Cyclic Cohomology Groups of Some Self-similar Sets

Takashi Maruyama

Contents

1 Introduction

In this thesis, by exploiting cyclic cohomology theory and the Young integration, we develop a generalisation of the *de Rham homology theory* for a certain class of self-similar sets and also exhibit some examples of its application.

Fractal sets introduced by Mandelbrot [21] behave in a complicated way, and their behaviour makes it difficult to analyse fractal sets themselves. For instance, the Cantor sets take different values as their Hausdorff dimensions even though they are homeomorphic to each other. The Hausdorff dimension is considered as an invariant of fractal sets, that is stable under bi-Lipschitz transformations but not under arbitrary homeomorphisms. Thus, it is difficult for a (co)homology theory to detect fractal invariants such as the Hausdorff dimension and the Minkowski content.

Then, Connes introduced cyclic cohomology theory [5]. He proposed Quantised calculus in [6] and exploits the Dixmier trace as a non-smooth analogue of the integration on manifolds. In particular, he applied it to the Cantor sets and succeeded to recover their Minkowski contents as a certain value of the Dixmier trace; see [6] for the details. Thus one can expect that cyclic cohomology theory is a highly capable tool to analyse fractal sets.

On the other hand, cyclic cohomology theory is also known as a generalisation of the de Rham homology theory. Let us briefly review cyclic cohomology theory here. It is a generalised cohomology for an arbitrary algebra A over a ring R, and a cyclic k-cocycle ϕ of A is an R-linear map to R from the $(k+1)$-fold tensor product of A satisfying the following two conditions:

(a) $$\sum_{i=0}^{k}(-1)^i\phi(a_0,\cdots,a_ia_{i+1},\cdots,a_{k+1})+(-1)^{k+1}\phi(a_{k+1}a_0,a_1,\cdots,a_k)=0,$$

(b) $$\phi(a_0,a_1,\cdots,a_k)=(-1)^k\phi(a_k,a_0,a_1,\cdots,a_{k-1}).$$

A typical example of cyclic cocycles is the integration along a submanifold or a simplicial cycle contained in an oriented smooth manifold V. More precisely, for a given k-dimensional cycle C of V, we get the following cyclic k-cocycle:

$$\phi_C = \int_C : \underbrace{C^\infty(V)\otimes\cdots\otimes C^\infty(V)}_{k+1} \to \mathbb{C}, \quad \phi_C(f_0, f_1, \cdots, f_k) = \int_C f_0 df_1 \ldots df_k.$$

Here, $C^\infty(V)$ denotes the algebra of smooth functions on V and d the exterior derivation. The cocycle ϕ_C satisfies the conditions (a) and (b) due to the Stokes theorem and the skew derivation of the differential forms. Connes proved that the above cocycles essentially exhaust all classes of the de Rham homology group. Namely,

Theorem 1.1. [5, Theorem 46] *Let V be a compact smooth manifold and $C^\infty(V)$ the algebra of smooth functions on V topologised by the Fréchet topology. Then, for each $k \in \mathbb{Z}_{\geqslant 0}$, the k-th cyclic cohomology group $HC^k(C^\infty(V))$ is canonically isomorphic to the direct sum*

$$\ker b_k \,\oplus\, H_{k-2}(V;\mathbb{C}) \,\oplus\, H_{k-4}(V;\mathbb{C}) \,\oplus\, \cdots,$$

where b_k denotes the k-th boundary map of the de Rham homology theory and $H_k(V;\mathbb{C})$ the k-th de Rham homology group of V. In particular, with $k > \dim(V)$, it follows that:

$$HC^k(C^\infty(V)) \cong H_k(V;\mathbb{C}) \oplus H_{k-2}(V;\mathbb{C}) \,\oplus\, H_{k-4}(V;\mathbb{C}) \,\oplus\, \cdots.$$

The element of the direct summand $H_{k-2j}(V;\mathbb{C})$ is obtained by the cyclic cocycle described above. In other words, given a $(k-2j)$-cycle C with $[C] \in H_{k-2j}(V;\mathbb{C})$, the corresponding cyclic cohomology class in $HC^k(C^\infty(V))$ is given by ϕ_C (plus S-stabilisation in general).

Theorem 1.1 proves that cyclic cohomology theory can be considered as a generalisation of the de Rham homology theory. It also suggests a possibility to extend the de Rham theory even to a fractal set, where no notion of *smooth functions* is established, by exploiting cyclic cohomology theory. In order to do this, one has to find out a suitable subalgebra associated with fractal sets that replaces the role of $C^\infty(V)$.

There is already an attempt due to Moriyoshi and Natsume [23] in this direction. They exploit the algebra $C^{Lip}(SG)$ of Lipschitz functions on the Sierpinski gasket SG to construct a cyclic 1-cocycle ϕ on $C^{Lip}(SG)$. They also show that, when Lipschitz functions are considered as 1-Hölder continuous functions with $\alpha = 1$, the domain of the cyclic cocycle ϕ can be extended to a larger algebra $C^\alpha(SG)$, the algebra of complex-valued α-Hölder continuous functions on SG, where α is greater than the half of the Hausdorff dimension: $\alpha > \dim_H(SG)/2$. Those results suggest that $C^\alpha(SG)$ can be a candidate of the non-smooth analogue of $C^\infty(V)$ and that cyclic cohomology theory could capture a deeper topological quantity such as the Hausdorff dimension of the Sierpinski gasket.

In order to define the cyclic 1-cocycle ϕ, Moriyoshi and Natsume exploit the Young integration, which played the key role to relate $C^\alpha(SG)$ and $\dim_H(SG)$ in [23]. The Young integration on the unit interval $I = [0,1]$ was originally developed in [34]. This is a bilinear function from the product $W_\alpha \times W_\beta$ of the Wiener classes such that $\alpha + \beta > 1 = \dim_H(I)$ to complex numbers:

$$Y : W_\alpha \times W_\beta \to \mathbb{C};$$

see [34] for the details. In particular, the map Y is well-defined if it is restricted to the algebra $C^\alpha(I)$ of α-Hölder continuous functions for $2\alpha > 1 = \dim_H(I)$. We note that $C^\alpha(I)$ contains the algebra $C^\infty(I)$ of smooth functions. Moreover, the Young integration coincides with the integration of a smooth 1-form fdg with $f, g \in C^\infty(I)$:

$$Y(f,g) = \int_I f dg.$$

Therefore, the Young integration is a generalisation of the integration for 1-forms that may not come from smooth functions. The Young integration is easily extended to a Jordan curve C composed of a finite number of unit intervals, and $Y(f,g)$ along C turns out to be a cyclic 1-cocycle if $2\alpha > 1 = \dim_H(C)$:

$$Y : C^\alpha(C) \times C^\alpha(C) \to \mathbb{C}.$$

Thus one has $[Y] \in HC^1(C^\alpha(C))$ and it gives rise to an element of the *generalised de Rham homology group*. Motivated by those results, Moriyoshi and Natsume define a cyclic 1-cocycle ϕ of $C^{Lip}(SG)$ by exploiting the Young integration.

In this thesis, we extend the cocycle of the Sierpinski gasket defined in [23] to a certain class of self-similar sets by exploiting the Young integration as an analogue of the integration on manifolds, and show that the cocycle can be applied to a variety of examples. More detailed and precise statements are given as follows.

We first define cellular self-similar sets, the preliminary notions of which are given in Section 3.1 below. A cellular self-similar set $K_{|X|}$ is a self-similar set that is a projective limit of a sequence of certain cell complexes in $\mathbb{R}^2$, and the unit square is a prototype of cellular self-similar sets. The precise definition is as follows:

Definition 1.2 (Definition 5.1)**.** Let $|X|$ be a 2-dimensional finite convex linear cell complex and $\{F_j\}_{j\in S}$ a set of similitudes $F_j : |X| \to |X|$ indexed by a finite set S. We also let $|X_1| = \bigcup_{j\in S} F_j(|X|)$. The triple $(|X|, S, \{F_j\}_{j\in S})$ is called a *cellular self-similar structure* if it satisfies

a) $\partial|X| \subset \partial|X_1|$, and

b) $\operatorname{int} F_i(|X|) \cap \operatorname{int} F_j(|X|) = \emptyset$, for all $i \neq j \in S$.

Let $(|X|, S, \{F_j\}_{j\in S})$ be a cellular self-similar structure. Then $(|X|, S, \{F_j\}_{j\in S})$ yields a sequence $\{|X_n|\}_{n\in\mathbb{N}}$ of 2-dimensional cell complexes, and, by Theorem 3.2

below, the sequence gives rise to the cellular self-similar set $K_{|X|}$ with respect to $(|X|, S, \{F_j\}_{j \in S})$.

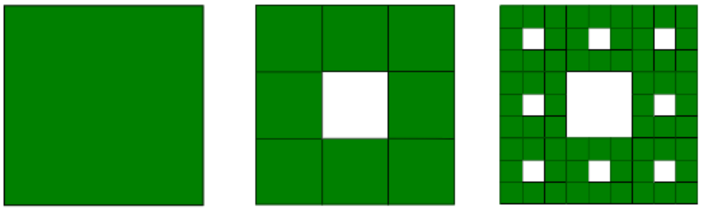


Figure 1: A part of a sequence whose projective limit is the Sierpinski carpet: $|X_0|$, $|X_1|$, $|X_2|$

For every $n \in \mathbb{N}$, $|X_n|$ is subdivided into a simplicial complex $|X_n^s|$ by Lemma 1 of Chapter 1 in [35]. From the resulting simplicial complex, we get a 1-cycle $I_n \in \tilde{S}_1(X_n^s; \mathbb{C})$ whose geometric incarnation is a union of all lacunas in $|X_n^s|$; see also Section 5.1.

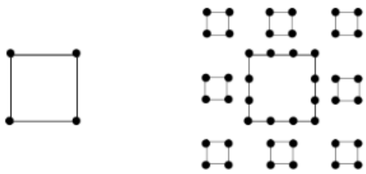


Figure 2: I_0, I_1, I_2

On the other hand, the algebra $C^\alpha(|X_n^s|)$ of complex-valued α-Hölder continuous functions defined on $|X_n^s|$ is a subspace of the function space $F^0(|X_n^s|; \mathbb{C}) = \{f : |X_n^s| \to \mathbb{C}\ \}$ as a $\mathbb{C}$-vector space. Therefore, $C^\alpha(|X_n^s|)$ can generate a $\mathbb{C}$-vector space $C^{\alpha,1}(|X_n^s|)$ with the differential and cup product of the Alexander-Spanier cochain complex [27]. The space $C^{\alpha,1}(|X_n^s|)$ consists of $f \cup \delta g - g \cup \delta f$ for any $f, g \in C^\alpha(|X_n^s|)$, and this is a subspace of $\mathrm{Hom}_{\mathbb{C}}(\tilde{S}_1(|X_n^s|; \mathbb{C}), \mathbb{C})$; see Section 5.2 for the details. For f and $g \in C^\alpha(K_{|X|})$ we have a cochain $f \cup \delta(g) - g \cup \delta(f) \in C^{\alpha,1}(|X_n^s|)$ for any $n \in \mathbb{N}$, which is denoted by $\omega_n(f, g)$. Finally we set $\phi_n(f, g)$ as $\omega_n(f, g)(I_n)$ and call the sequence $\{\phi_n(f, g)\}_{n \in \mathbb{N}}$ the approximating cyclic 1-cocycle of f and g, the definition of which is given in Section 5.2. We remark that, when f, g are α-Hölder continuous functions on the unit interval I, the limit $\lim_{n\to\infty} \phi_n(f, g)$ turns out to be the Young integration on I.

The first main theorem states that if $2\alpha > \dim_H(K_{|X|})$, we can define a bilinear map $\phi : C^\alpha(K_{|X|}) \times C^\alpha(K_{|X|}) \to \mathbb{C}$ by taking the limit of the approximating cyclic1-cocycle $\{\phi_n(f, g)\}_{n\in\mathbb{N}}$. This implies that the bilinear map ϕ may be seen as a generalisation of the classical Young integration on the unit interval.

Theorem 1.3 (Theorem 5.11, Existence theorem)**.** *Let $(|X|, S, \{F_j\}_{j\in S})$ be a cellular self-similar structure with $|X| \neq |X_1|$ and $K_{|X|}$ the cellular self-similar set with respect to $(|X|, S, \{F_j\}_{j\in S})$. We also let $C^\alpha(K_{|X|})$ be the algebra of α-Hölder continuous functions on $K_{|X|}$. If $2\alpha > \dim_H(K_{|X|})$, then for any $f,\ g \in C^\alpha(K_{|X|})$ the approximating cyclic1-cocycle $\{\phi_n(f, g)\}$ is a Cauchy sequence.*

The map ϕ was originally defined by Moriyoshi and Natsume [23] for the algebra $C^{Lip}(SG)$ of complex-valued Lipschitz functions on the Sierpinski gasket SG, and the construction is based on the classical Young integration on the unit interval. They use the simplexes I_n to prove the existence of the cyclic cocycle of the Sierpinski gasket. An obstacle to extend the construction to cellular self-similar sets is that, for each $n \in \mathbb{N}$, the lengths of 1-simplices belonging in $|X_n|$ are not equal. The key technical ingredient to overcome the difficulty is the existence of 2-dimensional simplicial complex $|K^s_{n,n+1}|$ whose boundary is a disjoint union of $\partial(|X_n|)$ and $\partial(|X_{n+1}|)$. By properties of cellular self-similar sets, we can prove that lengths of 1-simplices of $|K_{n,n+1}|$ have an upper bound which tends to 0 as $n \to \infty$. This property plays a crucial role to prove that $\{\phi_n(f, g)\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

The proof of the above theorem immediately yields the following theorem. This theorem proves that the bilinear map ϕ is a non-commutative representation of the Young integration.

Theorem 1.4 (Theorem 5.13). *For any $f,\ g \in C^{\alpha}(K_{|X|})$ with $2\alpha > \dim(K_{|X|})$, we have*

$$\phi(f,g) = -2 \cdot Y(f,g)|_{\partial|X|} = -2 \cdot \text{ (Young integration of } f \text{ and } g \text{ along } \partial|X|).$$

In particular, if $|X| \neq |X_1|$, for 1 and $x := id \in C^{\alpha}(K_{|X|})$, we get

$$\phi(1,x) = -2 \cdot Y(1,x)|_{\partial|X|} = -2 \cdot (\text{length of } \partial|X|).$$

After we define ϕ of the cellular self-similar set $K_{|X|}$, we prove that ϕ is a cyclic 1-cocycle of $C^{\alpha}(K_{|X|})$ and represents a nontrivial element in the first cyclic cohomology group $HC^1(C^{\alpha}(K_{|X|}))$. This theorem shows that ϕ may be seen as a non-commutative generalisation of the integration on manifolds.

Theorem 1.5 (Theorem 5.15). *Under the assumption of the existence theorem* :

a) The bilinear map ϕ is a cyclic 1-cocycle of $C^{\alpha}(K_{|X|})$.

b) The cocycle ϕ represents a non-trivial element $[\phi]$ in $HC^1(C^{\alpha}(K_{|X|}))$.

For the proof of the first statement, we need to use the Leibniz rule of the cup product defined on the Alexander-Spanier cochain complex. Theorem 1.4 immediately completes the proof of the second statement since $1 \otimes x$ represents an element in the Hochschild homology group, the definition of which is given in [18].

By Theorem 1.5, we find that the cocycle ϕ has the following additional properties: ϕ can detect the Hausdorff dimensions of cellular self-similar sets and distinguish them by their dimensions. For instance, we get the cocycles ϕ of the Sierpinski gasket *SG* and the Sierpinski carpet *SC*, whose thresholds of the well-definedness are different. Namely, their thresholds are $\dim_H(SG) = \log_2 3$ and $\dim_H(SC) = \log_3 8$, and since bi-Lipschitz transformations preserve the Hausdorff dimension, the cocycles can also prove that *SG* and *SC* are not bi-Lipschitz homeomorphic. Moreover, if we have a bi-Lipschitz transformation between cellular self-similar sets $K_{|X|}$ and

$K_{|X'|}$, the algebra $C^{\alpha}(K_{|X|})$ is isomorphic to $C^{\alpha}(K_{|X'|})$. Therefore, we further get the following property: the cocycle ϕ is invariant under bi-Lipschitz transformations. These are noteworthy properties which the classical (co)homology theories could not detect.

After the proof of the main results, we apply the results to some examples of cellular self-similar sets and some variants of them.

