Then $g(z) = f(z)/f(1)$ also satisfies the λ-three point condition and $g(e^n) \to 0$ as $n \to -\infty$. Since $g(1) = 1$, applying Proposition 2.6.1, it turns out that g is M_λ-biLipschitz. For the same reason as in the proof of Theorem B, g admits a K-quasiconformal extension $G : \mathbb{C} \to \mathbb{C}$ where K is a constant depending only on λ. We obtain a K-quasiconformal extension $F(z) = f(1)G(z)$ $(z \in \mathbb{C})$ of f.

(3) ⇒ (1) also follows for the same reason as in the proof of Theorem B. □

REMARK 2.6.5. *In condition (2), $f(e^n) \to 0$ as $n \to -\infty$ is necessary. More precisely, the λ-three point condition does not imply this property. In fact, $f : E \to E$, $e^n \mapsto e^{-n}$ satisfies the 1-three point condition, but $f(e^n) \to \infty$ as $n \to -\infty$.*

CHAPTER 3

Periodic quasiconformal deformations

3.1. Periodic quasiconformal deformations of $\mathbb{C} \setminus \mathbb{Z}$

Let R be a Riemann surface, and let $\mathrm{Aut}(R)$ be the group of all biholomorphic automorphisms of R. For $h \in \mathrm{Aut}(R)$, the order of h, denoted by $\mathrm{ord}(h)$, is the minimum positive integer k such that the k-times iteration $h^k = \overbrace{h \circ \cdots \circ h}^{k} = \mathrm{id}_R$. Further, if there are no such integers, we define $\mathrm{ord}(h) = \infty$.

Let us consider the Riemann surface $\mathbb{C} \setminus \mathbb{Z}$. By the removable isolated singularity theorem, we have $\mathrm{Aut}(\mathbb{C}\setminus\mathbb{Z}) = \langle -z \rangle \ltimes \langle z+1 \rangle$, $\mathrm{ord}(-z) = 2$ and $\mathrm{ord}(z+1) = \infty$. Let $R_n = (\mathbb{C} \setminus \mathbb{Z})/\langle z+n \rangle$ and $p_n : \mathbb{C} \setminus \mathbb{Z} \to R_n$ be the projection. Notice that R_n is an $(n+2)$-punctured Riemann sphere, that is, R_n is of finite type.

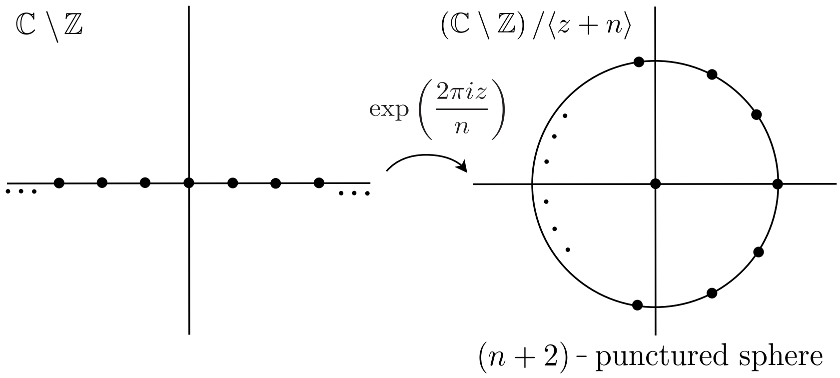

Figure 3

Let $f : R_n \to f(R_n)$ be an orientation preserving quasiconformal mapping from R_n to another Riemann surface $f(R_n)$. By the removable singularity theorem and the uniformization theorem, we may assume $f(R_n) = \widehat{\mathbb{C}} \setminus N$, $f(0) = 0$ and $f(\infty) = \infty$, where $\widehat{\mathbb{C}}$ denotes the Riemann sphere and $N \subset \widehat{\mathbb{C}}$ satisfies $\#N = n+2$ and $0, \infty \in N$. Let $\widehat{q} : \mathbb{C} \to \mathbb{C} \setminus \{0\}, z \mapsto e^{2\pi i z}$ and let $A = \widehat{q}^{-1}(N)$. Then,

$q = \widehat{q}|_{\mathbb{C}\setminus A} : \mathbb{C} \setminus A \to \mathbb{C} \setminus N$ is a covering with the covering transformation group $\mathrm{Deck}(q) = \langle z+1 \rangle$. Further, composing $z \mapsto -z$ if necessary, the mapping $f : R_n \to \mathbb{C} \setminus N$ lifts to an orientation preserving quasiconformal mapping $F : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C} \setminus A$ such that $F(z+n) = F(z) + 1$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$, that is, F is a periodic quasiconformal deformation of $\mathbb{C} \setminus \mathbb{Z}$ with period n.

