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Theories of non-gravitational massive spin two particles

有質量重力子とは異なる有質量スピン２粒子理論について

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Introduction

In the modern physics, special relativity, quantum mechanics and general relativity play crucial roles to explain many phenomena in the universe. Special relativity shows up at the high energy scales and quantum mechanics is necessary for understanding of the microscopic world. General relativity (GR) successfully describes the macroscopic world from motion of stars to the large scale structure of the universe. Relativistic quantum field theory (QFT), which is regarded as the unification of special relativity and quantum mechanics, is indispensable to probe the fundamental physics where both effects become important simultaneously. (The celebrated standard model of particle physics is actually constructed in this frame work.) Furthermore, while it is often said that GR is not compatible with quantum mechanics due to its nonrenormalizability, it can be regard as an effective field theory and can be constructed as the theory of massless spin-two fields within the framework of QFT. Therefore, we might say that the fundamental law of nature including gravity is based on QFT.

Although QFT is very powerful and universal, there are still unknown aspects. The famous example is the theory of massless higher spin fields. According to the Weinberg's no-go theorem, interactions of higher spin particles are prohibited in Minkowski spacetime due to the requirement of the Lorentz invariance under several assumptions. In spite of this fact, there are many attempts to find interactions for massless higher spin particles in nontrivial spacetime backgrounds inspired by string theory. For massive higher spin particles, there does not exist such a kind of theorem but other problems arise. Despite the absence of the no-go theorem coming from the Lorentz invariance, it is well known that it is nontrivial to construct free theories of massive higher spin particles which interact consistently with gravity. The minimal coupling of the free theories to gravity generates a ghost particle which could make the interpretation of quantum mechanics difficult. Furthermore, self-interactions also generally induce ghost particles in nontrivial background fields even if the interactions do not contain any derivatives. It is unclear that the latter property is generally crucial for massive higher spin theories but, as for gravitational massive spin-two particles (massive gravitons), it is asserted that the property is devastating because gravitational theories should be consistent even in nontrivial backgrounds.

Among these issues, through the establishment of massive gravity, there is progress in the ghost problem originating from nonlinearity. Although the free field theory called the Fierz-Pauli model was formulated over 70 years ago, theories of interacting massive gravitons had not been formulated due to the presence of the ghost until recently. The discovery of the late time acceleration of the universe, however, has given motivations to study seriously massive gravity for the explanation of the tiny value of the cosmological constant, which, as a result, leads to some excellent works. Especially, the Dvali-Gabadadze-Porrati brane world model (DGP model) and a field theoretical approach of massive gravity by Arkani-Hamed *et al.* contain great insights for the resolution of the ghost problem and, finally, de Rham, Gabadadze and Tolley have formulated the first ghost-free massive gravity called the dRGT massive gravity obeying the guiding principle which says that the interacting theory should have the same degree of freedom as the particles contained in the action.

Motivated by the guiding principle which leads to the construction of the dRGT model, we study properties of interacting non-gravitational massive spin-two particles. For non-gravitational theories, we do not need to assume universal couplings to all fields, which means that the kinetic term for massive spin-two particles is not necessary to be the fully nonlinear Einstein-Hilbert term $\sqrt{-g}R$. Actually, there are some previous works which do not regard massive spin-two particles as massive gravitons. While some discussed the consistency of the massive spin-two field as an alternative gravity theory from the late 1950s to the mid 1970s, Federbush worked on construction of a field theoretical model describing the dynamics of the charged massive spin-two particles. Federbush constructed the model of charged massive spin two particles coupled to photons by replacing partial derivatives of the complexified Fierz-Pauli Lagrangian with $U(1)$ covariant derivatives. His study revealed that the noncommutativity of the covariant derivatives gives an ambiguity to the definition of the kinetic term but the requirement on the degree of freedom of the system uniquely determines the theory.

In this paper, we build interacting theories of non-gravitational massive spin-two particles based on the lesson from the dRGT model and the Federbush model. Since there is no reason why we assume the fully nonlinear Einstein-Hilbert action for nongravitational particles, we choose the Fierz-Pauli Lagrangian as a starting point. Due to the requirement on the degree of freedom, the form of interaction terms is highly restricted and the interacting theory is uniquely determined. Thanks to the special properties of the interactions, the models could have stable nontrivial vacua, which motivates us also to investigate the behavior of the theories around nontrivial vacua. As a result, we reveal proerties of the vacua and obtain the parameter region for the theory to have at least one locally stable vacuum. Furthermore, it turns out that the theory can be defined only around the trivial vacuum if the $U(1)$ charge is assigned to the massive spin-two field.

In addition to the analysis in the Minkowski background, we also consider the self-interacting massive spin-two model in curved spacetime. As mentioned above, the coupling of free higher spin particles to gravity is subtle and the nonminimal coupling is required to eliminate a ghost from the free field theory. Thus, we study whether or not our self-interacting model of non-gravitational massive spin two particles reintroduce the ghost by using the Lagrangian analysis. Furthermore, we investigate another possibility of new interaction terms which keeps the degree of freedom on the curved background through the same analysis.

The organization of this paper is as follows. In Chap. 1, we review the representation theory of spin-one and spin-two particles and find the Lagrangian which describes their dynamics. In Chap. 2, we review massive gauge theories and show that field theoritical approaches leads to the establishment of the dRGT massive gravity. In Chap. 3, we propose a new model of self-interacting massive spin-two particles imposing Z_2 symmetry. Moreover, we also investigate properties of nontrivial vacua and the stability against quantum corrections. In Chap. 4, we propose the $U(1)$ invariant model by extending the Z_2 invariant theory. Then, the same analysis is carried out. We find that the properties of the two model are essentially different in nontrivial vacua. In Chap. 5, putting the new model on nontrivial spacetime, we see if the special interactions can keep their properties and find new interactions which only exist on nontrivial backgrounds.

Conventions and definitions: Throughout this paper, we assume four dimensional spacetime and take the mostly plus metric signature convention $g_{\mu\nu} = (-,+,+,+)$. The Greek indicies are used for space and time components and the Latin indicies are used for spatial components only. The definitions of christoffel symbols, curvature, and related quantities are given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\kappa} (g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\mu\nu,\kappa})$$

$$R^{\mu}{}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} + \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\rho\nu}$$

$$R_{\mu\nu} = R^{\mu}{}_{\nu\mu\sigma}.$$

Contents

Chapter 1

Free theory of massless and massive spin two particles

Quantum field theory is a useful tool to impose the Lorentz invarinace and the locality. On the other hand, due to the difference between the number of degrees of freedom (DOF) of a particle and components of the corresponding field, some trick is needed for the construction of a theory describing the dynamics of the particle in the language of the field. In this chapter, we review the representation theory of Poincaré group which is used to define a "one particle state" and show that the gauge redundancy is necessary for the theory of massless, finite spin particles. Furthermore, we mention that another technique is required to construct a theory of massive particles with spin greater than two. The discussion here is based on [1] and [2].

1.1 Poincaré algebra

In quantum field theory, one particle states are defined to be irreducible representations of Poincaré group. Therefore, in this section, we briefly review Poincaré algebla.

The Poincaré transformation consists of four spacetime translations and the Lorentz transformation. The commutation relation for the Lorentz generator is given as follows:

$$i[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\sigma\mu}M^{\rho\nu} + \eta^{\sigma\nu}M^{\rho\mu} \tag{1.1}$$

We can read off the generators of the Lorentz boost K_i and the spatial rotation J_i from (1.1) by identifying $K^i = M^{i0}$ and $J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}$. Including the generator of the spacetime translation, we obtain the commuation relations for the Poincaré generator:

$$\begin{aligned} i[M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\sigma\mu}M^{\rho\nu} + \eta^{\sigma\nu}M^{\rho\mu} && (1.2)\\ i[P^{\mu}, M^{\rho\sigma}] &= \eta^{\mu\rho}P^{\sigma} - \eta^{\mu\sigma}P^{\rho} && (1.3)\\ [P^{\mu}, P^{\rho}] &= 0. && (1.4) \end{aligned}$$

Here P^μ corresponds to the spacetime translation.

For later convenience, let us consider the linear combination of K^i and J^i as

$$N_i = \frac{1}{2}(J_i - iK_i), \qquad N_i^\dagger = \frac{1}{2}(J_i + iK_i). \tag{1.5}$$

The commutation relations for generators N_i and ${N^\dagger}_i$ are given as follows:

$$[N_i, N_j] = i\epsilon_{ijk}N_k \tag{1.6}$$
$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger \tag{1.7}$$
$$[N_i, N_j^\dagger] = 0 \tag{1.8}$$

From (1.6), (1.7) and (1.8), N_i and ${N^\dagger}_i$ are identified with $SU(2)$ generators and they are comutative with each other. In other words, the Lorentz group can be decomposed into $SU(2) \times SU(2)$.

1.2 One particle state

In this section, we classify one particle states by considering the irreducible representation of the Poincaré group. Before the discussion, we have to clarify the definition of a "one particle state." In quantum mechanics, the free particle is characterized as the simultaneous eigenstate of the Hamiltonian and its three-momentum. Thus, it is quite natural to define the eigenstate of the four-momentum as a one particle state $|p, \sigma\rangle$ where σ labels other physical states.

Since the unitary operator for the spacetime translation is given by $\exp(-iP^\mu a_\mu)$, we easily find the transformation rule as follows:

$$\exp(-iP^\mu a_\mu)\,|p, \sigma\rangle = (-ip^\mu a_\mu)\,|p, \sigma\rangle. \tag{1.9}$$

On the other hand, for the Lorentz transformation, we have to take into account the fact that the Lorentz generator is not commutative with the momentum operators. As the Lorentz transformation maps p^μ into $\Lambda^\mu{}_\nu p^\nu$, the state with the eigenvalue p^μ is also transformed into the state with the eigenvalue $\Lambda^\mu{}_\nu p^\nu$. Therefore, the transformation rule for one particle states is given by

$$U(\Lambda)|p, \sigma\rangle = \sum_{\sigma'} G(\Lambda, p)_{\sigma'\sigma}|\Lambda p, \sigma'\rangle. \tag{1.10}$$

Hence, what we have to do for the classification of the particle is to find the irreducible representation for $G(\Lambda)_{\sigma'\sigma}$.

For this purpose, let us introduce the momentum k which can be mapped into some arbitrary momentum p by the Lorentz transformation $L^\mu{}_\nu(p)$. (The inverse transformation of $L^\mu{}_\nu(p)$ sends back the momentum p to k.) Then, the eigenstate with the eigenvalue p is obtained from the eigenstate of the standard momentum:

$$|p, \sigma\rangle = N(p)U(L(p))|k, \sigma\rangle. \tag{1.11}$$

Here $N(p)$ is a normalization factor. Taking (1.11) into account, we find that implementation of the unitary operator $U(\Lambda)$ on $|p, \sigma\rangle$ gives

$$U(\Lambda)|p, \sigma\rangle = N(p)U(\Lambda L(p))|k, \sigma\rangle \tag{1.12}$$

The right hand side (1.12) can be rewritten as

$$N(p)U(\Lambda L(p))|k, \sigma\rangle = N(p)U(L(\Lambda p))U(L^{-1}(\Lambda p)\Lambda L(p))|k, \sigma\rangle. \tag{1.13}$$

The remarkable point in (1.13) is the unitary operator $U(L^{-1}(\Lambda p)\Lambda L(p))$ in the right hand side. It is easy to check that the operation of $L^{-1}(\Lambda p)\Lambda L(p)$ on the standard momentum leaves k invariant. Therefore, $U(L^{-1}(\Lambda p)\Lambda L(p))|k,\sigma\rangle$ is expressed as the linear combination of the ket $|k,\sigma\rangle$:

$$U(L^{-1}(\Lambda p)\Lambda L(p))|k,\sigma\rangle = \sum_{\sigma'} K(\Lambda,p)_{\sigma'\sigma}|k,\sigma'\rangle. \tag{1.14}$$

Combining the results from (1.12) to (1.14), we find another expression for (1.10).

$$\begin{aligned} U(\Lambda)|p,\sigma\rangle &= N(p)U(L(\Lambda p))\sum_{\sigma'} K(\Lambda,p)_{\sigma'\sigma}|k,\sigma'\rangle \\ &= \frac{N(p)}{N(\Lambda p)}\sum_{\sigma'} K(\Lambda,p)_{\sigma'\sigma}|\Lambda p,\sigma'\rangle \end{aligned} \tag{1.15}$$

The transformation which keeps some standard momentum invariant forms a subgroup called the little group and the above expression (1.15) clearly shows that we obtain the irreducible representation of the Lorentz group by finding the irreducible representation of the little group. Since the standard momentum is different for massless particles and massive particles, we consider the representation of the little group seperately.

1.2.1 Little group for massless particles

For massless particles, there does not exist any frame where the particles are at rest. Thus, we take $k^\mu = (1,0,0,1)$ as the standard momentum. In this case, the group keeping k^μ invariant is called $ISO(2)$ and massless particles are labeled by eigenvalues of the $ISO(2)$ generator.

Since $ISO(2)$ consists of the translation with two parameters and the rotation in two dimensions, the operator on the Fock space is given by the following form with three parameters α,β, and θ:

$$U(\alpha,\beta,\theta) := 1 + i\alpha A + i\beta B + i\theta J_3. \tag{1.16}$$

A and B represent the generator of the translation while J_3 corresponds to the two-dimensional rotation and their eigenvalue could characterize one particle states. The eigenvalues for the translation generators, however, should be strictly zero for physical states because their values are continuous. As we do not observe continuous DOF for massless particles, it is plausible to interpret zero eigenvalues for A and B as the necessary condition for all physical states of massless particles. Therefore, physical states for massless particles are classified by the eigenvalue σ of J_3.

The helicity σ seems to take arbitrary values other than integers or half-integers. Actually, we cannnot argue that the eigenvalue is restricted to be an integer or half-integer from the above discussion only. The point is that the Lorentz group is not simply connected but doubly connected. This fact imposes further restriction on the value σ and , as a result, the helicity σ is restricted to be an integer or half-integer.

$$K_{\sigma'\sigma} = \exp(i\theta\sigma)\delta_{\sigma'\sigma}, \qquad \sigma = 0, \pm\frac{1}{2}, \pm 1, \dots \tag{1.17}$$

Therefore, massless particles with spin have two DOF.

1.2.2 Little group for massive particles

For massive particles, there exists a frame where the particles are at rest and we can choose the standard momentum as $k^\mu = (m, 0, 0, 0)$. This fact suggests that the little group which keeps k^μ is identified with $SO(3)$ (the three dimensional (spatial) rotation). Therefore, the irreducible representation of the Poincaré group for massive particles is labeled by an integer or half-integer j with the dimensionality $2j + 1$.

1.3 Irreducible representation of fields

Now, we have classified one particle states by the spin j and mass m. We, however, need to find also irreducible representations for the corresponding fields to construct equations of motion. For this purpose, it is convenient to use the fact that the Lorentz group can be expressed as $SU(2) \times SU(2)$. Since the irreducuble representation of $SU(2)$ is characterized by an integer or half-integer k, the irreducible representaion of the Lorentz algebra is labeled by two indicies k and k' and the dimensionality of the representation is given by $(2k + 1)(2k' + 1)$. Furthermore, from the quations (1.5), $N_i + N_i^\dagger$ gives the generators of the three dimensional rotation J_i, which means that irreducible representations can also be expressed in terms of lablels j for irreducible representations angular momenta J_i with the usual procedure of spin addition.

In the following, some examples of irreducible representaion of fields are shown:

- Scalar ϕ ($k = k' = 0, \quad j = 0$)
- Left-handed spinor ψ^a ($k = \frac{1}{2}$, $k' = 0, \quad j = \frac{1}{2}$)
- Right-handed spinor $\psi^{\dot{a}}$ ($k = 0$, $k' = \frac{1}{2}, \quad j = \frac{1}{2}$)
- Four-vector A_μ ($k = \frac{1}{2}$, $k' = \frac{1}{2}, \quad j = 0, 1$)
- Traceless symmetric tensor $h_{\mu\nu}^{\text{traceless}}$ ($k = 1$, $k' = 1, \quad j = 0, 1, 2$).

Since each field contains irreducible representations of J_i, fields could correspond to massive particles defined as irreducible representations of $SO(3)$ while subtleties arise for massless particles defined as irreducible representations of $ISO(2)$.

1.4 Lagrangian

We construct theories to describe the dynamics of massless and massive particles. For this purpose, we have to introduce first class constraints for massless theories and second class constraints for massive theories to fill the gap between the number of components of the field and DOF of the particle. In this section, we consider exclusively theories of spin-one and spin-two particles.

1.4.1 Free massless spin one particles

In the previous sections, we find that all massless particles with finite spin have two DOF while the number of conponents of fields is more than two in general. Furthermore, there does not exist any field containing representation of $ISO(2)$. To resolve this problem, we need to introduce some redandancy for the theory called the gauge redandancy (symmetry), that is, we identify the field A_μ with the one shifted by some arbitrary function $A_\mu + \partial_\mu\theta$. Remarkably, this requrement fixes the free field theory up to an overall normalization.

The free theory should consist of quadratic terms in the field and contain two derivatives to yield the Klein-Gordon equation. In this case, candidates for a kinetic term are following (Here we neglect the term which is equivalent to the second term up to a total derivative.):

$$\partial_\mu A_\nu \partial^\mu A^\nu, \quad \partial_\mu A_\nu \partial^\nu A^\mu. \tag{1.18}$$

Thus, the general form of a kinetic term is given by the linear combination of these two terms

$$\mathcal{L}_{\text{kin}} \sim c_1 \partial_\mu A_\nu \partial^\mu A^\nu + c_2 \partial_\mu A_\nu \partial^\nu A^\mu. \tag{1.19}$$

Let us impose the "gauge redundancy" on the kinetic term. Since the A_μ and the $A_\mu + \partial_\mu\theta$ should describe the same physics, the variation of (1.19) under the gauge transformation should vanish:

$$\begin{aligned}\delta\mathcal{L}_{\text{kin}} &\sim c_1 \partial_\mu \delta A_\nu \partial^\mu A^\nu + c_2 \partial_\mu \delta A_\nu \partial^\nu A^\mu \\ &\sim c_1 \partial_\mu (\partial_\nu\theta) \partial^\mu A^\nu + c_2 \partial_\mu (\partial_\nu \theta) \partial^\nu A^\mu = 0.\end{aligned}$$

This requires that the coefficients should be $c_1 = -c_2$. As nonderivative terms inevitably violate the gauge symmetry, the free Lagrangian is uniquely determined with the appropriate normalization factor as

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.20}$$

1.4.2 Free massless spin two particles

The same logic holds for massless spin-two particles. However, we use a symmetric, but not traceless tensor instead of a symmetric traceless tensor in Sec.1.3 for technical convenience. (In order to construct a Lagrangian with the traceless symmetric tensor, we have to introduce an auxiliary field.) For the second rank tensor, the possible kinetic terms are

$$\partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu}, \quad \partial_\rho h_{\mu\nu} \partial^\nu h^{\mu\rho}, \quad \partial_\nu h^{\mu\nu} \partial^\mu h, \quad \partial_\mu h \partial^\mu h. \tag{1.21}$$

Here we neglect the term which is equivalent to the second term up to a total derivative. The general form of the kinetic term is given by

$$\mathcal{L}_{\text{kin}} \sim c_1 \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + c_2 \partial_\rho h_{\mu\nu} \partial^\nu h^{\mu\rho} + c_3 \partial_\nu h^{\mu\nu} \partial^\mu h + c_4 \partial_\mu h \partial^\mu h. \tag{1.22}$$

We require that the field $h_{\mu\nu}$ is identified with $h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ in the same way as in the case of spin-one particles. Thus, the Lagrangian should be invariant under this transformation:

$$\delta\mathcal{L}_{\text{kin}} \sim (4c_1 + 2c_2) \partial_\rho \partial_\mu \xi_\nu \partial^\rho h^{\mu\nu} + (2c_2 + 2c_3) \partial_\mu \partial_\nu \xi_\rho \partial^\rho h^{\mu\nu} + (4c_4 + 2c_3) \partial_\mu \partial_\rho \xi^\rho \partial^\mu h = 0. \tag{1.23}$$

From (1.23), the relative coefficients are completely fixed and we obtain the linearized Einsten-Hilbert term:

$$\mathcal{L}_{EH} = -\frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda} \partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\lambda h \partial^\lambda h. \tag{1.24}$$

1.4.3 Free massive spin one particles

Massive spin j particles have $2j+1$ DOF belonging to irreducible representations of $SO(3)$. On the other hand, each irreducible representation of fields also contains spin j representaion of $SO(3)$. Thus, the relation between one particle states and fields is more obvious than in the case of the theory of massless particles.

For $j=1$ particles, the simplest field that can describe the dynamics of massive spin-one particles is a four-vector A_μ. For our purpose, it is necessary to make the time component nondynamical because A_0 corresponds to spin $j=0$ representation. Fortunately, we have already known the kinetic term satisfying this requrement: $F_{\mu\nu}F^{\mu\nu}$. The time component of the vector field is nondynamical thanks to the antisymmetric property. Therefore, the kinetic term is completely same as before.

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.25}$$

For massive particles, we should add a mass term in addition to the kinetic term. As there is no "symmetry" prohibiting nonderivative term, we can add a quadratic term without derivatives and, in the case of the vector field, such a term is uniquely determined:

$$\mathcal{L}_{\text{mass}} \sim A^\mu A_\mu. \tag{1.26}$$

As a result, we obtain the free theory of massive spin-one particles.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \tag{1.27}$$

1.4.4 Free massive spin two particles

The field containing spin $j=2$ representation is a tarceless, symmetric tensor. However, as in the case of massless spin-two particles, we use a symmetric, but not traceless tensor $h_{\mu\nu}$ instead. Thus, we must make five componets of $h_{\mu\nu}$ nondynamical because the massive spin-two particle should have five DOF.

If we assume that the kinetic term for massive spin-two particles is given by the kinetic term for massless theory as in the previous subsection, four components $h_{0\mu}$ are guranteed to be nondynamical due to the absence of time derivatives. This means that one more dynamical DOF has to be removed from the system but we temporarily neglect this difficulty and let us consider mass terms.

For the second rank symmetric tensor field, there are two candidates for a mass term and the general form is given as follows:

$$\mathcal{L}_{\text{mass}} \sim h_{\mu\nu}h^{\mu\nu} - (1-a)h^2. \tag{1.28}$$

Since there exists an arbitrary coefficient a, it seems that the mass term is not uniquely determined for massive spin-two particles. This arbitrariness, however, disappears when the sixth DOF mentined above is eliminated from the system. In fact, the existence of the sixth DOF is tightly related to the form of the mass term of massive spin-two particles. Here, we confirm this statement with the Stückelberg trick which gives us the transparent way to deal with the vector mode and the scalar mode of the tensor field $h_{\mu\nu}$.

First, we introduce auxiliary fields inspired by the gauge transformation:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu B_\mu + \partial_\nu B_\mu \tag{1.29}$$

where B_μ is called the Stückelberg field. Actually, this replacement does not change the form of the kinetic term because the Einstein-Hilbert term is gauge invariant. The deformation of the theory arises from the mass term which is not gauge-invariant.

$$\begin{aligned} &h_{\mu\nu}h^{\mu\nu} - (1-a)h^2 \to \\ &\qquad h_{\mu\nu}h^{\mu\nu} - (1-a)h^2 + 2\partial_\mu B_\nu \partial^\mu B^\nu \\ &\qquad + 2(2a-1)\partial_\mu B_\nu \partial^\nu B^\mu + 4(h_{\mu\nu}\partial^\mu B^\nu - (1-a)h\partial_\mu B^\mu) \end{aligned} \tag{1.30}$$

Thanks to the Stückelberg fields, "gauge symmetry" is restored and the mass term (1.30) is invariant under the following gauge transformation

$$\delta h_{\mu\nu} = \partial_\mu \xi_\mu + \partial_\nu \xi_\mu, \quad \delta B_\mu = -\xi_\mu. \tag{1.31}$$

Furthermore, we can introduce one more Stückelberg field by

$$B_\mu \to B_\mu + \partial_\mu \phi. \tag{1.32}$$

for the system to have additional $U(1)$ gauge symmetry

$$\delta B_\mu = \partial_\mu \theta, \quad \delta\phi = -\theta. \tag{1.33}$$

Then, we obtain

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{\rm EH} &- \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - (1-a)h^2) - m^2\partial_\mu B_\nu \partial^\mu B^\nu - m^2(2a-1)\partial_\mu B_\nu \partial^\nu B^\mu \\ &- 2m^2(h_{\mu\nu}\partial^\mu B^\nu - (1-a)h\partial_\mu B^\mu) - 4m^2 a \partial_\mu B_\nu \partial^\mu \partial^\nu \phi - 2m^2 a \partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi \\ &- 2m^2(h_{\mu\nu}\partial^\mu \partial^\nu \phi - (1-a)h\partial_\mu \partial^\mu \phi) \end{aligned} \tag{1.34}$$

and this Lagrangian is invariant under the transformations (1.31) and (1.33). The reason for introducing these auxiliary fields to restore the gauge symmetry is that these fields mimic the behavior of the vector mode and the scalar mode of $h_{\mu\nu}$ in the high energy limit based on the equivalence theorem. Therefore, we can see more easily the effect of the mass term on each mode of $h_{\mu\nu}$ and discuss the link between the form of mass term and the dynamics of the system.

Let us count the number of DOF of the system in the high energy limit where the mass parameter m is negligible. Since, in this limit, $h_{\mu\nu}$, B_μ and ϕ correspond to massless spin-two, massless spin-one and massless scalar particles respectively, the system should have five dynamical DOF in total according to Sec.1.2. There exists, however, the fourth derivative with respect to ϕ in (1.34). The existence of the higher derivative term strongly suggests that an extra DOF appears in general and, to make matter worse, the extra mode is identified with a ghost which breaks the consistency as a quantum theory. Therefore, the sixth DOF is not only annoying because massive spin-two particles should have five DOF, but also really dengerous for quantum field theory.

In order to remove this mode, we have to eliminate the higher derivative term and this is very easy to achieve by choosing the parameter a to be zero. As a result, the mass term is determined uniquely by the requirement that the sixth DOF should be absence. This Lagrangian is called the Fierz-Pauli Lagrangian [3] and the explicit form is given as follows:

$$\mathcal{L}_{FP} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \tag{1.35}$$

1.5 Hamiltonian analysis

In previous section, we have constructed field theories of massless and massive spin-two particles by introducing the "gauge symmetry." However, it is not obvious how extra modes are eliminated from the therory. Therefore, in this section, we confirm that the Einstein-Hilbert term and the Fierz-Pauli Lagrangian both realize the correct number of DOF of the particles through the Hamiltonian analysis and see how constraints eliminate extra modes.

1.5.1 The Einstein Hilbert action

First of all, we define canonical momenta for $h_{\mu\nu}$. Because of absence of the time derivatives w.r.t h_{00} and h_{0i}, the canonical momenta for h_{00} and h_{0i} are identically zero.

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{h}_{00}} = 0 \tag{1.36}$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{h}_{0i}} = 0 \tag{1.37}$$

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}^{ij} - \dot{h}^k{}_k \delta^{ij} - 2\partial^{(i} h^{j)}{}_0 + 2\partial_k h^k{}_0 \delta^{ij} \tag{1.38}$$

Therefore, h_{00} and h_{0i} cannot be solved in terms of π and π^i, which means that $\pi = 0$ and $\pi^i = 0$ are regarded as primary constraints. By introducing Lagrange multipliers λ and λ_i, we formally obtain the Hamiltonian density for the system of the massless spin-two field.

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} + \pi \dot{h}_{00} + \ \pi^i \dot{h}_{0i} - \mathcal{L} + \lambda\pi + \lambda_i \pi^i \tag{1.39}$$

Here $\dot{h}_{ij}$ is given by

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{2}\pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0}. \tag{1.40}$$

As we include the four constraints $\pi = \pi^i = 0$ into the Hamiltonian using the Lagrange maultipliers, we can neglect the second and third terms which vanish on the hypersurface specified by the four constraints. Then, substituting (1.40) into (1.39), we find the Hamiltonian density expressed by the canonical variables and Lagrange multipliers:

$$\mathcal{H} = \mathcal{H}_0 + \lambda\pi + \lambda_i\pi^i \tag{1.41}$$

where we define $\mathcal{H}_0$ as

$$\begin{aligned} \mathcal{H}_0 = & \frac{1}{2}\pi^{ij}\pi_{ij} - \frac{1}{4}\pi^k{}_k \pi^l{}_l \\ & + \frac{1}{2}\partial_i h_{jm}\partial^i h^{jm} - \partial_i h_{jm}\partial^j h^{im} + \partial_i h^i{}_j \partial^j h^m{}_m - \frac{1}{2}\partial_m h^i{}_i \partial^m h^j{}_j \\ & - 2h_{0i}\partial_j \pi^{ij} - h_{00}(\nabla^2 h^j{}_j - \partial_i\partial^j h^i{}_j). \end{aligned}$$

Hence, the Hamiltonian of the sytem is given by

$$H = \int \mathcal{H} d^3x = H_0 + \int \left(\lambda\pi + \lambda_i\pi^i\right) d^3x. \tag{1.42}$$

Since two primary constraints $\pi = 0$ and $\pi^i = 0$ should be consistent with time evolution of the system, we require

$$\{\pi, H\} \approx 0, \quad \{\pi^i, H\} \approx 0 \tag{1.43}$$

where $\{\ ,\ \}$ means Poisson bracket defined by

$$\{F, G\} = \int d^3z \left[\frac{\delta F}{\delta h_{lm}(t,z)}\frac{\delta G}{\delta \pi^{lm}(t,z)} - \frac{\delta F}{\delta \pi^{lm}(t,z)}\frac{\delta G}{\delta h_{lm}(t,z)}\right]. \tag{1.44}$$

Thus, Poisson brackets among canonical variables are given by

$$\{h_{00}(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})\} = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{1.45}$$

$$\{h_{0i}(t,\boldsymbol{x}), \pi^j(t,\boldsymbol{y})\} = \delta_i{}^j\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{1.46}$$

$$\{h_{ij}(t,\boldsymbol{x}), \pi^{kl}(t,\boldsymbol{y})\} = \delta_{(i}{}^k\delta_{j)}{}^l\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{1.47}$$

From (1.63), we obtain four secondary constraints:

$$\phi := \{\pi, H\} = \{\pi, H_0\} = \nabla^2 h_{jj} - \partial_i\partial_j h_{ij} \approx 0 \tag{1.48}$$

$$\phi^{(i)} := \{\pi^i, H\} = \{\pi^i, H_0\} = 2\partial_j\pi^{ij} \approx 0 \tag{1.49}$$

Then, we redefine new Hamiltonian (density) $\hat{\mathcal{H}}$ including the four secondary constraints with corresponding Lagrange multipliers Λ and Λ_i as

$$\hat{\mathcal{H}} = \mathcal{H}_0 + \lambda\pi + \lambda_i\pi^i + \Lambda\phi + \Lambda_i\phi^i = \mathcal{H} + \Lambda\phi + \Lambda_i\phi^i. \tag{1.50}$$

We repeat the process from (1.63) to (1.50) until any new constraint does not emerge. Therefore, the quantities we have to consider in the next step are following:

$$\{\phi, \hat{H}\}, \quad \{\phi^i, \hat{H}\} \tag{1.51}$$

where $\hat{H} = \int \hat{\mathcal{H}} d^3x$. Calculating explicitly (1.68) gives

$$\begin{aligned}
\{\phi, \hat{H}\} &= \{\phi, H_0\} \\
&= \{\nabla^2 h_{jj} - \partial_i\partial_j h_{ij}, H_0\} = \{\nabla^2 h_{jj}, H_0\} - \{\partial_i\partial_j h_{ij}, H_0\} \\
&= \int d^3y \left[\nabla_x^2\delta^{ij}\{h_{ij}(\boldsymbol{x}), \mathcal{H}_0(\boldsymbol{y})\} - \partial_x^i\partial_x^j\{h_{ij}(\boldsymbol{x}), \mathcal{H}_0(\boldsymbol{y})\}\right] \\
&= -\partial_i\partial_j\pi^{ij} = -\partial_i\phi^i \approx 0
\end{aligned} \tag{1.52}$$

$$\begin{aligned}
\{\phi^i, \hat{H}\} &= \{\phi^i, H_0\} \\
&= 2\partial_j^x \int d^3y \{\pi^{ij}(\boldsymbol{x}), \mathcal{H}_0(\boldsymbol{y})\} = 0.
\end{aligned} \tag{1.53}$$

In (1.52), the quantity $\{\phi, \hat{H}\}$ does not vanish algebraically, but is expressed as a spatial derivative of the secondary constraint ϕ^i. Therefore, (1.52) vanishes on the hypersuface, which means that ϕ and ϕ^i are consistent with the time evolution and no more constraint appears.