Organisation of the Thesis

The content of this thesis is largely divided into two parts. Sections 2, 3 and 4 are devoted to the recollection of key notions for this thesis, the Young integration, self-similar sets, cyclic cohomology groups and K-theory. We begin, in Section 2, with a quick review of the definition of the Young integration and recall the sufficient condition for the existence of the integration. In Section 3, we explain key notions of self-similar sets and give some examples of self-similar sets. Most of them are in a class of cellular self-similar sets, which are introduced in Section 5. In Section 4, we recall the definition of the Hochschild cohomology, cyclic cohomology and K-theory. After that, we also review theorems on the pairing of K-theory and cyclic cohomology theory.

Section 5 is the main part of this thesis. First, we define cellular self-similar sets and study some properties in Section 5.1. In Section 5.2, we define the sequences of complex numbers for given α-Hölder continuous functions, which are key ingredients to define cyclic cocycles on the algebra of Hölder continuous functions. After that, we prove the main theorems in Section 5.3. Last few sections are devoted to the application of the main results to a variety of cellular self-similar sets and variants of cellular self-similar sets.

Conventions

We assume that algebras have unit unless otherwise stated, and the base ring of an algebra is the field $\mathbb{C}$ of complex numbers. The Euclidean space $\mathbb{R}^n$ is endowed with the standard Euclidean metric.

- $\otimes = \otimes_{\mathbb{C}}$.
- $\mathbb{Z}_{\geqslant 0} = \mathbb{N} \cup \{0\}$.
- ∂X : the boundary of a topological space X.
- $C^{\alpha}(X)$: the algebra of complex-valued α-Hölder continuous functions on a metric space X whose sum and multiplication are given by the pointwise sum and multiplication.
- $C^{Lip}(X)$: the algebra of complex-valued Lipschitz functions on a metric space X whose sum and multiplication are given by the pointwise sum and multiplication.

Acknowledgements

I would like to thank my supervisor Hitoshi Moriyoshi for patient advice about mathematical materials. His help and ideas improved this thesis as well as my mathematical skills. I also thank him for allowing me to work as a research internship student of a company. Without his constant encouragement and support of my life in and outside the department, I would not have come this far.

I would also like to thank Thomas Geisser. Without his suggestions about mathematical literature, this thesis would not exist. Also, after he left Nagoya, his encouragement and advice have been getting me focus on my study.

I was greatly helped by many colleague mathematicians in and outside the department. I was very lucky to have many experiences of joining the mathematical meetings and meeting many colleague mathematicians. They gave me a lot of valuable comments on mathematics and helped me to study new materials.

I was also helped by many non-mathematicians, with whom I have been spending my daily life during the years of graduate school, with wonderful and unforgettable experiences. Shientshitsu-staffs always kindly helped me and their support made me feel comfortable with my life during the years of graduate school.

Finally, I would like to thank my family, especially my parents, for their continuous love and support.

2. Young Integration

2 Young Integration

We begin with a quick review of the Young integration basically following [34] except the slight changes of the notation.

Let I be the unit interval $[a,b]$ and f, g complex-valued functions defined on I. We make a subdivision χ of I

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and define

$$F(\chi) = \sum_{i=1}^{n} f(x_i)(g(x_i) - g(x_{i-1})).$$

Then $F(\chi)$ can be also written as

$$F(\chi) = \sum_{0<i\leqslant j\leqslant n} \delta(f)(x_{i-1},x_i) \cdot \delta(g)(x_{j-1},x_j) + f(a)(g(b) - g(a)).$$

Here $\delta(f)(x_{i-1},x_i)$ denotes $f(x_i) - f(x_{i-1})$. This notation is the coboundary map of the Alexander-Spanier cohomology theory; see Chapter 6.4 in [27]. We also let $\alpha, \beta > 0$, and denote by

$$S_{\alpha,\beta}[a,b] = S_{\alpha,\beta}[a,b;f,g]$$

the upper bound of

$$\Big(\sum_i |\delta(f)(x_{i-1},x_i)|^{\frac{1}{\alpha}}\Big)^{\alpha} \Big(\sum_i |\delta(g)(x_{i-1},x_i)|^{\frac{1}{\beta}}\Big)^{\beta}$$

for every subdivision of I. Following lemmas of [34], if $\alpha + \beta > 1$ and $\xi \in [a,b]$ is a division point of χ, we have

$$\Big|F(\chi) - f(\xi)(g(1) - g(0))\Big| \leqslant (1 + \zeta(\alpha + \beta)) \cdot S_{\alpha,\beta}[0,1],$$

where $\zeta(\alpha+\beta)$ denotes the zeta function of $\alpha+\beta$.

This inequality yields a more general inequality for the sum associated to χ: for the given subdivision χ, let a point $x_{i-1} \leqslant \xi_i \leqslant x_i$ for each i. Applying this inequality for each interval $[x_{i-1}, x_i]$ and summing up, we get

$$\left|F(\chi) - \sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1}))\right| \leqslant \{1 + \zeta(\alpha+\beta)\} \cdot \sum_{i=1}^{n} S_{\alpha,\beta}[x_{i-1}, x_i \,;\, f, g].$$

Moreover if we have another subdivision χ' of I and subdivision points $x_{j-1} \leqslant \xi'_j \leqslant x_j$, then

$$\left|\sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1})) - \sum_{j=1}^{m} f(\xi'_j)(g(x'_j) - g(x'_{j-1}))\right|$$
$$\leqslant \{1+\zeta(\alpha+\beta)\} \cdot \left\{\sum_{i=1}^{n} S_{\alpha,\beta}[x_{i-1}, x_i \,;\, f, g] + \sum_{j=1}^{m} S_{\alpha,\beta}[x'_{j-1}, x'_j \,;\, f, g]\right\}.$$

Definition 2.1. We say that the *Stieltjes integral*

$$Y(f,g)$$

exists in the Riemann sense with the value J, if there exist $J \in \mathbb{C}$ and a function $\varepsilon_\delta > 0$ with respect to the variable $\delta > 0$ such that $\varepsilon_\delta \to 0$ as $\delta \to 0$, and if all the segments $[x_{i-1}, x_i]$ of a subdivision χ have lengths less than δ, then

$$\left| J - \sum_{i} f(\xi_i)(g(x_i) - g(x_{i-1}))\right| < \varepsilon_\delta.$$

We observe that, for the integrability in the Riemann sense, it is sufficient that the difference of any of two sums of the formula $\sum_i f(\xi_i)(g(x_i) - g(x_{i-1}))$ of Definition 2.1, for each of which the length of $[x_{i-1}, x_i]$ is less than δ, is less than ε_δ. By the inequality just before Definition 2.1, this is the case if for some $\alpha, \beta > 0$ such that

$\alpha + \beta > 1$ we have

$$\sum_{i=1}^{n} S_{\alpha,\beta}[x_{i-1}, x_i \,;\, f, g] < \varepsilon_\delta.$$

For the existence of the integrability, we define $W_\alpha(\delta)$ to be the set of functions such that the value $V_\alpha^{(\delta)}(f)$ defined below has an upper bound:

$$V_\alpha^{(\delta)}(f) = \sup_{|\chi| \leqslant \delta} \left\{ \left(\sum_i |f(x_i) - f(x_{i-1})|^{\frac{1}{\alpha}} \right)^\alpha \right\} < \infty.$$

Here $|\chi|$ denotes the maximum length of the intervals of χ, and the supremum runs over all subdivisions χ such that $|\chi|$ is less than or equal to δ. Finally we define the *Wiener class* W_α to be the set of functions f such that $V_\alpha^{(\delta)}(f)$ with respect to the variable δ has an upper bound.

Theorem 2.2 (Theorem on Stieltjes integrability)**.** *If $f \in W_\alpha$ and $g \in W_\beta$ where $\alpha, \beta > 0$ and $\alpha + \beta > 1$, have no common discontinuities, their Stieltjes integral exists in the Riemann sense.*

The Wiener class W_α is closed under the pointwise sum and scalar multiplication for $0 < \alpha < 1$. Therefore, if we regard the integration as a function from $W_\alpha \times W_\alpha$ to $\mathbb{C}$, this function turns out to be a bilinear function. On the other hand, it is clear from the definition that the set $C^\alpha(I)$ of complex-valued α-Hölder continuous functions defined on I is a subspace of W_α. Moreover, $C^\alpha(I)$ is closed under the pointwise multiplication in addition to the pointwise sum and scalar multiplication. The integration restricted to $C^\alpha(I)$ is referred to as the *Young integration*.

Remark 2.3. The Young integration is a special case of the Riemann-Stieltjes integration.

3. Self-similar Sets and Hausdorff Dimension

3 Self-similar Sets and Hausdorff Dimension

In this section we briefly recall the definition of self-similar sets and the Hausdorff dimension. This section is based on [17].

3.1 Self-similar Sets

We begin with the definition of some maps from a metric space (X,d) to itself.

Definition 3.1. Let (X,d) be a metric space.

a) A map $F : X \to X$ is a *contraction* if there exists a minimum real number $0 < r < 1$ such that $d(F(x),F(y)) \leqslant r \cdot d(x,y)$ for any x, $y \in X$. The real number r is called the *contraction ratio*.

b) A contraction $F : X \to X$ is a *similitude* if $d(F(x),F(y)) = r \cdot d(x,y)$ for any x, $y \in X$. We call r the *similarity ratio*.

For a finite set $\{F_j\}_{j\in S}$ of contractions defined on a complete metric space, there exists a unique compact subspace that is characterised by $\{F_j\}_{j\in S}$. Here is the precise statement of the existence of self-similar sets:

Theorem 3.2. *Let X be a complete metric space. We also let S be a finite set and $F_j : X \to X$ contractions indexed by S. We call the triple $(X,S,\{F_j\}_{j\in S})$ an* iterated function system *or* IFS. *Then, there exists a unique non-empty compact subset K_X of X that satisfies*

$$K_X = \bigcup_{j\in S} F_j(K_X).$$

The compact set K_X is called the *self-similar set with respect to* $(X,S,\{F_j\}_{j\in S})$.

Remark 3.3. In some literature the terminology *self-similar set* is used in a restricted sense. For instance, Hutchinson introduces the notion of *self-similar set* for a finite set of similitudes [14]. Self-similar sets defined in Theorem 3.2 are also referred to as *attractors* or *invariant sets*; see Section 9.1 in [9]. We employ Hutchinson's definition of self-similar sets in the last section to define cellular self-similar sets, the definition of which is given in Section 5.1 below.

For later use, we include an outline of a proof of Theorem 3.2. The proof is based on the following theorem.

Theorem 3.4 (Contraction principle)**.** *Let (X,d) be a complete metric space and $F : X \to X$ a contraction with respect to the metric. Then there exists a unique fixed point of F, in other words, there exists a unique solution to the equation $F(x) = x$. Moreover if x_* is the fixed point of F, then $\{F^n(a)\}_{n\geqslant 0}$ converges to x_* for all $a \in X$ where F^n is the n-th iteration of F.*

Let (X,d) be a metric space and $K(X)$ the set of non-empty compact subsets of X. We define the *Hausdorff metric* δ on $K(X)$ by

$$\delta(A,B) = \inf\{r > 0 : U_r(A) \subset B \text{ and } U_r(B) \subset A\},$$

where $U_r(A) = \{x \in X \ : \ d(x,A) \leqslant r\}$.

Lemma 3.5. *The pair $(K(X),\delta)$ forms a metric space. Moreover, if X is complete, $(K(X),\delta)$ is also complete.*

We now assume that the metric space (X,d) is complete. Define $F(A) = \bigcup_{j\in S} F_j(A)$ for $A \subset X$, and then $F : K(X) \to K(X)$ is a contraction with respect to the metric δ. Therefore, by applying Theorem 3.4 to $(K(X),\delta)$ and F, we get the self-similar set K_X with respect to $(X,S,\{F_j\}_{j\in S})$.

3.2 Hausdorff Dimension of Self-similar Sets

In the field of fractal geometry, the dimension of fractal sets is not as well-defined as the dimension of self-similar sets. Namely, there are several notions of *dimension*, like the Box-counting dimension, the Packing dimension, and so on; see Section 3 in [9] for the details. The Hausdorff dimension is a candidate for the dimension and widely used to analyse fractal sets. In this subsection we define the Hausdorff dimension, which plays a key role to define cyclic cocycles on cellular self-similar sets, the definition of which is given in Section 5 below.

Definition 3.6. Let (X,d) be a metric space. We also let $s > 0$ and $\delta > 0$. For any bounded set $A \subset X$, we define

$$\mathscr{H}^s_\delta(A) = \inf\left\{\sum_{i\geqslant 1}\mathrm{diam}(E_i)^s \ :\ A \subset \bigcup_{i\geqslant 1} E_i,\ \mathrm{diam}(E_i) \leqslant \delta\right\}.$$

Here the infimum runs over all the coverings $\{E_i\}$ of A which consist of sets, and $\mathrm{diam}(E_i)$ denotes the diameter of E_i. Also we define

$$\mathscr{H}^s(A) = \limsup_{\delta\downarrow 0}\mathscr{H}^s_\delta(A),$$

and we call $\mathscr{H}^s$ the *s-dimensional Hausdorff measure* of (X,d).

Remark 3.7. The s-dimensional Hausdorff measure is a complete Borel measure.

The following lemma shows that the measure $\mathscr{H}^s$ detects a critical point of any given subset of X.

Lemma 3.8. *For any subset $E \subset X$, we have*

$$\sup\{s \in \mathbb{R} \mid \mathscr{H}^s(E) = \infty\} = \inf\{s \in \mathbb{R} \mid \mathscr{H}^s(E) = 0\}.$$

Definition 3.9. The real number which satisfies Lemma 3.8 is called the *Hausdorff dimension* of E, and it is denoted by $\dim_H(E)$.

The Hausdorff dimension satisfies the following properties, which might be expected to hold for any reasonable definition of the dimension (see also Section 3 of [9]).

- *Monotonicity* : if $E \subset F$, then $\dim_H(E) \leqslant \dim_H(F)$.

- *Countable Stability* : if $F_1, F_2, \cdots$ is a countable sequence of sets, then $\dim_H(\bigcup_{i=1}^{\infty} F_i) = \sup_{1 \leqslant i < \infty} \dim_H(F_i)$.
- *Countable sets* : if F is countable, then $\dim_H(F) = 0$.
- *Open sets* : if $F \subset \mathbb{R}^n$ is open, then $\dim_H(F) = n$.
- *Smooth sets* : if F is a smooth m-dimensional submanifold of $\mathbb{R}^n$, then $\dim_H(F) = m$.

In general, the dimension is one of invariants of topological spaces. However, in the field of fractal geometry, the dimension is seen as a unique invariant of fractal sets.

Lemma 3.10. [9, Corollary 2.4] *Let $F \subset \mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a bi-Lipschitz transformation, that is, there exist $0 < c_1 \leqslant c_2 < \infty$ such that*

$$c_1 |x-y| \leqslant |f(x) - f(y)| \leqslant c_2 |x-y|,$$

then $\dim_H(f(F)) = \dim_H(F)$.

The lemma states that the Hausdorff dimension is invariant under bi-Lipschitz transformations. Moreover if we have a bi-Lipschitz transformation between metric spaces then the algebras of complex-valued α-Hölder continuous functions defined on the metric spaces are isomorphic.

In general, it is difficult to compute the Hausdorff dimension. Namely, the Hausdorff dimensions of a few self-similar sets have been computed. However, if we have a self-similar set K_X with respect to an IFS $(X, S, \{F_j\}_{j \in S})$ such that contractions are similitudes and the similitudes have "small" enough intersections, then we can compute the Hausdorff dimension of K_X by the following theorem:

Theorem 3.11. [22, Theorem II] *Let X be a compact subspace in $\mathbb{R}^n$ and $\{F_j : \mathbb{R}^n \to \mathbb{R}^n\}_{j \in S}$ a finite set of similitudes indexed with a finite set S. Suppose that the self-similar set K_X with respect to the IFS $(X, S, \{F_j\}_{j \in S})$ satisfies the* open set condition, *i.e., there exists a bounded non-empty open set* $O \subset \mathbb{R}^n$ *such that*

$$\bigcup_{j\in S} F_j(O) \subset O \quad and \quad F_i(O) \cap F_j(O) = \emptyset \ \ for\ any\ i \neq j \in S.$$

Then the Hausdorff dimension $\dim_H(K_X)$ *of the self-similar set* K_X *is the unique real number* α *such that the following relation holds*

$$\sum_{j\in S} r_j^{\alpha} = 1.$$

Here r_j *denotes the similarity ratio of* F_j*.*

Finally, we mention that the Hausdorff dimension of a self-similar set K_X with the open set condition coincides with the Box-counting dimension of K_X; see Section 9.2 of [9] for the details.