REMARK 3.1.1. *We can easily see that the above $A = \widehat{q}^{-1}(N)$ has the form; $A = \mathbb{Z} + \{a_j\}_{j=1}^n$ where each $a_j \in \mathbb{C}$ satisfies* $\mathrm{Re}(a_j) \in [0, 1)$.

In this chapter, we investigate these periodic deformations of $\mathbb{C} \setminus \mathbb{Z}$.

3.2. Preliminaries

3.2.1. Porous sets. A subset $A \subset \mathbb{C}$ is said to be c-porous in $\mathbb{C}$ for $c \geq 1$ if any closed disk $\overline{D}(z', r)$ with radius $r > 0$ centered at $z' \in \mathbb{C}$ contains z such that $D(z, r/c) \subset \mathbb{C} \setminus A$. It can be easily seen that;

- $\mathbb{Z} + i\mathbb{Z}$ is not porous in $\mathbb{C}$.
- Any subset of $\mathbb{R}$ is 1-porous in $\mathbb{C}$, particularly, $\mathbb{Z}$ is 1-porous in $\mathbb{C}$.
- $A_1 = \mathbb{Z} + i\,\{2^n \mid n = 0, 1, 2, \cdots\}$ is 8-porous.

Väisälä pointed out that the porosity in $\mathbb{C}$ is preserved by quasiconformal self-homeomorphisms of $\mathbb{C}$ in [**25**]. Thus it immediately follows that $\mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z})$ is not quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$. However, in this way, we cannot decide whether $\mathbb{C} \setminus A_1$ is quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$ or not. By Theorem D proved in the next section, it will be turns out that $\mathbb{C} \setminus A_1$ is not quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$.

3.2.2. Quasiconformal mappings and Extremal distances. Let $D \subset \mathbb{C}$ be a domain. For given continua $C_1, C_2 \subset D$,

$$\delta^D(C_1, C_2) = \mathrm{mod}(\mathscr{F}^D(C_1, C_2))$$

is called the extremal distance between C_1 and C_2 in D, where mod denotes the modulus of a curve family and $\mathscr{F}^D(C_1, C_2)$ denotes the family of all rectifiable curves which join C_1 and C_2 in D. The definition of the modulus of a curve family is given by

$$\mathrm{mod}(\mathscr{F}) := \inf_{\rho} \int_{\mathbb{C}} \rho(x+iy)^2 dxdy.$$

where the infimum is taken over all non-negative Borel functions $\rho : \mathbb{C} \to [0, \infty]$ with $\displaystyle\int_\gamma \rho |dz| \geq 1$ for all rectifiable $\gamma \in \mathscr{F}$. We remark that the modulus of a curve family coincides with the reciprocal of the extremal length introduced by Beurling–Ahlfors [**4**].

It is well known that a homeomorphism $f : D \to \mathbb{C}$ into $\mathbb{C}$ becomes a K-quasiconformal mapping for a constant $K \geq 1$ if and only if f satisfies the following inequality for any curve families $\mathscr{F}$ in D, see [**14**, Chapter IV];

$$\frac{1}{K}\mathrm{mod}(\mathscr{F}) \leq \mathrm{mod}(f(\mathscr{F})) \leq K\mathrm{mod}(\mathscr{F}).$$

The next useful lower bound for extremal distances was given by Vuorinen in [**27**, Lemma 4.7]; For each pair of disjoint continua $C_1, C_2 \subset \mathbb{C}$, it holds that

$$\delta^{\mathbb{C}}(C_1, C_2) \geq \frac{2}{\pi}\log\left(1 + \frac{\min_{i=1,2}\mathrm{diam}(C_i)}{\mathrm{dist}(C_1, C_2)}\right).$$