Since the eight constraints belong to the first class, we introduce gauge fixing conditions to determine the eight Lagramge multipliers. As a result, the system has 16 second class constraints in total and the massless spin-two field has two DOF. Needless to say, the quantum theory is obtained by replacement of the Poisson brakets with the commutation relations.

1.5.2 The Fierz Pauli Lagrangian

As in the case of the massless field, the canonical momenta for h_{00} and h_{0i} are identically zero.

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{h}_{00}} = 0 \tag{1.54}$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{h}_{0i}} = 0 \tag{1.55}$$

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}^{ij} - \dot{h}^k{}_k \delta^{ij} - 2\partial^{(i} h^{j)}{}_0 + 2\partial_k h^k{}_0 \delta^{ij} \tag{1.56}$$

By introducing Lagrange multipliers λ and λ_i, we obtain the Hamiltonian density for the system of the massive spin-two field.

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} + \lambda\pi + \lambda_i \pi^i \tag{1.57}$$

Here $\dot{h}_{ij}$ is given by (1.40) and the term $\pi\dot{h}_{00}$ and $\pi^i \dot{h}_{0i}$ are dropped due to the constraints. The substitution (1.40) into (1.57) gives

$$\mathcal{H} = \mathcal{H}_0 + \lambda\pi + \lambda_i \pi^i \tag{1.58}$$

where $\mathcal{H}_0$ is defined as

$$\mathcal{H}_0 = \frac{1}{2}\pi^{ij}\pi_{ij} - \frac{1}{4}\pi^k{}_k \pi^l{}_l \tag{1.59}$$

$$+ \frac{1}{2}\partial_i h_{jm}\partial^i h^{jm} - \partial_i h_{jm}\partial^j h^{im} + \partial_i h^i{}_j \partial^j h^m{}_m - \frac{1}{2}\partial_m h^i{}_i \partial^m h^j{}_j + \frac{1}{2}m^2(h_{ij}h_{ij} - h_{ii}^2) \tag{1.60}$$

$$- 2h_{0i}\partial_j \pi^{ij} - m^2 h_{0i}^2 - h_{00}(\nabla^2 h^j{}_j - \partial_i \partial^j h^i{}_j - m^2 h_{ii}). \tag{1.61}$$

Hence, the Hamiltonian of the sytem is given by

$$H = \int \mathcal{H} d^3x = H_0 + \int \left(\lambda\pi + \lambda_i \pi^i\right) d^3x. \tag{1.62}$$

For the consistency, two primary constraints $\pi = 0$ and $\pi^i = 0$ should satisfy

$$\{\pi, H\} \approx 0, \quad \{\pi^i, H\} \approx 0. \tag{1.63}$$

Then, we obtain one secondary constraint from the first equation in (1.63):

$$\phi := \{\pi, H\} = \nabla^2 h_{jj} - \partial_i \partial_j h_{ij} - m^2 h_{ii} \approx 0 \tag{1.64}$$

while the consistency condition for π^i gives

$$\{\pi^i, H\} = 2m^2 \delta^{ij} h_{0j} + 2\partial_j \pi^{ij} \approx 0 \tag{1.65}$$

which tells that h_{0i} are solved in terms of π^{ij}:

$$h_{0i} = -\frac{1}{m^2}\delta_{ik}\partial_j \pi^{kj}. \tag{1.66}$$

Substituting (1.66) into the Hamiltonian and including the secondary constraint, we find new Hamiltonian (density) $\hat{\mathcal{H}}$ as

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda\pi + \lambda_i\pi^i + \Lambda\phi = \mathcal{H} + \Lambda\phi. \tag{1.67}$$

Here,

$$\begin{aligned}\mathcal{H}_0 &= \frac{1}{2}\pi^{ij}\pi_{ij} - \frac{1}{4}\pi^k{}_k\pi^l{}_l \\ &+ \frac{1}{2}\partial_i h_{jm}\partial^i h^{jm} - \partial_i h_{jm}\partial^j h^{im} + \partial_i h^i{}_j\partial^j h^m{}_m - \frac{1}{2}\partial_m h^i{}_i\partial^m h^j{}_j + \frac{1}{2}m^2(h_{ij}h_{ij} - h_{ii}^2) \\ &+ \frac{1}{m^2}\partial_k\pi_{ik}\partial^l\pi^{il} - h_{00}(\nabla^2 h^j{}_j - \partial_i\partial^j h^i{}_j - m^2 h_{ii}).\end{aligned}$$

As the new hamiltonian is obtained, let us investigate the consistency condition for ϕ:

$$\{\phi, \hat{H}\} \tag{1.68}$$

where $\hat{H} = \int \hat{\mathcal{H}} d^3x$. Calculating explicitly (1.68) gives

$$\begin{aligned}\{\phi, \hat{H}\} &= \{\phi, H_0\} \\ &= \{\nabla^2 h_{jj} - \partial_i\partial_j h_{ij}, H_0\} = \{\nabla^2 h_{jj}, H_0\} - \{\partial_i\partial_j h_{ij}, H_0\} - \{m^2 h_{ii}, H_0\} \\ &= \int d^3y \left[\nabla_x^2\delta^{ij}\{h_{ij}(\boldsymbol{x}), \mathcal{H}_0(\boldsymbol{y})\} - \partial_x^i\partial_x^j\{h_{ij}(\boldsymbol{x}), \mathcal{H}_0(\boldsymbol{y})\} - \{m^2 h_{ii}(\boldsymbol{x}), \mathcal{H}_0(\boldsymbol{y})\}\right] \\ &= -\partial_i\partial_j\pi^{ij} - \frac{1}{2}m^2\pi_{ii} \approx 0\end{aligned} \tag{1.69}$$

Thus, we obtain one more constraint

$$\varphi := \{\phi, \hat{H}\} = -\partial_i\partial_j\pi^{ij} - \frac{1}{2}m^2\pi_{ii} \approx 0. \tag{1.70}$$

The new hamiltonian takes the following form:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \lambda\pi + \lambda_i\pi^i + \Lambda\phi + \Lambda'\varphi = \mathcal{H} + \Lambda\phi + \Lambda'\varphi. \tag{1.71}$$

Then, let us repeat the procedure by imposing the consistency conditions for all constraints with the new hamiltonian obtained above. As the conditions $\{\pi, \hat{H}\} \approx 0$ and $\{\pi^i, \hat{H}\} \approx 0$ do not determine any Lagrange multipliers, we consider the condition $\{\phi, H\} \approx 0$.

$$\begin{aligned}\{\phi, \hat{H}\} &= \{\phi, H_0\} + \int d^3y \Lambda'(\boldsymbol{y})\{\phi(\boldsymbol{x}), \varphi(\boldsymbol{y})\} \\ &= \{\phi, H_0\} + \frac{3}{2}m^4\Lambda' \approx 0\end{aligned}$$

Therefore, the Lagrange multiplier Λ' is given as

$$\Lambda' = -\frac{2}{3m^4}\varphi \tag{1.72}$$

The similar calculation holds for φ:

$$\begin{aligned}
\{\varphi, \hat{H}\} &= \{\varphi, H_0\} + \int d^3y \Lambda \{\varphi(\boldsymbol{x}), \phi(\boldsymbol{y})\} \\
&= \{\varphi, H_0\} - \frac{3}{2}m^4\Lambda \\
&= -\frac{1}{2}m^2\phi - \frac{3}{2}m^4 h_{ii} + \frac{3}{2}h_{00} - \frac{3}{2}m^4\Lambda \\
&\approx -\frac{3}{2}m^4 h_{ii} + \frac{3}{2}h_{00} - \frac{3}{2}m^4\Lambda \approx 0.
\end{aligned}$$

In the fourth line, we use the fact that ϕ vanishes on the hypersurface. As the above equation can be solved in terms of Λ, we see that no more constraint appears for the system of the massive spin-two field.

Thus, the Hamiltonian is given by (1.71) with

$$\Lambda = h_{00} - h_{ii}, \quad \Lambda' = -\frac{2}{3m^4}\varphi.$$

We note that h_{00} is absent in the Hamiltonian (1.71) due to $\Lambda = h_{00} - h_{ii}$, which means the dynamics of this system is determined completely in terms of h_{ij} and π^{ij}. Since the system has two constraints on the pair h_{ij} and π^{ij}, the Fierz-Pauli Lagrangian correctly realizes the degree of freedom which massive spin two particles should have.

Chapter 2

Gravitational massive spin two particles

Introduction of interactions between particles makes the theory more interesting. In quantum field theory, however, it is not true in general that any kind of interaction is allowed to be added to the theory because of the Lorentz invariance and the quantum mechanical consistency. In free theories of massless particles, the gauge symmetry (redundancy) should be respected in order for the theory to be Lorentz-invariant and, furthermore, coupling constants are also constrained by the Weinberg's theorem. As a result, interacting theories of massless particles are highly restricted, which leads to some kind of "uniqueness" of the theory. On the other hand, once particles are massive, it seems that many kinds of interactions can be added to the action even if we require that the full theory has the same degree of freedom as the quadratic part of the action. This is true for particles with spin lower than two but not true for higher spin particles. Needless to say, we do not know the importance of the requirement but, from a theoretical point of view, it is interesting that the interacting theory of massive higher spin particles is also restricted under this assumption. Furthermore, it turns out that the requirement is quite essential when we regard massive spin-two particles as massive "gravitons" and apply it to cosmology or other gravitational phenomena. In this chapter, we focus on this "gravitational massive spin-two particles" and obtain the full theory which has the same DOF as free massive spin-two particles do. Note that the discussion is based on [4].

2.1 Massive gravity

As mentioned in the previous chapter, the situation is very different from the case of massless spin-two particles: Interactions do not need to be gauge-invariant and massive spin-two particles do not need to couple with other particles with the same strength. Therefore, we have to assume some property which is appropriate to characterize "gravity." To obtain insight, let us remind the assumption which Einstein puts in the construction of general relativity; The equivalence principle and the general coordinate invariance. Since massive theories do not have any gauge symmetry, the latter should be abandoned in massive gravity. On the other hand, the equivalence principle does not conflict with the mass of gravitons. Thus, it is plausible to construct a theory of massive gravity keeping the equivalence principle, which enforces the kinetic term to be the Ricci scalar for the consistency of the system.

2.1.1 The simplest model I

As the simplest example, let us consider the following action of massive gravity:

$$S = \frac{1}{2\kappa^2}\int d^4x \left[\sqrt{-g}R - \frac{1}{4}m^2\eta^{(0)\mu\alpha}\eta^{(0)\nu\beta}(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta})\right] \tag{2.1}$$

where $\eta^{(0)}$ is called a fiducial metric and chosen to be the Minkowski metric here. $h_{\mu\nu}$ is defined as deviation from the flat metric $h_{\mu\nu} = g_{\mu\nu} - \eta^{(0)}_{\mu\nu}$. By expanding (2.1) around the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we easily confirm that this action consists of the Fierz-Pauli Lagrangian and derivative interactions in $h_{\mu\nu}$. Generally, the fiducial metric is not necessary to be a flat type and to belongs to a solution of the Einstein equation. Thus, we also have

$$S = \frac{1}{2\kappa^2}\int d^4x \left[\sqrt{-g}R - \sqrt{-g^{(0)}}\frac{1}{4}m^2 g^{(0)\mu\alpha}g^{(0)\nu\beta}(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta})\right]. \tag{2.2}$$

Here $h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$. We find the Fierz-Pauli Lagrangian in a curved spacetime described by ${g^{(0)}}_{\mu\nu}$ when the dynamical metric is expanded as $g_{\mu\nu} = g^{(0)}_{\nu\beta} + h_{\mu\nu}$.

2.1.2 The simplest model II

There is another way to formulate a theory of massive gravity. In the mass term of (2.2), indices of $h_{\mu\nu}$ are contracted with $g^{(0)\mu\nu}$ and the determinant is also constructed from $g^{(0)\mu\nu}$. We can obtain another model which has a very similar form to (2.2) by taking the contraction with the full metric $g^{\mu\nu}$ and replacing the determinant $\sqrt{-g^{(0)}}$ with $\sqrt{-g}$:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{1}{4}m^2 g^{\mu\alpha}g^{\nu\beta}(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta})\right]. \tag{2.3}$$

By expanding the dynamical metric around the fiducial metric $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, we find the action (2.3) reproduces the Fierz-Pauli Lagrangian at the quadratic level. The difference from the previous one is that the action (2.3) also contains nonlinear potential terms in addition to derivative interactions. This is because the inverse metric is expressed as the infinite series in terms of $h_{\mu\nu}$.

$$\begin{aligned} g^{\mu\nu} &= g^{(0)\mu\nu} - h^{\mu\nu} + h^{\mu\lambda}h_\lambda{}^\nu - h^{\mu\lambda}h_\lambda{}^\sigma h_\sigma{}^\nu + \cdots \\ \sqrt{-g} &= \sqrt{-g^{(0)}}\left[1 + \frac{1}{2}h - \frac{1}{4}\left(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right)\ldots\right] \end{aligned}$$

2.1.3 General model

In the previous subsections, we introduced two simplest models of massive gravity. Here, we add nonlinear terms in $h_{\mu\nu}$ whose indices are contracted with $g^{(0)\mu\nu}$ or $g^{\mu\nu}$ in order to build general models. If we use $g^{(0)\mu\nu}$ to contract the indices, the general action is

$$S = \frac{1}{2\kappa^2}\int d^4x \left[\sqrt{-g}R - \sqrt{-g^{(0)}}\frac{1}{4}m^2 V^{(0)}(h)\right] \tag{2.4}$$

where the potential is given by

$$V^{(0)}(h) = \sum_{i=2}^{\infty} v_i^{(0)}(h) \tag{2.5}$$

$$\begin{aligned}
v_2^{(0)}(h) &= [h^2] - [h]^2 \\
v_3^{(0)}(h) &= f_1[h^3] + f_2[h^2][h] + f_3[h]^3 \\
v_4^{(0)}(h) &= g_1[h^4] + g_2[h^3][h] + g_3[h^2][h]^2 + g_4[h^2][h^2] + g_5[h]^4 \\
v_5^{(0)}(h) &= h_1[h^5] + h_2[h^4][h] + h_3[h^3][h]^2 + h_4[h^3][h^2] + h_5[h^2][h]^3 + h_6[h^2]^2[h] + h_7[h]^5 \\
&\vdots
\end{aligned}$$

Here the bracket means a trace with the fiducial metric $g^{(0)\mu\nu}$.

If we use the dynamical metric $g^{\mu\nu}$ to contract the indices, another action is obtained as in the case of the simplest models:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \frac{1}{4} m^2 V(h) \right]. \tag{2.6}$$

The potential $V(h)$ has the same form to the previous action except that the tarce and the contraction is taken with the dynamical metric.

$$V(h) = \sum_{i=2}^{\infty} v_i(h) \tag{2.7}$$

$$\begin{aligned}
v_2(h) &= [h^2]_g - [h]_g^2 \\
v_3(h) &= J_1[h^3]_g + J_2[h^2]_g[h]_g + J_3[h]_g^3 \\
v_4(h) &= K_1[h^4]_g + K_2[h^3]_g[h]_g + K_3[h^2]_g[h]_g^2 + K_4[h^2]_g[h^2]_g + K_5[h]_g^4 \\
v_5(h) &= L_1[h^5]_g + L_2[h^4]_g[h]_g + L_3[h^3]_g[h]_g^2 + L_4[h^3]_g[h^2]_g + L_5[h^2]_g[h]_g^3 + L_6[h^2]_g^2[h]_g + L_7[h]_g^5 \\
&\vdots
\end{aligned}$$

We have to note that (2.4) and (2.6) are completely equivalent. In fact, we can compare these two actions using the relation

$$[h^n]_g = \sum_{l=0}^{\infty} {}_{l+n-1}C_l [h^n]_{g^{(0)}} \tag{2.8}$$

according to the famous review written by Hinterbichler [4].

2.2 The Boulware-Deser ghost

It seems that the simplest model works well because there is no higher derivative term which usually leads to a ghost mode. However, this is not true. There are several ways to see the existence of the ghost mode and, here, we use the Hamiltonian analysis through Arnowitt-Deser-Misner (ADM) decomposition. For the sake of completeness, we begin with the Hamiltonian analysis of the Einstein-Hilbert action.

2.2.1 Hamiltonian analysis for general relativity

General relativity is constructed based on the general coordinate invariance. This is a very important factor for the theory but it is a little confusing when we consider the time evolution of the metric because we should specify "time." For this, we have to consider some diffeomorphisim ϕ which maps space-time into $S \times \mathbb{R}$ where S represents a three dimensional space and $\mathbb{R}$ corresponds to time. Using the mapping ϕ, we can identify time and dynamical variables. Through this procedure, we can consider GR as the system where a space-like, three-dimensional hypersurface evolves in time. This method is called Arnowitt-Deser-Misner(ADM) formalism or 3+1 formalism [5, 6, 7, 8] and we can take arbitary space-like hypersurface by choosing an appropriate mapping.

In ADM formalism, we parametrize the four dimensional metric $g_{\mu\nu}$ as follows.

$$g_{00} = -N^2 + \gamma_{ij} N^i N^j, \quad g_{ij} = \gamma_{ij}, \quad g_{0i} = N_i \tag{2.9}$$

where N and N_i are called the lapse and the shift respectively, and γ_{ij} is the spatial metric. In the ADM variables, Ricci tensor is expressed as

$$R = {}^{(3)}R + K_{ij}K^{ij} - K^2 - 2\nabla_\nu (n^\mu \nabla_\mu n^\nu - n^\nu \nabla_\mu n^\mu) \tag{2.10}$$

where ${}^{(3)}R$ means Ricci tensor on the space-like hypersuface and K_{ij} is defined as

$$K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i N_j - D_j N_i\right) \tag{2.11}$$

and D_i is a covariant derivative with respect to the spatial metric γ_{ij}. K_{ij} is called extrinsic curvature and tells how the hypersurface is embedded in the four dimensional spacetime. The last term in (2.10) is a surface term and we ignore it here for simplicity. As a result, the Einstein-Hilbert action becomes

$$S_{\rm EH} = \int d^4x \mathcal{L}_{\rm EH} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R = \frac{1}{2\kappa^2}\int d^4x N\sqrt{\gamma}\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right). \tag{2.12}$$

Thanks to this decomposition, we can specify ten dynamical variables of the system N, N_i, and γ_{ij}. In order to obtain the Hamiltonian for general relativity, we define the canonical momenta for these variables.

$$\Pi = \frac{\partial \mathcal{L}_{\rm EH}}{\partial \dot{N}} = 0 \tag{2.13}$$

$$\Pi^i = \frac{\partial \mathcal{L}_{\rm EH}}{\partial \dot{N}_i} = 0 \tag{2.14}$$

$$\pi^{ij} = \frac{\partial \mathcal{L}_{\rm EH}}{\partial \dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{\gamma}\left(K^{ij} - K\gamma^{ij}\right) \tag{2.15}$$

As in the case of the linearized gravity, six components are only dynamical and remaining variables generate primary constraints. Introducing Lagrange multipliers λ and λ_i, we obtain the action with four primary constraints.

$$S_{\rm EH} = \int d^4x \left(\mathcal{L}_{\rm EH} - \lambda\Pi - \lambda_i\Pi^i\right) \tag{2.16}$$

By implementing the Legendre transformation, we find the formal expression of the Hamiltonian of the system.

$$H_{\mathrm{EH}} = \int d^3x \left(\pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}_{\mathrm{EH}} + \lambda\Pi + \lambda_i\Pi^i\right) \tag{2.17}$$

Here we ignore the term $\Pi\dot{N} + \Pi^i\dot{N}_i$ because it vanishes on the hypersurface in the phase space specified by four primary constraints.

Using the expression for $\dot{\gamma}_{ij}$ in terms of the canonical momenta π_{ij}

$$\dot{\gamma}_{ij} = \frac{4\kappa^2 N}{\sqrt{\gamma}}\left(\pi_{ij} - \frac{1}{2}\pi\gamma_{ij}\right) + D_iN_j + D_jN_i, \tag{2.18}$$

we find the hamiltonian which is written in terms of canonical variables explicitly:

$$H_{\mathrm{EH}} = \int d^3x \left\{N\mathcal{H} + N_i\mathcal{H}^i + \lambda\Pi + \lambda_i\Pi^i\right\} \tag{2.19}$$

where $\mathcal{H}$ and $\mathcal{H}_i$ are defined as

$$\mathcal{H} := \frac{2\kappa^2}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) - \frac{\sqrt{\gamma}}{2\kappa^2}{}^{(3)}R, \qquad \mathcal{H}^i := -2h^i{}_m D_j\pi^{jm}. \tag{2.20}$$

We have to note that a surface term is dropped here again.

As usual, we impose the consistency conditions for four primary constraints:

$$\{\Pi, H_{\mathrm{EH}}\} \approx 0, \quad \{\Pi_i, H_{\mathrm{EH}}\} \approx 0. \tag{2.21}$$

Since Poinsson brakets for canonical variables are given as

$$\{N(t,\boldsymbol{x}), \Pi(t,\boldsymbol{y})\} = \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{2.22}$$

$$\{N_i(t,\boldsymbol{x}), \Pi^j(t,\boldsymbol{y})\} = \delta_i{}^j\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{2.23}$$

$$\{h_{ij}(t,\boldsymbol{x}), \pi^{kl}(t,\boldsymbol{y})\} = \delta_{(i}{}^k\delta_{j)}{}^l\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \tag{2.24}$$

the calculation (2.21) is straightforward.

$$\mathcal{H} \approx 0, \quad \mathcal{H}^i \approx 0. \tag{2.25}$$

As we obtain new constraints, let us redefine the hamiltonian by introducing new Lagrange multipliers Λ and Λ_i

$$H_{\mathrm{EH}} = \int d^3x \left\{N\mathcal{H} + N_i\mathcal{H}^i + \lambda\Pi + \lambda_i\Pi^i + \Lambda\mathcal{H} + \Lambda_i\mathcal{H}^i\right\}. \tag{2.26}$$

Obviously, the first two terms vanishes due to the secondary constarints and the new Hamiltonian takes the following form:

$$H_{\mathrm{EH}} = \int d^3x \left\{\lambda\Pi + \lambda_i\Pi^i + \Lambda\mathcal{H} + \Lambda_i\mathcal{H}^i\right\}. \tag{2.27}$$

Since eight constraints Π, Π^i, $\mathcal{H}$ and $\mathcal{H}^i$ commute with each other, they are classified into the first class constraints. This means that we must add eight gauge fixing terms to make these constraints belong to the second class in order to determine the dynamics of this system. Consequently, we find that the spatial metric γ_{ij} has two dynamical DOF.

2.2.2 Hamiltonian analysis for the simplest model of massive gravity

One of the simplest models of massive gravity is given in (2.2) and we choose the Minkowski metric as the fiducial metric here:

$$S = \frac{1}{2\kappa^2}\int d^4x \left[\sqrt{-g}R - \frac{1}{4}m^2\eta^{(0)\mu\alpha}\eta^{(0)\nu\beta}(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta})\right].$$

There is no diffeomorphism in massive gravity but we formally apply the ADM decomposition to carry out Hamiltonian analysis. What we have to do is to rewrite the mass term in terms of the ADM variables because the Hamiltonian form of the Einstein-Hilbert term has already been given in the previous section. The mass term in terms of the ADM variables takes the following form:

$$\begin{aligned}\eta^{(0)\mu\alpha}\eta^{(0)\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta}\right) =& \delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij} \\ &- 2N^2\delta^{ij}h_{ij} - 2N_i\delta^{ij}N_i + 2\delta^{ij}h_{ij}\gamma_{lm}N^lN^m \end{aligned} \tag{2.28}$$

where $h_{ij} = \gamma_{ij} - \eta_{ij}$. Hence the hamiltonian is expressed as

$$\begin{aligned}H_{\mathrm{EH}} =& \int d^3x \Big\{N\mathcal{H} + N_i\mathcal{H}^i + \lambda\Pi + \lambda_i\Pi^i \\ &+ \frac{m^2}{8\kappa^2}\left[\delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij} - 2N^2\delta^{ij}h_{ij} - 2N_i\delta^{ij}N_i + 2\delta^{ij}h_{ij}\gamma_{lm}N^lN^m\right]\Big\}.\end{aligned} \tag{2.29}$$

Because of the nonlinearity of the lapse and the shift, the consistency conditions

$$\{\Pi, H_{\mathrm{EH}}\} \approx 0, \quad \{\Pi_i, H_{\mathrm{EH}}\} \approx 0, \tag{2.30}$$

lead to the following equations containing N and N_i:

$$\mathcal{H} - \frac{m^2}{2\kappa^2}\delta^{ij}h_{ij}N \approx 0, \quad \mathcal{H}^i - \frac{m^2}{2\kappa^2}\left(\delta^{il} - \delta^{km}h_{km}\gamma^{ij}\right)N_j \approx 0. \tag{2.31}$$

This means that the lapse and the shift are completely solved in terms of γ_{ij} and π^{ij} and no constraint appears anymore. Therefore, this system inevitably has the extra DOF. The result does not change if we consider the other simplest model (2.3). The ADM parametrization for this mass term yields

$$\begin{aligned}g^{\mu\alpha}g^{\nu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\nu\beta}\right) =& \left(\gamma^{ik} - \frac{N^iN^k}{N^2}\right)\left(\gamma^{jl} - \frac{N^jN^l}{N^2}\right)\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) \\ &- \frac{2}{N^2}\gamma\left(N_iN_j + (N^2 - \gamma^{lm}N_lN_m - 1)h_{ij}\right) + \cdots,\end{aligned} \tag{2.32}$$

which clearly shows that the action is highly nonlinear with respedt to the lapse and the shift. This fact also holds for the general model (2.4).

Certainly, this analysis is not enough to identify the extra mode with a ghost but, as shown later, this extra mode actually leads to instability of the system and is called the Boulware-Deser ghost for this reason [9].

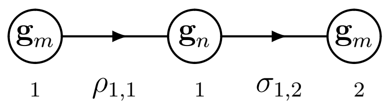

Figure 2.1: A moose diagram 2

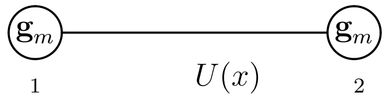

Figure 2.2: A moose diagram 3

2.3 Massive gravity as an effective field theory

Hamiltonian analysis tells us how many DOF the system has but nothing about their characters. If we find that a full theory has an extra degree of freedom compared to a free theory through the analysis, we cannot assert that the extra mode is a ghost. Furthermore, even if the extra mode is a ghost, the theory is still valid at energies far beneath the scale of the ghost mass. Thus, we should not only prove the existence of the ghost but also clarify the situation where the ghosty mode becomes active. One of useful tools for this purpose is the Sückelberg trick which was used in the construction of the Fierz-Pauli Lagrangan.

2.3.1 Theory space

We employ the Stückelberg trick in order to analyze the property of the simplest model of massive gravity. While it is possible to follow the step in Sec.1.4 where we constructed the Lagrangian of massive particles, we use the concept of "theory space" [10, 11, 12] here to introduce the Stückelberg fields because this idea gives us more transparent formulation of massive gravity.

We begin with the theory of massless spin-one particles with matter fields in the Minkowski spacetime to depict what theory space is. Theory space is represented by the diagram shown in Figure 2.1. The circles and the lines in the figure correspond to gauge groups and Weyl fermions respectively and indices 1 and 2 label circles where gauge groups live in. The direction of the line is very important: It expresses how the fields transform under the gauge transformation specified by each circle. If a line is directed away from a circle, the corresponding Weyl fermion transform as the fundamental representation of the gauge group in the circle. If a line is directed toward a circle, the fermion belongs to the complex conjugate of the fundamental representation.

Let us explain how we read off the theory defined by the diagram taking $\mathbf{g}_m = SU(m)$ and $\mathbf{g}_n = SU(n)$ in Figure 2.1. According to the rule explained above, two Weyl fermions $\rho_{1,1}$ and $\sigma_{1,2}$ transform as $(m, \bar{n}, 1)$ and $(1, n, \bar{m})$ respectively under the gauge group $SU_1(m) \times SU_1(n) \times SU_2(m)$. In addition to these matter fields, we also have gauge fields corresponding to three gauge groups. Therefore, the theory represented by the diagram consists of two Weyl fermions belonging to the bi-fundamental representation and three gauge fields. Here we suppose that the values of m and n are chosen for the theory to be anomaly-free and asymptotically free.

At this stage, it is not obvious why theory space is useful for our purpose. However, the link between theory space and Stückelberg fields becomes clear when the strongly coupled scale Λ_n of the gauge group $SU(n)$ is much larger than the scales Λ_m^1 and Λ_m^2 for $SU_1(m)$ and $SU_2(m)$. Under this assumption, the coupling of $SU(n)$ becomes strong around the scale of Λ_n while the two $SU(m)$ couplings remains weak. Therefore, below this enegy scales, the force between the two fermions gets to increase and, finally, the fermion condensation occurs. This means that the vaccum expectation values (VEV) of the fermion pair $\rho_{1,1}\sigma_{1,2}$ take non-zero values which are characterized by a $m \times m$ unitary matrix $U_{1,2}(x)$

$$\langle \rho_{1,1}\sigma_{1,2} \rangle \sim vU_{1,2}(x) \tag{2.33}$$

where $v \sim \Lambda_n$. The $m \times m$ unitary matrix is parametrized by Nambu-Goldstone (NG) boson fields or Sückelberg fields with the broken generators T^a with a decay constant f_n:

$$U_{1,2}(x) = e^{2i\pi^a(x)T^a/f_n}. \tag{2.34}$$

The theory space after the condensation is drawn in Figure 2.2. The line represents the unitary matrix which transforms under $SU_1(m) \times SU_2(m)$ as

$$U(x) \to g_1^{-1}U(x)g_2. \tag{2.35}$$

Therefore, this system is now described by two gauge bosons coupled to these Nambu-Goldstone bosons. The Lagrangian which is consistent with $SU_1(m) \times SU_2(m)$ is given by

$$\mathcal{L} = -\frac{1}{2g_1^2}\,\mathrm{tr}F_1^2 - \frac{1}{2g_2^2}\,\mathrm{tr}F_2^2 - f_n^2\,\mathrm{tr}(D_\mu U)(D^\mu U)^\dagger + \cdots \tag{2.36}$$

where $D_\mu U = \partial U + iA_{1\mu}U - iUA_{2\mu}$.

Now, it becomes clear that the fields $\pi^a(x)$ is actually identified with the Stückelberg fields. Let us set $U(x) = 1$ by using the gauge transformation. Then the theory in Figure 2.2 is not invariant under the $SU_1(m) \times SU_2(m)$ transformation anymore and, instead, invariant under the vector subgroup $SU_V(m) := SU_1(m) = SU_2(m)$. This is because one of the gauge field acquired a mass due to the fermion condensation and could not keep the $SU_1(m) \times SU_2(m)$ gauge symmetry without $\pi^a(x)$ fields. Conversely, this fact clearly illustrates that the $\pi^a(x)$ fields play a role of restoring the $SU_1(m) \times SU_2(m)$ symmetry as in the case of Sec.1.4. Furthermore, by taking the limit $g_1 \to 0$ after an appropriate normalization in (2.36) , the massless mode completely decouples. Therefore, we obtain the theory of massive spin-one particles with Stückelberg fields.

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}F^2 - f_n^2\,\mathrm{tr}(D_\mu U)(D^\mu U)^\dagger + \cdots \tag{2.37}$$

Here $g := g_2$ and $A_\mu := A_{2\mu}$. Taking the unitary gauge $U(x) = 1$, the Lagrangian takes the following form:

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}F^2 - \frac{m^2}{g^2}\,\mathrm{tr}(A_\mu A^\mu) + \cdots \tag{2.38}$$

where the mass paramter is defined as $m := gf$.