3.3 Examples of Self-similar Sets and Hausdorff Dimensions

In this subsection we give some examples of self-similar sets and their Hausdorff dimensions. For later use, we explain contractions of each self-similar set and give an IFS $(X, S, \{F_j\}_{j\in S})$ which gives rise to the self-similar set. We also provide figures for each self-similar set, that correspond to X, $F(X)(=\bigcup_{j\in S} F_j(X))$ and $F \circ F(X)$.

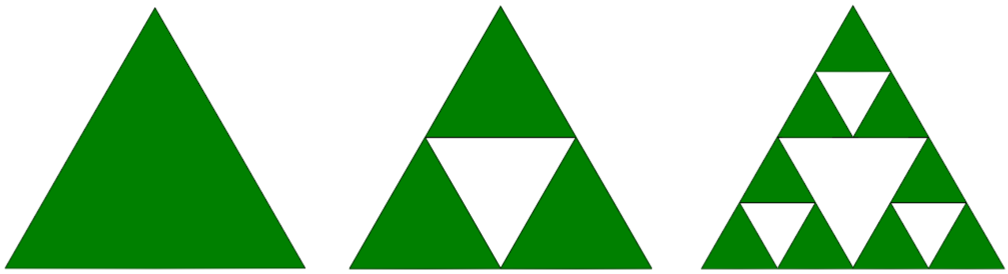
• **unit interval**

The unit interval $[0,1]$ can be thought as a self-similar set.

The left-hand side is the underlying space X that gives rise to the self-similar set, i.e., the unit interval $[0,1]$. The figure in the centre is the union of the images of two similitudes whose similarity ratios are $\frac{1}{2}$. The third one is obtained by applying the similitudes to the second figure. The triple $([0,1], S = \{1,2\}, \{F_j\}_{j \in S})$ forms an IFS, and it satisfies the open set condition since, for an open set $O = (0,1)$, we have $F_1(O) \cup F_2(O) \subset O$ and $F_1(O) \cap F_2(O) = \varnothing$. Therefore, by Theorem 3.11, the Hausdorff dimension of the resulting self-similar set $[0,1]$ is the root α given by $2 \cdot (\frac{1}{2})^\alpha = 1$, i.e., $\dim_H([0,1]) = \alpha = 1$. It also follows immediately that the Hausdorff dimension of I does not depend on the choices of similitudes and their similarity ratios if the triple satisfies the open set condition. More generally, for every n-dimensional unit cube I^n we have an IFS $(I^n, S, \{F_j\}_{j \in S})$ such that it gives rise to the self-similar set I^n and the Hausdorff dimension is $\dim_H(I^n) = n$.

• **Sierpinski gasket**

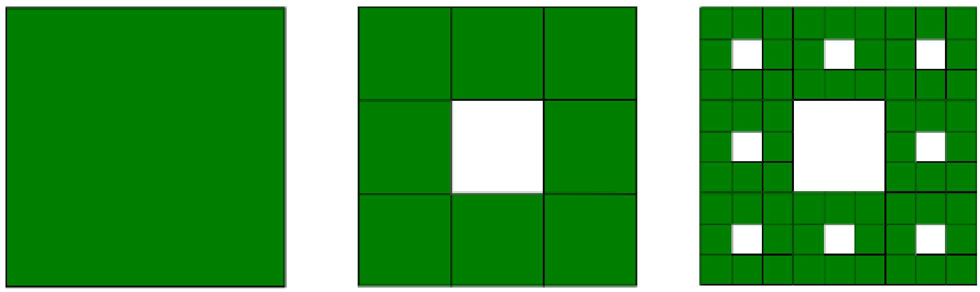
The Sierpinski gasket SG is a well-known example of self-similar sets. Here are the first 3 steps of a construction of the Sierpinski gasket:



The space of the left-hand side X is an equilateral triangle in $\mathbb{R}^2$. In the centre we have 3 equilateral triangles, the length of whose edges are a half of the ones of X. The similitudes F_1, F_2 and F_3 are defined by the 3 triangles, and the similarity ratios of F_j are $\frac{1}{2}$. The right-hand side is the space $F \circ F(X)$. Then, we get an IFS $(X, S = \{1,2,3\}, \{F_j\}_{j \in S})$, and it gives rise to SG. Moreover, SG satisfies the open set condition. Namely, we can choose an open set $O = \mathrm{int}(X)$, and we find that $\bigcup_{j \in S} F_j(O) \subset O$ and $F_i(O) \cap F_j(O) = \varnothing$ for any $i \neq j \in S$. Therefore, the Hausdorff dimension of SG is the root α given by the equation $\sum_{j \in S} (\frac{1}{2})^\alpha = 3 \cdot (\frac{1}{2})^\alpha = 1$, i.e., $\dim_H(SG) = \log_2 3$.

• **Sierpinski carpet**

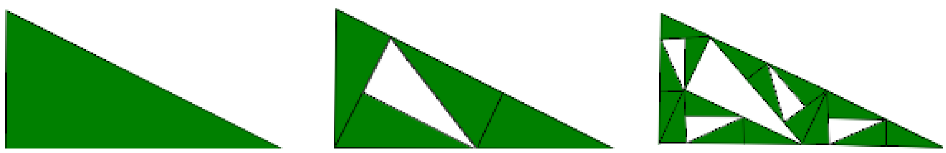
The Sierpinski carpet SC is defined by using a data that consists of a square X and 8 similitudes whose similarity ratios are $\frac{1}{3}$:



Then, we get an IFS $(X, S = \{1, 2, \cdots, 8\}, \{F_j\}_{j \in S})$, and SC satisfies the open set condition. Therefore the Hausdorff dimension of SC is the root α of $\sum_{j \in S} (\frac{1}{3})^\alpha = 8 \cdot (\frac{1}{3})^\alpha = 1$, i.e., $\dim_H(SC) = \log_3 8$.

• **Pinwheel fractal**

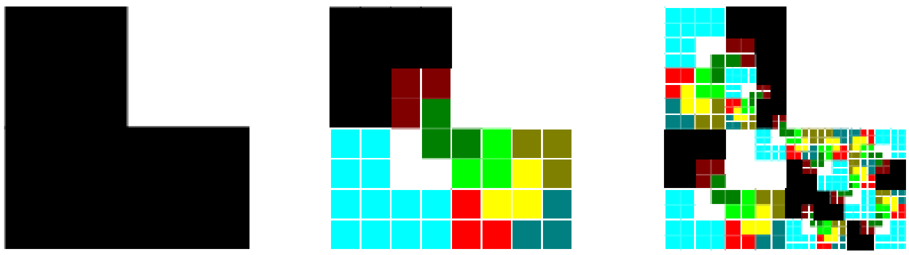
The Pinwheel fractal PW is a self-similar set which is modelled by the pinwheel tiling of the plane. There exist uncountably many pinwheel tilings, and therefore we can construct a self-similar set based on each given pinwheel tiling.



The figure gives rise to one of the pinwheel fractals based on the most well-known pinwheel tiling of $\mathbb{R}^2$. The pinwheel tiling was originally defined in [24]. The triangle of the left-hand side consists of 3 edges whose lengths are 1, 2 and $\sqrt{5}$. From the figure in the centre, we have 4 similitudes whose similarity ratios are $\frac{1}{\sqrt{5}}$. Therefore, we get an IFS $(X, S = \{1, \cdots, 4\}, \{F_j\}_{j \in S})$ which gives rise to PW. Since PW satisfies the open set condition, the Hausdorff dimension of the pinwheel fractal is given by the root of the equation $\sum_{j \in S} (\frac{1}{\sqrt{5}})^\alpha = 4 \cdot (\frac{1}{\sqrt{5}})^\alpha = 1$, i.e., $\dim_H(PW) = \log_{\sqrt{5}} 4$.

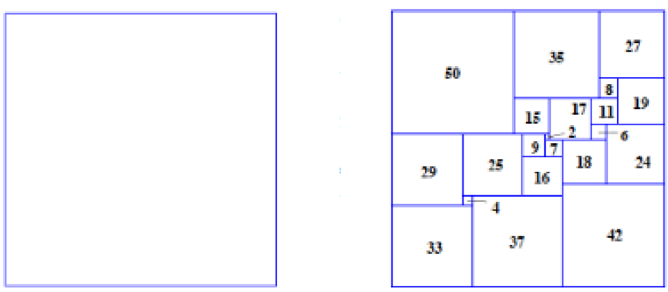
• **More self-similar sets**

The above examples are some of famous examples restricted to self-similar sets based on tiling methods. There exist some kinds of tilings, perfect tilings, partridge tilings, reptiles, irreptiles [10] and so on. A self-similar set induced by an IFS $(X, S, \{F_j\}_{j\in S})$ based on such a tiling satisfies the open set condition. Here are some kinds of such self-similar sets.



The above figure gives rise to a variant of the L-shape fractal. Black and cean L-shape spaces in the second figure are half sizes of the first figure X. The other figures with the other colours except the white space have quarter sizes of X. Since the resulting self-similar set satisfies the open set condition, the Hausdorff dimension of the L-shape fractal is the root α of $2 \cdot (\frac{1}{2})^{\alpha} + 7 \cdot (\frac{1}{4})^{\alpha} = 1$.

There exist some other types of tilings [3, 10, 28, 32], and one of which called *perfect tilings* correspond to electrical networks [3, 32]. The following figure corresponds to one of constructions of the perfect tilings of squares:



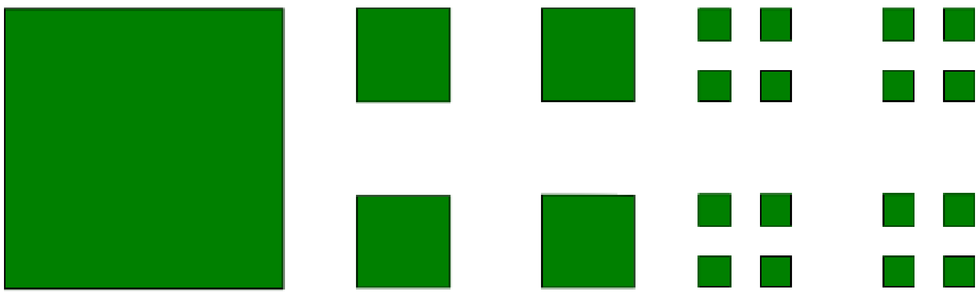
The numbers of squares represent the lengths of their edges, and the length of the edges of the underlying space X is 112. Suppose that the second figure lacks the square whose length of edges is 8. Then, we have 20 similitudes whose similarity ratios are given by numbers assigned to squares, and we get an IFS $(X, S = \{1, 2, \cdots, 20\}, \{F_j\}_{j \in S})$. The IFS satisfies the open set condition since we can choose a required open set O as the interior of X. Therefore, the Hausdorff dimension of the self-similar set is the root α given by the following equation:

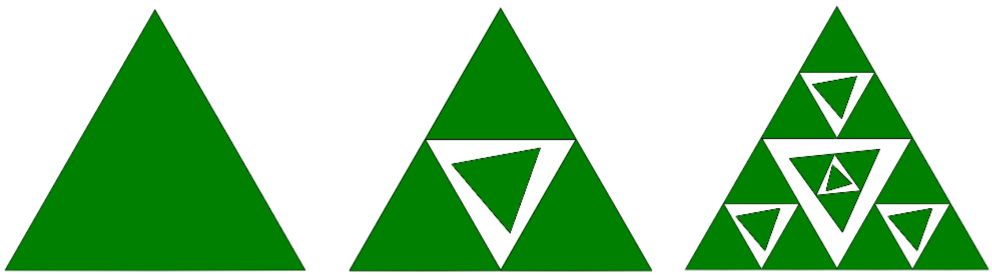
$$(\frac{50}{112})^\alpha + (\frac{42}{112})^\alpha + (\frac{37}{112})^\alpha + (\frac{35}{112})^\alpha + (\frac{33}{112})^\alpha + (\frac{29}{112})^\alpha + (\frac{27}{112})^\alpha +$$

$$(\frac{25}{112})^\alpha + (\frac{24}{112})^\alpha + (\frac{19}{112})^\alpha + (\frac{18}{112})^\alpha + (\frac{17}{112})^\alpha + (\frac{16}{112})^\alpha + (\frac{15}{112})^\alpha +$$

$$(\frac{11}{112})^\alpha + (\frac{9}{112})^\alpha + (\frac{7}{112})^\alpha + (\frac{6}{112})^\alpha + (\frac{4}{112})^\alpha + (\frac{2}{112})^\alpha = 1.$$

So far, we have looked at connected self-similar sets. However, there also exist non-connected self-similar sets:





The first row shows the first 3 iterations of a construction of the *Cantor dust*. The similitudes have the similarity ratio $\frac{1}{3}$. The second row represents the first 3-iteration of an IFS that consists of 4 similitudes, one of which has the similarity ratio $\frac{1}{3}$ and the rest have $\frac{1}{2}$.

4. Cyclic Cohomology and K-theory

4 Cyclic Cohomology and K-theory

In this section we review cyclic cohomology theory and K-theory. This section is based on [5] and [6] except slight changes of the notation.

4.1 Cyclic Cohomology Theory

Let $\mathscr{A}$ be a unital associative algebra over $\mathbb{C}$. For $n \in \mathbb{Z}_{\geqslant 0}$ we define $C_h^n(\mathscr{A})$ to be the $\mathbb{C}$-vector space of linear functions $\phi : \mathscr{A}^{\otimes n+1} \to \mathbb{C}$, and the *n-th Hochschild coboundary map* $b^n : C_h^n(\mathscr{A}) \to C_h^{n+1}(\mathscr{A})$ by

$$\begin{aligned} b^n(\phi)(a_0, a_1, \cdots, a_{n+1}) &= \sum_{i=0}^{n} (-1)^i \phi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\ &\qquad + (-1)^{n+1} \phi(a_{n+1} a_0, a_1, \cdots, a_n). \end{aligned}$$

Then the pair $(C_h^*(\mathscr{A}), b^*)$ forms a cochain complex and the cohomology group $HH^*(\mathscr{A})$ is called the *Hochschild cohomology group* of $\mathscr{A}$. We further define the subspace $C_\lambda^n(\mathscr{A})$ of $C_h^n(\mathscr{A})$ to be a $\mathbb{C}$-linear space of linear functions $\phi : \mathscr{A}^{\otimes n+1} \to \mathbb{C}$ satisfying the *cyclic condition*

$$\phi(a_0, a_1, \cdots, a_n) = (-1)^n \phi(a_n, a_0, a_1, \cdots, a_{n-1}).$$

The space $C_\lambda^n(\mathscr{A})$ also forms a cochain subcomplex $(C_\lambda^*(\mathscr{A}), b^*)$, and the cohomology group $HC^*(\mathscr{A})$ of $(C_\lambda^*(\mathscr{A}), b^*)$ is called the *cyclic cohomology group* of $\mathscr{A}$.

By construction of the Hochschild complex $C_h^*(\mathscr{A})$ and the cyclic complex $C_\lambda^*(\mathscr{A})$, the inclusion map $I : C_\lambda^*(\mathscr{A}) \to C_h^*(\mathscr{A})$ gives an exact sequence of cochain complexes

$$0 \to C_\lambda^*(\mathscr{A}) \xrightarrow{I} C_h^*(\mathscr{A}) \to C_h^*(\mathscr{A})/C_\lambda^*(\mathscr{A}) \to 0.$$

Note that the n-th cohomology group of $C_h^*(\mathscr{A})/C_\lambda^*(\mathscr{A})$ is $H^n(C_h^*(\mathscr{A})/C_\lambda^*(\mathscr{A})) \cong H^{n-1}(C_\lambda^*(\mathscr{A})) = HC^{n-1}(\mathscr{A})$; see [6] for the details. Thus we have a long exact sequence of the Hochschild cohomology group $HH^*(\mathscr{A})$ and the cyclic cohomology group $HC^*(\mathscr{A})$:

$$\cdots \to HC^n(\mathscr{A}) \xrightarrow{I} HH^n(\mathscr{A}) \xrightarrow{B} HC^{n-1}(\mathscr{A}) \xrightarrow{S} HC^{n+1}(\mathscr{A}) \xrightarrow{I} HH^{n+1}(\mathscr{A}) \to \cdots.$$

The sequence is called the *SBI-sequence* of $\mathscr{A}$.