3.3. Punctured planes with automorphisms of infinite order

Let $A \subset \mathbb{C}$ be a closed discrete subset. Suppose $\mathrm{Aut}(\mathbb{C} \setminus A)$ contains an automorphism h of infinite order. By the removable singularity theorem, h is the restriction of an Affine transformation of $\mathbb{C}$. Since $h(A) = A$ and A is closed and discrete in $\mathbb{C}$, h is conjugate to the parallel translation $z+1$. Therefore, we may assume A has the following form;

$$A = \mathbb{Z} + \{a_n\}_{n=1}^{k}$$

where $k \leq \infty$ and each $a_n \in \mathbb{C}$ satisfies $\mathrm{Re}(a_n) \in [0, 1)$.

Conversely, if A has this form, then $\mathrm{Aut}(\mathbb{C} \setminus A)$ contains the parallel translation $z + 1$.

THEOREM D. *Let $A \subset \mathbb{C}$ be a closed discrete subset which has the following form;*

$$A = \mathbb{Z} + \{a_n\}_{n=1}^{k}$$

where $k \leq \infty$ and each $a_n \in \mathbb{C}$ satisfies $\mathrm{Re}(a_n) \in [0, 1)$*. Then, $\mathbb{C} \setminus A$ is quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$ if and only if $k < \infty$.*

PROOF. (*Necessity*). To obtain a contradiction, assume $k = \infty$. Let $f : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal homeomorphism with $f(\mathbb{Z}) = A$, and by composing an Affine transformation, we may assume $0 \in A$, and $\sup\{\mathrm{Im}a_n \mid n \in \mathbb{N}\} = \infty$.

Under the above assumptions, we prove the following lemma:

LEMMA 3.3.1. $\sup_{m \in \mathbb{Z}} |\mathrm{Im}f(m) - \mathrm{Im}f(m+1)| = \infty.$

Proof. The subset A is c-porous for some $c \geq 1$ since $\mathbb{Z}$ is porous and f is quasiconformal. For any $r > 1$, we let $x = i\left\{(\sqrt{2}c+1)r+1\right\}$. Then by porousity of A, there exist $z \in \overline{D}(x, \sqrt{2}cr)$ such that $D(z, \sqrt{2}r) \subset \mathbb{C} \setminus A$. Then the square domain $\{w = u+iv \mid |u - \mathrm{Re}z| < r,\ |v - \mathrm{Im}z| < r\}$ does not intersect with A.

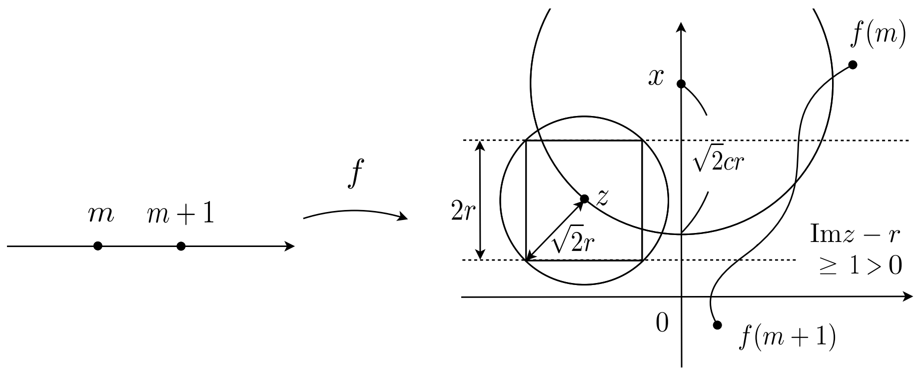

Figure 4

The followings are easily confirmed:

- $A \cap \{w \mid \mathrm{Im}z - r < \mathrm{Im}w < \mathrm{Im}z + r\} = \emptyset$, since $z+1 \in \mathrm{Aut}(\mathbb{C} \setminus A)$.
- $A \cap \{w \mid \mathrm{Im}w \geq \mathrm{Im}z + r\} \neq \emptyset$, since $\sup\{\mathrm{Im}a \mid a \in A\} = \infty$.
- $A \cap \{w \mid \mathrm{Im}z - r \geq \mathrm{Im}w\} \neq \emptyset$, since $0 \in A$ and $\mathrm{Im}z - r \geq 1$.