In the next subsection, we introduce Stückelberg fields for massive gravity by using the idea of theory space. However, before moving on to massive gravity, we would like to stress the usefulness of the Stückelberg field in effective field theories. For an illustration, let us consider the theory given in (2.38). Since the Lagrangian is not renormalizable,

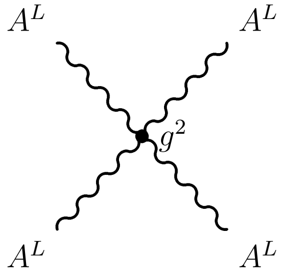

Figure 2.3: Four-point interaction

there exist some cutoff scale where perturbative unitarity breaks down. Naively, we can easily estimate this scale by considering $2 \to 2$ scattering process among the longitudinal modes A^L_μ because the polarization vector behaves as p^μ/m which potentially forces the scattering amplitude to exceed 1. The quartic term in (2.38) gives the diagram shown in Figure 2.3 and the straightforward calculation tells that the amplitude grows as $g^2(E/m)^4$ where E means the energy scale of this process. From this consideration, we may conclude that the perturbative unitarity breaks down around energies $E \sim m/\sqrt{g}$. Unfortunately, this is incorrect. Actually, another diagram constructed from the cubic interaction (Figure 2.4) cancels out the contribution from the four-point interaction. Then, the amplitude of this $2 \to 2$ scattering process grows as $g^2(E/m)^2$ and the true scale where the unitarity breaks down is given as m/g. Therefore, in order to correctly estimate a cutoff scale, we need to investigae all possible processes and see if some cancelations happen among them. The introduction of Stückelberg fields (2.37), however, makes the analysis much easier. As we have seen, the relevant contribution to the perturbative unitarity is the longitudinal mode of the vector field A_μ. On the other hand, according to the equivalence theorem [13, 14], Stückelberg fields mimic the behavior of the longitudinal mode of A_μ up to $\mathcal{O}(m/E)$. Therefore, we can estimate the cutoff scale by considering an amplitude of among Stückelber fields $\pi^a(x)$ in stead of the direct calculation. In this case, what we have to evaluate is the diagram 2.5 only. After the calculation, we find the amplitude grows as $g^2(E/m)^2$ which coincides the result obtained above.

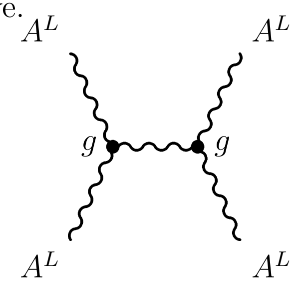

Figure 2.4: Exchange process

The point is that we can explicitly encode the longitudinal mode, which shows dangerous behavior at high energy scales in effective field theory, into Lagrangian through Sückelberg fields. Therefore, the Stückelberg trick is a very powerful tool not only for analysing effective field theories but also for constructing healthy field theories as we will see below.

The bottom line is that the Stückelberg trick gives us a transparent way to deal with effective field theories.

2.3.2 Sückelberg fields for massive gravity

We construct Sückelberg fields for massive gravity [12] using "theory space" introduced in the previous subsection. As with the example of spin-one particles, two metrics $g_{1\,\mu\nu}$ and $g_{2\,\mu\nu}$

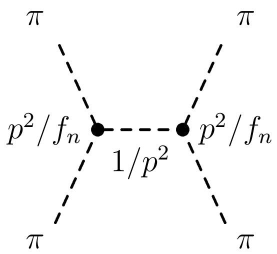

Figure 2.5: $\pi\pi \to \pi\pi$ process

live in each circle and a function $Y_{1,2}$ links the two diffeomorphisms in theory space (Figure 2.6). For internal symmetries $SU_1(m) \times SU_2(m)$, the unitary matrix $U(x)$, which links two gauge group, transforms as the bi-fundamental representation of $SU_1(m)$ and $SU_2(m)$. Thus, the function $Y_{1,2}$ should also transform under two diffeomorphisms. The task, however, is not straightforward because this statement seems to require a function depending on two sets of coordinates.

To clarify what kind of properties $Y_{1,2}$ should have, we would like to find an expression which enables us to compare a general coordinate transformation on $Y_{1,2}$ with a gauge transformation on $U(x)$. For the purpose, let us consider the general covariant tensor field $T_{\mu_1\nu_1\cdots\mu_n\nu_n}$ which transforms under a general coordinate transformation $x \to x' = f^{-1}(x)$ as

$$T_{\mu_1\nu_1\cdots\mu_n\nu_n}(x) \to T^{'}_{\mu_1\nu_1\cdots\mu_n\nu_n}(x^{'}) = \frac{\partial x^{\rho_1}}{\partial x'^{\mu_1}}\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\rho_n}}{\partial x'^{\mu_n}}\frac{\partial x^{\sigma_n}}{\partial x'^{\nu_n}}T_{\rho_1\sigma_1\cdots\rho_n\sigma_n}(x). \tag{2.39}$$

Rewriting (2.39) in terms of $x^{'}$ and dropping the prime of $x^{'}$, we also obtain another expression:

$$T_{\mu_1\nu_1\cdots\mu_n\nu_n}(x) \to T^{'}_{\mu_1\nu_1\cdots\mu_n\nu_n}(x) = \frac{\partial f^{\rho_1}(x)}{\partial x^{\mu_1}}\frac{\partial f^{\sigma_1}(x)}{\partial x^{\nu_1}}\cdots\frac{\partial f^{\rho_n}(x)}{\partial x^{\mu_n}}\frac{\partial f^{\sigma_n}(x)}{\partial x^{\nu_n}}T_{\rho_1\sigma_1\cdots\rho_n\sigma_n}(f(x)). \tag{2.40}$$

Unfortunately, it is still difficult to associate general coordinate transformations with gauge transformations. To get a clue, we apply the transformation rule (2.40) to a scalar field $\phi(x)$:

$$\phi(x) \to \phi'(x) = \phi(f(x)). \tag{2.41}$$

Since this is a form of function composition denoted by $\phi \circ f$, we reexpress (2.41) as

$$\phi \to \phi \circ f. \tag{2.42}$$

We extend this expression to the general $n-$form $\mathbf{T} := T_{\mu_1\cdots\nu_n}(x)dx^{\mu_1} \otimes \cdots \otimes dx^{\nu_n}$

$$\mathbf{T} \to \mathbf{T} \circ f \tag{2.43}$$

with the understanding that

$$\begin{aligned} dx^{\mu_1} \otimes dx^{\nu_1} \otimes \cdots dx^{\mu_n} \otimes dx^{\nu_n} &\to df^{\mu_1} \otimes df^{\nu_1} \otimes \cdots \otimes df^{\nu_n} \\ &= (\partial_{\rho_1} f^{\mu_1} \partial_{\sigma_1} f^{\nu_1} \cdots \partial_{\rho_n} f^{\mu_n} \partial_{\sigma_n} f^{\nu_n})\, dx^{\rho_1} \otimes dx^{\sigma_1} \otimes \cdots dx^{\rho_n} \otimes dx^{\sigma_n}. \end{aligned}$$

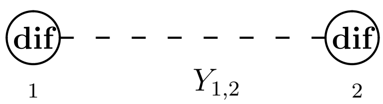

Figure 2.6: Theory space of massive gravity

In this notation, coordinate transformations are very similar to gauge transformations:

$$\mathbf{T} \to \mathbf{T} \circ f \Longleftrightarrow U \to Ug. \tag{2.44}$$

Taking advantage of this similarity, we construct $Y_{1,2}$. As the transformation rule for the unitary matrix $U_{1,2}$ is given by $g_1^{-1}Ug_2$, we assume that the link field $Y_{1,2}$ transforms under diffeo$_1$ $\times$ diffeo$_2$ in the completely same way :

$$Y_{1,2} \to f_1^{-1} \circ Y_{1,2} \circ f_2. \tag{2.45}$$

We see that the link field is a mapping from the circle 2 to the circle 1. More specifically, (2.45) can be expressed as

$$Y_{1,2}^{\mu} \to (f^{-1})^{\mu}(Y_{1,2}(f_2(x_2))). \tag{2.46}$$

Now, it is possible to construct a second rank covariant tensor $G_{2,\mu\nu}$ transforming under diffeo$_2$ from a second rank tensor living in the circle 1 by using the link field $Y_{1,2}$.

$$\mathbf{G}_2 = \mathbf{g}_1 \circ Y_{1,2}, \quad \text{or} \quad G_{2,\mu\nu}(x_2) = \frac{\partial Y_{1,2}^{\alpha}}{\partial x_2^{\mu}} \frac{\partial Y_{1,2}^{\beta}}{\partial x_2^{\nu}} g_{1,\alpha\beta}(Y_{1,2}(x_2)) \tag{2.47}$$

Therefore, we can write down a term which is invariant with under diffeo$_1$ $\times$ diffeo$_2$ with coefficients c_1 and c_2

$$c_1\sqrt{-g_2}g_2^{\mu\nu}(g_{2,\mu\nu} - G_{2,\mu\nu})g_2^{\rho\sigma}(g_{2,\rho\sigma} - G_{2,\rho\sigma}) + c_2\sqrt{-g_2}g_2^{\mu\rho}(g_{2,\mu\nu} - G_{2,\mu\nu})g_2^{\nu\sigma}(g_{2,\rho\sigma} - G_{2,\rho\sigma}) \tag{2.48}$$

which is a candidate for a mass term of massive gravity with Stückeberg fields.

Before going to the topic of the gauge invariant mass term, let us see explicitly how $G_{2,\mu\nu}(x_2)$ transforms under diffeo$_1$ $\times$ diffeo$_2$. Since $G_{2,\mu\nu}$ is clearly second rank covariant tensor in the circle 2, we concentrate on a coodinate transformation in the circle 1 $Y_{1,2}(x_2) \to f_1^{-1}(Y_{1,2}(x_2))$.

First of all, we consider $g_{1,\mu\nu}(Y_{1,2}(x_2))$. $g_{1,\mu\nu}(Y_{1,2}(x_2))$ lives in the circle 1, which means that it behaves as a tensor under the transformation. Thus, we have

$$g_{1,\mu\nu}(Y_{1,2}(x_2)) \to \frac{\partial f_1(Y)^{\rho}}{\partial Y_{1,2}^{\mu}} \frac{\partial f_1(Y)^{\sigma}}{\partial Y_{1,2}^{\nu}} g_{1,\rho\sigma}(f_1(Y_{1,2}(x_2))) \tag{2.49}$$

Furthermore, the link field $Y_{1,2}$ also transforms due to the rule $f_1^{-1} \circ Y_{1,2}$.

$$Y_{1,2}^{\mu}(x_2) \to (f_1^{-1})^{\mu}(Y_{1,2}(x_2)) \tag{2.50}$$

Combining (2.49) and (2.50), we find that $G_{2,\mu\nu}$ transforms as follows:

$$\begin{aligned} G_{2,\mu\nu}(x_2) &\to \frac{\partial f_1^{-1}(Y)^{\alpha}}{\partial x_2^{\mu}} \frac{\partial f_1^{-1}(Y)^{\beta}}{\partial x_2^{\nu}} \frac{\partial Y_{1,2}^{\rho}}{\partial f_1^{-1}(Y)^{\alpha}} \frac{\partial Y_{1,2}^{\sigma}}{\partial f_1^{-1}(Y)^{\beta}} g_{1\rho\sigma}(Y_{1,2}(x_2)) \\ &= \frac{\partial Y_{1,2}^{\rho}}{\partial x_2^{\mu}} \frac{\partial Y_{1,2}^{\sigma}}{\partial x_2^{\nu}} g_{1,\rho\sigma}(Y_{1,2}(x_2)) \\ &= G_{2,\mu\nu}(x_2). \end{aligned} \tag{2.51}$$

Hence, the transform rule for $G_{2,\mu\nu}$ under diffeo$_1$ $\times$ diffeo$_2$ is given by

$$G_{2,\mu\nu}(x_2) \to \frac{\partial f_2^{\rho_1}}{\partial x_2^\mu}\frac{\partial f_2^{\sigma_1}}{\partial x_2^\nu} G_{2,\rho\sigma}(f_2(x_2)). \tag{2.52}$$

Since the diffeo$_1$ $\times$ diffeo$_2$ invariant mass term in (2.48) has been obtained, we construct the diffeo$_1$ $\times$ diffeo$_2$ invariant Lagrangian. The remaining task is to find kinetic terms for this system. This work, however, is straightforward because it is clear that

$$\mathcal{L}_{\text{kin}} = \frac{1}{2\kappa_1^2}\sqrt{-g_1}R[g_1] + \frac{1}{2\kappa_2^2}\sqrt{-g_2}R[g_2] \tag{2.53}$$

is invariant under this transformation. Therefore, the gravitational action corresponding to the $SU_1(m) \times SU_2(m)$ gauge theory takes the following form:

$$\begin{aligned} S = &\frac{1}{2\kappa_2^2}\int d^4x_2\sqrt{-g_2}\left[R[g_2] - f^2 g_2^{\mu\nu} g_2^{\rho\sigma}(c_1 H_{\mu\nu}H_{\rho\sigma} + c_2 H_{\mu\rho}H_{\nu\sigma})\right] \\ &+ \frac{1}{2\kappa_1^2}\int d^4x_1\sqrt{-g_1}R[g_1] \end{aligned} \tag{2.54}$$

where $H_{\mu\nu}$ is defined as

$$\begin{aligned} H_{\mu\nu}(x_2) :&= g_{2,\mu\nu}(x_2) - G_{2,\mu\nu}(x_2) \\ &= g_{2,\mu\nu}(x_2) - \frac{\partial Y_{1,2}^\rho}{\partial x_2^\mu}\frac{\partial Y_{1,2}^\sigma}{\partial x_2^\nu} g_{1,\rho\sigma}(Y_{1,2}(x_2)). \end{aligned} \tag{2.55}$$

The action obtained here is called bi-gravity which consists of one massless graviton and one massive graviton. By increasing the number of circles and lines in theory space, we can build more general action describing a dynamics of one massless graviton and multiple massive gravitons [15, 16, 17, 18].

As with the case of (2.36), the massless graviton decouples from the theory in the limit $\kappa_1^2 \to 0$ and, as a result, the metric g_1 becomes completely nondynamical, which means the action (2.54) is reduced to

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R[g] - f^2 g^{\mu\nu} g^{\rho\sigma}(c_1 H_{\mu\nu}H_{\rho\sigma} + c_2 H_{\mu\rho}H_{\nu\sigma})\right] \tag{2.56}$$

where

$$H_{\mu\nu}(x) = g_{\mu\nu}(x) - \frac{\partial Y^\rho}{\partial x^\mu}\frac{\partial Y^\sigma}{\partial x^\nu} g_{1,\rho\sigma}(Y(x)) \tag{2.57}$$

Here we drop the labels of the dynamical metric $g_{2,\mu\nu}$ and the link field $Y_{1,2}^\mu$. By taking the unitary gauge $Y = \text{id}$ and choosing the coefficients c_1, c_2 appropriately, we find that the action (2.58) reproduces (2.3). Therefore, we finally obtained the simplest model of massive gravity with Stückelberg fields.

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{1}{4}m^2 g^{\mu\nu} g^{\rho\sigma}(H_{\mu\rho}H_{\nu\sigma} - H_{\mu\nu}H_{\rho\sigma})\right] \tag{2.58}$$

where $m^2 = 4f^2$.

2.3.3 Field theoretical analysis for massive gravity

We carry out a field theoretical analysis for massive gravity (2.58) as we done for the theory of massive spin-one particles with the assumption that $g_{1,\mu\nu} = \eta_{\mu\nu}$.

In the beginning, we expand the field $Y(x)$ around the unitary gauge, i.e.

$$Y^\mu(x) = x^\mu - B^\mu(x). \tag{2.59}$$

In order to make the anlysis easy, we introduce a $U(1)$ gauge symmetry in addition to the general coordinate invariance by replacing π_μ with $\pi_\mu + \partial_\mu \phi(x)$ to obtain

$$Y^\mu(x) = x^\mu - B^\mu(x) - \partial^\mu \phi(x). \tag{2.60}$$

Then, substituting (2.60) into (2.57) and dividing the metric $g_{\mu\nu}$ into $\eta_{\mu\nu}$ and $h_{\mu\nu}$, we find that

$$H_{\mu\nu} = h_{\mu\nu} + \partial_\mu B_\nu + \partial_\nu B_\mu + 2\partial_\mu\partial_\nu\phi - \partial_\mu B^\rho \partial_\nu B_\rho - \partial_\mu B^\rho \partial_\nu \partial_\rho \phi - \partial_\mu \partial^\rho \phi \partial_\nu B_\rho - \partial_\mu \partial^\rho \phi \partial_\nu \partial_\rho \phi. \tag{2.61}$$

The fields $B_\mu(x)$ and $\phi(x)$ are Nambu-Goldstone modes corresponding to the vector mode and the scalar mode of the massive spin-two particle respectively. The substitution of (2.61) gives the following terms:

$$\frac{m^2}{\kappa^2}(\partial^2\phi)^3, \quad \frac{m^2}{\kappa^2}(\partial^2\phi)^4, \quad \frac{m^2}{\kappa^2}\partial^2\phi(\partial B)^2. \tag{2.62}$$

In the effective field theoretical view point, these higher derivative terms are regarded as vertices which give a cutoff scale where the perturbative unitarity breaks down. In order to evaluate the scale, $\phi(x)$ and $\pi_\mu(x)$ should be canonically normalized. The kinetic term for $\phi(x)$ comes from the mixing between $h_{\mu\nu}(x)$ and $\phi(x)$ in the mass term and we can express this part schematically as $(m^2/\kappa^2)h\partial\partial\phi$. Since $\phi(x)$ field has mass dimension -2, it is plausible to redefine h as $h + m^2\phi$, which results in

$$\frac{m^2}{\kappa^2}h\partial\partial\phi \to \frac{m^4}{\kappa^2}\phi\partial\partial\phi. \tag{2.63}$$

Thus, we normalize the scalar field as

$$\phi(x) \to \frac{\kappa}{m^2}\phi(x). \tag{2.64}$$

Similarly, the kinetic term of $B_\mu(x)$ also comes from the mass term: $(m^2/\kappa^2)(\partial B)^2$, which leads to the following normalization for $B_\mu(x)$

$$B_\mu(x) \to \frac{\kappa}{m}B_\mu(x). \tag{2.65}$$

Then, we find correct scales of the interactions (2.62):

$$\frac{\kappa}{m^4}(\partial^2\phi)^3, \quad \frac{\kappa^2}{m^6}(\partial^2\phi)^4, \quad \frac{\kappa}{m^2}\partial^2\phi(\partial B)^2. \tag{2.66}$$

As the most dangerous contribution comes from the cubic self-interaction for ϕ in $2 \to 2$ scattering process, the cutoff scale (strongly coupled scale) Λ_5 is read off from the diagram Figure 2.7.

$$\left(\frac{p^6\kappa}{m^4}\right)^2 \frac{1}{p^2} = p^{10}\left(\frac{\kappa}{m^4}\right)^2 \sim 1 \quad \to \quad \Lambda_5 = \left(\frac{m^4}{\kappa}\right)^{1/5} \tag{2.67}$$

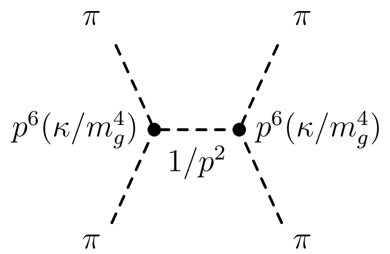

Figure 2.7: $\phi\phi \to \phi\phi$ process

Thus, the simplest model of massive gravity is valid up to Λ_5 as an effective field theory in Minkowski spacetime. Around the strongly coupled scale, quantum-induced operators suppressed by the scale Λ_5 get to dominate the theory. Thanks to the Stückelberg trick, we can roughly evaluate the structure of these induced operators by focusing on the scale Λ_5 with the limit $\Lambda_5 = \text{const}$, $\kappa \to 0$, $m \to 0$.

In this limit, the cubic self-interaction for ϕ only survives.

$$\frac{1}{\Lambda_5^5}(\partial^2\phi)^3 \tag{2.68}$$

Evidently, the scalar sector is invariant under an extended shift symmetry called the galilean symmetry $\partial_\mu\phi(x) \to \partial_\mu\phi(x) + c_\mu$. This indicates that induced operators must respect this global symmetry. With dimensional analysis, it is straghtforward to find the form of the operators:

$$\frac{\partial^m(\partial^2\phi)^n}{\Lambda_5^{3n+m-4}}. \tag{2.69}$$

Since the relation between the canonically normalized operator $\phi(x)$ and $h_{\mu\nu}(x)$ is given, we can write down the leading contributions of quantum corrections as follows:

$$\sum_{m,n} c_{m,n}\partial^m h^n, \qquad c_{m,n} \sim \frac{m^{2n}}{\Lambda_5^{3n+m-4}\kappa^n}. \tag{2.70}$$

The quantum correction to the mass parameter corresponds to $m = 0$, $n = 2$

$$\frac{\delta m^2}{\kappa^2} \sim \frac{m^2}{\kappa^2}\frac{m^2}{\Lambda^2}, \tag{2.71}$$

which shows that the mass parameter is technically natural.

2.3.4 Field theoretical analysis with sources

Although there exists the extra degree of freedom, it does not break the consistency of the theory as far as we regard the simplest model (2.58) as an effective field theory in a flat spacetime. The discussion, however, is not enough if we would like to apply to this model to gravitational phenomena because gravitational theories should also be consistent around nontrivial background spacetimes. Therefore, in this subsection, we include a source $T_{\mu\nu}$ and consider the behavior of the scalar field $\phi(x)$ which is most relevent to the consistency of the theory.

The action for $\phi(x)$ in the limit $\Lambda_5 = \text{const}$, $\kappa \to 0$, $m \to 0$ is written schematically as

$$\mathcal{L}_\phi = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\phi^3} \tag{2.72}$$

where

$$\mathcal{L}_{\phi^3} \sim \frac{1}{\Lambda_5^5}(\partial^2\phi)^3. \tag{2.73}$$

A coupling between the source and the scalar field is easy to obtain if the minimal coupling is assumed. For the field $h_{\mu\nu}$ with mass dimension 0, the minimal coupling to the source is defined as $h_{\mu\nu}T^{\mu\nu}$. On the other hand, in the process of canonically normalizing the scalar field, $h_{\mu\nu}$ is redefined as $h_{\mu\nu} + m^2\phi$, from which the coupling between $\phi(x)$ and the source $T_{\mu\nu}$ emerges.

$$h_{\mu\nu}T^{\mu\nu} \to \left(h_{\mu\nu} + m^2\phi\eta_{\mu\nu}\right)T^{\mu\nu}. \tag{2.74}$$

Therefore, we obtain

$$\mathcal{L}_\phi = \mathcal{L}_{\rm kin} + \mathcal{L}_{\phi^3} + \mathcal{L}_{\phi T} \tag{2.75}$$

$$\mathcal{L}_{\rm kin} \sim (\partial\phi)^2, \tag{2.76}$$

$$\mathcal{L}_{\phi^3} \sim \frac{1}{\Lambda_5^5}\left[(\Box\phi)^3 - (\Box\phi)(\partial_\mu\partial_\nu\phi)^2\right] \tag{2.77}$$

$$\mathcal{L}_{\phi T} \sim \kappa\phi T. \tag{2.78}$$

after the canonical normalization. We have to note that T is formally assumed to be infinity while a ratio of T to κ is kept in order for the coupling to survive in the limit $\Lambda_5 = \text{const}$, $\kappa \to 0$, $m \to 0$.

We would like to consider the scale where nonlinear effects become dominant for the later discussion. For a point source $T(x) \sim M\delta^3(\mathbf{x})$, the scalar field can have spherically symmetric solutions and, in the linear regime, $\phi(r)$ takes a familiar configurations:

$$\phi(r) \sim \kappa\frac{M}{r}. \tag{2.79}$$

Since the interaction terms $\mathcal{L}_{\phi^3}$ are suppressed by

$$\frac{\partial^4\phi}{\Lambda_5^5}, \tag{2.80}$$

compared to the kinetic term, we easily find that the linear approximated solution (2.79) breaks down when (2.80) becomes $\mathcal{O}(1)$:

$$\frac{\partial^4\phi}{\Lambda_5^5} \sim 1 \to r_V := (\kappa M)^{1/5}\frac{1}{\Lambda_5}. \tag{2.81}$$

r_V is called the Vainshtein radius [19] and characterizes the scale where nonlinear effects start to be dominant.

Next, we study the propagation of the scalar field on a nontrivial background. Expanding the scalar field around some spherically symmetric background as $\phi(x) = \Phi(r) + \varphi(x)$ and focusing quadratic terms of (2.75) to see structure of the propagator for φ, we obtain

$$\mathcal{L}_\phi^{(2)} \sim -(\partial\varphi)^2 + \frac{(\partial^2\Phi)}{\Lambda_5^5}(\partial^2\varphi)^2. \tag{2.82}$$

Remarkably, the theory on the nontrivial background contains a ghost which breaks the consistency as a quantum theory even if we consider (2.82) as an effective field theory because higher derivatives appear at the quadratic level. Actually, from the Lagrangian (2.82), we find the inverse propagator for φ:

$$\begin{aligned} p^2 + \frac{\partial^2\Phi}{\Lambda_5^5} p^4 &= \frac{\partial^2\Phi}{\Lambda_5^5}\left(\frac{\Lambda_5^5}{\partial^2\Phi}p^2 + p^4\right) \\ &\sim p^2\left(p^2 + \frac{\Lambda_5^5}{\partial^2\Phi}\right). \end{aligned} \tag{2.83}$$

Thus, the propagator for is given by

$$\frac{1}{p^2} - \frac{1}{p^2 + \frac{\Lambda_5^5}{\partial^2\Phi}}. \tag{2.84}$$

The second term proves the existence of the ghost and its r-dependent mass of the ghost particle is

$$m_{\text{ghost}}(r) \sim \frac{\Lambda_5^5}{\partial^2\Phi}. \tag{2.85}$$

This unhealthy degree of freedom is identified with the Boulware-Deser ghost mentioned in Section 2.2 because it originates from the nonlinearity of the theory. (The sixth DOF never appears in the vacuum as far as the effective field theoretical description holds.)

The expression (2.85) shows that the ghost mass becomes infinite when the background field is described by the trivial solution, which explains the reason why the ghost is absent when there is no source.

For the Lagrangian with $\Phi(r) \neq 0$ to be still valid as an effective field theory, the mass $m_{\text{ghost}}(r)$ must be above the cutoff scale Λ_5. Hence, conservative estimation gives the following relation

$$m_{\text{ghost}}(r) \sim \Lambda_5^2 \tag{2.86}$$

On the other hand, the nontrivial background configuration is given by $\Phi(r) \sim \kappa(M/r)$ outside the Vainshtein radius $r > r_V$. Therefore, the scale where the ghost becomes active is evaluated as

$$r_{\text{ghost}} \sim (\kappa M)^{1/3}\frac{1}{\Lambda_5}. \tag{2.87}$$

If the parameter M takes the value of the solar mass, the factor $(\kappa M)^{1/3}$ becomes very huge. This means that the simplest model loses its predictability at the scales which are much largar than the cutoff scale Λ_5^{-1}. Furthermore, more importantly, we have to emphasize that r_V is also larger than the Vainshtein radius. Therefore, the simplest model does not have any reliable region where nonlinear effects dominate the theory although the nonlinear terms are included in the classical action.

To conclude this subsection, we would like to roughly evaluate scales where quantum effects become important. Since the quantum-induced operators is given in (2.69) as

$$\frac{\partial^m(\partial^2\phi)^n}{\Lambda_5^{3n+m-4}} = \frac{\partial^{m+2}(\partial^2\phi)^{n-2}}{\Lambda_5^{3n+m-4}}(\partial\phi)^2, \tag{2.88}$$

what we have to do is to compare (2.88) with the kinetic term $(\partial\phi)^2$ and reveal scales where

$$\frac{\partial^{m+2}(\partial^2\phi)^{n-2}}{\Lambda_5^{3n+m-4}} \sim 1. \tag{2.89}$$

Substituting the linear solution of ϕ into the quantum-induced operators in (2.69), we obtain

$$r_{\mathrm{q}} \sim (\kappa M)^{\frac{n-2}{3n+m-4}} \frac{1}{\Lambda_5}. \tag{2.90}$$

The scales where quantum effects start to dominate the theory turns out to be

$$r_{\mathrm{quantum}} \sim (\kappa M)^{1/3} \frac{1}{\Lambda_5}. \tag{2.91}$$

It is worthwhile to note that two scales r_{ghost} and r_{quantum} coincide with each other. Thus, around the scales of the ghost mass, quantum corrections begin to be relevant and the effective field theory (2.58) is replaced by some UV completion.

From the above field theoretical observations, we conclude that the ghost originating from the nonlinearity is devastating for gravitational theories because it could break predictability in the nonlinear regime. The important lesson from this fact is that an extra DOF in the Hamiltonian analysis is not dangerous for effective field theories in a flat spacetime, but potentially dangerous for field theories in nontrivial backgrounds.

We should note that the above statement does not change even if we consider the general action (2.4) because quantities such as the cutoff scales are generally determined by the leading terms in the potential of $h_{\mu\nu}$.

2.4 Ghost-free massive gravity

In the previous section, we revealed that massive gravity has some intrinsic properties such as the cutoff scale. Among them, the fatal one is that the theory does not have any reliable region where nonlinearities are dominant because of the existence of the ghost, which seems to mean that massive gravity can not work as an alternative gravity. However, an important and interesting fact is also there: The value of the ghost mass is completely same as the scales where quantum corrections kick in. Therefore, we could expect that this situation might drastically change if we improve the cutoff scale by adjusting the parameters of the general model (2.6). Motivated by this observation, we try to construct the model which has cutoff scale larger than Λ_5 here. As a result, this attempt leads to the ghost-free massive gravity known as the dRGT massive gravity.

2.4.1 Raising up the cutoff scale

We consider if massive gravity having a cutoff scale larger than Λ_5 can be constructed. As shown in (2.6), the general action of massive gravity is given by

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \frac{1}{4} m^2 V(h) \right]. \tag{2.92}$$

where

$$V(h) = \sum_{i=2}^{\infty} v_i(h) \tag{2.93}$$

$$
\begin{aligned}
v_2(h) &= [h^2] - [h]^2 \\
v_3(h) &= J_1[h^3] + J_2[h^2][h] + J_3[h]^3 \\
v_4(h) &= K_1[h^4] + K_2[h^3][h] + K_3[h^2][h]^2 + K_4[h^2][h^2] + K_5[h]^4 \\
v_5(h) &= L_1[h^5] + L_2[h^4][h] + L_3[h^3][h]^2 + L_4[h^3][h^2] + L_5[h^2][h]^3 + L_6[h^2]^2[h] + L_7[h]^5 \\
&\vdots
\end{aligned}
$$

Note that indices of $h_{\mu\nu}$ are raised or lowered with the dynamical metric $g_{\mu\nu}$ and $[\cdots]$ means a trace.

Since the scale Λ_5 was obtained through the anlysis of the scalar sector of Stückelberg fields, it is reasonable to employ the Stückelberg trick again and focusing on the scalar component in order to obtain a clue to raise up the cutoff.

$$
h_{\mu\nu} \to H_{\mu\nu} = 2\partial_\mu\partial_\nu\phi - \partial_\mu\partial^\rho\phi\partial_\nu\partial_\rho\phi. \tag{2.94}
$$

Substituting this expression into the potential (2.93), we obtain a polynomial of infinite order in $(\partial^2\phi)$ and each term is suppressed by the scale Λ_5 or higher scales $\Lambda_n = (m^{n-1}/\kappa)^{1/n}$, $n \geq 4$. Thus, to improve the cutoff scale, we need to eliminate all terms suppressed by Λ_5. The most naive strategy is to choose parameters J_i, K_i, $L_i \cdots$ in order that coefficients of each power of $(\partial^2\phi)$ is equated to zero. This idea, however, does not work unfortunately.

Another idea comes from the following identity:

$$
\begin{aligned}
\int d^4x \mathcal{L}_n^{\mathrm{id}}(\Pi) &= \int d^4x \, \eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} \left(\Pi_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\cdots\Pi_{\mu_n\nu_n}\right) \\
&= \int d^4x \, \partial_{\mu_1}\left[\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\left(\partial_{\nu_1}\phi\Pi_{\mu_2\nu_2}\cdots\Pi_{\mu_n\nu_n}\right)\right] = 0.
\end{aligned} \tag{2.95}
$$

Here $\Pi_{\mu\nu} := \partial_\mu\partial_\nu\phi$ and $\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}$ means products of Minkowski metric $\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\cdots\eta^{\mu_n\nu_n}$ antisymmetrized over ν indices.