The map $S : HC^{n-1}(\mathscr{A}) \to HC^{n+1}(\mathscr{A})$, called the *periodicity map*, gives rise to the *periodic cohomology group $HP^*(\mathscr{A})$* of $\mathscr{A}$: for any $m \in \mathbb{N}$, we have

$$HP^*(\mathscr{A}) = \operatorname*{colim}_{\mathbb{Z}_{\geqslant 0} \sqcup \mathbb{Z}_{\geqslant 0}} HC^n(\mathscr{A}),$$

where the diagram $\mathbb{Z}_{\geqslant 0} \sqcup \mathbb{Z}_{\geqslant 0} \to \mathrm{Vect}_{\mathbb{C}}$ is the universal functor induced from the coproduct of the diagrams

$$(n_1 \leqslant n_1 + 1) \mapsto (HC^{2n_1}(\mathscr{A}) \xrightarrow{S} HC^{2(n_1+1)}(\mathscr{A})),$$

$$(n_2 \leqslant n_2 + 1) \mapsto (HC^{2n_2-1}(\mathscr{A}) \xrightarrow{S} HC^{2n_2+1}(\mathscr{A})).$$

Moreover, since the periodicity maps have degree 2, $HP^*(\mathscr{A})$ is decomposed into two parts. Namely, we have

$$HP^0(\mathscr{A}) \cong \operatorname*{colim}_{\mathbb{Z}_{\geqslant 0}} HC^{2n}(\mathscr{A}), \quad HP^1(\mathscr{A}) \cong \operatorname*{colim}_{\mathbb{Z}_{\geqslant 0}} HC^{2n-1}(\mathscr{A}).$$

So far we have defined cochain complexes for discrete algebras over $\mathbb{C}$. The cyclic bar construction still works for complete locally convex algebras over $\mathbb{C}$ when the tensor product is replaced with the *projective tensor product* $\hat{\otimes}_\pi$ [11] and the linear functions are assumed to be continuous. An well-known example of complete local convex algebras over $\mathbb{C}$ is the algebra of smooth functions $C^\infty(V)$ defined on a compact smooth manifold [5]. The cyclic cohomology group for a complete unital locally convex algebra $\mathscr{A}$ is called the *cyclic cohomology group* of $\mathscr{A}$, which we denote by $HC^*_{cont}(\mathscr{A})$.

One of spectacular results of the continuous Hochschild and cyclic cohomology groups is a result proved by Connes for the algebra of smooth functions defined on a compact smooth manifold. The result allows us to see the Hochschild cohomology group and the cyclic cohomology group as a generalisation of the space of de Rham currents and the de Rham homology group respectively: we let V be a compact smooth manifold and $C^\infty(V)$ the algebra of smooth functions defined on V. The algebra $C^\infty(V)$ admits a locally convex topology and we can take the projective tensor product $\hat{\otimes}_\pi$ to induce $HH^*_{cont}(C^\infty(V))$ and $HC^*_{cont}(C^\infty(V))$ of $C^\infty(V)$.

Theorem 4.1. [5, Lemma 45] *Let V be a compact smooth manifold, and consider $\mathscr{A} = C^\infty(V)$ as a locally convex topological algebra, then:*

1. *The continuous Hochschild cohomology group $HH^n_{cont}(\mathscr{A})$ is canonically isomorphic to the $\mathbb{C}$-vector space of de Rham currents of dimension n on V.*
2. *Under the isomorphism in* 1 *the operator $I \circ B : HH^n_{cont}(\mathscr{A}) \to HH^{n-1}_{cont}(\mathscr{A})$ is the de Rham boundary for currents.*

Theorem 4.2. [5, Theorem 46] *Let V be a compact smooth manifold, and $\mathscr{A} = C^\infty(V)$ as a locally convex algebra. Then:*

1. *For each $n \in \mathbb{Z}_{\geqslant 0}$, $HC^n_{cont}(\mathscr{A})$ is canonically isomorphic to the direct sum*

$$\ker b_n \oplus H_{n-2}(V;\mathbb{C}) \oplus H_{n-4}(V;\mathbb{C}) \oplus \cdots$$

   where $H_n(V;\mathbb{C})$ is the n-th de Rham homology group of V and b_n is the n-th de Rham boundary map.

2. *The periodic cohomology group $HP^0_{cont}(\mathscr{A}) \oplus HP^1_{cont}(\mathscr{A})$ is canonically isomorphic to the de Rham homology group $H_*(V;\mathbb{C})$.*

Remark 4.3. If we take the C^*-algebra $C(V)$ of continuous functions on V with the sup norm instead of $C^\infty(V)$, then the continuous Hochschild cohomology group of

$C(V)$ is trivial in dimension $\geqslant 1$ ([15]) and the *SBI*-sequence proves that the cyclic cohomology group is given by $HC^{2n}_{cont}(C(V)) = HC^0_{cont}(C(V))$ and $HC^{2n+1}_{cont}(C(V)) = 0$ for $n \in \mathbb{Z}_{\geqslant 0}$.

Now we assume again that $\mathscr{A}$ is a discrete algebra. Then there exists a notion of the *cup product* defined on $HC^*(\mathscr{A})$ (cf. [5]). The cup product is induced by the comodule structure on the cyclic homology group $HC_*(\mathscr{A})$ of $\mathscr{A}$ and the comodule structure is closely related to the structures defined on the homology groups of trivial S^1-spaces (see [18]). Therefore, $HC^*(\mathscr{A})$ can be thought as a group endowed with a dual of the comodule structures, and it turns out to be a graded algebra:

$$\# : HC^p(\mathscr{A}) \otimes HC^q(\mathscr{B}) \to HC^{p+q}(\mathscr{A} \otimes \mathscr{B}).$$

Example 4.4. Let $\mathscr{A}$ be a unital algebra over $\mathbb{C}$ and $\mathscr{B}$ the algebra of $n \times n$-matrices $M_n(\mathbb{C})$. We have a trace Tr as a cocycle in $HC^0(\mathscr{B})$. Note that $\mathscr{A} \otimes \mathscr{B} \cong M_n(\mathscr{A})$, and, by using the cup product, we then have a linear map

$$\#\mathrm{Tr} : HC^p(\mathscr{A}) \to HC^p(\mathscr{A} \otimes \mathscr{B}) \cong HC^p(M_n(\mathscr{A})).$$

Moreover, the map has an explicit expression: for $[\phi] \in HC^p(\mathscr{A})$ and $a^0, a^1, \cdots, a^p \in M_n(\mathscr{A})$,

$$\phi\#\mathrm{Tr}(a^0, \cdots, a^p) = \sum_{0 \leqslant j_0, \cdots, j_p \leqslant n} \phi(a^0_{j_0 j_1}, a^1_{j_1 j_2}, \cdots, a^p_{j_p j_0}).$$

As we see in the next subsection, $\#\mathrm{Tr}$ gives a pairing between K-theory and the cyclic cohomology groups.

4.2 K-theory and Pairing Between K-theory and Cyclic Cohomology Groups

In this subsection, we briefly recall the definitions of the algebraic and topological K-groups, and the paring between the algebraic K-groups and the cyclic cohomology groups.

Let A be a unital associative algebra over $\mathbb{C}$. Since we have an inclusion $j_{n,n+m}$: $M_n(A) \to M_{n+m}(A)$ defined by $j_{n,n+m}(a) = \mathrm{diag}(a,0)$, define $M_\infty(A) = \operatorname*{colim}_{\mathbb{N}} M_n(A)$ along the inclusion maps. Then, $M_\infty(A)$ is endowed with a *direct sum* $\oplus$: for $a \in M_n(A)$ and $b \in M_m(A)$, the sum $\oplus$ is defined as follows

$$(a,b) \mapsto a \oplus b = \mathrm{diag}(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+m}(A).$$

We let $\mathrm{Idem}_n(A)$ be the idempotent elements of $M_n(A)$. Then $\mathrm{Idem}_\infty(A) = \operatorname*{colim}_{\mathbb{N}} \mathrm{Idem}_n(A)$ is closed under the direct sum, and $\mathrm{Idem}_\infty(A)$ turns out to be a monoid under $\oplus$. We also write $GL_n(A)$ as the group of invertible elements in $M_n(A)$. We regard $GL_n(A) \subset GL_{n+1}(A)$ by a map

$$g \mapsto \mathrm{diag}(g,1),$$

and define $GL_\infty(A) = \operatorname*{colim}_{\mathbb{N}} GL_n(A)$. Note that $GL_\infty(A)$ acts by conjugation on $M_\infty(A)$ and $\mathrm{Idem}_\infty(A)$ and the sum $\oplus$ on $\mathrm{Idem}_\infty(A)$ is commutative up to conjugation. Therefore the coinvariants of $\mathrm{Idem}_\infty(A)$ by the conjugation of $GL_\infty(A)$

$$I(A) = (\mathrm{Idem}_\infty(A)_{GL_\infty(A)}, \oplus)$$

forms an abelian monoid. We define the *algebraic K_0-group* $K_0(A)$ of A as the group completion of $I(A)$. The example of a cyclic cocycle in Section 4.1 provides a pairing with the K_0-groups.

Theorem 4.5. [5, Proposition 14]

a) The following expression defines a bilinear pairing between $K_0(A)$ and $HP^0(A)$:

$$\langle [e], [\phi] \rangle = (2i\pi)^{-m}(m!)^{-1}(\phi \# \mathrm{Tr})(e, \cdots, e),$$

where $e \in \mathrm{Idem}_k(A)$ *and* $\phi \in Z_\lambda^{2m}(A) = \ker(b^{2m}) \cap C_\lambda^{2m}(A)$.

b) We have $\langle [e], [\phi] \rangle = \langle [e], S[\phi] \rangle$.

Remark 4.6. a) The subscripts k and m in *a)* of Theorem 4.5 have no links.

b) There exist other definitions of the algebraic K_0-group $K_0(A)$ of A, and one of them uses *finitely generated projective modules*. For an element $e \in M_n(A)$ the right action of $M_n(A)$ on $A^n = A^{\times n}$ induces a linear map $e : A^n \to A^n$. In addition if $e \in \mathrm{Idem}_n(A)$, the image eA^n is a finitely generated projective module. Thus we have a monoid map from $I(A)$ to the isomorphism class $P(A)$ of finitely generated projective modules over A. Moreover, for $e \in \mathrm{Idem}_n(A)$ and $f \in \mathrm{Idem}_m(A)$ the modules eA^n and fA^m are isomorphic if and only if e and f are in a same class of $I(A)$. Therefore the map is an isomorphism, and $P(A)$ has the same group completion of $I(A)$:

$$K_0(A) = G(I(A)) \cong G(P(A)).$$

We recall again that $GL_n(A)$ is the group of invertible elements in $M_n(A)$ and $i_n : GL_n(A) \to GL_{n+1}(A)$ an inclusion $i_n(a) = \mathrm{diag}(a, 1)$. Take the colimit along with the inclusion maps, and we get $GL_\infty(A) = \operatorname{colim}_{\mathbb{N}} GL_n(A)$. We also define $E_\infty(A)$ by the elementary matrix group $E_n(A)$ of $n \times n$-matrices and inclusion maps i_n. Note that the commutator subgroup $[GL_\infty(A), GL_\infty(A)]$ is equal to $E_\infty(A)$ by Whitehead's lemma. The *algebraic* K_1-group of A is defined to be

$$K_1(A) = GL_\infty(A)/E_\infty(A) = GL_\infty(A)/[GL_\infty(A), GL_\infty(A)].$$

A fundamental property of algebraic K_0-theory is that the functor

$$K_0 : (\text{Banach algebras}) \xrightarrow{\text{forget}} (\text{discrete rings}) \xrightarrow{K_0} (\text{abelian groups})$$

is homotopy invariant in the sense of the Banach topology. On the other hand, algebraic K_1-theory is *not* homotopy invariant as a functor from the category of Banach algebras. We now define the *topological* K_1-theory for Banach algebras, which is homotopy invariant: for a unital commutative Banach algebra A over $\mathbb{C}$, let $M_n(A)$ be the algebra of $n \times n$-matrices. We consider $M_n(A)$ as operators on $A^{\oplus n}$ whose norm is given by

$$||(a_1, \cdots, a_n)|| = ||a_1|| + \cdots + ||a_n||,$$

and the norm on $M_n(A)$ is defined by the operator norm. The general linear group $GL_n(A)$ of $n \times n$-matrices admits the induced topology of $M_n(A)$, and the inclusion map $i_{n,n+m} : GL_n(A) \to GL_{n+m}(A)$ turns out to be continuous. The *topological* K_1-group of A is defined by

$$K_1^{top}(A) = \pi_0(GL_\infty(A)) = \operatorname*{colim}_{\mathbb{N}} \pi_0(GL_n(A)).$$

The group structure is induced by the multiplication of $GL_\infty(A)$. We note that there exists a surjective homomorphism $id_* : K_1(A^\delta) \to K_1^{top}(A)$, so called *comparison map*, where A^δ is the algebra obtained by forgetting the Banach topology of A. The group $\pi_0(GL_\infty(A))$ is isomorphic to the quotient of $GL_\infty(A)$ by the normal subgroup $GL(A)_0$ of elements in $GL_\infty(A)$ to which there exist paths from $1 \in GL_\infty(A)$. Since $GL(A)_0$ includes $E(A)$, the continuous identity map $id : A^\delta \to A$ yields the comparison map.

Theorem 4.7. [5, Proposition 15] *Let A be a unital associative algebra over $\mathbb{C}$. Then:*

a) The following expression defines a bilinear pairing between $K_1(A)$ and $HP^1(A)$:

$$\langle [u], [\phi] \rangle = (2i\pi)^{-m} 2^{-(2m+1)} \frac{1}{(m-\frac{1}{2})\cdots\frac{1}{2}} \phi \# \mathrm{Tr}(u^{-1}-1, u-1, u^{-1}-1, \cdots, u-1),$$

where $\phi \in Z_\lambda^{2m-1}(A) = \ker(b^{2m-1}) \cap C_\lambda^{2m-1}(A)$ and $u \in GL_k(A)$.

b) We have $\langle [u], [\phi] \rangle = \langle [u], S[\phi] \rangle$.

Remark 4.8. a) The subscripts k and m in *a)* of Theorem 4.7 have no links.

b) There are similar notions of pairings which are defined between topological K-theory and the cyclic cohomology groups. Recall that the cyclic cohomology group $HC^*_{cont}(A)$ of a unital Banach algebra A is defined by using the projective tensor product, and we always have a forgetful map from $HC^*_{cont}(A)$ to $HC^*(A^\delta)$. On the other hand, there exists a fundamental fact which states a comparison between algebraic K-theory and topological K-theory: for a unital Banach algebra A over $\mathbb{C}$, there exists a natural map $\mathbb{K}(A^\delta) \to \mathbb{K}^{top}(A)$ of the algebraic K-theory spectrum and the topological K-theory spectrum, here A^δ is the discrete algebra attained by forgetting the topology of A. For those spectra, we can take the stable homotopy group functors π_0^s and π_1^s and get $\pi_i^s(\mathbb{K}(A^\delta)) = K_i(A^\delta)$ and $\pi_i^s(\mathbb{K}^{top}(A)) = K_i^{top}(A)$. Moreover, the map between the spectra gives rise to the isomorphism $K_0(A^\delta) \to K_0^{top}(A)$ and a surjective homomorphism $K_1(A^\delta) \to K_1^{top}(A)$; see [25, 26] for the details. Therefore, we have a diagram that includes the parings of algebraic K-theory and topological K-theory:

$$\begin{array}{ccccc} HC^*_{cont}(A) & \longrightarrow & \mathrm{Hom}(K_*^{top}(A),\mathbb{C}) & \longrightarrow & \mathbb{C} \\ \Big\downarrow{\scriptstyle forget} & & \Big\downarrow & & \\ HC^*(A^\delta) & \longrightarrow & \mathrm{Hom}(K_*(A^\delta),\mathbb{C}) & \longrightarrow & \mathbb{C}. \end{array}$$

5. Main Theorem

5 Main Theorem

In this section we define cyclic cocycles on a certain subclass of self-similar sets and prove the main theorems.

5.1 Cellular Self-similar Structures

First we define a kind of self-similar sets on which we define cyclic cocycles. From now on, self-similar sets are assumed to be in $\mathbb{R}^2$.

Definition 5.1. Let $|X|$ be a 2-dimensional finite convex linear cell complex and $\{F_j\}_{j\in S}$ a set of similitudes $F_j : |X| \to |X|$ indexed by a finite set S. We also let $|X_1| = \bigcup_{j\in S} F_j(|X|)$. The triple $(|X|, S, \{F_j\}_{j\in S})$ is called a *cellular self-similar structure* if it satisfies

a) $\partial|X| \subset \partial|X_1|$, and

b) $\operatorname{int} F_i(|X|) \cap \operatorname{int} F_j(|X|) = \varnothing$, for all $i \neq j \in S$.

Since, by Theorem 3.2, we have a unique self-similar set $K_{|X|}$ with respect to the cellular self-similar structure $(X, S, \{F_j\}_{j\in S})$, we call $K_{|X|}$ the *cellular self-similar set with respect to* $(|X|, S, \{F_j\}_{j\in S})$. By construction, $K_{|X|}$ is a compact subset of $|X| \subset \mathbb{R}^2$.