Therefore considering the image of real line under f, we have an integer $m \in \mathbb{Z}$ such that $|\mathrm{Im}f(m) - \mathrm{Im}f(m+1)| \geq 2r$. ☐

Continuation of Proof of Theorem D. By Lemma 3.3.1, there exists $m \in \mathbb{Z}$ such that

$$\ell := |\mathrm{Im}f(m) - \mathrm{Im}f(m+1)| > \exp\left(\frac{K\pi^2}{\log 2}\right).$$

Let

$$\begin{aligned} C_1' &:= \{f(m)+t \mid t \in [0,1]\}, \quad & C_1 &:= f^{-1}(C_1') \\ C_2' &:= \{f(m+1)+t \mid t \in [0,1]\}, \quad & C_2 &:= f^{-1}(C_2'). \end{aligned}$$

Then we have,

1. by quasiconformality of f

$$\delta^{\mathbb{C}}(C_1, C_2) \leq K\delta^{\mathbb{C}}(C_1', C_2'),$$

2. since C_1' and C_2' are separeted by the annulus $\{w \mid 1 < |w - f(m)| < \ell\}$
$$\delta^{\mathbb{C}}(C_1', C_2') \leq \frac{2\pi}{\log \ell} < \frac{2}{K\pi} \log 2,$$
3. from Vuorinen's theorem,
$$\delta^{\mathbb{C}}(C_1, C_2) \geq \frac{2}{\pi} \log\left(1 + \frac{\min_{i=1,2} \operatorname{diam}(C_i)}{\operatorname{dist}(C_1, C_2)}\right) \geq \frac{2}{\pi} \log\left(1 + \min_{i=1,2} \operatorname{diam} C_i\right).$$

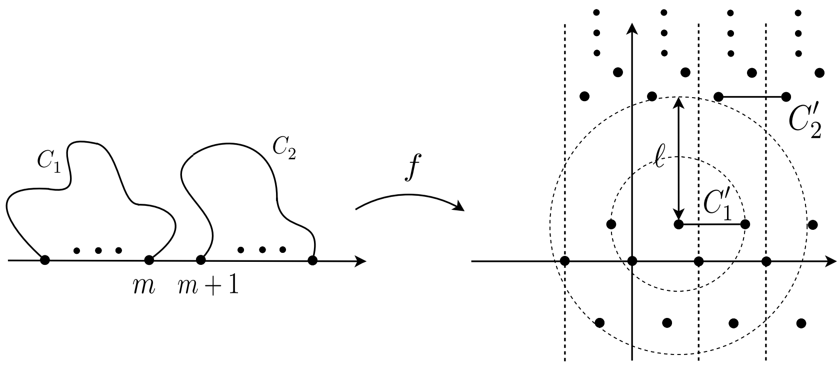

Figure 5

Combining the above inequalities, we obtain
$$\min_{i=1,2} \operatorname{diam} C_i < 1.$$
On the other hand, since each endpoints of C_i are in the integer set, $\operatorname{diam} C_i \geq 1$ $(i = 1, 2)$. This is a contradiction.

(*Sufficiency*). Since $(\mathbb{C} \setminus \mathbb{Z}) / \langle z+k \rangle$ and $(\mathbb{C} \setminus A) / \langle z+1 \rangle$ are $(k+2)$-punctured Riemann spheres, there exists a quasiconformal homeomorphism between them which fixes 0 and ∞. Then it can be lifted to a quasiconfromal homeomorphism between $\mathbb{C} \setminus \mathbb{Z}$ and $\mathbb{C} \setminus A$. $\square$

3.4. Teichmüller space of $\mathbb{C} \setminus \mathbb{Z}$ and periodic deformations

3.4.1. General definitions. We fix a Riemann surface R. Two orientation preserving quasiconformal deformations f and g of R are said to be Teichmüller equivalent to each other if there is a biholomorphic homeomorphism $h : f(R) \to g(R)$ such that $g \circ f^{-1}$ is homotopic to h relative to the ideal boundary $\partial f(R)$, see Nag [**19**]. The *Teichmüller space* $T(R)$ is the set of all Teichmüller equivalence classes

$[f]$ of all orientation preserving quasiconformal deformations f of R. Further, let $\mathrm{Mod}(R)$ be the group of all homotopy classes relative to ∂R of all orientation preserving quasiconformal self-homeomorphisms of R. The group $\mathrm{Mod}(R)$ is called the *Teichmüller modular group of* R, and $\mathrm{Mod}(R)$ acts on $T(R)$ by $[f]_* \cdot [g] = [g \circ f^{-1}]$ where $[f] \in \mathrm{Mod}(R)$ and $[g] \in T(R)$. Recall that this action is biholomorphic and isometric with respect to the Teichmüller distance $d_{T(R)}$, see Introduction.