The identity (2.95) tells that the cutoff scale is improved if we can put the potential into this form by specifying $i-1$ parameters for each $v_i(H)$. Actually, by imposing the following relations

$$
J_1 = 2J_3 + \frac{1}{2}, \quad J_2 = -3J_3 - \frac{1}{2},
$$

$$
\begin{aligned}
K_1 &= -6K_5 + \frac{1}{16}(24J_3 + 5), \quad K_2 = 8K_5 - \frac{1}{4}(6J_3 + 1), \\
K_3 &= 3K_5 - \frac{1}{16}(12J_3 + 1), \quad K_4 = -6K_5 + \frac{3}{4}J_3,
\end{aligned} \tag{2.96}
$$

we can eliminate a polynomial of $(\partial^2\phi)$ originating from $v_3(H)$ and $v_4(H)$. By putting similar relations for higher order terms $v_i(H)$, $i \geqq 5$, we obtain the theory whose cutoff scale is larger than Λ_5. Before studying the cutoff scale for this new theory, we discuss the number of parameters contaied the Lagrangian. Since we specify $i-1$ parameters only for each $v_i(H)$, it seems that there are infinite parameters in the Lagrangian. However, as $\mathcal{L}_n(\Pi)$ is identically zero for $n > 4$, the unfixed parameters in $v_i(H)\, i \geqq 5$ are completely irrelevant. As a result, the action of the new theory is given by

$$
S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{1}{4}m^2 V(h, J_3, K_3)\right] \tag{2.97}
$$

with (2.96).

2.4.2 The cutoff scale

From the above discussion, the action (2.97) has the cutoff scale higher than Λ_5 thanks to the nontrivial linear combination in $V(h, J_3, K_3)$ which eliminates all terms having the form of $(\partial^2\phi)^n$. Thus, in (2.97), operators in the potential terms should take the form of

$$\sim h^{a_1}(\partial A)^{a_2}(\partial^2\phi)^{a_3} \tag{2.98}$$

in the Sückelberg language. Since there is no term satisfying the condition $a_1 = 0, a_2 = 0, a_3 \neq 0$, the operators which give the lowest cutoff scale seems to be given by

$$\frac{1}{\Lambda_4}\partial B(\partial^2\phi)^2. \tag{2.99}$$

However, this term also becomes a total derivative and the cutoff scale turns out to be larger than Λ_4. Generally, the scalar-vector interactions having the following form

$$\frac{1}{\Lambda_n}\partial B(\partial^2\phi)^n, \qquad \Lambda_4 \leqq \Lambda_n < \Lambda_3. \tag{2.100}$$

are total derivatives in the action (2.97). This is because, as far as J_i and K_i satisfy (2.96), the explicit form of interactions (2.100) are

$$\partial_{\mu_1}B_{\nu_1}\frac{\delta}{\delta\Pi_{\mu_1\nu_1}}\mathcal{L}_n^{\text{id}}(\Pi) \tag{2.101}$$

and it is straightforward that

$$\partial_\mu\frac{\delta}{\delta\Pi_{\mu\nu}}\mathcal{L}_n^{\text{id}}(\Pi) = 0.$$

Therefore, all interactions suppressed by Λ_n $(4 \leqq n < 3)$ drop and the cutoff scale of the the action (2.97) is raised up to Λ_3.

2.4.3 dRGT massive gravity

We have constructed the theory having the cutoff Λ_3 by choosing parameters for each partial sum $v_i(h)$. Formally, this operation is possible, but we are faced with difficulties when the new theory is applied to cosmological or gravitational phenomena because the potential terms consists of infinite series in $h_{\mu\nu}$. Therefore, we need another representation of the potential to calculate physical quantities. For this purpose, let us see the structure of the Stückelberg field introduced in (2.94).

$$H_{\mu\nu} = 2\partial_\mu\partial_\nu\phi - \partial_\mu\partial^\rho\phi\partial_\nu\partial_\rho\phi = 2\Pi_{\mu\nu} - \Pi_\mu{}^\rho\Pi_{\rho\nu}$$

Here $h_{\mu\nu}$ and B_μ are dropped, but this expression has a great implication. Rewriting the right hand side, we have

$$H^\mu{}_\nu = 2\Pi^\mu{}_\nu - \Pi^{\mu\rho}\Pi_{\rho\nu} = \delta^\mu{}_\nu - (\delta^\mu{}_\nu - \Pi^\mu{}_\nu)^2. \tag{2.102}$$

Solving (2.102) in terms of $\Pi^\mu{}_\nu$, we obtain

$$\Pi^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu}. \tag{2.103}$$

By definition, (2.103) only holds when $h_{\mu\nu}$ and B_μ are set to be zero. Thus, we should express correctly the relation (2.103) as

$$\Pi^\mu{}_\nu = \mathcal{K}^\mu{}_\nu|_{h=B=0} \tag{2.104}$$

by using a new tensor defined as

$$\mathcal{K}^\mu{}_\nu := \delta^\mu{}_\nu - \sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu}. \tag{2.105}$$

The expression (2.105) is very useful in order to reformulate the potential term because $\mathcal{K}^\mu{}_\nu|_{h=B=0}$ is linear in $\Pi^\mu{}_\nu$ unlike $H^\mu{}_\nu$ given in (2.94). Thanks to this property, we just put the potential in the following form to obatin the completely equivalent action to (2.97):

$$S_{\text{potential}} = \frac{m^2}{8\kappa^2}\int d^4x\sqrt{-g}\sum_{n=2}^{4} g^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\mathcal{K}_{\mu_1\nu_1}\mathcal{K}_{\mu_2\nu_2}\cdots\mathcal{K}_{\mu_n\nu_n} \tag{2.106}$$

where $g^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}$ means products of the dynamical metric $g^{\mu_1\nu_1}g^{\mu_2\nu_2}\cdots g^{\mu_n\nu_n}$ antisymmetrized over ν indices.

Remarkably, (2.106) is expressed as the finite series in $\mathcal{K}^\mu{}_\nu$, which enables us to actually consider cosmology or other gravitational phenomena with the action:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + \frac{1}{4}m^2\sum_{n=2}^{4}\alpha_n e_n(\mathcal{K})\right]. \tag{2.107}$$

where

$$e_n(\mathcal{K}) := g^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}\mathcal{K}_{\mu_1\nu_1}\mathcal{K}_{\mu_2\nu_2}\cdots\mathcal{K}_{\mu_n\nu_n}$$

and α_3 and α_4 are free parameters while α_2 is fixed to be four.

Theories of massive gravity belonging to this class (2.107) is called the de Rham-Gabadadze-Tolley (dRGT) massive gravity [20, 21].

2.4.4 Field theoretical analysis for dRGT model

We have already known the cutoff scale of the dRGT massive gravity. Thus, we just introduce the explicit form of the Lagrangian in the limit $\Lambda_3 = \text{const}$, $\kappa \to 0$, $m \to 0$ here. According to [4], the action in this limit is given by

$$S = \int d^4x\,\left[\frac{1}{2}h_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}h_{\alpha\beta} - \frac{1}{2}h^{\mu\nu}\left(-4X^{(1)}_{\mu\nu}(\phi) - \frac{3\alpha_3+4}{\Lambda_3^3}X^{(2)}_{\mu\nu}(\phi) - \frac{2(4\alpha_4+\alpha_3)}{\Lambda_3^6}X^{(3)}_{\mu\nu}(\phi)\right)\right] \tag{2.108}$$

where

$$X^{(n)}_{\mu\nu}(\Phi) := \frac{1}{n+1}\frac{\delta}{\delta\Pi_{\mu\nu}}\mathcal{L}^{\text{id}}_{n+1}(\Pi) \tag{2.109}$$

and $\mathcal{E}^{\mu\nu,\alpha\beta}$ denotes the kinetic operator for the massless spin-two field $h_{\mu\nu}$. Note that each field is rescaled in order to have correct mass dimension.

By replacing the field $h_{\mu\nu}$ with

$$h_{\mu\nu} + \phi\eta_{\mu\nu} - \frac{\frac{3}{2}\alpha_3+2}{\Lambda_3^3}\partial_\mu\phi\partial_\nu\phi,$$

we find the Lagrangian which has the kinetic term for the scalar field:

$$
\begin{aligned}
S = \int d^4x \Bigg[& \frac{1}{2} h_{\mu\nu} \mathcal{E}^{\mu\nu,\alpha\beta} h_{\alpha\beta} + \frac{4\alpha_4 + \alpha_3}{\Lambda_3^6} h^{\mu\nu} X^{(3)}_{\mu\nu} - 3(\partial\phi)^2 - \frac{\frac{9}{2}\alpha_3 + 6}{\Lambda_3^3} (\partial\phi)^2 \Box\phi \\
& - \frac{(\frac{3}{2}\alpha_3 + 2)^2 + 8\alpha_4 + 2\alpha_3}{\Lambda_3^6} (\partial\phi)^2 \left([\Pi]^2 - [\Pi^2] \right) \\
& - \frac{(3\alpha_3 + 4)(5\alpha_4 + \frac{5}{4}\alpha_3)}{\Lambda_3^9} (\partial\phi)^2 \left([\Pi]^3 - 3[\Pi^2][\Pi] + 2[\Pi^3] \right) \Bigg] . \qquad (2.110)
\end{aligned}
$$

The scalar self-interactions

$$
\begin{aligned}
\mathcal{L}_3 &= -\frac{1}{2} (\partial\phi)^2 [\Pi] , \\
\mathcal{L}_4 &= -\frac{1}{2} (\partial\phi)^2 \left([\Pi]^2 - [\Pi^2] \right) , \\
\mathcal{L}_5 &= -\frac{1}{2} (\partial\phi)^2 \left([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3] \right) . \qquad (2.111)
\end{aligned}
$$

are well known as the galileon terms which yield second order equations of motion although the presence of higher derivatives [22, 23, 24, 25]. Therefore, the new theory does not contain any ghost originating from nonlinearities, which gives us expectation that the dRGT massive gravity has a reliable, classical nonlinear regime as an effective field theory.

Lastly, we mention quantum corrections. As in the case of the simplest model, the galileon terms also have the Galilean symmetry $\partial_\mu\phi + c_\mu$, which indicates that the induced operaters have the following form:

$$
\frac{\partial^m (\partial^2\phi)^n}{\Lambda_3^{3n+m-4}} . \qquad (2.112)
$$

Furthermore, the galileon terms are famous for the stability against quantum corrections in the sense that their tree level parameters never be renormalized [26, 27, 28]. Hence, the form of the induced operators are further restricted.

2.4.5 Field theoretical analysis with sources

To confirm whether or not the dRGT model has the reliable non-linear regime, we introduce a source $T_{\mu\nu}$ and carry out the same field theoretical analysis as we have done for the simplest model of massive gravity.

As the action of ϕ with no source has been obtained in (2.110), what we have to do is to introduce a coupling with a source field. Putting the same assumption as in the previous case, the coupling between $h_{\mu\nu}$, whose mass dimension is zero, and $T_{\mu\nu}$ is given by $h_{\mu\nu}T^{\mu\nu}$ as usual. Thus, due to the completely same logic in (2.74), the scalar field aquires the coupling to the source ϕT. Canonical normalizations for $h_{\mu\nu}$ and ϕ give the following terms to (2.110):

$$
\mathcal{L}_{\text{coupling}} = \mathcal{L}_{hT} + \mathcal{L}_{\phi T} + \mathcal{L}_{\phi\partial\partial T} \qquad (2.113)
$$

$$
\mathcal{L}_{hT} \sim \kappa h_{\mu\nu} T^{\mu\nu}, \quad \mathcal{L}_{\phi T} \sim \kappa\phi T, \quad \mathcal{L}_{\phi\partial\partial T} \sim \kappa \frac{\frac{3}{2}\alpha + 2}{\Lambda_3^3} \partial_\mu\phi\partial_\nu\phi T^{\mu\nu} . \qquad (2.114)
$$

Here we assume that $T \to \infty$ while the combination κT is fixed in order for these coupling terms to survive.

Now that the coupling terms are turned out, let us calculate the Vainshtein radius for a point source $T \sim M\delta(\mathbf{x})$. Comparing the leading self-interaction to the kinetic term, it is suppressed by the factor

$$\sim \frac{1}{\Lambda_3^3}\Box\phi \tag{2.115}$$

Since the configuration of the scalar field $\phi(x)$ is given by a Newtonian potential

$$\phi(r) \sim \kappa\frac{M}{r}. \tag{2.116}$$

in the linear regime, the scale where the interaction is comparable to the kinetic part is estimated as

$$\frac{1}{\Lambda_3^3}\Box\phi \sim 1 \Longleftrightarrow r_V := (\kappa M)^{1/3}\frac{1}{\Lambda_3}. \tag{2.117}$$

As mentioned above, the theory (2.110) does not have any ghost, which means that we have to compare the Vainshtein radius r_V to the scale where the quantum-induced operators get to be dominant in order to discuss the validity of the dRGT model as an alternative gravity. Ignoring the subtleties which might arise from the nonrenormalization theorem of the galileon terms, the scales where the quantum corrections starting to be effective seem to be same as (2.91) except the cutoff Λ_5

$$r_{\text{quantum}} \sim (\kappa M)^{1/3}\frac{1}{\Lambda_3} \tag{2.118}$$

because the allowed induced operators is same due to the same shift symmetry. Then, we find that $r_V \sim r_{\text{quantum}}$, which shows that the dRGT model does not work as a theory of gravity. This argument, however, is completely wrong. What the relation $r_V \sim r_{\text{quantum}}$ actually means that the linear approximation for ϕ no longer holds around these scales and the estimation of r_{quantum} based on the assumption should not be trusted.

Thus, instead of the evaluation based on the linear approximation, we compare directly the quantum-induced operators (2.112) to the nonlinear terms in the tree-level Lagrangian (2.110)

$$\sim \frac{(\partial\phi)^2(\partial^2\phi)^n}{\Lambda_3^{3n}} \tag{2.119}$$

to investigate the true scale where some UV completion replaces the effective field theory. For the comparison, we formally rewrite (2.119) as follows:

$$\sim \frac{\partial^{-2}(\partial^2\phi)^{n+2}}{\Lambda_3^{3n}}. \tag{2.120}$$

Then, by relabeling $n+2$ as n

$$\sim \frac{\partial^{-2}(\partial^2\phi)^n}{\Lambda_3^{3n-6}}, \tag{2.121}$$

we easily compare the (classical) nonlinear terms (2.119) to the quantum originating operators, which tells us that the quantum effects are suppressed by

$$\left(\frac{\partial}{\Lambda_3}\right)^{n+2}. \tag{2.122}$$

Since (2.112) becomes dominant when $\partial/\Lambda_3 \sim 1$, the scale where the effective theoretical description breaks down is estimated to be

$$r_{\text{quantum}} \sim \frac{1}{\Lambda_3} \tag{2.123}$$

which is much smaller than the Vainshtein radius.

In conclusion, the dRGT model has the reliable nonlinear regime and satisfies the necessary condition as an alternative theory of gravity.

2.4.6 Hamiltonian analysis for dRGT massive gravity

We have shown that the dRGT model (2.107) does not have any ghost in the limit $\Lambda_3 =$ const, $\kappa \to 0$, $m \to 0$. As a result, classical nonlinearities get to be important before quantum-iduced operators become effective, which means that the dRGT massive gravity actually works as an alternative theory of gravity. In this subsection, we show that the theory does not suffer from the Boulware-Deser ghost problem in the full nonlinear level based on [29, 30, 32].

For convenience, we give another representation of the dRGT model. Rewriting the potential in terms of $\sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu}$, we have

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + \frac{1}{4}m^2\sum_{n=0}^{3}\beta_n\, e_n\left(\sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu}\right)\right] \tag{2.124}$$

β_n is related to α_n as follows:

$$\begin{aligned} &\beta_0 = 48 + 24\alpha_3 + 24\alpha_4\,, \qquad &&\beta_1 = -24 - 18\alpha_3 - 24\alpha_4\,,\\ &\beta_2 = 4 + 6\alpha_3 + 12\alpha_4\,, &&\beta_3 = -\alpha_3 - 4\alpha_4 \end{aligned} \tag{2.125}$$

Note that $e_4\left(\sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu}\right)$ does not contribute to the dynamics because

$$\begin{aligned} \sqrt{-g}e_4(\sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu}) &= \sqrt{-g}\det\left(\sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu}\right)\\ &= \sqrt{-g}\det\left(\sqrt{\delta^\mu{}_\nu - g^{\mu\rho}\left(g_{\rho\nu} - \partial_\rho Y^\alpha(x)\partial_\nu Y^\beta(x)\eta_{\alpha\beta}\right)}\right)\\ &= \sqrt{-g}\det\left(\sqrt{g^{\mu\rho}\left(\partial_\rho Y^\alpha(x)\partial_\nu Y^\beta(x)\eta_{\alpha\beta}\right)}\right)\\ &= \sqrt{-g}\sqrt{g^{-1}f} = \sqrt{-f} \end{aligned}$$

where $f_{\mu\nu} := \partial_\mu Y^\alpha(x)\partial_\nu Y^\beta(x)\eta_{\alpha\beta}$.

Let us carry out the Hamiltonian analysis on the action (2.124). As a first step, we consider the minimal model which is obtained by choosing the parameters $\beta_0 = 24$ and $\beta_1 = -8$

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\left[R - 2m^2\,(\,\mathrm{tr}\sqrt{g^{-1}f} - 3)\right]. \tag{2.126}$$

After taking the unitary gauge $f_{\mu\nu} = \eta_{\mu\nu}$, we parametrize the potential terms (2.126) by the ADM variables

$$g_{00} = -N^2 + \gamma_{ij}N^iN^j, \quad g_{ij} = \gamma_{ij}, \quad g_{0i} = N_i$$

to obtain

$$S_{\text{potential}} = -\frac{m^2}{\kappa^2}\int d^4x N\sqrt{\gamma}\,\left(\mathrm{tr}\sqrt{g^{-1}\eta}-3\right). \tag{2.127}$$

Since we have known the canonical momenta in (2.128)

$$\Pi = \frac{\partial\mathcal{L}_{\text{EH}}}{\partial\dot{N}} = 0 \tag{2.128}$$

$$\Pi^i = \frac{\partial\mathcal{L}_{\text{EH}}}{\partial\dot{N}_i} = 0 \tag{2.129}$$

$$\pi^{ij} = \frac{\partial\mathcal{L}_{\text{EH}}}{\partial\dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{\gamma}\left(K^{ij}-K\gamma^{ij}\right), \tag{2.130}$$

the Hamiltonian of the minimal model is given by

$$H = \int d^3x\left[N\mathcal{H}+N_i\mathcal{H}^i+\lambda\Pi+\lambda_i\Pi^i+\frac{m^2}{\kappa^2}\sqrt{\gamma}\,N\,\left(\mathrm{tr}\sqrt{g^{-1}\eta}-3\right)\right] \tag{2.131}$$

Here $\sqrt{g^{-1}\eta}$ is a square root of

$$(g^{-1}\eta)^{\mu}_{\ \nu} = \frac{1}{N^2}\begin{pmatrix} 1 & N^l\delta_{lj} \\ -N^i & (N^2\gamma^{il}-N^iN^l)\delta_{lj}\end{pmatrix}. \tag{2.132}$$

In order to discuss the structure of constraints, we have to find the explicit form of $\sqrt{g^{-1}\eta}$ but, unfortunately, it is technically difficult. Therefore, instead, we impose a condition for $N\sqrt{g^{-1}\eta}$ to be linear in the lapse, which is a necessary condition for the system to be ghost-free, and see whether any inconsistency arises [29].

From the assumption, $\sqrt{g^{-1}\eta}$ has the following form:

$$N\sqrt{g^{-1}\eta}^{\mu}_{\ \ \nu} = A^{\mu}{}_{\nu}+NB^{\mu}{}_{\nu} \tag{2.133}$$

with matrices A and B. Squaring the both sides of (2.133), we compare

$$g^{-1}\eta^{\mu}{}_{\nu} = \frac{1}{N^2}A^{\mu}{}_{\rho}A^{\rho}{}_{\nu}+\frac{1}{N}(A^{\mu}{}_{\rho}B^{\rho}{}_{\nu}+B^{\mu}{}_{\rho}A^{\rho}{}_{\nu})+B^{\mu}{}_{\rho}B^{\rho}{}_{\nu} \tag{2.134}$$

to the expression (2.132). Introducing a new variable n^i as

$$N^i = (\delta^i{}_j+ND^i{}_j)n^j \tag{2.135}$$

with some matrix $D^i{}_j$, we obtain

$$A^{\mu}{}_{\nu} = \frac{1}{\sqrt{1-n^r\delta_{rs}n^s}}\begin{pmatrix} 1 & n^i\delta_{ij} \\ -n^i & -n^in^k\delta_{kj}\end{pmatrix}, \tag{2.136}$$

$$B^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{(\gamma^{ik}-D^i{}_ln^lD^k{}_mn^m)\delta_{kj}}\end{pmatrix}, \tag{2.137}$$

and find the following consistency condition for the matrix $D^i{}_j$:

$$(\sqrt{1-n^r\delta_{rs}n^s})\,D^i{}_j = \sqrt{(\gamma^{ik}-D^i{}_ln^lD^k{}_mn^m)\delta_{kj}}\,. \tag{2.138}$$

Since (2.138) can be solved in terms of ${D^i}_j$, the potential (2.127) is actually linear in the lapse and the Hamiltonian has the following form:

$$H = \int d^3x \left[N\mathcal{H} + (\delta^i_j + N{D^i}_j)n^j\mathcal{H}_i + \lambda\Pi + \lambda_i\Pi^i_{\mathbf{n}} \right.$$
$$\left. + \frac{m^2}{\kappa^2}\sqrt{\gamma}\left[\sqrt{1-n^r\delta_{rs}n^s} + N\,\mathrm{tr}\left(\sqrt{(\gamma^{ik}\delta_{kj} - {D^i}_l n^l {D^k}_m n^m \delta_{kj})}\right) - 3N\right]\right]. \tag{2.139}$$

where $\Pi^i_{\mathbf{n}}$ denotes the canonical momentum for the new shift.

As usual, we impose the consistency conditions:

$$\{\Pi, H\} \approx 0, \quad \{{\Pi_{\mathbf{n}}}^i, H\} \approx 0. \tag{2.140}$$

Due to the linearity of the lapse, the first Poisson bracket yields a constraint $\phi^{(1)}$. On the other hand, the second one is expressed as

$$\left(\mathcal{H}_i - \frac{m^2}{\kappa^2}\frac{\sqrt{\gamma}\,n^l\delta_{li}}{\sqrt{1-n^r\delta_{rs}n^s}}\right)\left[\delta^i_k + N\frac{\partial}{\partial n^k}\left({D^i}_j n^j\right)\right] \approx 0 \tag{2.141}$$

and $[\cdots]$ never vanishes because it is the Jacobian of the transformation (2.135). Thus, n^i is solved in terms of γ_{ij} and π^{ij}.

$$n^i = -\mathcal{H}_j\delta^{ji}\left[\frac{m^4}{\kappa^4}\det\gamma + \mathcal{H}_k\delta^{kl}\mathcal{H}_l\right]^{-1/2} \tag{2.142}$$

The one more constraint $\phi^{(2)}$ arises from $\{\phi^{(1)}, \hat{H}\}$ where $\hat{H} := H + \Lambda\phi^{(1)}$. As $\{\phi^{(1)}, \phi^{(2)}\}$ is not commutative, the Hamiltonian which determines the dynamics is given by

$$H = \int d^3x \left[N\mathcal{H} + (\delta^i_j + N{D^i}_j)n^j\mathcal{H}_i + \lambda\Pi + \Lambda\phi^{(1)} + \Lambda'\phi^{(2)} \right.$$
$$\left. + \frac{m^2}{\kappa^2}\sqrt{\gamma}\left[\sqrt{1-n^r\delta_{rs}n^s} + N\,\mathrm{tr}\left(\sqrt{(\gamma^{ik}\delta_{kj} - {D^i}_l n^l {D^k}_m n^m \delta_{kj})}\right) - 3N\right]\right]. \tag{2.143}$$

Note that the lapse N is canceled out by a Lagrange multiplier implicitly as in the case of the Fierz-Pauli Lagrangian in Section 1.5. Now that the Hamiltonian (2.143) is completely written in terms of γ_{ij} and π^{ij} with 2 constraints, the minimal model has exactly five degrees of freedom and evades the Boulware-Deser ghost.

The extention of the discussion to the general model (2.124) is straightforward. Let us consider $e_2(\sqrt{g^{-1}\eta})$ whose explicit form is

$$e_2(\sqrt{g^{-1}\eta}) = \frac{1}{2}\left[(\,\mathrm{tr}\sqrt{g^{-1}\eta})^2 - \,\mathrm{tr}g^{-1}\eta\right]. \tag{2.144}$$

Parametrizing $e_2(\sqrt{g^{-1}\eta})$ in terms of n^i and using the relation $\,\mathrm{tr}A^k = (\,\mathrm{tr}A)^k$, we have

$$N\,e_2(\sqrt{g^{-1}\eta}) = \frac{1}{2}\left[2(\,\mathrm{tr}A\,\mathrm{tr}B - \,\mathrm{tr}AB\,) + N\left((\mathrm{tr}B)^2 - \,\mathrm{tr}B^2\right)\right], \tag{2.145}$$

which shows that the polynomial $Ne_2(\sqrt{g^{-1}\eta})$ is linear in the lapse. Similarly, we easily see that $Ne_3(\sqrt{g^{-1}\eta})$ is also linear in N. Then, the consistency condition for the primary constraints gives the following relation:

$$\begin{aligned}\mathcal{H}_i + \frac{m^2}{8\kappa^2}\sqrt{\gamma}\bigg(\beta_1\frac{n^l\delta_{li}}{\sqrt{1-n^r\delta_{rs}n^s}} &+ \beta_2 n^l\left[\delta_{li}D^k{}_k - \delta_{lk}D^k{}_i\right] \\ &+ \beta_3(\sqrt{1-n^r\delta_{rs}n^s})\, n^l\delta_{lk}\Big[D^k{}_m D^m{}_i - D^k{}_i D^m{}_m \\ &+ \tfrac{1}{2}D^m{}_m D^j{}_j\delta^k_i - \tfrac{1}{2}D^m{}_j D^j{}_m\delta^k_i\Big]\bigg) = 0. \end{aligned} \tag{2.146}$$

From this expression, we see that n^i does not contain the lapse. Thus, the dRGT massive gravity is free from the Boulware-Deser ghost problem.

Chapter 3

Non-gravitational massive spin two particles I

Although the absence of the gauge symmetry and the nonrenormalizability, it has been turned out that the theory of gravitational massive spin-two particels is also highly constrained under the following assumption: The interacting theory has to have the same degree of freedom as the free particles. Otherwise, massive gravity is nonsense as an alternative theory of gravity. Thus, in the construction of the dRGT model, this requirement plays a role of a guiding principle. As mentioned in the previous chapter, it is incompletely understood whether or not such a guiding principle is necessary for higher spin fields but, theoretically, it is still interesting. There are three reasons for this: First, this requirement constrains allowed interactions severely, which could determine theories uniquely, not to mention massive gravity. In fact, over 70 years ago, Federbush constructed the theory of charged massive spin-two particles interacting through $U(1)$ gauge fields [33] following the "guiding principle." In the work, he found that this assumption eliminates the ambiguity coming from the noncomutativity of the $U(1)$ covariant derivatives and the theory is determined uniquely. Second, the interaction keeping degree of freedom basically does not generate a ghost mode even if background fields take nontrivial configurations: The form of interactions ensure the absence of the Boulware-Deser type ghost. Third, the allowed interactions could raise cutoff scales of EFT. Actually, the Ferderbush model is obtained if we choose appropriate interactions so that the theory has the largest values of the cutoff.

On the other hand, the new derivative interaction, which keeps the degrees of freedom of the system, for the Fierz-Pauli theory have been proposed recently [34, 35]. In the work [35], Hinterbichler also have shown that the leading terms of the potential in the dRGT theory does not change the degrees of freedom of free massive spin-two particles.

These facts motivate us to consider a new interacting spin-two models which have five DOF. Therefore, using these Boulware-Deser type ghost-free interactions for the Fierz-Pauli theory, we propose theories of non-gravitational massive spin-two particles in accordance with the "guiding principle" and study their proerties in the following chapters.

3.1 Interactions for the Fierz-Pauli theory

In this section, we introduce the "ghost-free" interactions which does not induce a new degree of freedom for the Fierz-Pauli Lagrangian, which is given by

$$\mathcal{L}_{\rm FP} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h - \frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) . \quad (3.1)$$

Remember that the relative sign among the quadratic potential terms is essential for the theory to be consistent.

Folkerts *et al.* and Hinterbichler pointed out that new interaction terms can be added to this model without any additional DOF by taking the specific linear combination [34, 35]. In four dimensions, there exist only three kinds of ghost-free interactions:

$$\mathcal{L}_3^{\rm d} \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \quad (3.2)$$

$$\mathcal{L}_3 \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} , \quad (3.3)$$

$$\mathcal{L}_4 \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} . \quad (3.4)$$

Here $\eta^{\mu_1\nu_1\cdots\mu_n\nu_n}$ is given by the product of n $\eta_{\mu\nu}$ which is antisymmetrized over the indices ν_1, ν_2, $\cdots$ ν_n and, for examples,

$$\begin{aligned}\eta^{\mu_1\nu_1\mu_2\nu_2} \equiv& \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1} , \\ \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \equiv& \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3} - \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_3\nu_2} + \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_3}\eta^{\mu_3\nu_1} \\ &- \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1}\eta^{\mu_3\nu_3} + \eta^{\mu_1\nu_3}\eta^{\mu_2\nu_1}\eta^{\mu_3\nu_2} - \eta^{\mu_1\nu_3}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_1} . \end{aligned} \quad (3.5)$$

The reason for the absence of higher-order interactions is apparent: Antisymmetric tensors constructed from more than four metrics are identically zero.

We note that, for illustration, the Fierz-Pauli Lagrangian is expressed with the tensors $\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}$ as

$$\mathcal{L}_{\rm FP} = \frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}h_{\mu_2\nu_2}\partial_{\nu_1}h_{\mu_3\nu_3} + \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} . \quad (3.6)$$

3.1.1 Hamiltonian analysis

Let us confirm that the above interactions does not introduce an extra degree of freedom through the Hamiltonian analysis. Here we assume the following Lagrangian:

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}h_{\mu_2\nu_2}\partial_{\nu_1}h_{\mu_3\nu_3} + \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} \\ &- \mu_1\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} - \mu_2\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \\ &- \mu_3\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}\end{aligned}$$

For simplicity, we set $\mu_2 = 0$ here. From the discussion in Chap. 1, we have already known the definition of canonical momenta for the variables $h_{\mu\nu}$:

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{h}_{00}} = 0 \quad (3.7)$$

$$\pi^i = \frac{\partial\mathcal{L}}{\partial\dot{h}_{0i}} = 0 \quad (3.8)$$

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = \dot{h}^{ij} - \dot{h}^k{}_k\delta^{ij} - 2\partial^{(i}h^{j)}{}_0 + 2\partial_k h^k{}_0\delta^{ij} \quad (3.9)$$

With Lagrange multipliers λ and λ_i, the Hamiltonian density is given by

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} + \lambda\pi + \lambda_i\pi^i. \tag{3.10}$$

By solving π^{ij} in terms of $\dot{h}_{ij}$, we have the explicit form of the Hamiltonian density.