Remark 5.2. a) The dimension of $|X|$ can be extended to any $n \in \mathbb{N}$.
b) In [29, 30], Strichartz introduces the notion of cell to give examples of *fractafold*. What Strichartz calls cell in [29, 30] differs from the notion of cellular introduced in Definition 5.1.

Example 5.3. All the examples of IFSs given in Section 3.3, except the unit interval and the Cantor dust, are cellular self-similar structures.

Lemma 5.4. *Any cellular self-similar structure* $(|X|, S, \{F_j\}_{j\in S})$ *satisfies the open set condition.*

Proof. The lemma follows immediately from the definition of cellular self-similar structures. Namely, $O = \mathrm{int}(|X|)$ is a required open set in $\mathbb{R}^2$. $\square$

Let $(|X|, S, \{F_j\}_{j\in S})$ be a cellular self-similar structure. For any $n \in \mathbb{N}$, we define a cell complex $|X_n|$ as follows: first, for $\omega = (j_1, \cdots, j_n) \in S^{\times n}$, we write

$$F_\omega = F_{j_1} \circ \cdots \circ F_{j_n}.$$

We define $|X_n|$ by the following skelton filtration:

- $sk_0(|X_n|) = \bigcup_{\omega\in S^{\times n}} F_\omega(sk_0(|X|)),$
- $sk_1(|X_n|) = \bigcup_{\omega\in S^{\times n}} F_\omega(sk_1(|X|)),$
- $sk_2(|X_n|) = \bigcup_{\omega\in S^{\times n}} F_\omega(sk_2(|X|)) = \bigcup_{\omega\in S^{\times n}} F_\omega(|X|).$

A 1-cell in $|X_n|$ is defined to be the closure of a connected component in $sk_1(|X_n|) - sk_0(|X_n|)$. The definition of a cellular self-similar structure yields

$$|X_{n+1}| = \bigcup_{j\in S} F_j(\bigcup_{\omega\in S^{\times n}} F_\omega(|X|)) = \bigcup_{j\in S} F_j(|X_n|).$$

Therefore we have an inclusion map $i_{n,n+1} : |X_{n+1}| \hookrightarrow |X_n|$ for every $n \in \mathbb{Z}_{\geqslant 0}$, and then $K_{|X|}$ is written as the inverse limit of inclusion maps $\{i_{n,n+1} : |X_{n+1}| \hookrightarrow |X_n|\}$, that is,

$$K_{|X|} = \bigcap_{n=1}^{\infty} |X_n|.$$

From this point of view, we also have a canonical inclusion map $i_n : K_{|X|} \hookrightarrow |X_n|$ for each $n \in \mathbb{Z}_{\geqslant 0}$.

For $n \in \mathbb{N}$ and a 1-cell $|\sigma|$ in $\partial|X_n|$, we define E^n_σ to be the set of 1-cells of $|X_{n+1}|$ which are subspaces of $|\sigma|$. Then, we have

$$|\sigma| = \bigcup_{|\tau|\in E^n_\sigma} |\tau|.$$

Lemma 5.5. *There exists $M \in \mathbb{N}$ that satisfy the following condition: for any $n \in \mathbb{N}$ and a* 1*-cell $|\sigma|$ in $\partial|X_n|$ we have $\#E^n_\sigma \leqslant M$.*

Proof. For every 1-cell $|\sigma|$ in $\partial|X_n|$, there exists a unique $\omega \in S^{\times n}$ and a unique 1-cell $|\tilde{\sigma}|$ in $F_\omega(|X|)$ such that $|\sigma| \subset |\tilde{\sigma}|$. Since $|X_{n+1}|$ is obtained by replacing each 2-cell $F_\omega(|X|)$ by $F_\omega(|X_1|) = F_\omega(\bigcup_{j\in S} F_j(|X|))$, $|\tilde{\sigma}|$ is subdivided by at most $\#S$ 2-cells. This completes the proof of the lemma. □

Now, since every 2-cell in $|X_n|$ is a convex linear cell complex, we can associate an abstract simplicial complex X^s_n by employing a lemma in [35]:

Lemma 5.6. [35, Lemma 1, Chapter I] *A convex linear cell complex can be subdivided into a simplicial complex without introducing any more vertices.*

For any simplicial complex $|X^s_n|$ and $p \geqslant 0$, we define $S_p(X^s_n)$ to be a set of $(p+1)$-tuples of points of $sk_0(X^s_n)$ such that all the points are contained in a simplex of X^s_n, that is,

$$S_p(X^s_n) = \left\{(x_0, \cdots, x_p) \in sk_0(X^s_n)^{\times(p+1)} \mid \text{there exists a } p\text{-simplex } \sigma \in X^s_n \ s.t.\ x_i \in \sigma \text{ for } \forall i\right\}.$$

We also define face maps $d_i : S_p(X^s_n) \to S_{p-1}(X^s_n)$ for $0 \leqslant i \leqslant p$, and the pair $(S_*(X^s_n), d_*)$ forms a semi-simplicial set; see the definition in [8]. We note that, for $p \geqslant 1$, $S_p(X^s_n)$ contains a *degenerate simplex* $(x_0, \cdots, x_p)$, that is, a simplex $(x_0, \cdots, x_p) \in S_p(X^s_n)$ such that there exist distinct indexes i and j such that $x_i = x_j$. Now, we define $\tilde{S}_p(X^s_n; \mathbb{C})$ to be the free $\mathbb{C}$-module generated by $S_p(X^s_n)$ and a map $\tilde{\partial}_p : \tilde{S}_p(X^s_n; \mathbb{C}) \to \tilde{S}_{p-1}(X^s_n; \mathbb{C})$ by

$$\tilde{\partial}_p(x_0, \cdots, x_p) = \sum_{j=0}^{p} (-1)^j d_j(x_0, \cdots, x_p) = \sum_{j=0}^{p} (-1)^j (x_0, \cdots, \hat{x}_j, \cdots, x_p).$$

We call the resulting chain complex $(\tilde{S}_*(X^s_n; \mathbb{C}), \tilde{\partial}_*)$ the *ordered chain complex* of $|X^s_n|$ whose coefficients are in $\mathbb{C}$, see also [27].

Then we have a commutative diagram:

$$\begin{array}{ccc} \tilde{S}_p(X_n^s;\mathbb{C}) & \xrightarrow{\tilde{\partial}_p} & \tilde{S}_{p-1}(X_n^s;\mathbb{C}) \\ \pi \big\downarrow & & \big\downarrow \pi \\ C_p(X_n^s;\mathbb{C}) & \xrightarrow[\partial_p]{} & C_{p-1}(X_n^s;\mathbb{C}), \end{array}$$

where $C_p(X_n^s;\mathbb{C})$ is the p-th simplicial chain group of X_n^s whose coefficients are in $\mathbb{C}$, ∂_p the p-th simplicial boundary map and π the quotient map.

Remark 5.7. The chain map π is a chain equivalence; see Theorem 8 in Chapter 4.3 of [27] for the details.

We now assign the counterclockwise orientation on each 2-simplex in every $|X_n^s|$, and choose a basis $B_n = \{[\sigma]\}$ of $C_2(X_n^s;\mathbb{C})$ consisting of non-degenerate 2-simplexes σ in X_n^s. We assume that each element $[\sigma]$ of B_n represents the counterclockwise orientation and define simplicial chains for every $n \in \mathbb{Z}_{\geqslant 0}$: let

$$c_n = \sum_{[\sigma]\in B_n} [\sigma] \in C_2(X_n^s;\mathbb{C}).$$

Then $\partial_2(c_n) \in C_1(X_n^s;\mathbb{C})$ is the sum of all 1-simplices which lie on $\partial|X_n|$, and we can choose $s_n \in \pi^{-1}(c_n)$ so that s_n has no degenerate simplexes and each summand of $\tilde{\partial}_2(s_n) \in \tilde{S}_1(X_n^s;\mathbb{C})$ lies on $\partial|X_n|$. Now we define a boundary chain $b_n \in \tilde{S}_1(X_n^s;\mathbb{C})$ by

- $b_n = \tilde{\partial}_2(s_n)$.

We next let $\varepsilon(b_n)$ be the subset of 1-simplices in $S_1(X_n^s)$ which are direct summands of b_n. Since any $\sigma \in \varepsilon(b_n)$ is non-degenerate, we can take the geometric realisation $|\sigma| \subset |X_n^s|$. We also define a subset $\varepsilon(o_n) \subset \varepsilon(b_n)$ by

$$\varepsilon(o_n) = \Big\{\sigma \in \varepsilon(b_n) \ : \ |\sigma| \subset \partial|X|\Big\}.$$

For each $\sigma \in \varepsilon(o_n)$, we have the sign of σ in b_n and denote it by $\operatorname{sgn}(\sigma)$. We now define

- $o_n = \sum_{\sigma \in \varepsilon(o_n)} \mathrm{sgn}(\sigma) \cdot \sigma \in \tilde{S}_1(X_n^s; \mathbb{C})$,

- $I_n = b_n - o_n \in \tilde{S}_1(X_n^s; \mathbb{C})$.

Let $\varepsilon(I_n) = \varepsilon(b_n) \backslash \varepsilon(o_n)$. We also define $|\varepsilon(I_n)| = \bigcup_{\sigma \in \varepsilon(I_n)} |\sigma|$ and $\varepsilon(I_n \backslash I_{n-1})$ by

$$\varepsilon(I_n \backslash I_{n-1}) = \left\{ \sigma \in \varepsilon(b_n) \ : \ |\sigma| \subset \overline{|\varepsilon(I_n)| \backslash |\varepsilon(I_{n-1})|} \right\}.$$

Finally we define a 1-chain by

- $I_n \backslash I_{n-1} = \sum_{\sigma \in \varepsilon(I_n \backslash I_{n-1})} \mathrm{sgn}(\sigma) \cdot \sigma \in \tilde{S}_1(X_n^s; \mathbb{C})$.

Example 5.8.

For each example, we give spaces that represent $\varepsilon(b_0)$, $\varepsilon(b_1)$, $\varepsilon(b_2)$, and $\varepsilon(I_0)$, $\varepsilon(I_1)$, $\varepsilon(I_2)$. The first row corresponds to $\varepsilon(b_i)$, and the second corresponds to $\varepsilon(I_i)$. The dots in spaces denote the vertices of 1-simplices, i.e., 0-simplices.

- Sierpinski gasket

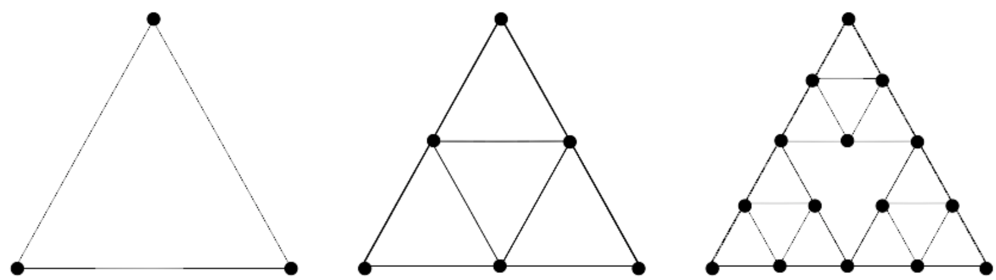


Figure 3: $\varepsilon(b_0)$, $\varepsilon(b_1)$, $\varepsilon(b_2)$

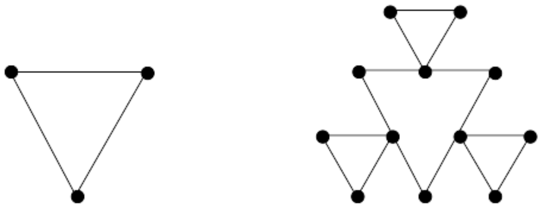


Figure 4: $\varepsilon(I_0)$, $\varepsilon(I_1)$, $\varepsilon(I_2)$

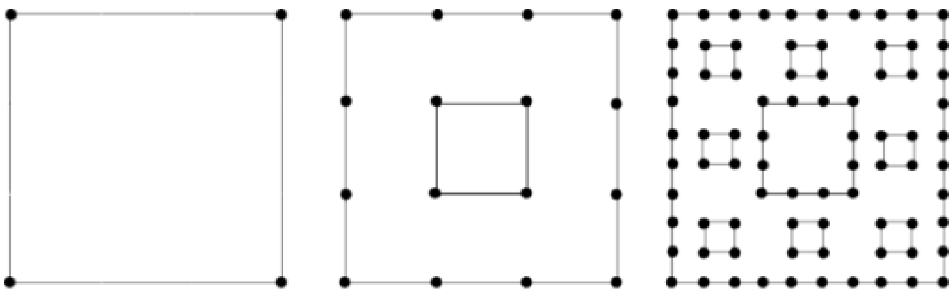


Figure 5: $\varepsilon(b_0)$, $\varepsilon(b_1)$, $\varepsilon(b_2)$

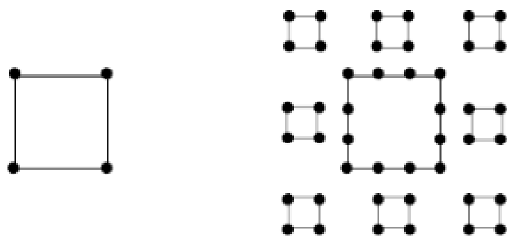


Figure 6: $\varepsilon(I_0)$, $\varepsilon(I_1)$, $\varepsilon(I_2)$

• Pinwheel fractal

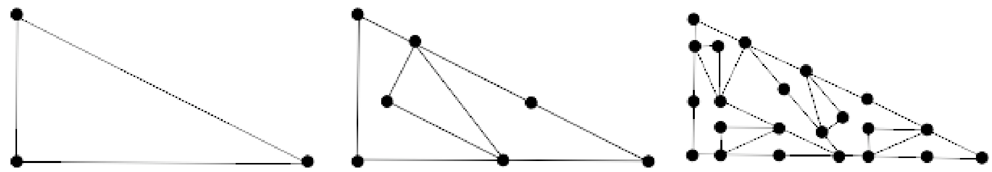


Figure 7: $\varepsilon(b_0)$, $\varepsilon(b_1)$, $\varepsilon(b_2)$

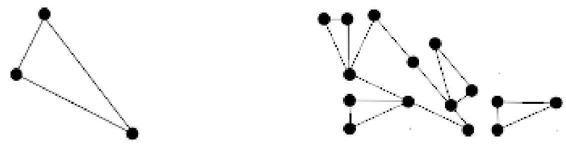


Figure 8: $\varepsilon(I_0)$, $\varepsilon(I_1)$, $\varepsilon(I_2)$

Next, for every $n \in \mathbb{Z}_{\geqslant 0}$, we define a 2-dimensional cell complex $|K_{n,n+1}|$. For every $n \in \mathbb{Z}_{\geqslant 0}$ we endow

$$|X_{n,n+1}| = \overline{|X_n| - |X_{n+1}|} \text{ (= the closure of } |X_n| - |X_{n+1}|),$$

with a cell complex structure, whose structure is defined by the following skelton filtration:

- $sk_0(|X_{n,n+1}|) = sk_0(\partial|X_{n+1}|) \cap |X_{n,n+1}|$
- $sk_1(|X_{n,n+1}|) = \partial|X_{n,n+1}|$
- $sk_2(|X_{n,n+1}|) = |X_{n,n+1}|$

We also define a subspace $|K_{n,n+1}|$ in $\mathbb{R}^3$ to be

$$|K_{n,n+1}| = [0,1] \times \partial|X_{n+1}| \cup \{1\} \times |X_{n,n+1}|.$$

We use z as the variable of the first coordinate of $|K_{n,n+1}|$. We now endow $|K_{n,n+1}|$ with a 2-dimensional cell complex structure as follows: let $p_1 : |K_{n,n+1}| \to |K_{n,n+1}||_{z=1}$ be a projection defined by $p_1(t,x) = (1,x)$. We define

- $sk_0(|K_{n,n+1}|) = \{0\} \times sk_0(\partial|X_{n+1}|) \cup \{1\} \times (sk_0(\partial|X_n|) \cup sk_0(|X_{n,n+1}|))$
- $sk_1(|K_{n,n+1}|) = \{0\} \times sk_1(\partial|X_{n+1}|) \cup \{1\} \times (sk_1(\partial|X_n|) \cup sk_1(\partial|X_{n,n+1}|) \cup |E_{n,n+1}|)$
- $sk_2(|K_{n,n+1}|) = |K_{n,n+1}|$.