3.4.2. The space of periodic deformations of $\mathbb{C}\setminus\mathbb{Z}$. By considering the liftings, the projection $p_n : \mathbb{C}\setminus\mathbb{Z} \to R_n$ induces the embedding $p_n^* : T(R_n) \to T(\mathbb{C}\setminus\mathbb{Z})$. We remark that the induced embedding is independent of the choice of liftings.

Let $T_n = p_n^*(T(R_n)) \subset T(\mathbb{C}\setminus\mathbb{Z})$. By the arguments of Section 3.1, the subspace T_n describes periodic quasiconformal deformations of $\mathbb{C}\setminus\mathbb{Z}$ with period n. More precisely, every $[f] \in T_n$ contains a quasiconformal deformation $F : \mathbb{C}\setminus\mathbb{Z} \to \mathbb{C}\setminus A$ such that A has the form $A = \mathbb{Z} + \{a_j\}_{j=1}^n$ ($\mathrm{Re}(a_j) \in [0,1)$), and $F(z+n) = F(z)+1$ for all $z \in \mathbb{C}\setminus\mathbb{Z}$, see Remark 3.1.1. We call such a deformation as F, a *regular representative of* $[f]$. By the McMullen theorem [**18**, Corollary 1.2], T_n is a geodesically convex subspace of $T(\mathbb{C}\setminus\mathbb{Z})$ with respect to $d_{T(\mathbb{C}\setminus\mathbb{Z})}$.

REMARK 3.4.1. *Regular representatives cannot be uniquely determined at all, since any quasiconformal deformations of R_n lift to regular representatives.*

Let $T_0 = \bigcup_{n=1}^{\infty} T_n$. By definition, T_0 is the space which describes all periodic quasiconformal deformations of $\mathbb{C}\setminus\mathbb{Z}$.

LEMMA 3.4.2. *T_0 is geodesically convex and separable.*

PROOF. Let $n, m \in \mathbb{N}$. If m divides by n, then there is a covering $p_{n,m} : R_m \to R_n$ such that $p_n = p_{n,m} \circ p_m$. Thus we have $T_n \subset T_m$. Let $c_n = \prod_{k=1}^n k$. Then we have an increasing sequence $T_{c_1} \subset T_{c_2} \subset T_{c_3} \subset \cdots$ of subspaces of $T(\mathbb{C}\setminus\mathbb{Z})$. Clearly, it follows that;

$$T_0 = \bigcup_{n=1}^{\infty} T_{c_n}.$$

Since any two points $[f], [g] \in T_0$ are contained in some T_{c_n}, and since T_{c_n} is geodesically convex, there is a geodesic of $T(\mathbb{C}\setminus\mathbb{Z})$ which joins $[f]$ and $[g]$ in $T_{c_n} \subset T_0$. Thus T_0 is geodesically convex. Further, each T_{c_n} are finite dimensional, in particular, separable. Therefore T_0 is also separable. □

3.4.3. Periodic deformations and limit set of $\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})$**.** Let T_∞ be the set of all $[f] \in T(\mathbb{C}\setminus\mathbb{Z})$ such that $\mathrm{Aut}(f(\mathbb{C}\setminus\mathbb{Z}))$ contains an automorphism of infinite order. Since the stabilizer $\mathrm{Stab}_{\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})}([f])$ is isomorphic to $\mathrm{Aut}(f(\mathbb{C}\setminus\mathbb{Z}))$ for any $[f] \in T(\mathbb{C}\setminus\mathbb{Z})$, see [**7**, Lemma 2], if $[f] \in T_\infty$ then $\#\mathrm{Stab}_{\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})}([f]) = \infty$. Namely, $\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})$ does not act properly discontinuously on $\overline{T_\infty} \subset T(\mathbb{C}\setminus\mathbb{Z})$. As a corollary of Theorem D, we have the following:

Theorem E. $T_0 = \bigcup_{n=1}^{\infty} T_n$ *is separable and geodesically convex. Further, the following equality holds;*

$$T_\infty = \bigcup_{[f]\in\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})} [f]_*(T_0).$$

Proof. The first statement follows from Lemma 3.4.2. Thus we prove the above equality. By the definition of T_0, clearly $\bigcup_{[f]\in\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})}[f]_*(T_0) \subset T_\infty$. Conversely, let $[g] \in T_\infty$. Then we may assume $g(\mathbb{C}\setminus\mathbb{Z}) = \mathbb{C}\setminus A$, where $A = \mathbb{Z} + \{a_n\}_{n=0}^{k}$, $k < \infty$ and $\mathrm{Re}(a_n) \in [0,1)$ by Theorem D, see also Section 3.3. Then by the proof of the sufficiency part of Theorem D, there is a periodic deformation $[h] \in T_0$ such that $h(\mathbb{C}\setminus\mathbb{Z}) = \mathbb{C}\setminus A = g(\mathbb{C}\setminus\mathbb{Z})$. Let $f = g^{-1}\circ h$. Then $[f] \in \mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})$ and

$$[f]_*\cdot[h] = [h\circ f^{-1}] = [g].$$

Thus, we have the converse inclusion. $\square$

Theorem F. *Let* $[f] \in \mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})$*. If* $[f]_*(T_0)\cap T_0 \neq \emptyset$*, then* $[f]_*(T_0) = T_0$.

Proof. Let $[g],[h] \in T_0$ such that $[f]_*\cdot[g] = [h]$. We may assume that g and h are regular representatives of $[g]$ and $[h]$ respectively. Namely,

$$g : \mathbb{C}\setminus\mathbb{Z} \to \mathbb{C}\setminus A \quad (A = \mathbb{Z} + \{a_j\}_{j=1}^{n},\ \mathrm{Re}(a_j) \in [0,1)),$$
$$h : \mathbb{C}\setminus\mathbb{Z} \to \mathbb{C}\setminus B \quad (B = \mathbb{Z} + \{b_j\}_{j=1}^{m},\ \mathrm{Re}(b_j) \in [0,1)),$$

with $g(z+n) = g(z)+1$, $h(z+m) = h(z)+1$ for all $z \in \mathbb{C}\setminus\mathbb{Z}$.

Since $[f]_*\cdot[g] = [g\circ f^{-1}] = [h]$, there is a biholomorphic homeomorphism $\phi : \mathbb{C}\setminus A \to \mathbb{C}\setminus B$ which is homotopic to $h\circ(g\circ f^{-1}) = h\circ f\circ g^{-1}$. Let $F = h^{-1}\circ\phi\circ g : \mathbb{C}\setminus\mathbb{Z} \to \mathbb{C}\setminus\mathbb{Z}$. Then F is homotopic to f, that is, $[F] = [f]$ in $\mathrm{Mod}(\mathbb{C}\setminus\mathbb{Z})$. By the removable singularity theorem, ϕ extends to a biholomorphic automorphism of $\mathbb{C}$ such that $\phi(A) = B$. Thus $\phi(z) = az+b$ for some $a,b \in \mathbb{C}$ ($a \neq 0$). Further, we can see $a = \pm n/m$ by elementary arguments. If $a = -n/m$, replacing g by $j\circ g : \mathbb{C}\setminus\mathbb{Z} \to \mathbb{C}\setminus j(A)$, and ϕ by $\phi\circ j : \mathbb{C}\setminus j(A) \to \mathbb{C}\setminus B$ where

$j(z) = -z$, we may assume $a = n/m$ (remark that $[g] = [j \circ g]$). Let $N = nm$. Then we have,

$$\begin{aligned} F(z+N) &= h^{-1} \circ \phi(g(z)+m) \\ &= h^{-1}\left(\frac{n}{m}g(z)+n+b\right) \\ &= h^{-1}\left(\frac{n}{m}g(z)+b\right)+nm = F(z)+N. \end{aligned}$$

Take $[G] \in T_0$, and assume G is a regular representative so that $G(z+t) = G(z)+1$ where t is a positive integer. Then for any $z \in \mathbb{C}\setminus\mathbb{Z}$,

$$\begin{aligned} G \circ F^{-1}(z+Nt) &= G(F^{-1}(z)+Nt) \\ &= G \circ F^{-1}(z) + N. \end{aligned}$$

Thus $G \circ F^{-1}$ induces the quasiconformal deformation of $R_{Nt} = (\mathbb{C} \setminus \mathbb{Z})/\langle z+Nt\rangle$, that is, $[f]_* \cdot [G] = [F]_* \cdot [G] = [G \circ F^{-1}] \in T_0$. We have $[f]_*(T_0) \subset T_0$.