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} + \lambda\pi + \lambda_i\pi^i \tag{3.11}$$

Here the first term $\mathcal{H}_0$ corresponds to the free field theory and has been obtained in (1.59). The second term $\mathcal{H}_{\text{int}}$ denotes the interaction part of the Lagrangian (3.7):

$$\mathcal{H}_{\text{int}} = \mu_1\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} + \mu_3\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \tag{3.12}$$

Then, we require the consistency condition for the constraints:

$$\{\pi, H\} \approx 0, \quad \{\pi^i, H\} \approx 0. \tag{3.13}$$

It is clear that h_{0i} is highly nonlinear due to the interactions, which leads to

$$\{\pi^i, H\} = 2m^2\delta^{ij}h_{0j} + 2\partial_j\pi^{ij} + \mathcal{O}(h_{0i}^2, h_{0i}^3) \approx 0. \tag{3.14}$$

We have to notice that $\mathcal{O}(h_{0i}^2, h_{0i}^3)$ which consists of quadratic and cubic terms in h_{0i} never contains h_{00} due to the antisymmetric property of $\eta^{\mu_1\nu_1\cdots\mu_n\nu_n}$. Thus, h_{0i} can be solved in terms of h_{ij} and π^{ij}. On the other hand, since h_{00} is linear despite the presence of the interactions due to the anti-symmetric property, we have

$$\begin{aligned}\{\pi, H\} = {}& \nabla^2 h_{jj} - \partial_i\partial_j h_{ij} - m^2 h_{ii} \\ & - 3\mu_1\eta^{00ijkl}h_{ij}h_{kl} - 4\mu_3\eta^{00ijklmn}h_{ij}h_{kl}h_{mn} \approx 0.\end{aligned} \tag{3.15}$$

Thus, the system has one primary constraint

$$\begin{aligned}\phi := {}& \nabla^2 h_{jj} - \partial_i\partial_j h_{ij} - m^2 h_{ii} \\ & - 3\mu_1\eta^{00ijkl}h_{ij}h_{kl} - 4\mu_3\eta^{00ijklmn}h_{ij}h_{kl}h_{mn} \approx 0.\end{aligned} \tag{3.16}$$

The Hamiltonian including this constraint is defined with a Lagrange multiplier Λ as

$$\hat{H} := \int d^3x \left(\mathcal{H} + \Lambda\phi\right). \tag{3.17}$$

Then, the consistency condition for ϕ also gives one more constraint:

$$\varphi := \{\phi, \hat{H}\} = -\partial_i\partial_j\pi^{ij} - \frac{1}{2}m^2\pi_{ii} + f(h_{ij}, \pi^{ij}) \approx 0. \tag{3.18}$$

Here $f(h_{ij}, \pi^{ij})$ is nonlinear h_{ij} but linear in π^{ij}.

It is clear from the discussion in Chap. 1 that two constraints ϕ and φ are not commutative with each other. Thus the complete Hamiltonian is given by

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{int}} + \lambda\pi + \lambda_i\pi^i + \Lambda\phi + \Lambda'\varphi. \tag{3.19}$$

Now that we have two constraints on 12 dynamical variables h_{ij} and π^{ij}, the number DOF of this system is five, which is completely same as the Fierz-Pauli Lagrangian.

3.2 Self-interacting massive spin two particles

We construct the self-interacting massive spin-two model using the ghost-free interactions in accordance with the guiding principle. Furthermore, to emphasize that our model is essentially different from the theory of massive gravity, we impose Z_2 symmetry, which prohibits the Einstein-Hilbert term.

Under these assumption, the Lagrangian of the interacting massive spin-two model [36] is given by

$$\begin{aligned}\mathcal{L} =& -\frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\left(\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2}\right)h_{\mu_3\nu_3} + \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} \\ &+ \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \\ =& -\frac{1}{2}\left(h\Box h - h^{\mu\nu}\Box h_{\mu\nu} - h\partial^\mu\partial^\nu h_{\mu\nu} - h_{\mu\nu}\partial^\mu\partial^\nu h + 2h_\nu{}^\rho\partial^\mu\partial^\nu h_{\mu\rho}\right) \\ &+ \frac{m^2}{2}\left(h^2 - h_{\mu\nu}h^{\mu\nu}\right) \\ &+ \frac{\lambda}{4!}\left(h^4 - 6h^2h_{\mu\nu}h^{\mu\nu} + 8hh_\mu{}^\nu h_\nu{}^\rho h_\rho{}^\mu - 6h_\mu{}^\nu h_\nu{}^\rho h_\rho{}^\sigma h_\sigma{}^\mu + 3\left(h_{\mu\nu}h^{\mu\nu}\right)^2\right) . \end{aligned} \tag{3.20}$$

Here λ is a dimensionless parameter. We cannot decide the sign of λ, because it is non trivial to learn which sign for λ stabilizes this system.

Although the model (3.20) is power counting renormalizable, the model is not renormalizable because the propagator behaves as $\mathcal{O}\left(p^2\right)$ for large momentum p instead of the naive expectation $\mathcal{O}\left(p^{-2}\right)$. In fact, the propagator has the following form:

$$D^m_{\alpha\beta,\rho\sigma} = -\frac{1}{2\left(p^2+m^2\right)}\left\{P^m_{\alpha\rho}P^m_{\beta\sigma} + P^m_{\alpha\sigma}P^m_{\beta\rho} - \frac{2}{3}P^m_{\alpha\beta}P^m_{\rho\sigma}\right\} , \tag{3.21}$$

$$P^m_{\mu\nu} := \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} . \tag{3.22}$$

Then when p^2 is large, the propagator behaves as $D^m_{\alpha\beta,\rho\sigma} \sim \mathcal{O}\left(p^2\right)$ due to the projection operator $P^m_{\mu\nu}$, which makes the behavior for large p^2 worse and therefore the model should not be renormalizable.

3.3 Classical stability condition

Since the spin-two field can have Lorentz invariant vacuum expectation values and the "ghost-free" potential does not introduce an extra degree of freedom, this new theory could have stable, nontrivial vacuum where the particle description holds. Thus, it is quite interesting to find nontrivial solutions and ask whether or not the particle description holds in nontrivial vacua.

Before carrying these analysis, let us clarify the criterion for the particle discreption to hold in each vacuum. For the purpose, we consider the Fierz-Pauli Lagrangian:

$$\mathcal{L}_{\rm FP} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h - \frac{1}{2}m^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) . \tag{3.23}$$

Formally substituting the Lorentz invariant vacuum ansatz $h_{\mu\nu} = C\eta_{\mu\nu}$ into the Fierz-Pauli Lagrangian, we find the potential for C

$$\mathcal{L}_{\rm FP} = -V(C) = -(-6m^2C^2). \tag{3.24}$$

Here C is constant. Solutions of (3.24) determines a vacuum where "particles" are defined.

Remarkably, the potential $V(C)$ is not bounded from below and the unique solution $C = 0$ corresponds to the local maximum instead of the local minimum. As we know, however, that the massive spin-two field is stable around the local maximum. Thus, for theoreis of massive spin-two fields, the necessary condition for the particle description to hold is that the solutions of $V(C)$ corresponds to the local maximum.

The reason for such a contradiction to the intuition occurs is C does not correspond to the propagating mode unlike in the case of scalar fields. To show this fact, let us assume that C is not constant and see the structure of the equation of motion for $C(x)$. The equations of motion for the Fierz-Pauli Lagrangian is given by

$$\begin{aligned}\frac{\delta S}{\delta h^{\mu\nu}} &= \Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda{}_\nu - \partial_\lambda \partial_\nu h^\lambda{}_\mu + g_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - g_{\mu\nu}\Box h \\ &\quad - m^2(h_{\mu\nu} - g_{\mu\nu}h) = 0 \,. \end{aligned} \tag{3.25}$$

The substitution of $h_{\mu\nu} = C(x)\eta_{\mu\nu}$ gives

$$0 = \eta^{\mu\nu}\left(-2\Box C + 3m^2 C\right) + 2\partial^\mu\partial^\nu C \,. \tag{3.26}$$

Then when $\mu \neq \nu$ in (3.26) gives

$$\partial_\mu\partial_\nu C = 0 \,, \tag{3.27}$$

which tells that C is given by a sum of the functions of each of coordinates $C = \sum_\mu C^{(\mu)}(x^\mu)$. Eq. (3.26) also gives

$$\eta^{\mu\mu}\partial_\mu^2 C = \eta^{\nu\nu}\partial_\nu^2 C \,. \tag{3.28}$$

In Eq. (3.28), the indicies μ in the left hand side and ν in the right hand side are not summed up. From Eq. (3.28), we find that C takes the following form: $C = \sum_{\mu,\nu}\frac{c}{2}\eta_{\mu\nu}x^\mu x^\nu + \sum_\mu c_\mu x^\mu + C_0$ where c, c_μ's and C_0 are all constants. By substituting this expression into (3.26), we find $c = 0$ and $c_\mu = 0$, which means C must be a constant. This tells that even if C is on the local maximum of the potential (3.30), C does not roll down.

We emphasize that this property purely comes from the structure of the kinetic term: The Fierz-Pauli tuning is completely irrelevant to the above statement. We also note that $C(x)$ is still constant even if nonderivative interactions in (3.3) and (3.4) are turned on.

3.4 Classical vacuum solutions in new theory of massive spin two field

Using the criterion obtained in the previous section, we find classical vacuum solutions of the interacting massive spin-two model having Z_2 symmetry and study their stability [37].

The criterion tells that the "stable" vacuum where the particle description holds should be the local maximum of the potential. Since the potential term for the model (3.20) is given by

$$V(h) = -\frac{1}{2}m^2\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} - \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}, \tag{3.29}$$

the substitution of the ansatz $h_{\mu\nu} = C\eta_{\mu\nu}$ gives

$$V(C) = -6m^2C^2 - \lambda C^4 \,. \tag{3.30}$$

Potential extrema are obtained from the following equation:

$$V'(C) = -12m^2C - 4\lambda C^3 = 0. \tag{3.31}$$

The solutions for (3.31) are given by

$$C = 0\,, \quad \pm\sqrt{-\frac{3m^2}{\lambda}}. \tag{3.32}$$

For the nontrivial solutions to take real values, the parameters are constrained to be

$$\begin{cases} \lambda > 0 \quad \text{for} \quad m^2 < 0 \\ \\ \lambda < 0 \quad \text{for} \quad m^2 > 0 \end{cases}.$$

Otherwise, there exists the trivial solution $C = 0$ only. Thus, for the parameter region $\lambda < 0, m^2 < 0$, we can not define the theory because it is obvious that there is no local maximum. This means that if the theory does not allow the existence of nontrivial vacua, the allowed parameter is given by

$$\lambda > 0, \quad m^2 > 0. \tag{3.33}$$

If $V(C)$ does not vanish, the potential $V(C)$ could be the vacuum energy. Then it is interesting to investigate the (in)stability of the classical solution and their energy spectrum corresponding to the extrema of the potential based on the above criterion. In the following, we consider each allowed parameter region.

(a) $\lambda > 0$ and $m^2 < 0$

In this parameter region, the trivial vacuum $C = 0$ corresponds to the local minima and is not stable becuase the particle description does not hold. On the other hand, the nontrivial vacuum solutions is real

$$C_\pm = \pm\sqrt{\frac{3|m^2|}{\lambda}} \tag{3.34}$$

and it is obvious that the substitution of (3.34) into $V''(C)$ yields

$$V''(C) = 12(|m^2| - \lambda C^2) = -24|m^2| < 0, \tag{3.35}$$

which means the nontrivial solutions correspond to local maxima. Thus, due to our criterion, the particle description holds in nontrivial vacua. Furthermore, these facts indicate that each nontrivial vacuum has the positive energy whose value is given by

$$V(C) = 6|m^2|C^2 + \lambda C^4 = \frac{27|m^2|^2}{\lambda} > 0. \tag{3.36}$$

(b) $\lambda < 0$ and $m^2 > 0$

In this parameter region, the trivial vacuum $C = 0$ corresponds to the local maxima and is stable in the sense that the particle description does hold. In contrast, the nontrivial vacuum solutions yield

$$V''(C) = 12(|m^2| - \lambda C^2) = +24m^2 > 0, \tag{3.37}$$

which shows the nontrivial solutions correspond to local minima. Thus, the particle description never holds in nontrivial vacua. Moreover, the energy of each nontrivial vacuum is negative in this case and takes the following value:

$$V(C) = 6|m^2|C^2 + \lambda C^4 = -\frac{9m^4}{|\lambda|} < 0. \tag{3.38}$$

(3.33) and the analysis (a) and (b) tell that the locally stable vacua are not the lowest energy state of the system. This is because, as mentioned in the previous section, the model of massive spin-two particle is that the vacuum where the potential is convex upward is stable but the vacuum where the potential is convex downward is unstable. We may think that the system could be ultimately unstable by the quantum tunneling from the stable "false" vacua to the unstable "true" vacuum. In case of the scalar field theory, this speculation could be true. In case of the massive spin-two field, however, it is not clear if the system is unstable or not because the potential does not correspond to the propagating modes, which is not the scalar mode but the massive spin-two mode. If we consider the tunneling for the massive spin-two mode by, say, the WKB approximation, we need to consider inhomogeneous and anisotropic intermediate states, which makes the situation very complex. Therefore at least at present, we do not know how we should discuss the global stability and we only concentrate on the arguments about the local stability.

3.5 Decoupling limit and Stability against quantum correction

In this section, we study the behavior of the theory around the perturbative cutoff scale and the quantum stability. First, we introduce the Stuckelberg field.

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu \partial_\nu \phi. \tag{3.39}$$

After the diagonalizing the quadratic mixing terms between $h_{\mu\nu}$ and ϕ and canonically normalizing ϕ, we find the most dangerous interactions for the perturbative unitarity,

$$\begin{aligned}
&\sim \frac{\lambda}{m^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} h_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}\Pi_{\mu_4\nu_4},\\
&\sim \frac{\lambda}{m^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} h_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}\Pi_{\mu_4\nu_4},\\
&\sim \frac{\lambda}{m^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \partial_{\mu_1}\phi\partial_{\nu_1}\phi\Pi_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}.
\end{aligned}$$

Here we define $\Pi_{\mu\nu}$ as $\partial_\mu\partial_\nu\phi$. The tree level amplitude for $\phi\phi \to \phi\phi$ scattering at energy E goes as $\mathcal{M} \sim \frac{\lambda E^6}{m^6}$. Thus, the theory becomes strongly coupled at the energy $E \sim m/\lambda^{\frac{1}{6}}$. We focus on the strongly coupled scale $\Lambda := m/\lambda^{\frac{1}{6}}$ by taking the decoupling limit $m \to 0$, $\lambda \to 0$, while $\Lambda = m/\lambda^{\frac{1}{6}}$ is fixed.

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}h_{\mu_2\nu_2}\partial_{\nu_1}h_{\mu_3\nu_3} + 2\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}\Pi_{\mu_2\nu_2} \\ &+ \frac{1}{3}\frac{1}{\Lambda^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}\Pi_{\mu_4\nu_4} \end{aligned} \tag{3.40}$$

We diagonalize the quadratic term to obtain the kinetic term for the scalar field by redefining the field $h_{\mu\nu} \to h_{\mu\nu} + \phi\eta_{\mu\nu}$.

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}h_{\mu_2\nu_2}\partial_{\nu_1}h_{\mu_3\nu_3} + \frac{1}{3}\frac{1}{\Lambda^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}\Pi_{\mu_4\nu_4} \\ &- 6\partial_\mu\phi\partial^\mu\phi - \frac{1}{3}\frac{1}{\Lambda^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}\phi\partial_{\nu_1}\phi\Pi_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}, \end{aligned} \tag{3.41}$$

Note that, in [28], de Rham *et al.* show that the tree level Lagrangian belonging to this type of scalar tensor theories never be renormalized. This fact suggests that quantum corrections to the Lagrangian (3.20) are proportional to m and λ. Let us roughly estimate the corrections to (3.20) using the information obtained from (3.41). Due to the Galilean symmetry, the induced operators to the Lagrangian (3.41) is expected to take the following form:

$$\frac{\partial^q(\partial^2\phi)^p}{\Lambda^{3p+q-4}}. \tag{3.42}$$

Therefore, the relevant operator for the mass correction can be expected to take the form of $\frac{1}{\Lambda^2}(\partial\partial\phi)^2$. Then, considering the relation between h and ϕ, we find the correction is given by $\delta m^2 \sim \left(\frac{m^2}{\Lambda^2}\right) m^2 = \lambda^{1/3}m^2$ and the value of the mass is technically natural. On the other hand, the quantum effect might induce a ghost having a mass lower than the cutoff scale. As the general mass term of the massive spin-two field is given by the form of

$$-\frac{1}{2}m^2(h^{\mu\nu}h_{\mu\nu} - (1-a)h^2), \tag{3.43}$$

the the scale of the ghost mass m_g is roughly estimated as $m_g^2 \sim \frac{m^2}{a}$. Therefore, if the quantum correction breaks the Fierz-Pauli tuning, the ghost mass is comparable to the cutoff scale and this model is consistent as an effective field theory.

Fortunately, by explicitly calculating the one loop correction to the mass term in the model (3.20), we find the Fierz-Pauli tuning does not break down at one loop level [38], which indicates that the ghost mass is larger than Λ.

Chapter 4

Non-gravitational massive spin two particles II

Since the dRGT massive gravity is considered as the general action containing all ghost-free interaction terms between neutral spin-two particles, it is expected that the more general charged spin-two action which keeps DOF of the system can be obtained from the dRGT massive gravity. de Rham, Matas, Ondo and Tolley attempted this kind of extension in [39], but they proved that the Einstein-Hilbert action is not compatible with $U(1)$ symmetry and the Einstein-Hilbert term should be modified. Unfortunately, according to [40], the modification necessarily leads to the undesirable ghost mode. Therefore, we cannot write down the $U(1)$ invariant massive gravity action. On the other hand, our model proposed in the previous chapter consists of the linearized Einstein-Hilbert term and interaction terms only. This suggests that we could potentially construct the $U(1)$ invariant classical action which has exactly five DOF by extending the model in [36].

4.1 New model of massive spin two particle

In Chap. 3, we construct the new theory of the massive spin-two particle which is invariant under Z_2 transformation. The free part of the Lagrangian consists of the linearized Einstein-Hilbert action and the Fierz-Pauli mass term,

$$\mathcal{L}_{\mathrm{FP}} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2). \tag{4.1}$$

Due to the Z_2 symmetry, the only interaction which does not generate an extra DOF is given by

$$\mathcal{L}_4 \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}\,. \tag{4.2}$$

Here $\eta^{\mu_1\nu_1\cdots\mu_n\nu_n}$ is the product of n $\eta_{\mu\nu}$ given by antisymmetrizing the indices ν_1, ν_2, $\cdots$, and ν_n Thus, the Z_2 invariant theory having five DOF takes the follwing form:

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}h_{\mu_2\nu_2}\partial_{\nu_1}h_{\mu_3\nu_3} \\ &+ \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} + \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}.\end{aligned} \tag{4.3}$$

We have seen that the particle description also holds in nontrivial vacua in some region of the parameter space spanned by m^2 and λ thanks to the property of the interactions.

4.2 Global $U(1)$ theory

We build the model of massive spin-two particles by replacing the real field with the complex field for the Z_2 invariant model. For the theory to have $U(1)$ symmetry, the cubic interaction is not allowed. The explict expression of the Lagrangian is given by

$$\mathcal{L} = \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}h^\dagger_{\mu_2\nu_2}\partial_{\nu_1}h_{\mu_3\nu_3} + m^2\eta^{\mu_1\nu_1\mu_2\nu_2}h^\dagger_{\mu_1\nu_1}h_{\mu_2\nu_2} + \frac{\lambda}{3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h^\dagger_{\mu_1\nu_1}h_{\mu_2\nu_2}h^\dagger_{\mu_3\nu_3}h_{\mu_4\nu_4}. \tag{4.4}$$

$m^2 > 0$ ensures the stability around the trivial vacuum. From the knowledge in Chap. 3, the theory does not have any nontrivial vacuum and is stable only around the trivial vacuum when m^2 and λ are both positive.

The complex field $h_{\mu\nu}$ can be parametrized with two real fields $a_{\mu\nu}$, $b_{\mu\nu}$ as usual:

$$h_{\mu\nu} = \frac{1}{\sqrt{2}}\left(a_{\mu\nu} + ib_{\mu\nu}\right). \tag{4.5}$$

Then, the action (4.4) becomes the interacting real massive spin-two field theory having $SO(2)$ symmetry.

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}a_{\mu_2\nu_2}\partial_{\nu_1}a_{\mu_3\nu_3} + \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}a_{\mu_1\nu_1}a_{\mu_2\nu_2}\\ &+ \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}a_{\mu_1\nu_1}a_{\mu_2\nu_2}a_{\mu_3\nu_3}a_{\mu_4\nu_4} + \frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}b_{\mu_2\nu_2}\partial_{\nu_1}b_{\mu_3\nu_3}\\ &+ \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}b_{\mu_1\nu_1}b_{\mu_2\nu_2} + \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}b_{\mu_1\nu_1}b_{\mu_2\nu_2}b_{\mu_3\nu_3}b_{\mu_4\nu_4}\\ &+ \frac{\lambda}{2\cdot 3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}a_{\mu_1\nu_1}a_{\mu_2\nu_2}b_{\mu_3\nu_3}b_{\mu_4\nu_4}\end{aligned} \tag{4.6}$$

Let us briefly show that the theory does not have an extra DOF since the procedure of the Hamiltonian analysis is almost same as the Z_2 theory. The canonical momenta are given as usual:

$$\begin{aligned}\pi_a &= \frac{\partial\mathcal{L}}{\partial\dot{a}_{00}} = 0, \quad \pi_b = \frac{\partial\mathcal{L}}{\partial\dot{b}_{00}} = 0\\ \pi_a^i &= \frac{\partial\mathcal{L}}{\partial\dot{a}_{0i}} = 0, \quad \pi_b^i = \frac{\partial\mathcal{L}}{\partial\dot{b}_{0i}} = 0\\ \pi_a^{ij} &= \frac{\partial\mathcal{L}}{\partial\dot{a}_{ij}} = \dot{a}^{ij} - \dot{a}^k{}_k\delta^{ij} - 2\partial^{(i}a^{j)}{}_0 + 2\partial_k a^k{}_0\delta^{ij}\\ \pi_b^{ij} &= \frac{\partial\mathcal{L}}{\partial\dot{b}_{ij}} = \dot{b}^{ij} - \dot{b}^k{}_k\delta^{ij} - 2\partial^{(i}b^{j)}{}_0 + 2\partial_k b^k{}_0\delta^{ij}.\end{aligned}$$

The primary constraints are given by the canonical momenta for a_{00}, a_{0i}, b_{00}, and b_{0i}

$$\pi_a := \frac{\partial\mathcal{L}}{\partial\dot{a}_{00}} = 0, \quad \pi_a^i := \frac{\partial\mathcal{L}}{\partial\dot{a}_{0i}} = 0, \quad \pi_b := \frac{\partial\mathcal{L}}{\partial\dot{b}_{00}} = 0, \quad \pi_b^i := \frac{\partial\mathcal{L}}{\partial\dot{b}_{0i}} = 0.$$

Thus, using eight Lagrange multipliers λ_a, $\lambda_{a\,i}$, λ_b and $\lambda_{b\,i}$, the Hamiltonian takes the following form:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} + \lambda_a\pi_a + \lambda_{a\,i}\pi_a^i + \lambda_b\pi_b + \lambda_{b\,i}\pi_b^i \tag{4.7}$$

where $\mathcal{H}_0$ represents the contribution of the Fierz-Pauli Lagrangian and $\mathcal{H}_{\rm int}$ is given by

$$\begin{aligned}\mathcal{H}_{\rm int} = &-\frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}a_{\mu_1\nu_1}a_{\mu_2\nu_2}a_{\mu_3\nu_3}a_{\mu_4\nu_4} - \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}b_{\mu_1\nu_1}b_{\mu_2\nu_2}b_{\mu_3\nu_3}b_{\mu_4\nu_4} \\ &- \frac{\lambda}{2\cdot 3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}a_{\mu_1\nu_1}a_{\mu_2\nu_2}b_{\mu_3\nu_3}b_{\mu_4\nu_4}. \end{aligned} \tag{4.8}$$

The remarkable property is that the Hamiltonian (4.7) is linear in a_{00} and b_{00}.

The consistency conditions for π_a^i and π_b^i give two linear equations in a_{0i} and b_{0i}:

$$\begin{aligned}\left\{\pi_a^i, H\right\} &= f^i(a_{ij}, b_{ij}, \pi_a^{ij}, \pi_b^{ij}, a_{0i}, b_{0j}) \approx 0, \\ \left\{\pi_a^i, H\right\} &= g^i(a_{ij}, b_{ij}, \pi_a^{ij}, \pi_b^{ij}, a_{0i}, b_{0j}) \approx 0,\end{aligned}$$

which means that a_{0i} and b_{0i} can be expressed in terms of other variables and never contain a_{00} and b_{00}:

$$\begin{aligned}a_{0i} &= a_{0i}(a_{ij}, b_{ij}, \pi_a^{ij}, \pi_b^{ij}) \\ b_{0i} &= b_{0i}(a_{ij}, b_{ij}, \pi_a^{ij}, \pi_b^{ij}).\end{aligned}$$

Therefore, the substitution of the explicit form of a_{0i} and b_{0i} into the Hamiltonian does not spoil the linearity of a_{00} and b_{00}. Then, the consistency conditions for π_a and π_b give the two constraints:

$$\phi_a^{(1)} := -\frac{\lambda}{3!}\eta^{i_1j_1i_2j_2i_3j_3}a_{i_1j_1}b_{i_2j_2}b_{i_3j_3} - \frac{\lambda}{3!}\eta^{i_1j_1i_2j_2i_3j_3}a_{i_1j_1}a_{i_2j_2}a_{i_3j_3} - m^2\eta^{ij}a_{ij} + \eta^{i_1j_1i_2j_2}\partial_{i_1}\partial_{j_1}a_{i_2j_2} = 0, \tag{4.9}$$

$$\phi_b^{(1)} := -\frac{\lambda}{3!}\eta^{i_1j_1i_2j_2i_3j_3}b_{i_1j_1}a_{i_2j_2}a_{i_3j_3} - \frac{\lambda}{3!}\eta^{i_1j_1i_2j_2i_3j_3}b_{i_1j_1}b_{i_2j_2}b_{i_3j_3} - m^2\eta^{ij}b_{ij} + \eta^{i_1j_1i_2j_2}\partial_{i_1}\partial_{j_1}b_{i_2j_2} = 0. \tag{4.10}$$

Again, by imposing the consitency condition on $\phi_a^{(1)}$ and $\phi_b^{(1)}$, we obtain the two more constraints. As a result, effectively, the system has the 20 dimensional phase space spanned by a_{ij}, b_{ij}, π_a^{ij} and π_b^{ij}. Therefore, we can conclude that this system has the same DOF as the free field theory.

4.3 Decoupling limit and quantum stability

In this section, we carry out the same analysis as in the previous section. Introducing the Stuckelberg fields

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu + 2\partial_\mu\partial_\nu\phi \tag{4.11}$$

clarifies the most dangerous interactions for the perturbative unitarity:

$$\begin{aligned}&\sim \frac{\lambda}{m^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h^\dagger_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\Pi^\dagger_{\mu_3\nu_3}\Pi_{\mu_4\nu_4}, \\ &\sim \frac{\lambda}{m^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}\Pi^\dagger_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}\Pi^\dagger_{\mu_4\nu_4}, \\ &\sim \frac{\lambda}{m^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}\phi^\dagger\partial_{\nu_1}\phi\Pi^\dagger_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}.\end{aligned}$$

We have to note that all fields have been canonically normalized in addition to the diagonalization of the kinetic term for the scalar field. Due to the completly same logic in Sec.3.5, we find that the tree level unitarity breaks down at the energy $E \sim m/\lambda^{\frac{1}{6}}$. Furthermore, by taking the limit $m \to 0$, $\lambda \to 0$, $\Lambda = m/\lambda^{\frac{1}{6}} = \text{const}$, we focus on the high energy behavior of the theory. After the diagonalization of the kinetic term, the Lagrangian takes the following form:

$$\begin{aligned}\mathcal{L} =&\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}h^\dagger_{\mu_2\nu_2}\partial_{\nu_1}h_{\mu_3\nu_3} + \frac{16}{3!}\frac{1}{\Lambda^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h^\dagger_{\mu_1\nu_1}\Pi_{\mu_2\nu_2}\Pi^\dagger_{\mu_3\nu_3}\Pi_{\mu_4\nu_4}\\ &+\frac{16}{3!}\frac{1}{\Lambda^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}\Pi^\dagger_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}\Pi^\dagger_{\mu_4\nu_4} - 6\partial_\mu\phi^\dagger\partial^\mu\phi\\ &-\frac{32}{3!}\frac{1}{\Lambda^6}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}\phi^\dagger\partial_{\nu_1}\phi\Pi^\dagger_{\mu_2\nu_2}\Pi_{\mu_3\nu_3}.\end{aligned} \tag{4.12}$$

The nonrenormalization theorem also holds for this action [28], which suggests that quantum corrections to the tree level Lagrangian (4.4) is proportional to m^2 and λ as in the Z_2 model.

We also note that the Fierz-Pauli tuning also does not break down at one loop level in the $U(1)$ model.

4.4 The behavior of the theory around vacua

In the previous chapter, we found that the Z_2 model has multiple stable vacua for $\lambda > 0$, $m^2 < 0$ or $\lambda < 0\, m^2 > 0$. In this section, we show the essential difference from the Z_2 model through the stability analysis on nontrivial vacua. Since nontrivial vacua are not stable for $\lambda < 0$, $m^2 > 0$ in the Z_2 model, we exlusively investigate here the $U(1)$ model with $\lambda > 0$, $m^2 < 0$. Then, the field acquires vacuum expectation value (VEV) whose value is given by

$$h^{\text{VEV}}_{\mu\nu} = \frac{C\mathrm{e}^{i\theta}}{\sqrt{2}}\eta_{\mu\nu} = \frac{1}{\sqrt{2}}\sqrt{\frac{3|m^2|}{\lambda}}\mathrm{e}^{i\theta}\eta_{\mu\nu}. \tag{4.13}$$

Here θ is a parameter of degenetated vacua.

We consider the fluctuation around the VEV to obtain the Lagrangian in the broken phase:

$$h_{\mu\nu} = h^{\text{VEV}}_{\mu\nu} + H_{\mu\nu} \tag{4.14}$$

The mass term takes the following form.

$$\begin{aligned}\mathcal{L}_{\text{mass}} =& -|m^2|\eta^{\mu_1\nu_1\mu_2\nu_2}h^\dagger_{\mu_1\nu_1}h_{\mu_2\nu_2}\\ =& -6|m^2|C^2 - \frac{3}{\sqrt{2}}C|m^2|H - \frac{3}{\sqrt{2}}C|m^2|H^\dagger - |m^2|\eta^{\mu_1\nu_1\mu_2\nu_2}H^\dagger_{\mu_1\nu_1}H_{\mu_2\nu_2},\end{aligned}$$

where H and $H^\dagger$ represent $\eta^{\mu\nu}H_{\mu\nu}$ and $\eta^{\mu\nu}H^\dagger_{\mu\nu}$ respectively. The interaction term in the broken phase are given as

$$\begin{aligned}
\mathcal{L}_{\rm int} =&3|m^2|C^2+\frac{3}{\sqrt{2}}C|m^2|H+\frac{3}{\sqrt{2}}C|m^2|H^\dagger+2|m^2|\eta^{\mu_1\nu_1\mu_2\nu_2}H^\dagger_{\mu_1\nu_1}H_{\mu_2\nu_2}\\
&+\frac{|m^2|}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}H_{\mu_1\nu_1}H_{\mu_2\nu_2}+\frac{|m^2|}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}H^\dagger_{\mu_1\nu_1}H^\dagger_{\mu_2\nu_2}\\
&+\sqrt{\frac{\lambda|m^2|}{6}}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}H_{\mu_1\nu_1}H^\dagger_{\mu_2\nu_2}H_{\mu_3\nu_3}+\sqrt{\frac{\lambda|m^2|}{6}}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}H^\dagger_{\mu_1\nu_1}H_{\mu_2\nu_2}H^\dagger_{\mu_3\nu_3}\\
&+\frac{\lambda}{3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}H^\dagger_{\mu_1\nu_1}H_{\mu_2\nu_2}H^\dagger_{\mu_3\nu_3}H_{\mu_4\nu_4}. \qquad (4.15)
\end{aligned}$$

Thus, the total Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{\rm BP} =&\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}H^\dagger_{\mu_2\nu_2}\partial_{\nu_1}H_{\mu_3\nu_3}+|m^2|\eta^{\mu_1\nu_1\mu_2\nu_2}H^\dagger_{\mu_1\nu_1}H_{\mu_2\nu_2}\\
&+\frac{|m^2|}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}H_{\mu_1\nu_1}H_{\mu_2\nu_2}+\frac{|m^2|}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}H^\dagger_{\mu_1\nu_1}H^\dagger_{\mu_2\nu_2}\\
&+\sqrt{\frac{\lambda|m^2|}{6}}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}H_{\mu_1\nu_1}H^\dagger_{\mu_2\nu_2}H_{\mu_3\nu_3}+\sqrt{\frac{\lambda|m^2|}{6}}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}H^\dagger_{\mu_1\nu_1}H_{\mu_2\nu_2}H^\dagger_{\mu_3\nu_3}\\
&+\frac{\lambda}{3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}H^\dagger_{\mu_1\nu_1}H_{\mu_2\nu_2}H^\dagger_{\mu_3\nu_3}H_{\mu_4\nu_4} \qquad (4.16)
\end{aligned}$$

Needless to say, the Lagrangian in the broken phase does not have the Boulware Deser type ghost and is not $U(1)$ invariant.