Here,

$$E_{n,n+1} = \Big\{(x,y) \mid x \in \{1\} \times sk_0(\partial|X_n|) \text{ or } x \in \{1\} \times sk_0(|X_{n,n+1}|),$$

$$y \in \{0\} \times sk_1(\partial|X_{n+1}|) \; s.t. \; p_1(x) = y\Big\},$$

$$|E_{n,n+1}| = \bigcup_{(x,y)\in E_{n,n+1}} |(x,y)|.$$

By construction of $|K_{n,n+1}|$, we have

$$\partial|K_{n,n+1}| = \{0\} \times \partial|X_{n+1}| \cup \{1\} \times \partial|X_n|$$

as a cell complex in $\mathbb{R}^3$. By employing Lemma 5.6 again, the cell complex $|K_{n,n+1}|$ is subdivided into a 2-dimensional simplicial complex $|K^s_{n,n+1}|$, and we may therefore choose chains $s_{n,n+1}$, $\tilde{s}_{n,n+1}$ and $\tilde{\tilde{s}}_{n,n+1} \in \tilde{S}_2(K^s_{n,n+1};\mathbb{C})$ so that the chains consist of non-degenerate simplexes:

$$\tilde{\partial}_2(s_{n,n+1}) = b_n - b_{n+1}, \quad \tilde{\partial}_2(\tilde{s}_{n,n+1}) = I_n - I_{n+1}, \quad \tilde{\partial}_2(\tilde{\tilde{s}}_{n,n+1}) = I_{n+1} \backslash I_n.$$

We define the sets $\varepsilon(s_{n,n+1})$, $\varepsilon(\tilde{s}_{n,n+1})$ and $\varepsilon(\tilde{\tilde{s}}_{n,n+1})$ in a manner similar to the definition of $\varepsilon(b_n)$ and assume that $\tilde{s}_{n,n+1}$ and $\tilde{\tilde{s}}_{n,n+1}$ are summands of $s_{n,n+1}$, in other words,

$$\varepsilon(\tilde{s}_{n,n+1}), \; \varepsilon(\tilde{\tilde{s}}_{n,n+1}) \subset \varepsilon(s_{n,n+1}).$$

By a *Jordan cycle* z in $I_{n+1}\backslash I_n$ we mean a subset z of $\varepsilon(I_{n+1}\backslash I_n)$ such that $\bigcup_{\sigma\in z}|\sigma|$ is homomorphic to S^1, and denote $\bigcup_{\sigma\in z}|\sigma|$ by $|z|$. We also denote by $\mathrm{cyc}(I_{n+1}\backslash I_n)$ the set of Jordan cycles in $I_{n+1}\backslash I_n$, and define $\tilde{z} = \sum_{\sigma\in z}\mathrm{sgn}(\sigma)\cdot\sigma \in \tilde{S}_1(K^s_{n,n+1};\mathbb{C})$ for $z \in \mathrm{cyc}(I_{n+1}\backslash I_n)$. Then, for every Jordan cycle z in $I_{n+1}\backslash I_n$, there exists a non-degenerate 2-chain $\tilde{\tilde{s}}_z \in \tilde{S}_2(K^s_{n,n+1};\mathbb{C})$ such that $\tilde{\partial}_2(\tilde{\tilde{s}}_z) = \tilde{z}$.

For $n = 0$, we define

$$|\tilde{K}_{0,1}| = [0,1]\times\partial(|X|-|X_1|)\cup\{1\}\times|X_{0,1}|$$

and then $|\tilde{K}_{0,1}|$ is written as

$$|\tilde{K}_{0,1}| = \bigcup_{z\in\mathrm{cyc}(I_1\backslash I_0)}|\varepsilon(\tilde{\tilde{s}}_z)|$$

since $\partial(|X|-|X_1|) = |\varepsilon(I_1\backslash I_0)|$. Moreover, since we have an inclusion map $i_\omega : \partial(|X|-|X_1|)\hookrightarrow F_\omega(\partial|X_1|)$ for every $\omega\in S^{\times n}$, there exists a family $\{\tilde{i}_\omega\}_{\omega\in S^{\times n}}$ of inclusion maps $\tilde{i}_\omega : |\tilde{K}_{0,1}|\hookrightarrow|K_{n,n+1}|$ such that

$$\tilde{i}_\omega|_{z=0} = i_\omega.$$

Finally we fix a subdivision of $|\tilde{K}_{0,1}|$ and assume that a subdivision of the images of the inclusion maps $\tilde{i}_\omega$ are given by the subdivision of $|\tilde{K}_{0,1}|$.

5.2 Approximating cyclic1-cocycles

In this subsection, we define a sequence of complex numbers for given Hölder continuous functions, that we call an approximating cyclic1-cocycle. In order to define the sequence, we first recall a cochain complex which gives rise to the Alexander-Spanier cohomology theory.

Let R be a ring. We also let X be a set and $X^{(p+1)}$ the $(p+1)$-fold product of

X. We define $F^p(X;R)$ to be the abelian group of functions from $X^{(p+1)}$ to R, whose sum is given by the pointwise sum. A coboundary homomorphism $\delta : F^p(X;R) \to F^{p+1}(X;R)$ is defined by

$$(\delta\phi)(x_0, \cdots, x_{p+1}) = \sum_{j=0}^{p+1} (-1)^j \phi(x_0, \cdots, \hat{x_j}, \cdots, x_{p+1}).$$

We also define the cup product on the complex $(F^*(X;R), \delta)$: for $\phi_1 \in F^p(X;R)$ and $\phi_2 \in F^q(X;R)$, the cup product $\phi_1 \cup \phi_2 \in F^{p+q}(X;R)$ is defined by

$$(\phi_1 \cup \phi_2)(x_0, \cdots, x_{p+q}) = \phi_1(x_0, \cdots, x_p)\phi_2(x_p, \cdots, x_{p+q}).$$

The Leibniz rule holds for the cup product: for $\phi_1 \in F^p(X;R)$ and $\phi_2 \in F^q(X;R)$,

$$\delta(\phi_1 \cup \phi_2) = \delta\phi_1 \cup \phi_2 + (-1)^p \phi_1 \cup \delta\phi_2.$$

Remark 5.9. The cochain complex $(F^*(X;R), \delta)$ does *not* give proper cohomology theory because the complex includes *locally zero* cochains; see Chapter 6.4 of [27]. Due to locally zero cochains, if X is a nonempty set, then the p-th cohomology group $H^p((F^*(X;R), \delta))$ is R for $p \geqslant 0$. However, if X is a topological space, then we can take the quotient of $(F^*(X;R), \delta)$ by locally zero cochains, and the quotient complex yields a cohomology theory of X, that is called the *Alexander-Spanier cohomology theory*. Since the cup product defined above gives rise to a cup product on the cohomology group, the Alexander-Spanier cohomology group turns out to be a graded algebra over R. Moreover, the cup product is compatible with the one defined on singular cohomology theory; see Chapter 6 of [27] for the details.

Now, we define a cochain subcomplex of $(F^*(X;R), \delta)$: we assume that X is a metric space and R the field of complex numbers $\mathbb{C}$. We also let $C^\alpha(X)$ be the space of complex-valued α-Hölder continuous functions on X. Then, $C^\alpha(X)$ is a subspace of $F^0(X;\mathbb{C})$, and for each $p \in \mathbb{Z}_{\geqslant 0}$ we define the subspace $C^{\alpha,p}(X)$ of $F^p(X;\mathbb{C})$

generated by $C^{\alpha}(X) \subset F^{0}(X;\mathbb{C})$ with the coboundary maps and the cup products.

We now apply the construction for a cellular self-similar structure $(|X|, S, \{F_j\}_{j\in S})$: let $C^{\alpha}(K_{|X|})$ be the α-Hölder continuous functions defined on $K_{|X|}$. For each $n \in \mathbb{N}$, we endow $sk_0(|X_n|)$ with the induced metric of $\mathbb{R}^2$. Since we have an inclusion map $j_n : sk_0(|X_n|) \hookrightarrow K_{|X|}$ for every $n \in \mathbb{Z}_{\geqslant 0}$, we have a commutative diagram of cochain complexes

$$\begin{array}{ccc} F^{p}(K_{|X|};\mathbb{C}) & \xrightarrow{\ j_n^{*}\ } & F^{p}(sk_0(|X_n^s|);\mathbb{C}) \\ \big\uparrow & & \big\uparrow \\ C^{\alpha,p}(K_{|X|}) & \xrightarrow[\ j_n^{*}\ ]{} & C^{\alpha,p}(sk_0(|X_n^s|)) \end{array}$$

The cochain complex $F^{p}(sk_0(|X_n^s|);\mathbb{C})$ is seen as the set of complex-valued functions $\mathrm{Func}(S_p(\Delta^{\# sk_0(|X_n^s|)}),\mathbb{C})$ defined on $S_p(\Delta^{\# sk_0(|X_n^s|)}) := sk_0(|X_n^s|)^{\times p+1}$. In a manner similar to the definition of the face maps d_i of $S_p(X_n^s)$, we define the face maps on $S_*(\Delta^{\# sk_0(|X_n^s|)})$, and then the pair $(S_*(\Delta^{\# sk_0(|X_n^s|)}), \{d_i\})$ turns out to be a semi-simplicial set; the definition of which is given in [8]. Since the inclusion map $S_*(|X_n^s|) \hookrightarrow S_*(\Delta^{\# sk_0(|X_n^s|)})$ is a map of semi-simplicial sets, we therefore get the following commutative diagram:

$$\begin{array}{ccccc} F^{p}(K_{|X|};\mathbb{C}) & \xrightarrow{\ j_n^{*}\ } & F^{p}(sk_0(|X_n^s|);\mathbb{C}) & \xrightarrow[\text{linearly}]{\text{extend}} & \mathrm{Hom}_{\mathbb{C}}(\tilde{S}_p(\Delta^{\# sk_0(|X_n|)};\mathbb{C}),\mathbb{C}) \\ \big\uparrow & & \big\uparrow & & \big\downarrow \text{restrict} \\ C^{\alpha,p}(K_{|X|}) & \xrightarrow[\ j_n^{*}\ ]{} & C^{\alpha,p}(sk_0(|X_n^s|)) & \xrightarrow[\ r\ ]{} & \mathrm{Hom}_{\mathbb{C}}(\tilde{S}_p(|X_n^s|;\mathbb{C}),\mathbb{C}). \end{array}$$

Now we define $C^{\alpha,p}(|X_n^s|) = \mathrm{im}(r)$. For any $f,\, g \in C^{\alpha}(K_{|X|})$ and $p = 1$, we have a 1-cochain $\omega_n(f,g) = (f \cup \delta g) - (g \cup \delta f)$ in $C^{\alpha,1}(|X_n^s|)$ for every $n \in \mathbb{N}$.For every $n \in \mathbb{N}$, we set $\phi_n(f,g)$ as the evaluation of $\omega_n(f,g)$ with $I_n \in \tilde{S}_1(|X_n^s|;\mathbb{C})$:

$$\phi_n(f,g) = \omega_n(f,g)(I_n).$$

Definition 5.10. Let $f,\, g \in C^{\alpha}(K_{|X|})$. We call the sequence $\{\phi_n(f,g)\}_{n\in\mathbb{N}}$ the *approximating cyclic1-cocycle* for f and g.

5.3 Non-trivial Cyclic 1-cocycles

We first state again the main result of this section, called existence theorem, and prove the theorem.

Theorem 5.11 (Existence theorem). *Let* $(|X|, S, \{F_j\}_{j\in S})$ *be a cellular self-similar structure with* $|X| \neq |X_1|$ *and* $K_{|X|}$ *the cellular self-similar set with respect to* $(|X|, S, \{F_j\}_{j\in S})$. *If* $2\alpha > \dim_H(K_{|X|})$*, then the approximating cyclic1-cocycle* $\{\phi_n(f,g)\}_{n\in\mathbb{N}}$ *is a Cauchy sequence for any* f, $g \in C^\alpha(K_{|X|})$.

Proof. We first endow $|K_{n,n+1}|$ with a quasi-metric by $d((t,x),(t',x')) = |x-x'|_{\mathbb{R}^2}$. Let f, $g \in C^\alpha(K_{|X|})$. Since we have an inclusion map $j_{n+1} : sk_0(|X_{n+1}|) \hookrightarrow K_{|X|}$ for every $n \in \mathbb{Z}_{\geqslant 0}$, we can define a map $f_n : sk_0(|K_{n,n+1}|) \to \mathbb{C}$ by $f_n(t,x) = f(x)$. Then, for any (t,x) and $(t',x') \in sk_0(|K_{n,n+1}|)$, the map f_n satisfies

$$d_{\mathbb{C}}(f_n(t,x), f_n(t',x')) \leqslant c_f \cdot d((t,x),(t',x')) = c_f \cdot |x-x'|_{\mathbb{R}^2}.$$

We also let, for $h, k \in C^\alpha(sk_0(|K_{n,n+1}|))$, $\omega_n(h,k) = (h \cup \delta k) - (k \cup \delta h)$ be a 1-cochain in $C^{\alpha,1}(|K^s_{n,n+1}|)$. Then, we have

$$\begin{aligned}
|\phi_n(f,g) - \phi_{n+1}(f,g)| &= |\omega_n(f_n,g_n)(I_n - I_{n+1})| \\
&= |\omega_n(f_n,g_n)(\tilde{\partial}_2(\tilde{s}_{n,n+1}))| \\
&\leqslant |\omega_n(f_n,g_n)(\tilde{\partial}_2(\tilde{s}_{n,n+1}))| + |\omega_n(f_n,g_n)(\tilde{\partial}_2(s_{n,n+1} - \tilde{s}_{n,n+1}))| \\
&\leqslant \sum_{\sigma\in\varepsilon(s_{n,n+1})} |\omega_n(f_n,g_n)(\tilde{\partial}_2(\sigma))| \\
&= \sum_{\sigma\in\varepsilon(s_{n,n+1})} |(\delta f_n \cup \delta g_n)(\sigma) - (\delta g_n \cup \delta f_n)(\sigma)|. \qquad (1)
\end{aligned}$$

We note that every $\sigma \in \varepsilon(s_{n,n+1})$ is given by $\sigma = (x,y,z)$ for some x, y, $z \in sk_0(|K_{n,n+1}|)$.

Therefore, (1) may be written as

$$
\begin{aligned}
(1) \quad &= \sum_{(x,y,z)\in\varepsilon(s_{n,n+1})} |(\delta f_n \cup \delta g_n)(x,y,z) - (\delta g_n \cup \delta f_n)(x,y,z)| \\
&= \sum_{(x,y,z)\in\varepsilon(s_{n,n+1})} \Big|(f_n(y)-f_n(x))(g_n(z)-g_n(y)) - (g_n(y)-g_n(x))(f_n(z)-f_n(y))\Big| \\
&\leqslant \sum_{(x,y,z)\in\varepsilon(s_{n,n+1})} 2\cdot c_f \cdot c_g \cdot |y-x|^{\alpha}|z-y|^{\alpha}, \qquad (2)
\end{aligned}
$$

where c_f and c_g are the Hölder constants of f and g, respectively.

We now define a map to estimate the term (2). For any $\sigma \in \varepsilon(s_{n,n+1})\backslash\varepsilon(\tilde{\tilde{s}}_{n,n+1})$ there exists a unique $\omega = (j_1,\cdots,j_n) \in S^{\times n}$ such that $p_1(|\sigma|) \subset \partial F_\omega(|X|)$. By this assignment, we can define a map $\rho : \varepsilon(s_{n,n+1})\backslash\varepsilon(\tilde{\tilde{s}}_{n,n+1}) \to S^{\times n}$, and let $\tilde{S}^{\times n}$ be $\mathrm{im}(\rho)$. We note that, by Lemma 5.3, there exists $M \in \mathbb{N}$ such that $\#\rho^{-1}(\omega) < M$ for any $\omega \in \tilde{S}^{\times n}$. Moreover, since $p_1(|\sigma|) \subset \partial F_\omega(|X|)$ we have an inequality

$$\mathrm{diam}(|\sigma|) = \mathrm{diam}(p_1(|\sigma|)) \leqslant r_{j_1}\cdot\cdots\cdot r_{j_n}\cdot d_{K_{|X|}} = \mathrm{diam}(F_\omega(|X|)),$$

where $(j_1,\cdots,j_n) = \omega \in \tilde{S}^{\times n}$, r_j are the similarity ratios of F_j and $d_{K_{|X|}} = \mathrm{diam}(K_{|X|})$ is the diameter of $K_{|X|}$.