Applying the same arguments to $[f^{-1}]_*$, we have the converse inclusion $T_0 \subset [f]_*(T_0)$. □

3.5. Appendix: Another example: $\mathbb{C}^* \setminus \{e^n\}_{n\in\mathbb{Z}}$

Let $E = \{e^n\}_{n\in\mathbb{Z}}$ and let us consider the Riemann surface $\mathbb{C}^* \setminus E$ where $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. We can easily show that $\mathrm{Aut}(\mathbb{C}^*\setminus E) = \langle 1/z\rangle \ltimes \langle ez\rangle$. Let $S_n = (\mathbb{C}^* \setminus E)/\langle e^n z\rangle$ for $n \in \mathbb{N}$, and let $q_n : \mathbb{C}^* \setminus E \to S_n$ be the projection. Notice that S_n is an n-punctured torus.

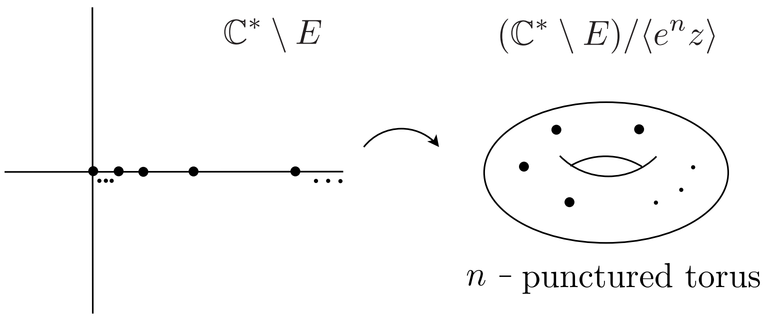

Figure 6

For the same reason as in the arguments for $\mathbb{C} \setminus \mathbb{Z}$, for every quasiconformal deformation of $\mathbb{C}^* \setminus E$, there is a closed discrete subset $A \subset \mathbb{C}^*$ such that $f(\mathbb{C}^* \setminus E)$ is biholomorphic to $\mathbb{C}^* \setminus A$. In this case, the following theorem which corresponds to Theorem D holds. This can be proved far more easily than the case of $\mathbb{C} \setminus \mathbb{Z}$, because of the relative compactness of the fundamental domain of $\langle e^n z \rangle$.

THEOREM 3.5.1. *Let $A \subset \mathbb{C}^*$ be a closed discrete infinite subset. If $\mathrm{Aut}(\mathbb{C}^* \setminus A)$ contains an automorphism of infinite order, then the followings hold:*

1. *$\mathbb{C}^* \setminus A$ is quasiconformally equivalent to $\mathbb{C}^* \setminus E$.*
2. *For any $h \in \mathrm{Aut}(\mathbb{C}^* \setminus A)$ with $\mathrm{ord}(h) = \infty$, the quotient space $(\mathbb{C}^* \setminus A)/\langle h \rangle$ is a finitely often punctured torus.*

PROOF. Let $h \in \mathrm{Aut}(\mathbb{C}^* \setminus A)$ such that $\mathrm{ord}(h) = \infty$. By the removable singularity theorem, h extends to a biholomorphic automorphism of $\mathbb{C}^*$ such that $h(A) = A$. Thus $h(z) = \lambda z$ or $h(z) = \lambda/z$ for some $\lambda \in \mathbb{C}^*$. Since $\mathrm{ord}(h) = \infty$, the latter case cannot occur and λ is not a root of unity. If $|\lambda| = 1$, then for any $a \in A$, the orbit $\langle h \rangle(a) = \{h^k(a) \mid k \in \mathbb{Z}\} \subset A$ has an accumulation point in $\mathbb{C}^*$. This contradicts that A is closed and discrete in $\mathbb{C}^*$. If $|\lambda| < 1$, then the inverse of h is $\lambda^{-1}z$. Thus, we may assume $h(z) = \lambda z$ with $|\lambda| > 1$. Since the fundamental domain $D = \{z \in \mathbb{C}^* \mid 1 < |z| < |\lambda|\}$ of $\langle h \rangle$ is relatively compact in $\mathbb{C}^*$, $\overline{D} \cap A$ contains at most a finite number of points (and contains at least one point). Thus $((\mathbb{C}^* \setminus A)/\langle h \rangle$ is a finitely often punctured torus.