Apparently, this looks that the system could contain one Nambu-Goldstone (NG) boson corresponding to the broken generator of $U(1)$ group. To study whether or not this expectation is right, we concentrate on the quadratic part of the Lagrangian,

$$\begin{aligned}
\mathcal{L}^{(2)}_{\rm BP} =& \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}H^\dagger_{\mu_2\nu_2}\partial_{\nu_1}H_{\mu_3\nu_3}\\
&+|m^2|\eta^{\mu_1\nu_1\mu_2\nu_2}H^\dagger_{\mu_1\nu_1}H_{\mu_2\nu_2}+\frac{|m^2|}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}H_{\mu_1\nu_1}H_{\mu_2\nu_2}+\frac{|m^2|}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}H^\dagger_{\mu_1\nu_1}H^\dagger_{\mu_2\nu_2}. \qquad (4.17)
\end{aligned}$$

Let us parametrize the field $H_{\mu\nu}$ in terms of two real fields $A_{\mu\nu}$ and $B_{\mu\nu}$ as in (4.5).

$$H_{\mu\nu}=\frac{1}{\sqrt{2}}\left(A_{\mu\nu}+iB_{\mu\nu}\right) \qquad (4.18)$$

Then, we find

$$\begin{aligned}
\mathcal{L}^{(2)}_{\rm BP} =&\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}H^\dagger_{\mu_2\nu_2}\partial_{\nu_1}H_{\mu_3\nu_3}\\
&+|m^2|\eta^{\mu_1\nu_1\mu_2\nu_2}H^\dagger_{\mu_1\nu_1}H_{\mu_2\nu_2}+\frac{|m^2|}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}H_{\mu_1\nu_1}H_{\mu_2\nu_2}+\frac{|m^2|}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}H^\dagger_{\mu_1\nu_1}H^\dagger_{\mu_2\nu_2}\\
=&\frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}A_{\mu_2\nu_2}\partial_{\nu_1}A_{\mu_3\nu_3}+\frac{1}{2}m_A^2\eta^{\mu_1\nu_1\mu_2\nu_2}A_{\mu_1\nu_1}A_{\mu_2\nu_2}\\
&+\frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}B_{\mu_2\nu_2}\partial_{\nu_1}B_{\mu_3\nu_3}. \qquad (4.19)
\end{aligned}$$

Here $m_A^2=2|m^2|$.

From Goldstone's theorem, the massless mode should correspond to the oscillation along the flat direction of the potential. Therefore, if the field $B_{\mu\nu}$ has a oscillating mode along the direction, the massless spin-two field is regarded as the NG field. To see whether this is the case, we need to investigate the flat direction of the potential. Since nontrivial, degenerated vacua in this $U(1)$ model are given by

$$h_{\mu\nu}^{\mathrm{VEV}} = \frac{C\mathrm{e}^{i\theta}}{\sqrt{2}}\eta_{\mu\nu}, \tag{4.20}$$

the infinitesimal difference between two vacua, which corresponds to the "flat direction", is given by

$$\delta h_{\mu\nu}^{\mathrm{VEV}} = \frac{i\theta}{\sqrt{2}}C\eta_{\mu\nu}. \tag{4.21}$$

This clearly means that $B_{\mu\nu}$ should contain a scalar mode if it is the NG field. The field $B_{\mu\nu}$, however, is traceless and does not have such a mode as long as the perturbative description (particle description) is assumed. Therefore, we find that the Nambu-Goldstone mode is absent. In addition to this fact, this model has nonderivative self-interaction terms for $B_{\mu\nu}$ although the field is interpreted as the massless spin-two field from the form of the quadratic Lagrangian. Thus, in the broken phase, the degree of freedom of the quadratic Lagrangian never coincides with the degree of freedom of the full Lagrangian.

$$\begin{aligned}\mathcal{L}_{\text{interactions}} =& \sqrt{\frac{\lambda}{24}}m_A\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}A_{\mu_1\nu_1}A_{\mu_2\nu_2}A_{\mu_3\nu_3} + \sqrt{\frac{\lambda}{24}}m_A\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}A_{\mu_1\nu_1}B_{\mu_2\nu_2}B_{\mu_3\nu_3}\\ &+ \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}B_{\mu_1\nu_1}B_{\mu_2\nu_2}B_{\mu_3\nu_3}B_{\mu_4\nu_4} + \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}A_{\mu_1\nu_1}A_{\mu_2\nu_2}A_{\mu_3\nu_3}A_{\mu_4\nu_4}\\ &+ \frac{\lambda}{3!\cdot 2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}A_{\mu_1\nu_1}A_{\mu_2\nu_2}B_{\mu_3\nu_3}B_{\mu_4\nu_4}\end{aligned}$$

This fact strongly suggests that this $U(1)$ model is not valid as an effective field theory as in the discussion of [35] and the perturbative picture assumed in the above analysis should break down. This is the reason why the system seems not to have the Nambu-Goldstone mode: In the broken phase, the Nambu-Goldstone mode would exist, but the model does not have enough power to describe the dynamics of the massless scalar particle as an effective field theory. This explanation is completely consistent with the statement that the NG boson is absent as long as the perturbative description is assumed.

In conclusion, the $U(1)$ model cannot be defined around the nontrivial vacua but is defined only around the trivial vacuum instead. This situation is quite different from the case of the neutral massive spin-two model (4.3) where the perturbative discription still holds in nontrivial vacua.

Chapter 5

Non-gravitational massive spin two particles III

To clarify the difference between the dRGT massive gravity and the new model we proposed, we also considered our model in a curved spacetime with the assumption that the spin-two field is not a deviation from some background metric and prove that the model with the new interactions is consistent only if the background spacetime has the maximal symmetry as in the case of the Fierz-Pauli theory in a curved spacetime [41, 42]. Furthermore, we introduce the general interactions allowed in the maximally symmetric spacetime.

5.1 Interactions for non-gravitational massive spin two particles

We start with the Lagrangian of the Fierz-Pauli theory on a flat spacetime:

$$\mathcal{L}_{\mathrm{FP}} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h\partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2)\,. \tag{5.1}$$

In four dimensions, we have seen that only three kinds of interactions, which do not change DOF of the system, exist:

$$\mathcal{L}_3^{\mathrm{d}} \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \tag{5.2}$$

$$\mathcal{L}_3 \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}\,, \tag{5.3}$$

$$\mathcal{L}_4 \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \tag{5.4}$$

The detailed property of $\eta^{\mu_1\nu_1\cdots\mu_n\nu_n}$ is summarized in Appendix D.

5.2 Lagrangian analysis

In this chapter, we consider the model where the massive spin-two field couples with gravity and count DOF by employing the lagrangian formalism as in [41].

First of all, we explain the procedure of the Lagrangian analysis. Let us consider the Lagrangian consisting of set of N fields ϕ^a, $a = 1, 2, \cdots$. If the equations of motion are only defined for $r < N$ fields, the remaining equations $N - r$ are regarded as primary constraints for the system. Then, as in the case of the Hamiltonian analysis, we impose the consistency condition for the time evolution on the primary constraints, which could define the second order time derivatives for the remaining fields whose dynamics is not determined from the equations of motion. We continue this manipulation until the dynamics of all fields are completely determined. As a result, all constrains are time-independent by definition and the dymamics of the system is completely fixed by N dynamical fields and costraints obtained through this procedure.

Since this process is a little bit complicated, as a warm up, we begin with counting of DOF on the flat space-time. First, just for simplicity, we only include the cubic interactions only. Thus, the Lagrangian we consider is given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\partial_{\mu_1}h_{\mu_2\nu_2}\partial_{\nu_1}h_{\mu_3\nu_3} + \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} - \frac{\mu}{3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} . \tag{5.5}$$

We find the equations of motion $E_{\mu\nu}$ by taking the variation with respect to $h_{\mu\nu}$:

$$0 = E_{\mu\nu} = -\eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}\partial^{\mu_1}\partial^{\nu_1}h^{\mu_2\nu_2} + m^2\eta_{\mu\nu\mu_1\nu_1}h^{\mu_1\nu_1} - \frac{\mu}{2}\eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}h^{\mu_1\nu_1}h^{\mu_2\nu_2} . \tag{5.6}$$

There are equations which contain the first order derivative with respect to time in (5.6), but do not have the second order derivative,

$$0 = E_{0\nu} = -\eta_{(0\nu)\mu_1\nu_1\mu_2\nu_2}\partial^{\mu_1}\partial^{\nu_1}h^{\mu_2\nu_2} + m^2\eta_{0\nu\mu_1\nu_1}h^{\mu_1\nu_1} - \frac{\mu}{2}\eta_{(0\nu)\mu_1\nu_1\mu_2\nu_2}h^{\mu_1\nu_1}h^{\mu_2\nu_2} . \tag{5.7}$$

Thanks to the antisymmetric property of the tensor $\eta_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}$, these equations do not include any term including the second order derivatives nor the first order derivatives of h_{00} with respect to time, which indicates $h_{0\mu}$ are regarded as auxiliary fields. The remaining equations in (5.6) have the second order derivative with respect to time,

$$0 = E_{ij} = \eta_{(ij)kl}\ddot{h}_{kl} + (\text{terms without } \ddot{h}) , \tag{5.8}$$

where we used the following identity,

$$\eta_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2\mu_3\nu_3} + \eta_{\mu_1\nu_2}\eta_{\mu_2\nu_3\mu_3\nu_1} + \eta_{\mu_1\nu_3}\eta_{\mu_2\nu_1\mu_3\nu_2} . \tag{5.9}$$

For later convenience, we now solve Eq. (5.8) in terms of $\ddot{h}_{ij}$. As the inverse of the coefficient matrix $A_{ij,kl} := \eta_{(ij)kl}$ in (5.8) is given by

$${A^{-1}}_{kl,mn} = -\eta_{m(k}\eta_{l)n} + \frac{1}{2}\eta_{kl}\eta_{mn} , \tag{5.10}$$

$\ddot{h}_{ij}$ can be written by using the terms which do not contain the second order derivative with respect to time.

$$0 = \left(-\eta_{m(i}\eta_{j)n} + \frac{1}{2}\eta_{ij}\eta_{mn}\right)E_{ij} = \ddot{h}_{mn} + (\text{terms without } \ddot{h}) \tag{5.11}$$

In order to count DOF of this system, we require the equations in (5.7), which are regarded as primary constraints, are consistent with the time evolution. Then, the original equations $E_{0\mu}$ get to become time-independent and are regarded as the "constraints" on the initial values.

$$0 \approx E_{0\nu} \equiv \phi^{(1)}_\nu . \tag{5.12}$$

Here $\approx$ means equivalence up to constraints. From the requirement $\dot{\phi}_\mu = 0$, we obtain

$$0 = -\dot{\phi}^{(1)}_\nu \approx m^2 \eta_{(\mu\nu)\mu_1\nu_1}\partial^\mu h^{\mu_1\nu_1} - \mu\eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}\partial^\mu h^{\mu_1\nu_1}\cdot h^{\mu_2\nu_2} = \partial^\mu E_{\mu\nu} \equiv \phi^{(2)}_\nu \,. \tag{5.13}$$

Here the constraints (5.7) and the equations (5.8) are used. The derivation of the above equation is a little bit cumbersome but if we use the equation

$$\partial^\mu E_{\mu\nu} = m^2 \eta_{(\mu\nu)\mu_1\nu_1}\partial^\mu h^{\mu_1\nu_1} - \mu\eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}\partial^\mu h^{\mu_1\nu_1}\cdot h^{\mu_2\nu_2} \tag{5.14}$$

from the beginning, we find

$$-E_{i\nu,i} + m^2 \eta_{(\mu\nu)\mu_1\nu_1}\partial^\mu h^{\mu_1\nu_1} - \mu\eta_{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}\partial^\mu h^{\mu_1\nu_1}\cdot h^{\mu_2\nu_2} = -\dot{E}_{0\nu} = -\dot{\phi}^{(1)}_\nu \,. \tag{5.15}$$

The terms $E_{i\nu,i}$ can be ignored up to (5.7) and (5.8). Thus, we obtain the functions $\phi^{(2)}_\nu$, which is identical with (5.13) without tedious calculations. Now, the primary constraints (5.7) are time-independent and hold all the time. However, since $\phi^{(2)}_\nu = 0$ is the equation including only the first order differential equation with respect to time, we also impose the consistency condition on $\phi^{(2)}_\nu$. Needless to say, we can directly calculate $\dot{\phi}^{(2)}{}_\nu$ and require $\dot{\phi}^{(2)}{}_\nu = 0$. However, we can easily obtain the expression by using

$$\begin{aligned}
0 &= \partial^\mu\partial^\nu E_{\mu\nu} + \frac{m^2}{2}\eta^{\mu\nu}E_{\mu\nu} - \mu h^{\mu\nu}E_{\mu\nu} \\
&= -\frac{3\mu m^2}{2}\eta_{\mu\nu\mu_1\nu_1}h^{\mu\nu}h^{\mu_1\nu_1} + \frac{\mu^2}{2}\eta_{\mu\nu\mu_1\nu_1\mu_2\nu_2}h^{\mu\nu}h^{\mu_1\nu_1}h^{\mu_2\nu_2} \\
&+ \frac{3m^4}{2}h - \mu\eta_{\mu\nu\mu_1\nu_1\mu_2\nu_2}\partial^\mu h^{\mu_1\nu_1}\partial^\nu h_{\mu_2\nu_2} \,.
\end{aligned} \tag{5.16}$$

Actually, we easily find

$$\begin{aligned}
0 \approx& -\dot{\phi}^{(2)}_0 = -\partial_0\partial^\mu E_{\mu 0} \\
\approx& \partial^\mu\partial^\nu E_{\mu\nu} + \frac{m^2}{2}\eta^{\mu\nu}E_{\mu\nu} - \mu h^{\mu\nu}E_{\mu\nu} \\
=& -\frac{3\mu m^2}{2}\eta_{\mu\nu\mu_1\nu_1}h^{\mu\nu}h^{\mu_1\nu_1} + \frac{\mu^2}{2}\eta_{\mu\nu\mu_1\nu_1\mu_2\nu_2}h^{\mu\nu}h^{\mu_1\nu_1}h^{\mu_2\nu_2} \\
&+ \frac{3m^4}{2}h - \mu\eta_{\mu\nu\mu_1\nu_1\mu_2\nu_2}\partial^\mu h^{\mu_1\nu_1}\partial^\nu h_{\mu_2\nu_2} \equiv \phi^{(3)} \,.
\end{aligned} \tag{5.17}$$

Notice that Eq. (5.17) does not contain any time-derivative of h_{00} and the second order time-derivative of h_{0i} and h_{ij}, which means that $\phi^{(3)}$ also works as a constraint.

On the other hand, from requirement of the conservation of the constraints $\phi^{(2)}_i$, we find the second order derivative equations for h_{0i} up to the equation (5.8),

$$\dot{\phi}^{(2)}_i = (m^2\eta_{ij} - \mu\eta_{ijkl}h_{kl})\ddot{h}_{0j} + (\text{terms without } \ddot{h}) = 0 \,. \tag{5.18}$$

Therefore, the dynamics of h_{0i} is determined by (5.18), which also guarantees that the constraints $\phi^{(2)}_i$ hold all the time. Actually, except the special configurations of fields where the matrix $M_{ij} = m^2\eta_{ij} - \mu\eta_{ijkl}h_{kl}$ has any vanishing eigenvalue, we can solve the equations (5.18) with respect to $\ddot{h}_{0i}$ as follows,

$$0 = \frac{1}{m^2}\left[\eta_{ij} + \sum_{n=1}^{\infty}(\mathbf{H}^n)_{ij}\right]\dot{\phi}^{(2)}_j = \ddot{h}_{0i} + \Big(\text{terms without } \ddot{h}\Big) \,. \tag{5.19}$$

Here, $(\mathbf{H}^n)_{ij}$ is defined by

$$(\mathbf{H}^n)_{ij} \equiv H_{ik_1} H_{k_1 k_2} \cdots H_{k_{n-1} j}\,, \quad H_{ij} \equiv \frac{\mu}{m^2} \eta_{ijkl} h_{kl}\,. \tag{5.20}$$

Now, let us consider the condition for the conservation of the constraint $\phi^{(3)}$. Because $\phi^{(3)}$ does not contain any derivative of h_{00} nor the second order derivative of h_{0i}, h_{ij} with respect to time, $\dot{\phi}^{(3)}$ is going to have the first order derivative of h_{00} and the second order derivative of h_{0i} and h_{ij} with respect to time. As we have seen, $\ddot{h}_{ij}$ and $\ddot{h}_{0i}$ can be eliminated by using Eqs. (5.11) and (5.19). Hence, we find one more constraint which does not contain the terms including the second order derivative with respect ot time,

$$\dot{\phi}^{(3)} = (\text{terms without } \ddot{h}) \equiv \phi^{(4)} \approx 0\,. \tag{5.21}$$

Though we do not give explicit form of this constraint, we can see that the consistency condition for the constraint (5.21) does not yield constraints any more , which can be found as follows. Focusing only on the linear terms, we found the consistency condition $\phi^{(4)}$ is given by

$$0 = \dot{\phi}^{(4)} = \frac{3m^4}{2} \ddot{h} + \mathcal{O}(h^2)\,. \tag{5.22}$$

We should stress that the first term cannot be eliminated by $\mathcal{O}(h^2)$ terms, which indicates that this equation defines the dynamics of h_{00}. Therefore, constraints obtained until now are all time-independent, which is ensured by the equations (5.18) and (5.22).

Finally, we have ten dynamical variables $h_{\mu\nu}$ and ten time-independent constraints $\phi^{(1)}_\mu$, $\phi^{(2)}_\mu$, $\phi^{(3)}$, $\phi^{(4)}$. As a result, the theory (5.5) has $(20 - 10)/2 = 5$ degrees of freedom on the flat space.

5.3 Pseudo-linear theory on curved space

Before the discussion of the new interactions on curved space-time, let us briefly review the Fierz-Pauli theory on curved space-time. In [41], Buchbinder *et al.* showed that the Fierz-Pauli theory on the non-trivial background require the non-minimal coupling terms and the maximally symmetric spacetime in order to keep the consistency if the action consists of finite terms. The action is given by

$$\begin{aligned} S = \int d^4x \sqrt{-g} \bigg\{ &\frac{1}{2} \nabla_\mu h \nabla^\mu h - \frac{1}{2} \nabla_\mu h_{\nu\rho} \nabla^\mu h^{\nu\rho} - \nabla^\mu h_{\mu\nu} \nabla^\nu h + \nabla_\mu h_{\nu\rho} \nabla^\rho h^{\nu\mu} \\ &+ \frac{m^2}{2} g^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} + \frac{\xi}{4} R h_{\alpha\beta} h^{\alpha\beta} + \frac{1-2\xi}{8} R h^2 \bigg\}\,, \end{aligned} \tag{5.23}$$

This suggests that these non-minimal coupling terms should be added when we consider the model consisting of the Fierz-Pauli Lagrangian and the new interctions $\mathcal{L}_3^{\rm d}$, $\mathcal{L}_3$, $\mathcal{L}_4$ on curved space-time. Now, let us concentrate on nonderivative interactions $\mathcal{L}_3$, $\mathcal{L}_4$ and consider the following model:

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\,\Big\{ & \frac{1}{2}\nabla_\mu h \nabla^\mu h - \frac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho} - \nabla^\mu h_{\mu\nu}\nabla^\nu h + \nabla_\mu h_{\nu\rho}\nabla^\rho h^{\nu\mu} \\
& + \frac{m^2}{2} g^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} + \frac{\xi}{4} R h_{\alpha\beta} h^{\alpha\beta} + \frac{1-2\xi}{8} R h^2 \\
& - \frac{\mu}{3!} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} - \frac{\lambda}{4!} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \Big\}\,. \qquad (5.24)
\end{aligned}
$$

Here the metric is chosen to be the Einstein manifold, where the curvatures satisfy the following condition:

$$
R_{\mu\nu} = \frac{R}{4} g_{\mu\nu}\,. \qquad (5.25)
$$

As a first step, we count DOF of the system ignoring the quartic potential $\lambda = 0$.

$$
\begin{aligned}
0 = E_{\mu\nu} &= g^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{\mu\nu} - g_{\mu\nu} g^{\alpha\beta} g^{\gamma\delta}\nabla_\alpha\nabla_\beta h_{\gamma\delta} + g_{\mu\nu} g^{\alpha\gamma} g^{\beta\delta}\nabla_\alpha\nabla_\beta h_{\gamma\delta} - 2 g^{\sigma\rho}\nabla_\sigma\nabla_{(\mu} h_{\nu)\rho} \\
&\quad + g^{\alpha\beta}\nabla_\mu\nabla_\nu h_{\alpha\beta} + m^2 g_{(\mu\nu)}{}^{\alpha\beta} h_{\alpha\beta} + \frac{\xi}{2} R h_{\mu\nu} + \frac{1-2\xi}{4} R g^{\alpha\beta} g_{\mu\nu} h_{\alpha\beta} - \frac{\mu}{2} g_{(\mu\nu)}{}^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} \\
&= -g_{(\mu\nu)}{}^{\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}\nabla_{\nu_1} h_{\mu_2\nu_2} + (\text{terms without } \nabla\nabla h) \\
&= -g_{i(\mu} g_{\nu)j} g^{ij00\mu_2\nu_2}\nabla_0\nabla_0 h_{\mu_2\nu_2} + (\text{terms without } \nabla_0\nabla_0 h)\,. \qquad (5.26)
\end{aligned}
$$

The equations which do not include $\nabla_0\nabla_0 h$ (or $\partial_0\partial_0 h$) is considered as constraints as in the previous section. Unlike the case in a flat spacetime, however, $E_{0\mu}$ contain the second order derivative terms with respect to time. Thus, we consider the linear combinations of $E_{\mu\nu}$ as follows,

$$
\begin{aligned}
E^0{}_\nu &= g^{00} E_{0\nu} + g^{0i} E_{i\nu} \\
&= -g_{\nu\sigma} g^{(0\sigma)\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}\nabla_{\nu_1} h_{\mu_2\nu_2} + (\text{terms without } \nabla\nabla h) \\
&= (\text{terms without } \nabla_0\nabla_0 h) \equiv \phi^{(1)}_\nu \approx 0\,, \qquad (5.27)
\end{aligned}
$$

which enables us to regard $\phi^{(1)}_\nu \equiv E^0{}_\nu$ as primary constraints. Then spatial components of Eq. (5.26) have the following forms:

$$
0 = E_{ij} = -g_{m(i} g_{j)n} g^{mn00kl}\nabla_0\nabla_0 h_{kl} + (\text{terms without } \nabla_0\nabla_0 h)\,. \qquad (5.28)
$$

In order to solve Eq. (5.28) in terms of $\nabla_0\nabla_0 h_{ij}$, we use the ADM variables defiend as

$$
g^{00} = -\frac{1}{N^2}\,, \quad g_{0k} = N_k\,, \quad g_{ij} = e_{ij}\,, g_{00} = N^k N_k - N^2\,, \qquad g^{0i} = \frac{N^i}{N^2}\,, \quad g^{ij} = e^{ij} - \frac{N^i N^j}{N^2}\,.
$$

Here e_{ij} is a three dimensional metric field and has the following properties,

$$
e^{ij} e_{ij} = \delta^i{}_j\,, \quad N^i \equiv e^{ij} N_j\,, \quad e^{ij} = g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}}\,. \qquad (5.29)
$$

By using the ADM varibles, the coefficient matrix in equations (5.28) can be expressed as (see (D.7) in AppendixD),

$$
A_{ij}{}^{,kl} \equiv -g_{m(i} g_{j)n} g^{mn00kl} = \frac{1}{N^2} e_{(ij)}{}^{kl}\,. \qquad (5.30)
$$

Here

$$e_{i_1j_1i_2j_2\cdots i_nj_n} \equiv e_{i_1j_1}e_{i_2j_2}\cdots e_{i_nj_n} - e_{i_1j_2}e_{i_2j_1}\cdots e_{i_nj_n} + \cdots \,. \tag{5.31}$$

Note that the indices in $e_{i_1j_1\cdots i_nj_n}$ are raised or lowered by e^{ij} and e_{ij}. The inverse of the matrix (5.30) is expressed as

$$\begin{aligned} {A^{-1}}_{kl}{}^{,mn} &= N^2\left(\frac{1}{2}e_{kl}e^{mn} - e_{(k}{}^{m}e_{l)}{}^{n}\right) = \frac{1}{g^{00}}\left\{g_{(k}{}^{m}g_{l)}{}^{n} - \frac{1}{2}g_{kl}(g^{mn} - \frac{g^{0m}g^{0n}}{g^{00}})\right\} \,, \\ A_{ij}{}^{,kl}{A^{-1}}_{kl}{}^{,mn} &= \delta^k_{(i}\delta^l_{j)} \,. \end{aligned} \tag{5.32}$$

Then, Eq. (5.28) can be solved in terms of $\nabla_0\nabla_0 h_{ij}$ as follows:

$$0 = \frac{1}{g^{00}}\left\{g_{(k}{}^{i}g_{l)}{}^{j} - \frac{1}{2}g_{kl}\left(g^{ij} - \frac{g^{0i}g^{0j}}{g^{00}}\right)\right\}E_{ij} = \nabla_0\nabla_0 h_{ij} + (\text{terms without } \nabla_0\nabla_0 h) \,. \tag{5.33}$$

Because Eq. (5.28) gives 6 independent equations containing the second order time-derivative and are also independent of the primary constraints $\phi^{(1)}_\nu$, (5.28) describe the dynamics of h_{ij}. In order to obtain the consitency conditions for the primary constraints easily, we use the following relations:

$$\begin{aligned} \nabla^\mu E_{\mu\nu} =& \frac{R}{4}g^{\alpha\beta}\nabla_\nu h_{\alpha\beta} - \frac{R}{2}g^{\sigma\rho}\nabla_\sigma h_{\rho\nu} + m^2 g_{\nu\nu_1}g^{\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}h_{\mu_2\nu_2} \\ &+ \frac{\xi}{2}Rg^{\sigma\rho}\nabla_\sigma h_{\rho\nu} + \frac{1-2\xi}{4}Rg^{\alpha\beta}\nabla_\nu h_{\alpha\beta} - \mu g_{\nu\nu_1}g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3}\left(\nabla_{\mu_1}h_{\mu_1\nu_1}\right)h_{\mu_2\nu_2} \\ =& \left(\frac{1-\xi}{2}R + m^2\right)g_{\nu\nu_1}g^{\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}h_{\mu_2\nu_2} - \mu g_{\nu\nu_1}g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3}\left(\nabla_{\mu_1}h_{\mu_2\nu_2}\right)h_{\mu_3\nu_3} \,. \end{aligned} \tag{5.34}$$

Then the secondary constraints are obtained as

$$\partial_0\phi^{(1)}_\nu = \partial_0 E^0{}_\nu \approx \nabla^\mu E_{\mu\nu} \equiv \phi^{(2)}_\nu \approx 0 \,. \tag{5.35}$$

For convenience, we choose independent constraints as follows,

$$\phi^{(2)0} \equiv g^{00}\phi^{(2)}_0 + g^{0i}\phi^{(2)}_i \approx 0 \,, \quad \phi^{(2)}_i \approx 0 \,. \tag{5.36}$$

Furthermore, by using the following relation:

$$\begin{aligned} &\nabla^\mu\nabla^\nu E_{\mu\nu} + \frac{m^2}{2}g^{\mu\nu}E_{\mu\nu} - \mu h^{\mu\nu}E_{\mu\nu} + \frac{1-\xi}{4}Rg^{\mu\nu}E_{\mu\nu} \\ =& h\left(\frac{3m^4}{2} + \frac{5-6\xi}{4}m^2 R + \frac{(1-\xi)(2-3\xi)}{8}R^2\right) \\ &- \frac{3\mu m^2}{2}g^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} - \mu g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\left(\nabla_{\mu_1}h_{\mu_2\nu_2}\right)\nabla_{\nu_1}h_{\mu_3\nu_3} \\ &+ \frac{\mu^2}{2}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} - \frac{7-9\xi}{12}\mu Rg^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} - \mu C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta} \,, \end{aligned} \tag{5.37}$$

we find one more constraint:

$$\partial_0\phi^{(2)0} \approx \nabla^\mu\nabla^\nu E_{\mu\nu} + \frac{m^2}{2}g^{\mu\nu}E_{\mu\nu} - \mu h^{\mu\nu}E_{\mu\nu} + \frac{1-\xi}{4}Rg^{\mu\nu}E_{\mu\nu} \equiv \phi^{(3)} \approx 0 \,. \tag{5.38}$$

We have to notice that the non-minimal coupling terms in (5.37) are crucial for the existence of the constraint $\phi^{(3)}$. Since the term $\frac{R}{4}g^{\alpha\beta}\nabla_\nu h_{\alpha\beta} - \frac{R}{2}g^{\sigma\rho}\nabla_\sigma h_{\rho\nu}$ in (5.37) contains the derivatives of h_{00} with respect to time, if there was no such a term, the system would not have the appropriate number of constraints. The contribution from the non-minimal couplings, however, eliminates these dangerous terms and makes the system have five DOF.

On the other hand, contrary to the situation of the kinetic part, the term accompanied by the curvature tensor never appears from the potential in $\nabla^\mu E_{\mu\nu}$, which leads to the fact that $\nabla^\mu E_{\mu\nu}$ does not contain any derivative of h_{00} with respect to time. Needless to say, the time-derivative of h_{00} appears when another covariant derivative is acted on $\nabla^\mu E_{\mu\nu}$, but this term is eliminated by the term $h^{\mu\nu}E_{\mu\nu}$ in (5.37). This indicates that additional non-minimal coupling terms is not required for the full action to have five DOF.

We have to note that a new non-minimal coupling term $Rg^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}$ can be added to the system without additional DOF. This fact, however, does not change the following analysis because the Ricci scalar takes constant values in the Einstein manifold and we can eliminate the effect by redefining the coupling constant μ.

By using the ADM decomposition, the conditions for the conservation of $\phi_i^{(2)}$ have the following forms:

$$\begin{aligned}
&\partial_0\phi_i^{(2)} \approx \nabla_0\nabla^\mu E_{\mu i} = B_i{}^j\nabla_0\nabla_0 h_{0j} + C_i{}^{kl}\nabla_0\nabla_0 h_{kl} + (\text{terms without } \nabla_0\nabla_0 h) = 0\,,\\
&B_i{}^j \equiv \frac{1}{N^2}\left[\left(\frac{1-\xi}{2}R + m^2\right)\delta_i{}^j - \mu e^j{}_i{}^{mn}h_{mn}\right]\,,\\
&C_i{}^{kl} \equiv -\frac{1}{N^2}\left[N^k\delta_i{}^l - \mu\left\{e^{kl}{}_i{}^j h_{0j} - N^k e_i{}^{lmn}h_{mn} - N^m e_i{}^{nkl}h_{mn}\right\}\right]\,,
\end{aligned}\tag{5.39}$$

where (D.6) and (D.7) have been used. Then $\nabla_0\nabla_0 h_{kl}$ can be eliminated from the first equation in (5.39) by using Eq. (5.33).

$$\partial_0\phi_i^{(2)} \approx \nabla_0\nabla^\mu E_{\mu i} - C_i{}^{kl}A^{-1}{}_{kl}{}^{,mn}E_{mn} = B_i{}^a\nabla_0\nabla_0 h_{0a} + (\text{terms without}\nabla_0\nabla_0 h) = 0 \tag{5.40}$$

Now, it is obvious that equation the (5.40) describes the dynamics of h_{0i} so that the constraints $\phi_i^{(2)}$ are conserved. Except the special case, the equations in (5.40) are solved in terms of $\nabla_0\nabla_0 h_{0i}$ as follows:

$$\begin{aligned}
&B^{-1}{}_k{}^i\left[\nabla_0\nabla^\mu E_{\mu i} - C_i{}^{kl}A^{-1}{}_{kl}{}^{,mn}E_{mn}\right] = \nabla_0\nabla_0 h_{0k} + (\text{terms without } \nabla_0\nabla_0 h) = 0\,,\\
&B^{-1}{}_i{}^j = \frac{N^2}{\frac{1-\xi}{2}R + m^2}\left[\delta_i{}^j + \sum_{n=1}^{\infty}(\mathbf{H}^n)_i{}^j\right]\,,\\
&(\mathbf{H}^n)_i{}^j \equiv H_{ik_1}e^{k_1l_1}H_{l_1k_2}e^{k_2l_2}\cdots H_{l_{n-1}k_n}e^{k_nj}\,,\quad H_{ij} \equiv \frac{\mu}{\frac{1-\xi}{2}R + m^2}e_{ij}{}^{mn}h_{mn}\,.
\end{aligned}\tag{5.41}$$

As in the case of the the flat spacetime, the constraint obtained from the consistency condition of $\phi^{(3)}$ has the following form:

$$\partial_0\phi^{(3)} \approx (\text{terms without } \nabla_0\nabla_0 h) \equiv \phi^{(4)} \approx 0 \tag{5.42}$$

and the linear terms of the consistency condition $\dot{\phi}^{(4)}$ defines the dynamics of h_{00}. As a result, the pseudo-linear theory described by action (5.24) with $\lambda = 0$ has five DOF on the Einstein manifold.