On the other hand, we let $L = \#\mathrm{cyc}(I_1\backslash I_0)$ be the number of Jordan cycles in $I_1\backslash I_0$. At the $(n+1)$-step, for every $\omega \in S^{\times n}$, there exist L Jordan cycles in $F_\omega(\bigcup_{j\in S} F_j(|X|)) = F_\omega(|X_1|)$. We recall that, for every Jordan cycle z in $I_{n+1}\backslash I_n$, there is a 2-chain $\tilde{\tilde{s}}_z \in \tilde{S}_2(K^s_{n,n+1})$ such that $\varepsilon(\tilde{\tilde{s}}_z) \subset \varepsilon(\tilde{\tilde{s}}_{n,n+1})$ and $\tilde{\partial}_2(\tilde{\tilde{s}}_z) = \tilde{z}$; see also Section 5.1. Therefore, $\tilde{\tilde{s}}_{n,n+1}$ is decomposed into

$$\tilde{\tilde{s}}_{n,n+1} = \sum_{\omega\in S^{\times n}} \sum_{1\leqslant i\leqslant L} \tilde{\tilde{s}}_{\omega,z_i}.$$

We also recall from Section 5.1 that, for every $\omega \in S^{\times n}$, we have an inclusion map

$\tilde{\imath}_\omega : |\tilde{K}^s_{0,1}| \hookrightarrow |K_{n,n+1}|$ and

$$\mathrm{im}(\tilde{\imath}_\omega) = \bigcup_{z\in\mathrm{cyc}(I_{n+1}\setminus I_n) \text{ s.t. } |z|\subset F_\omega(|X_1|)} |\varepsilon(\tilde{\tilde{s}}_z)|.$$

Therefore, since the subdivision of the images $\mathrm{im}(\tilde{\imath}_\omega)$ are induced by the fixed subdivision of $|K^s_{0,1}|$, we may define

$$\overline{M} = \sup_{z\in\mathrm{cyc}(I_{n+1}\setminus I_n)} \{\#\varepsilon(\tilde{\tilde{s}}_z)\} = \sup_{z\in\mathrm{cyc}(I_1\setminus I_0)} \{\#\varepsilon(\tilde{\tilde{s}}_z)\}.$$

From these arguments, (2) is now decomposed into two parts:

$$\begin{aligned}
(2) &= \sum_{(x,y,z)\in\varepsilon(s_{n,n+1})\setminus\varepsilon(\tilde{\tilde{s}}_{n,n+1})} 2\cdot c_f\cdot c_g\cdot|y-x|^\alpha|z-y|^\alpha \\
&\qquad + \sum_{(x,y,z)\in\varepsilon(\tilde{\tilde{s}}_{n,n+1})} 2\cdot c_f\cdot c_g\cdot|y-x|^\alpha|z-y|^\alpha \\
&\leqslant \sum_{(j_1,\cdots,j_n)\in\tilde{S}^{\times n}} 2\cdot c_f\cdot c_g\cdot\#\rho^{-1}(\omega)\cdot(r_{j_1}^{2\alpha}\ \cdots\ r_{j_n}^{2\alpha}\cdot d^{2\alpha}_{K_{|X|}}) \\
&\qquad + \sum_{(j_1,\cdots,j_n)\in S^{\times n}}\sum_{1\leqslant i\leqslant L} 2\cdot c_f\cdot c_g\cdot\#\varepsilon(\tilde{\tilde{s}}_{z_i})\cdot(r_{j_1}^{2\alpha}\ \cdots\ r_{j_n}^{2\alpha}\cdot d^{2\alpha}_{K_{|X|}}) \\
&\leqslant \sum_{(j_1,\cdots,j_n)\in S^{\times n}} 2\cdot c_f\cdot c_g\cdot M\cdot(r_{j_1}^{2\alpha}\ \cdots\ r_{j_n}^{2\alpha}\cdot d^{2\alpha}_{K_{|X|}}) \\
&\qquad + \sum_{(j_1,\cdots,j_n)\in S^{\times n}} 2\cdot c_f\cdot c_g\cdot L\cdot\overline{M}\cdot(r_{j_1}^{2\alpha}\ \cdots\ r_{j_n}^{2\alpha}\cdot d^{2\alpha}_{K_{|X|}}) \\
&= 2\cdot c_f\cdot c_g\cdot d^{2\alpha}_{K_{|X|}}\cdot(M+L\cdot\overline{M})\cdot(\sum_{j\in S} r_j^{2\alpha})^n.
\end{aligned}$$

We denote $2\cdot c_f\cdot c_g\cdot d^{2\alpha}_{K_{|X|}}\cdot(M+L\cdot\overline{M})$ by K, and then we have

$$\begin{aligned}
|\phi_{n+k}(f,g)-\phi_n(f,g)| &\leqslant \sum_{1\leqslant i\leqslant k} |\phi_{n+i}(f,g)-\phi_{n+i-1}(f,g)| \\
&\leqslant \sum_{1\leqslant i\leqslant k} K\cdot(\sum_{j\in S} r_j^{2\alpha})^{n+i-1} \\
&= K\cdot(\sum_{j\in S} r_j^{2\alpha})^n\cdot\sum_{1\leqslant i\leqslant k}(\sum_{j\in S} r_j^{2\alpha})^{i-1}. \qquad (3)
\end{aligned}$$

Since $2\alpha > \dim_H(K_{|X|})$ and $\dim_H(K_{|X|})$ is computed by the formula in Theorem 3.11, the term $(\sum_{j\in S} r_j^{2\alpha})$ is less than 1. Therefore the term $\sum_{1\leqslant i\leqslant k}(\sum_{j\in S} r_j^{2\alpha})^{i-1}$ also converges to a finite value as k tends to ∞. Hence, we have

$$(3) \quad \leqslant \quad K\cdot\Big\{\sum_{i=1}^{\infty}\big(\sum_{j\in S} r_j^{2\alpha}\big)^{i-1}\Big\}\cdot\big(\sum_{j\in S} r_j^{2\alpha}\big)^n,$$

and the right hand side converges to 0 as n tends to ∞. This completes the proof of Theorem 5.11. □

From now on, we assume that $2\alpha > \dim_H(K_{|X|})$, and define a bilinear map

$$\phi : C^{\alpha}(K_{|X|}) \times C^{\alpha}(K_{|X|}) \to \mathbb{C}, \qquad \phi(f,g) = \lim_{n\to\infty} \phi_n(f,g).$$

Lemma 5.12. *The map ϕ is independent of the choice of I_n.*

Proof. In order to check the mentioned property of the bilinear map $\phi : C^{\alpha}(K_{|X|}) \times C^{\alpha}(K_{|X|}) \to \mathbb{C}$, we have to show that the approximating cyclic1-cocycle converges to the same value regardless of the choice of I_n which represents the given orientation. Let I_n, $I'_n \in \pi^{-1}([I_n])$ such that $|\varepsilon(I_n)| = |\varepsilon(I'_n)|$ and $\phi'_n(f,g) = (f \cup \delta g)(I'_n) - (g \cup \delta f)(I'_n)$. Then there exists a 2-dimensional simplicial complex J_n such that $|J_n| = |\varepsilon(I_n)| \times [0,1]$, and we choose $\hat{s}_n \in \tilde{S}_2(J_n;\mathbb{C})$ such that $\tilde{\partial}_2(\hat{s}_n) = I_n - I'_n$. We endow $|J_n|$ with a quasi-metric similar to the one on $|K_{n,n+1}|$, and then we have

$$\begin{aligned}
|\phi_n(f,g) - \phi'_n(f,g)| &= |\delta\omega_n(f_n,g_n)(\hat{s}_n)| \\
&\leqslant 2\sum_{(x,y,z)\in\varepsilon(\hat{s}_n)} c_f\cdot c_g\cdot|y-x|^{\alpha}\cdot|z-y|^{\alpha} \\
&\leqslant 2\sum_{(x,y)\in\varepsilon(I_n)} 2\cdot c_f\cdot c_g\cdot|y-x|^{2\alpha} \\
&\leqslant 2\sum_{(j_1,\cdots,j_n)\in\tilde{S}^{\times n}} 2\cdot c_f\cdot c_g\cdot d_{K_{|X|}}^{2\alpha}\cdot r_{j_1}^{2\alpha}\cdot\dots\cdot r_{j_n}^{2\alpha} \\
&\leqslant 4\cdot c_f\cdot c_g\cdot d_{K_{|X|}}^{2\alpha}\cdot\big(\sum_{j\in S} r_j^{2\alpha}\big)^n.
\end{aligned}$$

This completes the proof of Lemma 5.12. □

Based on the proof of the existence theorem, we can prove the following theorem.

Theorem 5.13. *For any $f,\ g \in C^{\alpha}(K_{|X|})$ with $2\alpha > \dim(K_{|X|})$, we have*

$$\phi(f,g) = -2 \cdot Y(f,g)|_{\partial|X|} \ = -2 \cdot \text{ (Young integral along } \partial|X|).$$

In particular, if $|X| \neq |X_1|$, for 1 *and $x := id \in C^{\alpha}(K_{|X|})$, we get*

$$\phi(1,x) = -2 \cdot Y(1,x)|_{\partial|X|} \ = \ -2 \cdot (\text{length of } \partial|X|).$$

Proof. By the construction of the approximating cyclic1-cocycle of $f,\ g \in C^{\alpha}(K_{|X|})$, we have

$$\phi_n(f,g) = \omega_n(f,g)(I_n) = -\omega_n(f,g)(o_n) + \omega_n(f,g)(b_n).$$

The proof of Theorem 5.11 yields directly that the sequence $\{\omega_n(f,g)(b_n)\}_{n \in \mathbb{Z}_{\geqslant 0}}$ converges to 0 if $2\alpha > \dim_H(K_{|X|})$. Since $\{\omega_n(f,g)(o_n)\}_{n \in \mathbb{Z}_{\geqslant 0}}$ provides the Young integration along $\partial|X|$ which is the finite union of closed segments, we get the mentioned equalities. □

Remark 5.14. There exists the dual notion of the Hochschild cohomology group $HH^*(A)$ and cyclic cohomology group $HC^*(A)$ of a unital commutative algebra A, known as the *Hochschild homology group $HH_*(A)$* and the *cyclic homology group $HC_*(A)$* of A respectively [18]. There is also a well-known fact, known as the Hochschild-Kostant-Rosenberg theorem, that the group $HH_*(A)$ is canonically isomorphic to the de Rham cochain complex $\Omega^*(A)$ of A if the given algebra A is smooth in the sense of algebraic geometry [13]. Moreover, when we focus on $HH_1(A)$ the assumption that the algebra A is smooth is not required. Under the isomorphism of $HH_1(A)$ and $\Omega^1(A)$, the Hochschild 1-cycle $1 \otimes x$ corresponds to $dx \in \Omega^1(A)$. However, we do not know the case when $\phi(1 \otimes x)$ gives the proper volume of $K_{|X|}$.

Theorem 5.15. *Under the assumption of Theorem 5.11*:

a) The bilinear map ϕ is a cyclic 1-cocycle of $C^{\alpha}(K_{|X|})$.

b) The cocycle ϕ represents a non-trivial element $[\phi]$ in $HC^1(C^{\alpha}(K_{|X|}))$.

Proof. We have a linear map $\phi : C^{\alpha}(K_{|X|}) \otimes C^{\alpha}(K_{|X|}) \to \mathbb{C}$. It follows immediately that the cocycle ϕ satisfies the cyclic condition since $\phi_n(f,g)$ satisfies the cyclic condition for any $n \in \mathbb{Z}_{\geqslant 0}$. Accordingly, it remains to show that ϕ is a Hochschild 1-cocycle. For f, g, $h \in C^{\alpha}(K_{|X|})$, we may write $b\phi(f,g,h)$ as

$$\begin{aligned} b\phi(f,g,h) &= \phi(fg,h) - \phi(f,gh) + \phi(hf,g) \\ &= \lim_{n\to\infty} \phi_n(fg,h) - \lim_{n\to\infty} \phi_n(f,gh) + \lim_{n\to\infty} \phi_n(hf,g) \\ &= \lim_{n\to\infty} \Big(\phi_n(fg,h) - \phi_n(f,gh) + \phi_n(hf,g) \Big) \\ &= \lim_{n\to\infty} b\phi_n(f,g,h). \end{aligned}$$

Therefore, to prove $b\phi(f,g,h) = 0$ is equivalent to prove $\lim_{n\to\infty} b\phi_n(f,g,h) = 0$. Using $\delta(\eta \cup \tau) = \delta\eta \cup \tau + (-1)^{\deg(\eta)} \eta \cup \delta\tau$, we have

$$\begin{aligned} \phi_n(fg,h) &= \Big(fg \cup \delta h - h \cup \delta(fg) \Big)(I_n) \\ &= \Big((f \cup g \cup \delta h) - (h \cup \delta f \cup g) - (h \cup f \cup \delta g) \Big)(I_n). \end{aligned}$$

Similarly,

$$\phi_n(f,gh) = \Big((f \cup \delta g \cup h) + (f \cup g \cup \delta h) - (g \cup h \cup \delta f) \Big)(I_n),$$

$$\phi_n(hf,g) = \Big((h \cup f \cup \delta g) - (g \cup \delta h \cup f) - (g \cup h \cup \delta f) \Big)(I_n).$$

Therefore,

$$b\phi_n(f,g,h) = -\Big((h \cup \delta f \cup g) + (f \cup \delta g \cup h) + (g \cup \delta h \cup f) \Big)(I_n). \tag{4}$$

Since

$$
\begin{aligned}
-(h \cup \delta f \cup g)(I_n) &= \Big((\delta h \cup f \cup g) + (h \cup f \cup \delta g) - (\delta(hfg))\Big)(I_n) \\
&= \Big((\delta h \cup f \cup g) + (h \cup f \cup \delta g)\Big)(I_n),
\end{aligned}
$$

we have

$$
\begin{aligned}
(4) &= \Big((\delta h \cup f \cup g) + (h \cup f \cup \delta g) - (f \cup \delta g \cup h) - (g \cup \delta h \cup f)\Big)(I_n) \\
&= \sum_{(x,y)\in\varepsilon(I_n)} \pm\Big((\delta h \cup f \cup g) + (h \cup f \cup \delta g) - (f \cup \delta g \cup h) - (g \cup \delta h \cup f)\Big)(x,y) \\
&= \sum_{(x,y)\in\varepsilon(I_n)} \pm(h(y)-h(x))(g(y)-g(x))(f(y)-f(x)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
|b\phi_n(f,g,h)| &\leqslant \sum_{(x,y)\in\varepsilon(I_n)} |h(y)-h(x)|\cdot|g(y)-g(x)|\cdot|f(y)-f(x)| \\
&= \sum_{(x,y)\in\varepsilon(I_n)} c_f\cdot c_g\cdot c_h\cdot|x-y|^{3\alpha} \\
&\leqslant c_f\cdot c_g\cdot c_h\cdot d_{K_{|X|}}^{3\alpha}\cdot M \sum_{(j_1,\cdots,j_n)\in S^{\times n}} r_{j_1}^{3\alpha}\cdot\cdots\cdot r_{j_n}^{3\alpha} \\
&= c_f\cdot c_g\cdot c_h\cdot d_{K_{|X|}}^{3\alpha}\cdot M\cdot(\sum_{j\in S} r_j^{3\alpha})^n \\
&\to 0, \quad \text{as } n\to\infty.
\end{aligned}
$$

This completes the proof of (a).

We now prove (b). We note that we have the pairing

$$HH_1(C^\alpha(K_{|X|}))\times HH^1(C^\alpha(K_{|X|}))\to\mathbb{C}.$$

As seen in Theorem 5.13, we know that $\phi(1\otimes x)\neq 0$, and this completes the proof of (b). □

We recall from Section 3.2 that for cellular self-similar structures $(|X|, S, \{F_j\}_{j\in S})$ and $(|X^{'}|, S^{'}, \{F_j^{'}\}_{j\in S^{'}})$, if there exists a bi-Lipschitz function between the cellular self-similar sets $K_{|X|}$ and $K_{|X'|}$, the algebras of α-Hölder continuous functions on $K_{|X|}$ and $K_{|X'|}$ are isomorphic and their Hausdorff dimensions coincide. Therefore, under this assumption, the cyclic cohomology groups of the algebras of α-Hölder continuous functions on $K_{|X|}$ and $K_{|X'|}$ are isomorphic and the thresholds for the well-definedness of the cocycles are same.

Remark 5.16. a) The cocycles stated in the above theorem may be extended to certain variants of cellular self-similar sets. In particular, Strichartz introduces the notion of *fractafolds* [29, 30], and the cocycle ϕ showed in Theorem 5.15 may be extended on some fractafolds. We look at cocycles on variants of cellular self-similar sets in the last subsection.

b) The algebra $C^{\alpha}(X)$of α-Hölder continuous functions on a compact metric space admits a Banach topology, and $C^{\alpha}(X)$ turns out to be a Banach algebra. However, we do not know whether or not the cocycle ϕ is continuous in the sense of a map between Banach algebras.

5.4 Examples

In this subsection we examine the cyclic cocycle on some cellular self-similar sets. The spaces on which the cocycles are examined are basically the examples given in Section 3.3.