Further, we assume that $(\mathbb{C}^* \setminus A)/\langle h \rangle$ is an n-punctured torus. Since $(\mathbb{C}^* \setminus E)/\langle e^n z \rangle$ is also an n-punctured torus, there exists a quasiconformal mapping $f : (\mathbb{C}^* \setminus E)/\langle e^n z \rangle \to (\mathbb{C}^* \setminus A)/\langle h \rangle$, and f lifts to a quasiconformal mapping between $\mathbb{C}^* \setminus E$ and $\mathbb{C}^* \setminus A$. Thus we have the claim. □

Similarly, we set $T_n(\mathbb{C}^* \setminus E) = q_n^*(T(S_n))$, $T_0(\mathbb{C}^* \setminus E) = \bigcup_{n=1}^{\infty} T_n(\mathbb{C}^* \setminus E)$, and set $T_\infty(\mathbb{C}^* \setminus E)$ be the set of all $[f] \in T(\mathbb{C}^* \setminus E)$ such that $\mathrm{Aut}(f(\mathbb{C}^* \setminus E))$ contains an automorphism of infinite order. Then, every $T_n(\mathbb{C}^* \setminus E)$ is geodesically convex by the McMullen theorem. Thus, we have analogous theorems as follows;

THEOREM 3.5.2. *$T_0(\mathbb{C}^* \setminus E)$ is separable and geodesically convex. Further, the following equality holds;*

$$T_\infty(\mathbb{C}^* \setminus E) = \bigcup_{[f]\in \mathrm{Mod}(\mathbb{C}^* \setminus E)} [f]_*(T_0(\mathbb{C}^* \setminus E)).$$

THEOREM 3.5.3. *Let $[f] \in \mathrm{Mod}(\mathbb{C}^* \setminus E)$. If $[f]_*(T_0(\mathbb{C}^* \setminus E)) \cap T_0(\mathbb{C}^* \setminus E) \neq \emptyset$, then $[f]_*(T_0(\mathbb{C}^* \setminus E)) = T_0(\mathbb{C}^* \setminus E)$.*

3.6. Appendix: Natural question

By the above observations, a natural question arises; Can analogous arguments be applied to Riemann surfaces which have the following properties?

1. It has an automorphism of infinite order.
2. For any automorphism of infinite order, the quotient space by the action of its cyclic group is of finite type.

For example, the Riemann surface R_0 defined by $w^2 = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$ has the above properties.

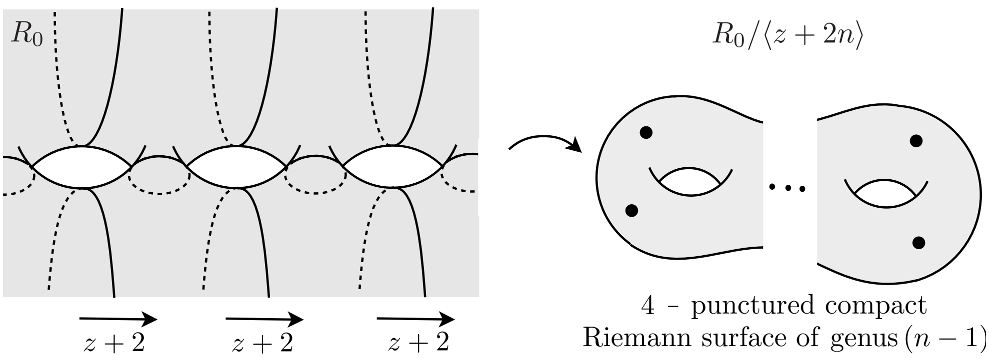

Figure 7

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