5.4 $\lambda \neq 0$ case

We now investigate more general case where λ is not zero and reveal whether the discussion for $\lambda = 0$ can be applied.

Because of the quartic coupling, the equations are changed as follows,

$$\begin{aligned} 0 = E_{\mu\nu} =& g^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{\mu\nu} - g_{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\nabla_\alpha\nabla_\beta h_{\gamma\delta} + g_{\mu\nu}g^{\alpha\gamma}g^{\beta\delta}\nabla_\alpha\nabla_\beta h_{\gamma\delta} - 2g^{\sigma\rho}\nabla_\sigma\nabla_{(\mu}h_{\nu)\rho} \\ &+ g^{\alpha\beta}\nabla_\mu\nabla_\nu h_{\alpha\beta} + m^2 g_{(\mu\nu)}{}^{\alpha\beta}h_{\alpha\beta} + \frac{\xi}{2}Rh_{\mu\nu} + \frac{1-2\xi}{4}Rg^{\alpha\beta}g_{\mu\nu}h_{\alpha\beta} \\ &- \frac{\mu}{2}g_{(\mu\nu)}{}^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} - \frac{\lambda}{3!}g_{(\mu\nu)}{}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}\,. \end{aligned} \tag{5.43}$$

The primary condition takes the following form:

$$E^0{}_\nu \equiv \phi^{(1)}_\nu \approx 0\,. \tag{5.44}$$

By using the conservation of the constraint $\phi^{(1)}_\nu$, the secondary constraints are also found:

$$\begin{aligned} \nabla^\mu E_{\mu\nu} =& \left(\frac{1-\xi}{2}R + m^2\right) g_{\nu\nu_1}g^{\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}h_{\mu_2\nu_2} - \mu g_{\nu\nu_1}g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3}\left(\nabla_{\mu_1}h_{\mu_2\nu_2}\right)h_{\mu_3\nu_3} \\ &- \frac{\lambda}{2}g_{\nu\nu_1}g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\left(\nabla_{\mu_1}h_{\mu_2\nu_2}\right)h_{\mu_3\nu_3}h_{\mu_4\nu_4} \equiv \phi^{(2)}_\nu \approx 0\,. \end{aligned} \tag{5.45}$$

The consistency condition for the constraints $\phi^{(2)}_i$ yields three equations determining the dynamics of h_{0i}. The explicit form is given as

$$\begin{aligned} &B_i{}^j\nabla_0\nabla_0 h_{j0} + (\text{terms without } \nabla_0\nabla_0 h) = 0\,, \\ &B_i{}^j \equiv \frac{1}{N^2}\left[\left(\frac{1-\xi}{2}R + m^2\right)\delta_i{}^j - \mu e^j{}_i{}^{mn}h_{mn} - \frac{\lambda}{2}e_{ii_1}e^{(i_1j)i_2j_2i_3j_3}h_{i_2j_2}h_{i_3j_3}\right]\,. \end{aligned} \tag{5.46}$$

The matirix $B_i{}^j$, can be eliminated thanks to the existence of the inverse matrix:

$$\begin{aligned} &B^{-1}{}_i{}^j = \frac{N^2}{\frac{1-\xi}{2}R + m^2}\left[\delta_i{}^j + \sum_{n=1}^{\infty}\left(\mathbf{H}^n\right)_i{}^j\right]\,, \\ &\left(\mathbf{H}^n\right)_i{}^j \equiv H_{ik_1}e^{k_1l_1}H_{l_1k_2}e^{k_2l_2}\cdots H_{l_{n-1}k_n}e^{k_nj}\,, \\ &H_{ij} \equiv \frac{1}{\frac{1-\xi}{2}R + m^2}\left[\mu e_{ij}{}^{mn}h_{mn} + \frac{\lambda}{2}e_{(ij)}{}^{klmn}h_{kl}h_{mn}\right]\,. \end{aligned} \tag{5.47}$$

Moreover, the conservation of $\phi^{(2)\,0}$ yields a constraint:

$$\begin{aligned} &\nabla^\mu\nabla^\nu E_{\mu\nu} + \frac{m^2}{2}g^{\mu\nu}E_{\mu\nu} + \frac{1-\xi}{4}Rg^{\mu\nu}E_{\mu\nu} - \mu h^{\mu\nu}E_{\mu\nu} + \frac{\lambda}{2}g^{00ijklmn}A^{-1,ab}_{ij}E_{ab}h_{kl}h_{mn} \\ &= -\mu g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3}\left(\nabla_{\mu_1}h_{\mu_2\nu_2}\right)\nabla_{\nu_1}h_{\mu_3\nu_3} - \lambda g^{(\mu_1\nu_1)\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\left(\nabla_{\mu_1}h_{\mu_2\nu_2}\right)\left(\nabla_{\nu_1}h_{\mu_3\nu_3}\right)h_{\mu_4\nu_4} \\ &\quad - \lambda g^{(0i)\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\left(\nabla_0\nabla_i h_{\mu_2\nu_2}\right)h_{\mu_3\nu_3}h_{\mu_4\nu_4} + \frac{\lambda}{2}g^{00ijklmn}A^{-1,ab}_{ij}\left(-2g_{ab}{}^{(0c)\mu\nu}\nabla_0\nabla_c h_{\mu\nu}\right)h_{kl}h_{mn} \\ &\quad + (\text{terms without any time derivatives of } h) \equiv \phi^{(3)} \approx 0\,. \end{aligned} \tag{5.48}$$

Here $A^{-1,kl}_{ij}$ is defined by (5.32). By using the expression (5.48), it turns out that the derivative of $\phi^{(3)}$ with respect to time does not have the second order time-derivatives of h_{00} while the second order time-derivatives of h_{0i} and h_{ij} are present. As the second order derivatives of h_{0i} and h_{ij} with respect to time are eliminated, there emerges one more constraint:

$$\phi^{(4)} \approx 0\,. \tag{5.49}$$

Therefore, the model with $\lambda \neq 0$ actually has five DOF on the Einstein manifold.

5.5 A new non-minimal coupling term

In [41], in order to eliminate a ghost mode appearing at the quatratic level in $h_{\mu\nu}$, non-minimal coupling terms had to be added. In this section, we show the exsitence of another non-minimal coupling which does not induce an extra degree of freedom. We should emphasize that the constraint $\phi^{(3)}$ plays a crucial role in the elimination of the extra DOF.

Let us begin with the quadratic action to be more general form than that in [41] on the Einstein manifold:

$$\begin{aligned} S = \int d^4x\sqrt{-g}\,\bigg[&\frac{1}{2}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\nabla_{\mu_1}h_{\mu_2\nu_2}\nabla_{\nu_1}h_{\mu_3\nu_3} + \frac{m^2}{2}g^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} \\ &+\frac{\alpha}{2}Rh^2 + \frac{\beta}{2}Rh_{\mu\nu}h^{\mu\nu} + \frac{\gamma}{2}C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}\bigg]\,, \end{aligned} \tag{5.50}$$

and find the parameters which keeps five DOF. Note that the kinetic term in (5.50) is not identical with that in (5.23) due to the non-commutativity of the covariant derivatives. That is, the first term in (5.50) is expanded as follows:

$$\begin{aligned} \frac{1}{2}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\nabla_{\mu_1}h_{\mu_2\nu_2}\nabla_{\nu_1}h_{\mu_3\nu_3} &= \frac{1}{2}\nabla_\mu h\nabla^\mu h - \frac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho} - \nabla^\mu h_{\mu\nu}\nabla^\nu h + \nabla_\mu h_{\nu\rho}\nabla^\rho h^{\nu\mu} \\ &+ \frac{R}{4}h_{\alpha\beta}h^{\alpha\beta} - \frac{R}{8}h^2 - \frac{1}{2}C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta} + \frac{R}{12}g^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} \end{aligned} \tag{5.51}$$

Here the following relation which holds on the Einstein manifold (5.25),

$$R_{\mu\alpha\nu\beta} = C_{\mu\alpha\nu\beta} + \frac{R}{12}g_{\mu\nu\alpha\beta} \tag{5.52}$$

has been used. Here $C_{\mu\alpha\nu\beta}$ denotes the Weyl tensor. Thus, we have to subtract the terms accompanied by the curature tensor when we use $g^{\mu_1\nu_1\mu_2\nu_2\cdots}$ notation to express the kinetic term.

The contribution to the equation from the kinetic terms in the action (5.50) is given by

$$E_K^{\mu\nu} \equiv -g^{(\mu\nu)\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2} \quad = -g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2} + \frac{1}{2}g^{[\mu\nu]\mu_1\nu_1\mu_2\nu_2}R_{\nu_1\ \ \mu_1\mu_2}^{\ \ \sigma}h_{\sigma\nu_2}\,.$$

Here the identity

$$R_{\lambda\alpha\beta\gamma} + R_{\lambda\beta\gamma\alpha} + R_{\lambda\gamma\alpha\beta} = 0\,. \tag{5.53}$$

has been used. By applying (5.53) and the following identities to $E_K^{\mu\nu}$,

$$\begin{aligned} g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}C_{\nu_1\ \ \mu_1\mu_2}^{\ \ \sigma} &= (g^{\mu_1\nu_1}g^{\mu\nu\mu_2\nu_2} + g^{\mu\nu_1}g^{\mu_2\nu\mu_1\nu_2} + g^{\mu_2\nu_1}g^{\mu_1\nu\mu\nu_2})C_{\nu_1\ \ \mu_1\mu_2}^{\ \ \sigma} \\ &= 2C^{\mu\sigma\nu_2\nu}\,, g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}g_{\nu_1\mu_1\ \ \mu_2}^{\ \ \ \ \ \sigma} \\ &= 2g^{\mu\nu\mu_1\nu_1\mu_2\nu_2}g_{\nu_1\mu_1}g^{\sigma}_{\ \ \mu_2} = 4g^{\mu\nu\sigma\nu_2}\,, \end{aligned} \tag{5.54}$$

we find that there is a symmetry with respect of the exchange of the indices μ and ν. Hence, the last term in $E_K^{\mu\nu}$ vanishes. Then we find the following expression,

$$\begin{aligned}\nabla_\mu E_K^{\mu\nu} &= \frac{1}{2} g^{\mu\nu\mu_1\nu_1\mu_2\nu_2} R^\sigma_{\nu_1\ \ \mu_1\mu} \left[\nabla_\sigma h_{\mu_2\nu_2} - \nabla_{\nu_2} h_{\mu_2\sigma}\right] \\ &= -C^{\mu\alpha\nu\beta}\nabla_\mu h_{\alpha\beta} + \frac{R}{6} g^{\mu\nu\alpha\beta}\nabla_\mu h_{\alpha\beta} \qquad (5.55)\end{aligned}$$

where in the first line, we have applied $g^{\mu\nu\mu_1\nu_1\mu_2\nu_2} R_{\mu_2\ \ \mu\mu_1}^{\ \ \sigma} = 0$, which is obtained from (5.53), and in the second line, we have used (5.54). Let us remind here that there exists the constraint $\phi^{(3)}$ if $\nabla_\mu E^{\mu\nu}$ does not contain the time-derivative of h_{00} in the previous sections although this is not the sufficient condition for DOF to be kept.

From (5.55), we find that condition for the existence of $\phi^{(3)}$ is satisfied in $\nabla_\mu E_K^{\mu\nu}$ and also satisfied in the mass terms. Thus, the model does not have ghost even if we set all parameters zero.

On the other hand, we may add extra terms with non-minimal coupling which do not induce ghost. This extra term really exists if we choose $\beta = -\alpha$,

$$\begin{aligned}S = \int d^4x\sqrt{-g}\,\Big[&\frac{1}{2} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\nabla_{\mu_1} h_{\mu_2\nu_2}\nabla_{\nu_1} h_{\mu_3\nu_3} + \frac{m^2}{2} g^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} \\ &+ \frac{\alpha}{2} R g^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} + \frac{\gamma}{2} C^{\mu\alpha\nu\beta} h_{\mu\nu} h_{\alpha\beta}\Big] \,. \qquad (5.56)\end{aligned}$$

On the Einstein manifold, since R is constant, the terms which are proportional to α can be absorbed into the mass terms by redefinition, which means that this term is irrelevant to the extra DOF. Furthermore, the term proportional to γ change only the coefficient of the first term of the second line in (5.55) and therefore this term does not induce the ghost.

Finally we mention the relation between (5.56) and the non-minimal coupling in [41], which is given by

$$\frac{\xi}{4} R h_{\alpha\beta} h^{\alpha\beta} + \frac{1-2\xi}{8} R h^2 = \frac{R}{4} h_{\alpha\beta} h^{\alpha\beta} + \frac{R}{8} h^2 - \frac{\xi - 1}{4} R g^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} \,. \qquad (5.57)$$

The first two terms contributes as the shift of the mass. By comparing (5.57) with the non-minimal coupling terms (5.50) and (5.51), we find that (5.57) corresponds to the case that $\gamma = 1$ in (5.50). Thus, in general, we can add the following non-minimal coupling,

$$\frac{\gamma}{2} C^{\mu\alpha\nu\beta} h_{\mu\nu} h_{\alpha\beta} \,. \qquad (5.58)$$

This term vanish on the (anti-)de Sitter space-time, which is conformally flat, but this term gives non-trivial contribution on the Schwarzchild (anti-)de Sitter spce-time, etc.

5.6 Derivative interaction

Until now, we have not consider the derivative interaction $\mathcal{L}_3^d$. Thus, in this section, we study if the derivative interaction in a flat spacetime

$$l\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\partial_{\mu_1}\partial_{\nu_1} h_{\mu_2\nu_2} \cdot h_{\mu_3\nu_3} h_{\mu_4\nu_4} \,. \qquad (5.59)$$

still keeps DOF.

Including the derivative terms always generate the terms proportional to the curvature in the constraint $\phi^{(2)\nu}$ because the covariant derivatives are not commutative. Among these terms, there usually appear the terms containing the time-derivative of h_{00} which we need to cancel by including additional terms with non-minimal coupling to the action.

Unfortunately, since types of the non-minimal couplings are constrained, it is not trivial if the terms including the time-derivative of h_{00} are eliminated. Acutally, we fail to cancel the terms.

To focus on the effect of the derivative interaction term,

$$l g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2}\cdot h_{\mu_3\nu_3}h_{\mu_4\nu_4}\,, \tag{5.60}$$

we pick up the contribution of (5.60) to the equations of motion, which is shown as

$$E_D^{\mu\nu} \equiv 2l g^{(\mu\nu)\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2}\cdot h_{\mu_3\nu_3} + l g^{(\mu\nu)\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\nabla_{\mu_1}h_{\mu_2\nu_2}\cdot\nabla_{\nu_1}h_{\mu_3\nu_3}\,. \tag{5.61}$$

In the expression of (5.61), the terms containing the time-derivative of h_{00} are as follows:

$$\nabla_\nu E_D^{\mu\nu} \supset l g^{\mu\nu\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\left(-\frac{1}{2}\nabla_{\nu_2}h_{\mu_2\sigma} + \nabla_\sigma h_{\mu_2\nu_2} - \nabla_{\mu_2}h_{\sigma\nu_2}\right)R_{\mu_1\ \ \nu\nu_1}^{\ \ \sigma}h_{\mu_3\nu_3}\,. \tag{5.62}$$

In the first term in the parentheses $(\cdots)$, we have used (5.53). We now have following identities,

$$\begin{aligned}
g^{\mu\nu\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}C_{\mu_1\ \ \nu\nu_1}^{\ \ \sigma} &= -6C^{\nu_2\sigma(\mu\mu_2}g^{\mu_3)\nu_3} - 6C^{\nu_3\sigma(\mu\mu_3}g^{\mu_2)\nu_2}\,,\\
g^{\mu\nu\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}g_{\mu_1\nu\ \ \nu_1}^{\ \ \ \ \sigma} &= -2g^{\mu\sigma\mu_2\nu_2\mu_3\nu_3}\,,
\end{aligned} \tag{5.63}$$

In the first equation of (5.63), the parentheses $(\cdots)$ for upper indecies does not denote the symmetrization but summing up by changing the indices in cyclic way, for example,

$$T_{(\alpha\beta\gamma)} \equiv \frac{1}{3}\left(T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta}\right)\,. \tag{5.64}$$

Substituting (5.63) into (5.62) and, then, using (5.52), it turns out that the terms proportional to $g^{\mu\sigma\mu_2\nu_2\mu_3\nu_3}$ appear and do not include any time-derivative of h_{00}. On the other hand, the terms with the Weyl tensor $C^{\mu\nu\alpha\beta}$ have the time-derivative of h_{00}:

$$\nabla_\nu E_D^{\mu\nu} \supset l\Big\{-C^{\mu\alpha0\beta}g^{00} + C^{\alpha0\beta0}g^{\mu0} + C^{\mu00\alpha}g^{\beta0}\Big\}h_{\alpha\beta}\nabla_0 h_{00}\,, \tag{5.65}$$

which means that we need to eliminate (5.65) by adding the terms with the non-minimal couplings. The general candidate which could cancel out the above contribution (5.65) is

$$c_1 C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}h + c_2 C^{\mu\alpha\nu\beta}h_{\mu\nu}h_\alpha^{\ \lambda}h_{\lambda\beta}\,. \tag{5.66}$$

Then the contribution to $\nabla_\nu E^{\mu\nu}$ from the term (5.66) are given by

$$\begin{aligned}
\nabla_\mu E^{\mu\nu} \supset &\left\{(2c_1+c_2)C^{\mu\alpha0\beta}g^{00} + (2c_1+c_2)\,C^{0\alpha0\beta}g^{\mu0}\right\}h_{\alpha\beta}\nabla_0 h_{00}\\
&+ (\text{terms not including } \nabla_0 h_{00})\,,
\end{aligned} \tag{5.67}$$

which tells that there cannot be cancellation unfortunately. Therefore at least in the present formulation, the derivative interaction (5.60) inevitably introduces the extra DOF.

We should notice, however, that on the conformally flat space-time, where $C^{\mu\alpha\nu\beta} = 0$ holds, Eq. (5.62) contributes to $\nabla_\nu E^{\mu\nu}$ as follows:

$$\nabla_\nu E_D^{\mu\nu} \supset -\frac{R}{12}g^{\mu\nu\mu_2\nu_2\mu_3\nu_3}\nabla_\nu h_{\mu_2\nu_2}h_{\mu_3\nu_3}\,, \tag{5.68}$$

which clearly shows that the term (5.60) does keep DOF because Eq. (5.68) does not include $\nabla_0 h_{00}$.

5.7 Various non-minimal couplings

Lastly, we introduce many kinds of non-minimal couplings which actually does keep DOF of the dynamical system. The most trivial ones are constructed from the Ricci scalar and nonderivative terms in $h_{\mu\nu}$:

$$R^m g^{\mu_1\nu_1\cdots\mu_n\nu_n} h_{\mu_1\nu_1}\cdots h_{\mu_n\nu_n}\,. \tag{5.69}$$

This is because the Ricci scalar is constant on the Einstein manifold and potential terms in $h_{\mu\nu}$ do not violate the constraints.

More nontrivial terms are obtained from the Weyl tensor. Let us remind the term presented in (5.66)). As it is clear from (5.67), if we choose $c_2 = -2c_1$, the non-minimal coupling (5.66) does not change DOF of the system:

$$C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}h - 2C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\lambda}h^{\lambda}{}_{\beta}. \tag{5.70}$$

Rewriting this expression as

$$\begin{aligned}
&C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}h - 2C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\lambda}h^{\lambda}{}_{\beta}\\
&\quad= \left(C^{\mu_1\mu_2\nu_1\nu_2}g^{\mu_3\nu_3} + C^{\mu_1\mu_2\nu_2\nu_3}g^{\mu_3\nu_1} + C^{\mu_1\mu_2\nu_3\nu_1}g^{\mu_3\nu_2}\right) h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}\\
&\quad= \frac{1}{2\cdot 3!}\delta^{\mu_1}{}_{\rho_1}{}^{\mu_2}{}_{\rho_2}{}^{\mu_3}{}_{\rho_3}\delta^{\nu_1}{}_{\sigma_1}{}^{\nu_2}{}_{\sigma_2}{}^{\nu_3}{}_{\sigma_3}C^{\rho_1\rho_2\sigma_1\sigma_2}g^{\rho_3\sigma_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}\,.
\end{aligned} \tag{5.71}$$

gives us a clue to constructing general non-minimal interaction. In fact, the tensor $\delta^{\nu_1}{}_{\lambda_1}{}^{\nu_2}{}_{\lambda_2}{}^{\nu_3}{}_{\lambda_3}C^{\mu_1\mu_2\lambda_1\lambda_2}g^{\mu}$ has a similar structure to $g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}$. Thus, extending this expression, we obtain

$$\begin{aligned}
&\delta^{\mu_1}{}_{\rho_1}{}^{\mu_2\cdots\mu_{n+2}}{}_{\rho_2\cdots\rho_{n+2}}\delta^{\nu_1}{}_{\sigma_1}{}^{\nu_2\cdots\nu_{n+2}}{}_{\sigma_2\cdots\sigma_{n+2}}C^{\rho_1\rho_2\sigma_1\sigma_2}g^{\rho_3\sigma_3}\cdots g^{\rho_{n+2}\sigma_{n+2}}\\
&\sim \delta^{\mu_1}{}_{\rho_1}{}^{\mu_2\cdots\mu_{n+2}}{}_{\rho_2\cdots\rho_{n+2}}\delta^{\nu_1}{}_{\sigma_1}{}^{\nu_2\cdots\nu_{n+2}}{}_{\sigma_2\cdots\sigma_{n+2}}C^{\rho_1\rho_2\sigma_1\sigma_2}g^{\rho_3\sigma_3\cdots\rho_{n+2}\sigma_{n+2}}\,.
\end{aligned} \tag{5.72}$$

If we include the higher power of the curvature tensors, we obtain more kinds of the tensors. In four dimensions, for example, we have the following non-minimal coupling terms:

$$\begin{aligned}
&C^{\mu_1\mu_2\nu_1\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2}\,,\\
&\delta^{\mu_1}{}_{\rho_1}{}^{\mu_2}{}_{\rho_2}{}^{\mu_3}{}_{\rho_3}\delta^{\nu_1}{}_{\sigma_1}{}^{\nu_2}{}_{\sigma_2}{}^{\nu_3}{}_{\sigma_3}C^{\rho_1\rho_2\sigma_1\sigma_2}g^{\rho_3\sigma_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}\\
&\delta^{\mu_1}{}_{\rho_1}{}^{\mu_2}{}_{\rho_2}{}^{\mu_3}{}_{\rho_3}{}^{\mu_4}{}_{\rho_4}\delta^{\nu_1}{}_{\sigma_1}{}^{\nu_2}{}_{\sigma_2}{}^{\nu_3}{}_{\sigma_3}{}^{\nu_4}{}_{\sigma_4}C^{\rho_1\rho_2\sigma_1\sigma_2}g^{\rho_3\sigma_3\rho_4\sigma_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}\,.
\end{aligned} \tag{5.73}$$

Chapter 6

Summary

We reviewed the representation theory in Chap. 1. We have confirmed that massless particles and massive particles are irreducible representations of the Poincaré group. Then, through the construction of the free field theory of massless particles, we have seen that the concept of the gauge symmetry (redundancy) is quite essential for massless theories. Furthermore, for massive higher spin particles, we need a similar trick to describe the dynamics of particles by fields. Then, carrying out the Hamiltonian analysis on each case, we prove that the free field theory like the linearized Einstein-Hilbert term and the Fierz-Pauli Lagrangian actually realizes the correct number of DOF.

In Chap. 2, we considered gravitational massive spin-two particles. After we studied that the Boulware-Deser ghost inevitably emerges from the general action of massive gravity, we reviewed the field theoretical properties of massive gravity based on the work by Arkani-Hamed *et al.* We found that higher derivatives appear although the original action does not have explicitly such a higher derivative term. Then we have identified the origin of the Boulware-Deser ghost as the higher derivative terms and confirmed that the Boulware-Deser ghost is eliminated by taking the special linear combination of nonderivative interactions in $h_{\mu\nu}$. The interesting point is that massive gravity with the Boulware-Deser ghost does not have any predictability in nontrivial background while the theory is valid as an effective field theory in a flat spacetime. This is because nontrivial backgrounds activate the Boulware-Deser ghost. Motivated by the construction of the Boulware-Deser ghost-free massive gravity, we considered theories of non-gravitational massive spin two particles in the following chapters.

In Chap. 3, we proposed the Z_2 invariant model of interacting massive spin-two particles under the assumption that we only add interactions which does not induce any Boulware-Deser type ghost. After the confirmination of the ghost-free property through the Hamiltonian analysis, we considered the parameter region for the theory to have at least one stable vaccum. Then, nontrivial vacua were investigated. The peculiar property is that the vaccum where the particle description holds does not correspond to the lowest energy states. Furthermore, we also study the stability against quantum corrections. Due to the analysis, the tuning of the mass term is not broken at the one loop level.

We extend the Z_2 spin-two model to the charged $U(1)$ model. Basically, these two models share some characters but, the properties of vacua are very different. The charged massive spin-two model does not admit any nontrivial vacua where field theories can be defiend because DOF in the asymptotic region does not coincide with DOF of the full theory in the nontrivial vacua for any value of m^2 and λ.

In Chap. 5, we considered whether or not the interactions which do not change DOF of the system actually can keep their special properties on curved spacetime. By implementing the Lagrangian analysis, we found that nonderivative interactions do not introduce an extra DOF on the maximally symmetric spacetime while the derivative interaction generally induce the Boulware Deser type ghost. Furthermore, through the analysis, we discovered the completely new, notrivial nonminimal coupling terms on the Einstein manifold.

Appendix A

Christoffel symbols and Curvature in ADM variables

We consider 4D space-time and a space-like hypersurface evolving in time. In this case, 4D metric $g_{\mu\nu}$ is parametrized as follows :

$$g_{00} = -N^2 + \gamma_{ij} N^i N^j \qquad g^{00} = -\frac{1}{N^2}$$

$$g_{ij} = \gamma_{ij} \qquad g^{ij} = \gamma^{ij} - \frac{N^i N^j}{N^2}$$

$$g_{0i} = N_i \qquad g^{0i} = \frac{N^i}{N^2}$$

where N and N_i are the lapse and shifts. The index on the shifts are raised with the metric on the hypersurface γ^{ij} defined by $\gamma_{ik}\gamma^{kj} = \delta_i^j$.