- **unit square**

As we mentioned in Section 3.3, the unit square $I^2 = [0,1] \times [0,1]$ is a cellular self-similar set. Namely, by regarding I^2 as a union of 9 squares whose size are $\frac{1}{3}$ of I^2, we have a cellular self-similar structure that consists of 9-similitudes. However, for every $n \in \mathbb{N}$ the inner simplicial 1-chain I_n is 0 because $|X| = |X_1|$. Therefore, the cyclic cocycle ϕ defined on $C^{\alpha}(I^2)^{\otimes 2}$ turns out to be a trivial map. This is the case where a cellular self-similar structure does not satisfy the assumption $|X| \neq |X_1|$.

• **Sierpinski gasket**

The theorems in the Section 5.3 may be applied to the Sierpinski gasket SG, and the cyclic 1-cocycle ϕ on $C^{\alpha}(SG)$ is well-defined for $2\alpha > \dim_H(SG) = \log_2 3$. Moreover, the cocycle is non-trivial since $|X| \neq |X_1|$. This cocycle was originally given in [23].

• **Sierpinski carpet**

The cyclic 1-cocycle is well-defined if $2\alpha > \dim_H(SC) = \log_3 8$, and the cocycle is non-trivial since $|X| \neq |X_1|$. We recall that bi-Lipschitz transformation preserve the Hausdorff dimension. Therefore, the cocycles of SG and SC prove that there exists no bi-Lipschitz transformations between SG and SC. Similar argument also works for the other examples described below.

• **pinwheel fractal**

Pinwheel fractal PF may also be seen as a cellular self-similar set. The self-similar structure consists of 4 similitudes whose ratios are $\frac{1}{\sqrt{5}}$; see Section 3.3 for the details. The cyclic cocycle is well-defined if $2\alpha > \dim_H(PF) = \log_{\sqrt{5}} 4$ and non-trivial in HC^1.

• ***L*-shape fractal**

L-shape fractal set LSF in Section 3.3 is the limit set of a cellular self-similar structure that consists of 9 similitudes, two of which have similarity ratios $\frac{1}{2}$ and the rest of which have ratios $\frac{1}{4}$. Therefore $\dim_H(LSF) = \log_{\frac{1}{2}} \frac{2\sqrt{2}-1}{7}$ by Theorem 3.11, and the cocycle is well-defined on $C^{\alpha}(LSF)$ if $2\alpha > \log_{\frac{1}{2}} \frac{2\sqrt{2}-1}{7}$.

• **self-similar set based on perfect tiling**

We employ the self-similar structure based on the perfect tiling of the square in Section 3.3. The Hausdorff dimension of the cellular self-similar set $\dim_H(PT)$ is the root of the equation in Section 3.3. Then we have the non-trivial cyclic 1-cocycle on the self-similar set. Actually, there exist choices of cellular self-similar structures so that the resulting self-similar sets have different Hausdorff dimensions. For each choice there exists a cellular self-similar set on which the non-trivial cyclic 1-cocycle is defined.

- **Cantor dust**

Unfortunately, the theorem cannot be applied to the Cantor dust *CD* since *CD* is not a cellular self-similar set. However, a cyclic cocycle on the *Cantor set* is defined in [23], and the cocycle can detect the upper Minkowski content.

- **Infinite isolated Sierpinski gaskets**

The final example in Section 3.3, that we denote by *ISG*, gives a cellular self-similar structure. Therefore, the cyclic 1-cocycle may be defined on the space, and the cocycle is non-trivial when $2\alpha > \dim_H(ISG)$. From now until the end of this subsection, we discuss the structure of $HC^0(C^{Lip}(ISG))$.

By the self-similar structure of *ISG*, $\pi_0(ISG) = \bigoplus_{p\in\mathbb{N}} \mathbb{Z}$, each of whose summands corresponds to a connected component Y_p of *ISG*. Therefore *ISG* may be written as

$$ISG = \bigsqcup_{p\in\mathbb{N}} Y_p.$$

Then we have the canonical inclusion map

$$in_p : Y_p \to \bigsqcup_{p\in\mathbb{N}} Y_p = ISG$$

for any $p \in \mathbb{N}$. We now fix a base point $y_p \in Y_p$ for each $p \in \mathbb{N}$, and define a cyclic 0-cocycle ψ_p of $C^{Lip}(Y_p)$ by taking the evaluation of y_p for any $f \in C^{Lip}(Y_p)$. Therefore, the canonical inclusion map in_p induces the map of cyclic cohomology groups:

$$(in_p)_* : HC^0(C^{Lip}(Y_p)) \to HC^0(C^{lip}(ISG)).$$

We now let P be a finite subset of $\mathbb{N}$ and assume that $\Psi_P = \sum_{p\in P} \alpha_p (in_p)_*([\psi_p]) =$

0. We also define $c_p \in C^{Lip}(ISG)$ by

$$c_p(y) = \begin{cases} 1, & y \in Y_p \\ 0, & otherwise. \end{cases}$$

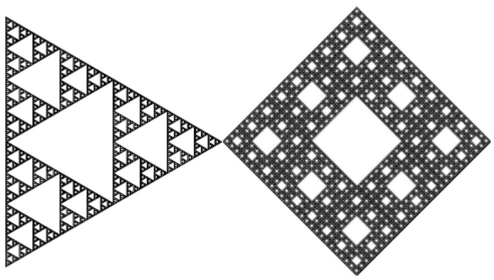
Then, for any $\tilde{p} \in P$, we have a pairing of the Hochschild homology group and the Hochschild cohomology group of $C^{Lip}(ISG)$:

$$0 = \langle \Psi_P,\ c_{\tilde{p}} \rangle = \sum_{p \in P} \alpha_p (in_p)_*([\psi_p])(c_{\tilde{p}}) = \alpha_{\tilde{p}},$$

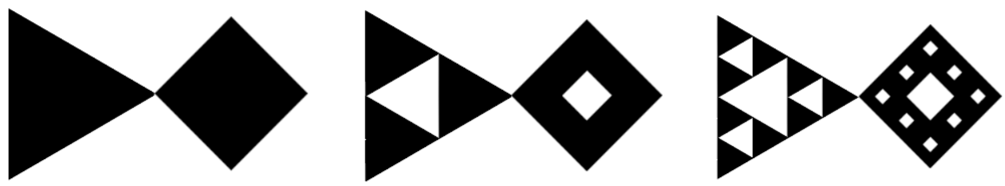
and which means that the set $\{(in_p)_*([\psi_p])\}_{p \in P}$ is a linearly independent set. Since this argument also works for any finite set P of $\mathbb{N}$, we can conclude that $\{(in_p)_*([\psi_p])\}_{p \in \mathbb{N}}$ forms a linearly independent set of $HC^0(C^{Lip}(ISG))$, and therefore $HC^0(C^{Lip}(ISG))$ contains $\bigoplus_{p \in \mathbb{N}} \mathbb{C}$ as a $\mathbb{C}$-vector space.

5.5 Further Work

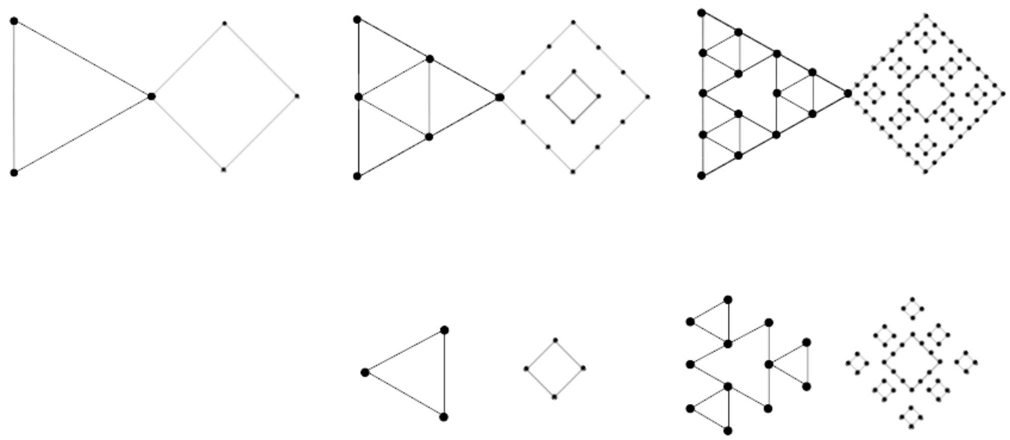
Strichartz proposed the notion of *fractafolds* [29, 30], and on which he examines fractal versions of the classical theories, for example, Hodge-de Rham theory, spectral theory, homotopy theory. In particular, the Laplacian on some kinds of self-similar sets has been extensively studied, and it is applied to various fields [2, 17, 29, 30]. Here, we give some examples of finite unions of cellular self-similar sets.



The first example is the wedge sum of a Sierpinski gasket and a Sierpinski carpet with base points at their corners. However, the space is neither a cellular self-similar set nor a fractafold. However, the theorem may be applied to this space. Namely, the space is seen as the projective limit of the following spaces:



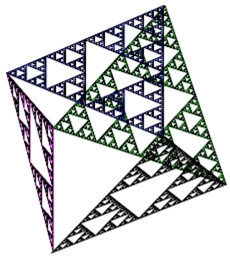
The figure is obtained by taking the wedge sum of the sequences which give rise to the Sierpinski gasket and the Sierpinski carpet. Similarly, we have sequences of boundary chains b_0, b_1, b_2 and inner chains I_0, I_1, I_2 respectively:



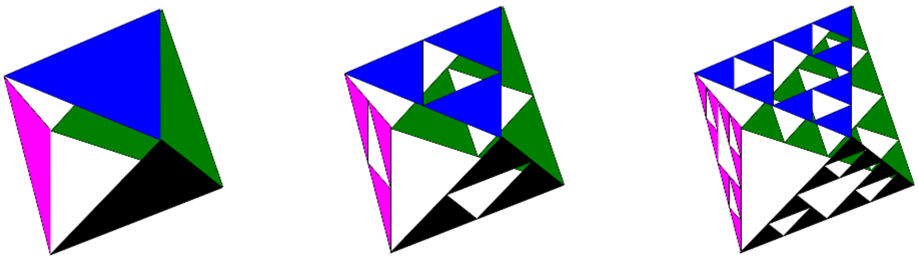
We therefore have an approximating cyclic1-cocycle, and it can be written by the element-wise sum of approximating cyclic1-cocycles of SG and SC. In order that approximating cyclic1-cocycle is a Cauchy sequence, it is enough that the Hölder index α satisfies $2\alpha > \dim_H(SC)$.

From this point of view, SG can be seen as a union of 3 Sierpinski gaskets, and therefore SG may be seen as a *fractafold with boundary*, see [29, 30] for the details. As defined in Section 5.4, we have a cyclic cocycle on SG.

Finally, we define a cyclic cocycle of the algebra of Lipschitz functions defined on a *fractafold* based on the Sierpinski gasket:



The space is a union of four copies of the Sierpinski gasket in $\mathbb{R}^3$ obtained by gluing the points at corners of a copy with each corner of the other Sierpinski gaskets. This space is one of examples of what Strichartz calls *fractafolds without boundaries*, and we denote it by FSG. The space FSG can be seen as the projective limit of a sequence of the spaces that is obtained by gluing copies of the sequence which gives rise to *SG*:



We therefore get, by applying the theorem to each Sierpinski gasket, a cyclic 1-cocycle on $C^{\alpha}(FSG)$ when $2\alpha > \log_2 3$.

Remark 5.17. Strichartz introduces the Hodge-de Rham theory for fractal graphs [1]. In [1], Laplacian on some fractal sets are defined by exploiting the Alexander-Spanier cochain complexes. However, we do not know whether or not there exist any relation between the cyclic 1-cocycle ϕ defined in the present thesis and the Laplacian of [1].

References

[1] S. Aaron, Z. Conn, R. Strichartz, H. Yu, *Hodge - de Rham theory on fractal graphs and fractals*, Commun. Pure Appl. Anal. 13 (2014), no. 2, p. 903-928.

[2] M. L. Barlow, *Random walks and diffusions on fractals*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), p. 1025-1035, Math. Soc. Japan, Tokyo, 1991.

[3] R. L. Brooks, C. A. B. Smith, A. H. Stone, W. T. Tutte, *The dissection of rectangles into squares*, Duke Math. J. 7, 1940, p. 312-340.

[4] E. Christensen, C. Ivan, E. Schrohe, *Spectral triples and the geometry of fractals*, J. Noncommut. Geom. 6 (2012), no. 2, p.249-274.

[5] A. Connes, *Non-commutative differential geometry*, Publications Mathéme-tiques de l'I.H.É.S., tome 62, n^{o} 2, 1985, p. 41-144.

[6] A. Connes, *Noncommutative Geometry*, Academic Press, Inc., San Diego, CA, 1994.

[7] J. H. Conway, C. Radin, *Quaquaversal tilings and rotations*, Inventions Mathematicae, vol. 132, 1998, p. 179-188.

[8] S. Eilenberg, J. A. Zilber, *Semi-simplicial complexes and singular homology*, Annals of Mathematics, Second Series, Vol. 51, No. 3, 1950, p. 499-513.

[9] K. Falconer, *Fractal geometry*, Mathematical foundations and applications, Second edition, John Wiley and Sons, Inc., Hoboken, NJ, 2003.

[10] E. Friedman, *Reptiles, partridges, and golden bees : tiling shapes with similar copies*, www2.stetson.edu/ efriedma/papers/reptiles.ppt

[11] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of the American Mathematical Society, No. 16, 1966.

[12] N. Higson, J. Roe, *Analytic K-homology*, Oxford university press, 2000.

[13] G. Hochschild, B. Kostant, A. Rosenberg, *Differential forms on regular affine algebras*, Transactions AMS 102, No.3, 1962, p. 383-408.

[14] J. E. Hutchinson, *Fractals and self similarity*, Indiana Unit. Math. J. 30, 1981, p. 713-747.

[15] B. Johnson, *Cohomology in Banach algebras*, Memoirs American Mathematical Societies 127, 1972.

[16] M. Karoubi, *K-theory*, Springer, 1978.

[17] J. Kigami, *Analysis on Fractals*, Cambridge Tracts in Mathematics(No. 143), Cambridge University Press, 2001.

[18] J. L. Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften, Volume 301, 1998.

[19] J. L. Loday, D. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helvetic 59, 1984, p. 565-591.

[20] R. MacPherson, B. Schweinhurt, *Measuring shape with topology*, Journal of Mathematical Physics, 2012.

[21] B. B. Mandelbrot, *The fractal geometry of nature*, W. H. Freeman and Co., San Francisco, Calif., 1982.

[22] P. A. P. Moran, *Additive functions of intervals and Hausdorff measure*, Mathematical Proceedings of the Cambridge Philosophical Society, Volume 42, Issue 01, 1946, p. 15-23.

[23] H. Moriyoshi, T. Natsume, *in preparation*.

[24] C. Radin, *The pinwheel tilings of the plane*, Annals of Mathematics 139 (3), p. 661-702, 1994.

[25] J. Rosenberg, *Comparison between algebraic and topological K-theories for Banach algebras and C^*-algebras*, Handbook of K-theory. Vol. 1, 2, p. 843-874, Springer, Berlin, 2005.

[26] J. Rosenberg, *The algebraic K-theory of operator algebras*, K-theory, 12(1), 1997, p. 75-99.

[27] E. H. Spanier, *Algebraic Topology*, Springer New York, 1966.

[28] R. Sprague, *Zur Abschatzung der Mindestzahl inkongruenter Quadrate, die ein gegebenes Rechteck ausfullen*, Math. Z. 46 (1940), p. 460-471.

[29] R. Strichartz, *Fractafolds based on the Sierpinski gasket and their spectra*, Transactions of the American mathematical society, vol. 355, Num. 10, 2003, p. 4019-4043.

[30] R. Strichartz, *Spectral analysis on infinite Sierpinski fractafolds*, arXiv:1011.1049v1 [math.FA] 4 Nov 2010.

[31] B. L. Tsygan, *Homology of matrix Lie algebras over rings and the Hochschild homology*, Translation in Russ. Math. Survey 38(2) (1983), p. 198-199.

[32] W. T. Tutte, *The Dissection of Equilateral Triangles into Equilateral Triangles*, Mathematical Proceedings of the Cambridge Philosophical Society, 44, 1948, p. 463-482.

[33] S. Winter, *Minkowski content and fractal curvatures of self-similar tilings and generator formulas for self-similar sets*, Advances in Mathematics 274, 2015, p. 285-322.

[34] L. C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Mathematica, Volume 67, issue 1, 1936, p. 251-282.

[35] E. C. Zeeman, *Seminar on combinatorial topology*, Institut de Hautes Etudes Scientifiques, 1963.