A.1 Christoffel symbol

$$\Gamma^0_{ij} = \frac{K_{ij}}{N} \tag{A.1}$$

$$\Gamma^0_{0j} = \frac{1}{N}(\partial_j N + N^k K_{kj}) \tag{A.2}$$

$$\Gamma^0_{00} = \frac{1}{N}(\partial_t N + N^i \partial_i N + N^i N^j K_{ij}) \tag{A.3}$$

$$\Gamma^i_{0j} = -\frac{N^i}{N}(\partial_j N + N^k K_{kj}) + N K^i{}_j + D_j N^i \tag{A.4}$$

$$\Gamma^i_{jk} = -\frac{N^i}{N} K_{jk} + {}^{(3)}\Gamma^i_{jk} \tag{A.5}$$

$$\begin{aligned}\Gamma^i_{00} &= N\partial^i N + 2NN^l K^i{}_l - \frac{N^i}{N}(\partial_t N + N^j \partial_j N + N^j N^k K_{jk}) \\ &\quad + \partial_t N^i + N^l D_l N^i\end{aligned} \tag{A.6}$$

where D_i represents a covariant derivative with γ_{ij} and K_{ij} is defined by

$$K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - D_i N_j - D_j N_i) \tag{A.7}$$

A.2 Riemann tensor

$$R_i{}^j{}_{kl} = {}^{(3)}R_i{}^j{}_{kl} + K_{ik}K^j{}_l - K_{il}K^j{}_k + \frac{N^j}{N}(D_kK_{il} - D_lK_{ik}) \tag{A.8}$$

$$R_i{}^0{}_{kl} = \frac{1}{N}(D_lK_{ik} - D_kK_{il}) \tag{A.9}$$

$$R_i{}^0{}_{j0} = \frac{1}{N}[\partial_tK_{ij} - D_j\partial_iN - \mathcal{L}_{\boldsymbol{N}}K_{ij} + N^k(D_kK_{ij} - D_jK_{ki})] - K_{kj}K^k{}_i \tag{A.10}$$

$$\begin{aligned} R_0{}^0{}_{0i} = {} & \frac{1}{N}\left\{N^j[-\partial_tK_{ji} + D_j\partial_iN + \mathcal{L}_{\boldsymbol{N}}K_{ji}] + N^jN^k[D_iK_{jk} - D_jK_{ik}]\right\} \\ & + N^kK_k{}^jK_{ji} \end{aligned} \tag{A.11}$$

$$\begin{aligned} R_0{}^j{}_{kl} = {} & \frac{N^jN^m}{N}(D_kK_{ml} - D_lK_{mk}) + N(D_lK^j{}_k - D_kK^j{}_l) \\ & + N^m({}^{(3)}R_m{}^j{}_{kl} + K_{mk}K^j{}_l - K_{ml}K^j{}_k) \end{aligned} \tag{A.12}$$

$$\begin{aligned} R_0{}^i{}_{0j} = {} & N(-\partial_tK^i{}_j + D_j\partial^iN + \mathcal{L}_{\boldsymbol{N}}K^i{}_j - 2N^mD_mK^i{}_j + N^mD^iK_{jm} + N^mD_jK^i{}_m) \\ & + N^lN^m({}^{(3)}R_l{}^i{}_{mj} + K^i{}_jK_{lm} - K^i{}_mK_{lj}) - N^iN^lK_{jk}K^k{}_l - \frac{N^iN^l}{N}(D_l\partial_jN + N^mD_jK_{lm} \\ & - N^mD_mK_{lj} + \mathcal{L}_{\boldsymbol{N}}K_{lj} - \partial_tK_{lj}) - N^2K^i{}_kK^k{}_j \end{aligned} \tag{A.13}$$

$$\begin{aligned} R_k{}^i{}_{0j} = {} & \frac{N^i}{N}(\partial_tK_{jk} - D_jD_kN - \mathcal{L}_{\boldsymbol{N}}K_{jk} + N^mD_mK_{jk} - N^mD_jK_{km}) \\ & - N^m({}^{(3)}R_k{}^i{}_{jm} + K_{jk}K_m{}^i - K_j{}^iK_{mk}) + N(D^iK_{jk} - D_kK_j{}^i) - N^iK_j{}^mK_{mk} \end{aligned} \tag{A.14}$$

A.3 Ricci tensor

$$
\begin{aligned}
R_{ij} = {} & \frac{1}{N}[\partial_t K_{ij} - D_i D_j N - \mathcal{L}_{\boldsymbol{N}} K_{ij}] + {}^{(3)}R_{ij} \\
& + K K_{ij} - 2K_{ik} K^k{}_j
\end{aligned} \tag{A.15}
$$

$$
\begin{aligned}
R_{0i} = {} & \frac{1}{N}\left[N^k \partial_t K_{ki} - N^k \mathcal{L}_{\boldsymbol{N}} K_{ki} - N^j D_j D_i N\right] + {}^{(3)}R_{im} N^m \\
& + 2N D_{[j} K^j{}_{i]} + N^k (K K_{ki} - 2K_k{}^j K_{ji})
\end{aligned} \tag{A.16}
$$

$$
\begin{aligned}
R_{00} = {} & \frac{1}{N}\left[N^i N^j K_{ij,0} - N^i N^j D_i D_j N - N^i N^j \mathcal{L}_{\boldsymbol{N}} K_{ij}\right] \\
& + N(-K_{,0} + D_i D^i N + 2N^l D_i K^i{}_l) + N^i D_i D_j N^j \\
& - {}^{(3)}\Gamma^i_{ij,0} N^j + K(N^i \partial_i N) \\
& + N^i N^j ({}^{(3)}R_{ij} + K K_{ij} - 2K_{ik} K^k{}_j) - N^2 K^i{}_j K^j{}_i \\
= {} & \frac{1}{N}\left\{N^i N^j [\partial_t K_{ij} - D_i \partial_j N - \mathcal{L}_{\boldsymbol{N}} K_{ij}]\right\} \\
& + N^i N^j [{}^{(3)}R_{ij} + K K_{ij} - 2K_{ik} K^k{}_j] \\
& - N[\gamma^{ij} \partial_t K_{ij} - D_i D^i N - \gamma^{ij} \mathcal{L}_{\boldsymbol{N}} K_{ij} - 2N^k D_i K^i{}_k + 2N^k \partial_k K] \\
& + N^2 K_{ij} K^{ij}
\end{aligned} \tag{A.17}
$$

$$
\begin{aligned}
= {} & \frac{1}{N}\left[N^i N^j \left(\partial_t K_{ij} - D_i \partial_j N - \mathcal{L}_{\boldsymbol{N}} K_{ij}\right)\right] \\
& + N^i N^j [{}^{(3)}R_{ij} + K K_{ij} - 2K_{ik} K^k{}_j] \\
& - N[\gamma^{ij} \partial_t K_{ij} - D_i D^i N + N^k \partial_k K - 2D_i N_j K^{ij} - 2N^k D_i K^i{}_k] \\
& + N^2 K_{ij} K^{ij}
\end{aligned} \tag{A.18}
$$

A.4 Ricci scalar

$$
\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + 2g^{0i} R_{0i} + g^{ij} R_{ij} \\
&= {}^{(3)}R + K^2 - 3K_{ik} K^{ik} + \frac{1}{N}\left[2\gamma^{ij} \partial_t K_{ij} - 2D_i \partial^i N - 2N^i \partial_i K - 4D_i N_j K^{ij}\right]
\end{aligned} \tag{A.19}
$$

This result coincides with the one derived from Gauss equation. Ricci scalar derived from Gauss equation is

$$
R = {}^{(3)}R + K_{ij} K^{ij} - K^2 - 2\nabla_\nu (n^\mu \nabla_\mu n^\nu - n^\nu \nabla_\mu n^\mu) \tag{A.20}
$$

The total derivative can be written in terms of ADM variables. First, we write $\nabla_\mu n^\nu$ in ADM variables.

$$
\nabla_0 n^i = \partial^i N + N^j K^i{}_j \tag{A.21}
$$

$$
\nabla_i n^j = K^j{}_i \tag{A.22}
$$

$$
\nabla_\mu n^\mu = K \tag{A.23}
$$

$$
(\text{otherwise}) = 0
$$

Thus, the surface term takes the form :

$$-2\nabla_\nu(n^\mu\nabla_\mu n^\nu - n^\nu\nabla_\mu n^\mu) = \frac{1}{N}(2\partial_t K - 2D_i\partial^i N - 2N^i\partial_i K) + 2K^2 \tag{A.24}$$

To see that (A.20) is equivalent to (A.19), some manupulation is required.

$$\partial_t K = K_{ij}\partial_t\gamma^{ij} + \gamma^{ij}\partial_t K_{ij} \tag{A.25}$$

From the definition of K_{ij}, K^{ij} is

$$K^{ij} = -\frac{1}{2N}(\partial_t\gamma^{ij} + D^i N^j + D^j N^i) \tag{A.26}$$

Using (A.26), the first term of (A.25) is

$$K_{ij}\partial_t\gamma^{ij} = -2NK_{ij}K^{ij} - 2D^i N^j K_{ij} \tag{A.27}$$

Thus, the surface term is

$$\begin{aligned}-2\nabla_\nu(n^\mu\nabla_\mu n^\nu - n^\nu\nabla_\mu n^\mu) &= \frac{1}{N}(2\partial_t K - 2D_i\partial^i N - 2N^i\partial_i K) + 2K^2 \\ &= \frac{1}{N}\big(-4NK_{ij}K^{ij} - 4D^i N^j K_{ij} \\ &\quad + 2\gamma^{ij}\partial_t K_{ij} - 2D_i\partial^i N - 2N^i\partial_i K\big) + 2K^2\end{aligned} \tag{A.28}$$

As a result, (A.20) is

$$R = {}^{(3)}R + K^2 - 3K_{ij}K^{ij} + \frac{1}{N}\left[2\gamma^{ij}\partial_t K_{ij} - 2D_i\partial^i N - 2N^i\partial_i K - 4D^i N^j K_{ij}\right] \tag{A.29}$$

(A.19) is the same as (A.20).

Appendix B

Equivalence theorem

Accordin to the equivalence theorem, the scattering amplitude of the longitudinal polalizations of the massive spin two field and the amplitude of the corresponding Nambu-Goldstone bosons is same up to $\mathcal{O}(m_{\text{graviton}}/E)$. Based on the work [14] which proves the theorem through Glashow -Weinberg-Salam (GWS) model using BRST symmetry, we see this theorem actually hold. Note that we adapt the following convention here: $\eta_{\mu\nu} = (+,-,-,-)$.

B.1 Glashow-Weinberg-Salam model

GWS model is known as the framework which unifies electromagnetic dynamics and the weak force. The Lagrangian has $SU(2) \times U(1)$ gauge symmetry and is given as follows:

$$\mathcal{L}_{GWS} = -\frac{1}{4}\,\mathrm{tr}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - \frac{1}{2}\lambda\left(\phi^\dagger\phi - \frac{\mu^2}{\lambda}\right)^2 + (\text{matter sector}). \tag{B.1}$$

For our purpose here, we concentrate the following part:

$$\mathcal{L}_{gauge,Higgs} = -\frac{1}{4}\,\mathrm{tr}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - \frac{1}{2}\lambda\left(\phi^\dagger\phi - \frac{\mu^2}{\lambda}\right)^2. \tag{B.2}$$

Here the explicit forms of $F^{\mu\nu}, B^{\mu\nu}, (D_\mu\phi)_i$ are defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu] \tag{B.3}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{B.4}$$

$$(D_\mu\phi)_i = \partial_\mu\phi_i - i\left[gA_\mu + g'\frac{1}{2}B_\mu\right]_i^{\ j}\phi_j \tag{B.5}$$

Note that ϕ is the $SU(2)$ doublet and $A_\mu$is the $SU(2)$ gauge field while B_μ is the $U(1)$ gauge field and Y is the $U(1)$ charge.

In this model, three Nambu-Goldstone (NG) bosons emerge when ϕ takes the vacuum expectation value (VEV) and breaks $SU(2)\times U(1)$ symmetry into $U(1)$ symmetry. By taking the unitary gauge, we confirm that these NG bosons are eaten by the gauge fields.

$$\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{B.6}$$

Subtstituting (B.6) into (B.5) yields

$$\mathcal{L}_{mass} = \frac{1}{2}\frac{v^2}{4}\left[g^2(A_\mu^1)^2 + g^2(A_\mu^2)^2 + (-gA^3 + g'B_\mu)^2\right]. \tag{B.7}$$

The diagonalization of the mass matrix gives physical fields and corresponding masses.

$$\begin{array}{llll} \text{W boson} & W^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \mp iA_\mu^2\right) & \text{mass} & M_W = \frac{gv}{2} \\ \text{Z boson} & Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(gA_\mu^3 - g'B_\mu\right) & \text{mass} & M_Z = \frac{\sqrt{g^2+g'^2}v}{2} \\ \text{photon} & A_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g'A_\mu^3 + gB_\mu\right) & \text{mass} & M_A = 0 \end{array}$$

Unfortunately, the unitary gauge is not appropriate for the calculation of loop diagrams because the propagators of the massive gauge fields do behave as $1/m$ at the high energy scales. Thus, R_ξ gauge is often adapted for the discussion in stead of the unitay gauge. Now, let us parametrize the $SU(2)$ doublets as

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^\square \end{pmatrix} = \begin{pmatrix} -i(\varphi^1 - i\varphi^2) \\ \frac{1}{\sqrt{2}}(v + H + i\chi) \end{pmatrix}. \tag{B.8}$$

Here $1/\sqrt{2}v$ means the VEV of the Higgs field and H denotes the physical Higgs particle. φ^1、φ^2、χ correspond to NG bosons. This parametrization gives a coupling term between the NG bosons and the gauge bosons. This term is a little bit annoying but R_ξ gauge eliminates the coupling from the theory.

The explicit form of the gauge fixing fuction corresponding to R_ξ gauge is

$$F_a = \begin{cases} \partial^\mu W_\mu^\pm - \xi M_W \varphi^\pm \\ \partial^\mu Z_\mu - \xi M_Z \chi \\ \partial^\mu A_\mu . \end{cases} \tag{B.9}$$

By adding Faddeev-Popov(FP) ghosts in addtion to the gauge fixing function, we get to treat GWS model quantum-mechanically. Then, the full Lagrangian consists of three parts:

$$\mathcal{L}_{tot} = \mathcal{L}_{\mathrm{GWS,BP}} + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{FP}}. \tag{B.10}$$

The first term denots GWS model in the broken phase and the second term represents the gauge fixing function. The remaining term denots FP ghosts.

B.2 BRST symmetry

Before the proof of the theorem, let us briefly review BRST symmetry. According to the addeev-Popov's method, we can quantize gauge theories by adding the gauge fixing term and the Faddeev-Popov term to the original Lagrangian. Naively, the full Lagrangian does not have symmetries anymore due to the fixing function. Contrary to this naive speculation, there actually exists the symmetry called BRST symmetry. The transformation rule can be obtain by replacing the parameters for the gauge transformation with products of Grassmann numbers and ghost fields. We should notice that the transformation rules superficially change depending on the unbroken phase or the broken phase.

$$\begin{aligned}
&\text{matter fields} \quad \Theta_i \;:\; \delta_B\Theta_i = -i\lambda g c^a [T^a]_{ij}\Theta_j && \text{(B.11)}\\
&\text{gauge fields} \quad G^a_\mu \;:\; \delta_B G^a_\mu = \lambda\left[\partial_\mu c^a - igf^{abc}G^b_\mu c^c\right], \quad G^a = A^a_\mu, B_\mu &&\\
&\text{FP ghosts} \quad c^a \;:\; \delta_B c^a = \frac{1}{2}\lambda g f^{abc}c^b c^c &&\\
&\text{anti-FP ghosts} \quad \bar{c}^a \;:\; \delta_B \bar{c}^a = i\lambda B^a &&\\
&\text{Nakanishi-Lautrap field} \quad B^a \;:\; \delta_B B^a = 0 && \text{(B.12)}
\end{aligned}$$

Here T^a denotes the $SU(2)$ generator and Grassmann numbers are represented by λ. The transformation rules for the ghost fiedls and Nakanishi-Lautrap are obtained from the following requirement:$\delta_B^2(\cdots) = 0$.

Finally, we have to mention that the all physical states $|\text{phys}\rangle$ are invariant under BRST transformation.

$$Q_B|\text{phys}\rangle = 0 \tag{B.13}$$

where Q_B is the BRST generator.

B.3 Derivation of identity

We derive the identity which plays an important role in the proof. Generally, BRST transformation on some arbitrary operator $\mathcal{O}$ is represented by

$$\delta_B\mathcal{O} = [i\lambda Q_B, \mathcal{O}]. \tag{B.14}$$

Thus, from the property (B.13), the following relation holds:

$$\langle\text{phys, out}|\;[i\lambda Q_B, T\{\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\}]\;|\text{phys, in}\rangle = 0, \tag{B.15}$$

which is equivalent to

$$\sum_{k=1}^{n}\langle\text{phys, out}|T\{\mathcal{O}_1(x_1)\ldots\delta_B\mathcal{O}_k(x_k)\cdots\mathcal{O}_n(x_n)\}|\text{phys, in}\rangle = 0. \tag{B.16}$$

From the relation (B.16), we would like to derive

$$\langle\text{phys, out}|\;T\{F_{a_1}(x_1)\cdots F_{a_n}(x_n)\}\;|\text{phys, in}\rangle_{\text{con}} = 0. \tag{B.17}$$

Here con means that connexced diagrams are only considered.

In order to derive (B.17), some preparations are needed. Let us begin with the derivation of the transformation rules for anti FP ghosts under BRST transformation in the broken phase. Generally, by using BRST transformation, we can reexpress $\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{FP}}$ as

$$\mathcal{L}_{GF+FP} = -i\boldsymbol{\delta}_B(\bar{c}^a\tilde{F}^a) = B^a\tilde{F}^a + i\bar{c}^a\boldsymbol{\delta}_B\tilde{F}^a. \tag{B.18}$$

Here we have to note that $\boldsymbol{\delta}_B$ is defined as $\lambda\boldsymbol{\delta}_B$. Thus, we learn how the anti FP ghosts transform by determines $\tilde{F}^a$ in order to reproduce (B.9). Then, we have

$$\tilde{F}^a = \partial^\mu A^a_\mu + \frac{1}{2}\xi^a B^a - \xi M_a\varphi^a. \tag{B.19}$$

Here the index a corresponds to the following fields and masses.

$$\begin{cases} A^a_\mu = W^\pm_\mu, \text{ or } A_\mu \\ \varphi^a = \varphi^\pm, \text{or } \chi \\ M_a = M_W,\ M_Z, \text{ or } 0 \end{cases} \tag{B.20}$$

Thanks to (B.19), the explicit form of (B.18) is given by

$$\mathcal{L}_{GF+FP} = B^a \partial^\mu A^a_\mu + \frac{1}{2}\xi B^a B^a + i\bar{c}^a \partial^\mu D^{ab}_\mu c^b. \tag{B.21}$$

By taking a variation with respect to B^a, we have

$$B^a = -\frac{1}{\xi}(\partial^\mu A^a_\mu - \xi M_a \varphi^a) = -\frac{1}{\xi} F^a, \tag{B.22}$$

which shows

$$\delta_B \bar{c}^a = i\lambda B^a = -i\frac{\lambda}{\xi} F^a. \tag{B.23}$$

Next, we find the transformation rule for F^a. This can be read off from (B.18).

$$\delta_B F^a = -i\lambda \frac{\partial \mathcal{L}}{\partial \bar{c}^a} = -i\lambda \left[\frac{\partial \mathcal{L}}{\partial \bar{c}^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{c}^a)} \right] = -i\lambda \mathcal{L}_{\bar{c}^a} \tag{B.24}$$

where $\mathcal{L}_{\bar{c}^a}$ is defined as follows:

$$\mathcal{L}_{\bar{c}^a} := \frac{\partial \mathcal{L}}{\partial \bar{c}^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{c}^a)}. \tag{B.25}$$

Finally, we prove

$$\langle \text{A, out}|\ T\left[\mathcal{L}_\Phi \mathcal{O}\right]\ |\text{B, in}\rangle = i \langle \text{A, out}|\ T\left[\frac{\partial \mathcal{O}}{\partial \Phi}\right]\ |\text{B, in}\rangle \tag{B.26}$$

using the Schwinger-Dyson equation. Here we define $\mathcal{L}_\Phi$ as

$$\mathcal{L}_\Phi := \frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}. \tag{B.27}$$

Furthermore, we assume that the states $|\text{A, out}\rangle$ and $|\text{B, in}\rangle$ do not contain any particle created by the operator Φ. The left hand side in (B.26) can be expressed in terms of a vacuum expectation value of some operators thanks to the Lehmann-Symanzik-Zimmermann(LSZ) reduction formula.

$$\langle \text{A, out}|\ T\left[\mathcal{L}_\Phi \mathcal{O}\right]\ |\text{B, in}\rangle \sim \langle 0|\ T\left[\mathcal{L}_\Phi \left\{\mathcal{O}\psi_1\psi_2\cdots\right\}\right]\ |0\rangle \tag{B.28}$$

Applying the Schwinger-Dyson equation to (B.28), we obtain

$$\begin{aligned} \langle 0|\ T\left[\mathcal{L}_\Phi \left\{\mathcal{O}\psi_1\psi_2\cdots\right\}\right]\ |0\rangle &= i\langle 0|\ T\left[\frac{\delta}{\delta\Phi}\left\{\mathcal{O}\psi_1\psi_2\cdots\right\}\right]\ |0\rangle \\ &= i\langle 0|\ T\left[\left(\frac{\delta}{\delta\Phi}\mathcal{O}\right)\psi_1\psi_2\cdots\right]\ |0\rangle. \end{aligned} \tag{B.29}$$

Then, by using the LSZ reduction formula again, (B.26) is proven:

$$i\langle 0|\ T\left[\left(\frac{\delta}{\delta\Phi}\mathcal{O}\right)\psi_1\psi_2\cdots\right]\ |0\rangle \sim i\langle \mathrm{A,\ out}|\ T\left[\frac{\delta}{\delta\Phi}\mathcal{O}\right]\ |\mathrm{B,\ in}\rangle. \tag{B.30}$$

In the following, we assume that the states A and B denote physical states.

Now that all relations to be needed have been obtained, we prove (B.17) by using mathematical induction.

- For $n = 1$ in (B.16), we have

$$\langle \mathrm{A,\ out}|\ T\left[\delta_B\mathcal{O}\right]\ |\mathrm{B,\ in}\rangle = 0. \tag{B.31}$$

  Setting $\mathcal{O}{=}\bar{c}^a$ and using $(B.23)$, we find

$$\langle \mathrm{A,\ out}|\ F_{a_1}\ |\mathrm{B,\ in}\rangle = 0. \tag{B.32}$$

  Thus, (B.17)$n = 1$.

- Similarly, for $n = 2$, we have

$$\langle \mathrm{A,\ out}|\ T\left[(\delta_B\mathcal{O}_1)\mathcal{O}_2\right]\ |\mathrm{B,\ in}\rangle + \langle \mathrm{A,\ out}|\ T\left[\mathcal{O}_1(\delta_B\mathcal{O}_2)\right]\ |\mathrm{B,\ in}\rangle = 0. \tag{B.33}$$

  Setting $\mathcal{O}_1 = F_{a_1}$, $\mathcal{O}_2 = \bar{c}_{a_2}$ gives

$$\langle \mathrm{A,\ out}|\ T\left[(\delta_B F_{a_1})\bar{c}_{a_2}\right]\ |\mathrm{B,\ in}\rangle + \langle \mathrm{A,\ out}|\ T\left[F_{a_1}(\delta_B\bar{c}_{a_2})\right]\ |\mathrm{B,\ in}\rangle = 0. \tag{B.34}$$

  From the transformation rules (B.23) and (B.24),

$$-\langle \mathrm{A,\ out}|\ T\left[\mathcal{L}_{\bar{c}_{a_1}}\ \bar{c}_{a_2}\right]\ |\mathrm{B,\ in}\rangle - \frac{1}{\xi}\langle \mathrm{A,\ out}|\ T\left[F_{a_1}\ F_{a_2}\right]\ |\mathrm{B,\ in}\rangle = 0. \tag{B.35}$$

  Since the ghosts never appear in physical states, we can apply (B.26) to the first term:

$$\langle \mathrm{A,\ out}|\ T\left[\mathcal{L}_{\bar{c}_{a_1}}\ \bar{c}_{a_2}\right]\ |\mathrm{B,\ in}\rangle = i\delta(x_1-x_2)\delta_{a_1a_2}\langle \mathrm{A,\ out}|\mathrm{B,\ in}\rangle. \tag{B.36}$$

  From this relation, (B.34) is written as

$$\frac{1}{\xi}\langle \mathrm{A,\ out}|\ T\left[F_{a_1}\ F_{a_2}\right]\ |\mathrm{B,\ in}\rangle + i\delta(x_1-x_2)\delta_{a_1a_2}\langle \mathrm{A,\ out}|\mathrm{B,\ in}\rangle = 0. \tag{B.37}$$

  We have to note that the first term can be divided into the connected part and the disconnected part.

$$\begin{aligned}\langle \mathrm{A,\ out}|\ T\left[F_{a_1}\ F_{a_2}\right]\ |\mathrm{B,\ in}\rangle &= \langle \mathrm{A,\ out}|\ T\left[F_{a_1}\ F_{a_2}\right]\ |\mathrm{B,\ in}\rangle_{\mathrm{con}} \\ &\quad + \langle \mathrm{A,\ out}|\ T\left[F_{a_1}\ F_{a_2}\right]\ |\mathrm{B,\ in}\rangle_{\mathrm{dis}}\end{aligned} \tag{B.38}$$

  As the disconnected part is expressed as

$$\langle \mathrm{A,\ out}|\ T\left[F_{a_1}\ F_{a_2}\right]\ |\mathrm{B,\ in}\rangle_{\mathrm{dis}} = \langle \mathrm{A,\ out}|\mathrm{B,\ in}\rangle\langle 0|\ T\left[F_{a_1}\ F_{a_2}\right]\ |0\rangle, \tag{B.39}$$

  we obatain the following two relations:

$$\langle 0|\ T\left[F_{a_1}\ F_{a_2}\right]\ |0\rangle = -i\xi\delta(x_1-x_2)\delta_{a_1a_2} \tag{B.40}$$

  and

$$\langle \mathrm{A,\ out}|\ T\left[F_{a_1}\ F_{a_2}\right]\ |\mathrm{B,\ in}\rangle_{\mathrm{con}} = 0. \tag{B.41}$$

  Therefore, (B.17) also holds for $n = 2$.

- Then, we prove that (B.17) for $n = N$ by assuming that the relation for $n = N - 1$ is correct. For $n = N$, we find

$$\sum_{k=1}^{N} \langle \mathrm{A, out}| T \left\{ \mathcal{O}_1(x_1) \ldots \delta_B \mathcal{O}_k(x_k) \cdots \mathcal{O}_n(x_n) \right\} |\mathrm{B, in}\rangle = 0 \tag{B.42}$$

from (B.16). In this case, we set operators are given by

$$\begin{aligned} &\mathcal{O}_i = F_{a_i}, \quad i = 1, 2, \ldots, N-1 \\ &\mathcal{O}_N = \bar{c}_{a_N}. \end{aligned} \tag{B.43}$$

Remaining manipulation is completely same as in the case of $n = 1$ and $n = 2$. Then, we have

$$\begin{aligned} &i\langle \mathrm{A}|\ T\left[F_{a_2} \cdots F_{a_{N-1}}\right]\ |\mathrm{B}\rangle \delta_{a_1 a_N} \delta(x_1 - x_N) + \cdots \\ &\quad + i\langle \mathrm{A}|\ T\left[F_{a_1} \cdots F_{a_{N-2}}\right]\ |\mathrm{B}\rangle \delta_{a_{N-1} a_N} \delta(x_{N-1} - x_N) + \frac{1}{\xi}\langle \mathrm{A}|T\left[F_{a_1} \cdots F_{a_N}\right]|\mathrm{B}\rangle = 0. \end{aligned} \tag{B.44}$$

As (B.17) holds for $n \leq N - 1$, the identity $n = N$

$$\langle \mathrm{A, out}|\ T\left\{F_{a_1}(x_1) \cdots F_{a_n}(x_n)\right\}\ |\mathrm{B, in}\rangle_{\mathrm{con}} = 0 \tag{B.45}$$

is proven.

B.4 Proof of equivalence theorem

Now, we prove the equivalence theorem using (B.17). Let us start with the equations of motion for the gauge fields.

$$\left\{ (\Box + M_a^2) g_{\mu\nu} + \left(\frac{1}{\xi} - 1 \right) \partial_\mu \partial_\nu \right\} W_a^\mu = -J_{\mu\, a} \tag{B.46}$$

For later convenience, we define $L_{\mu\nu}$ as

$$L_{\mu\nu} := (\Box + M_a^2) g_{\mu\nu} + \left(\frac{1}{\xi} - 1 \right) \partial_\mu \partial_\nu. \tag{B.47}$$

As an example, let us consider a scattering process including at least one gauge particle in the initial state: B→ A $+W_\mu^+$. According to the LSZ reduction formula, we have

$$\mathcal{M}_\mu(B \to A + W_\mu^+) = \mathrm{FT}\left\{ L_{\mu\nu} \langle \mathrm{A, out}|\ W_\mu^+\ |\mathrm{B, in}\rangle \right\}. \tag{B.48}$$

Multiplying the both side of (B.48) with ∂^μ, we find

$$\begin{aligned} ip^\mu \mathcal{M}_\mu(B \to A + W_\mu^+) &= \mathrm{FT}\left\{ \partial^\mu \mathrm{L}_{\mu\nu} \langle \mathrm{A, out}|\ \mathrm{W}^{+\,\mu}\ |\mathrm{B, in}\rangle \right\} &\text{(B.49)} \\ &= \mathrm{FT}\left\{ \left(\mathrm{M_W^2} + \frac{1}{\xi}\Box \right) \langle \mathrm{A, out}|\ \partial^\mu \mathrm{W}_\mu^+\ |\mathrm{B, in}\rangle \right\}. &\text{(B.50)} \end{aligned}$$

Using the relation given in (B.16)

$$\langle \mathrm{A,\ out}\ | F_{+}(x)\ | \mathrm{B,\ in} \rangle = 0, \tag{B.51}$$

we have the relation

$$\langle \mathrm{A,\ out}|\partial_\mu W_{+}^{\mu}|\mathrm{B,\ in}\rangle = \xi M_W \langle \mathrm{A,\ out}|\varphi^{+}|\mathrm{B,\ in}\rangle. \tag{B.52}$$

Substituting the above result into (B.50) gives

$$\begin{aligned} i\frac{p^\mu}{M_W}\mathcal{M}_\mu(B \to A + W_\mu^{+}) &= \mathrm{FT}\left\{\left(\Box + \xi M_W^2\right)\langle \mathrm{A,\ out}|\varphi^{+}|\mathrm{B,\ in}\rangle\right\} \\ &= \mathcal{M}(B \to A + \varphi^{+}). \end{aligned} \tag{B.53}$$

The left hand side in (B.53) is equivalent to the longitudinal gauge boson's amplitude up to $\mathcal{O}(m_{\text{gauge}}/E)$ where E stands for the center-mass energy of the process. This is because the longitudinal polarization vector $\epsilon_L^\mu(p)$ in the high energy limit behaves as

$$\epsilon_L^\mu(p) = \frac{p^\mu}{m} + \mathcal{O}\left(\frac{m}{E}\right) \tag{B.54}$$

when p^μ is given by $p^\mu = (E, 0, 0, p)$.

As a result, we have found the following powerful relation:

$$\epsilon_L^\mu(p)\mathcal{M}_\mu(B \to A + W_\mu^{+}) = \mathcal{M}(B \to A + \varphi^{+}) + O\left(\frac{m}{E_p}\right). \tag{B.55}$$

We can generalize this argument for amplitudes including multiple massive gauge bosons.

Appendix C

The detail on Hamiltonian analysis for the dRGT masive gravity

C.1 The derivation of (2.136),(2.137),(2.138)

We derive (2.136), (2.137) and (2.138) here. First of all, we arrange the expression (2.132) in order of $1/N$:

$$\begin{aligned}(g^{-1}\eta)^{\mu}{}_{\nu} &= \frac{1}{N^2}\begin{pmatrix} 1 & (\delta^l_k + ND^l{}_k)n^k\delta_{lj} \\ -(\delta^i_k + ND^i{}_k)n^k & \left\{N^2\gamma^{il} - (\delta^i_k + ND^i{}_k)n^k(\delta^l_m + ND^l{}_m)n^m\right\}\delta_{lj}\end{pmatrix} \\ &= \frac{1}{N^2}\begin{pmatrix} 1 & n^k\delta_{kj} \\ -n^i & -n^i n^k\delta_{kj}\end{pmatrix} + \frac{1}{N}\begin{pmatrix} 0 & D^l{}_k n^k\delta_{lj} \\ -D^i{}_k n^k & -(n^i D^l{}_m n^m\delta_{lj} + D^i{}_k n^k n^l\delta_{lj})\end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ik}\delta_{kj} - D^i{}_l n^l D^k{}_m n^m\delta_{kj}\end{pmatrix}. \end{aligned} \tag{C.1}$$

Equating (C.1) to the following

$$g^{-1}\eta = \frac{1}{N^2}A^2 + \frac{1}{N}(AB + BA) + B^2, \tag{C.2}$$

we find the expressions for A^2 and B^2 in terms of the (modified) ADM variables:

$$A^2 = \begin{pmatrix} 1 & n^k\delta_{kj} \\ -n^i & -n^i n^k\delta_{kj}\end{pmatrix}, \quad B^2 = \begin{pmatrix} 0 & 0 \\ 0 & (\gamma^{ik} - D^i{}_l n^l D^k{}_m n^m)\delta_{kj}\end{pmatrix}.$$

Hence, A and B are given by

$$A = \frac{1}{\sqrt{1 - n^r\delta_{rs}n^s}}\begin{pmatrix} 1 & n^k\delta_{kj} \\ -n^i & -n^i n^k\delta_{kj}\end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{(\gamma^{ik} - D^i{}_l n^l D^k{}_m n^m)\delta_{kj}}\end{pmatrix}. \tag{C.3}$$

From (C.3), the second term in (C.2) is calculated as

$$\begin{aligned}\frac{1}{N}(AB + BA) &= \frac{1}{N}\frac{1}{\sqrt{1 - n^r\delta_{rs}n^s}} \\ &\times \begin{pmatrix} 0 & n^r\delta_{rl}\sqrt{(\gamma^{lk} - D^l{}_m n^m D^k{}_s n^s)\delta_{kj}} \\ -\sqrt{(\gamma^{ik} - D^i{}_l n^l D^k{}_m n^m)\delta_{kr}}n^r & a^i{}_j\end{pmatrix}\end{aligned} \tag{C.4}$$

where

$$a^i{}_j = -n^i n^r \delta_{rl} \sqrt{(\gamma^{lk} - D^l{}_m n^m D^k{}_s n^s)\delta_{kj}} - \sqrt{(\gamma^{ik} - D^i{}_l n^l D^k{}_m n^m)\delta_{kr}} n^r n^s \delta_{sj}.$$

Then, comparing (C.4) and (C.1), we have

$$(\sqrt{1 - n^r \delta_{rs} n^s})\, D^i{}_j = \sqrt{(\gamma^{ik} - D^i{}_l n^l D^k{}_m n^m)\delta_{kj}}. \tag{C.5}$$

C.2 The derivation of (2.141)

C.2.1 Preparation I

To derive (2.141), we introduce a useful relation. For an arbitrary matrix M^{ij}, the relation in the following holds.

$$\sqrt{M^{il}\delta_{lk}}\delta^{kj} = \delta^{il}\sqrt{\delta_{lk}M^{kj}} \tag{C.6}$$

We prove (C.6). Let us write the matrix M as follows:

$$M^{ij} = M^{il}\delta_{lm}\delta^{mj} = \sqrt{M^{il}\delta_{lr}}\sqrt{M^{rs}\delta_{sm}}\delta^{mj}. \tag{C.7}$$

On the other hand, M is also expressed as

$$M^{ij} = \sqrt{M^{il}\delta_{lr}}\delta^{rm}\sqrt{\delta_{ms}M^{sj}}. \tag{C.8}$$

From (C.7) and (C.8), we obtain

$$\sqrt{M^{il}\delta_{lr}}\sqrt{M^{rs}\delta_{sm}}\delta^{mj} = \sqrt{M^{il}\delta_{lr}}\delta^{rm}\sqrt{\delta_{ms}M^{sj}}$$

$$\Longleftrightarrow \quad \sqrt{M^{il}\delta_{lr}}(\sqrt{M^{rs}\delta_{sm}}\delta^{mj} - \delta^{rm}\sqrt{\delta_{ms}M^{sj}}) = 0. \tag{C.9}$$

Therefore, we have shown that (C.6) holds for an arbitrary matrix.

C.2.2 Preparation II

Here we prove that the matrix $D^i{}_l\delta^{lj}$ is symmetric based on the important relation

$$(\sqrt{1 - n^r \delta_{rs} n^s})\, D^i{}_j = \sqrt{(\gamma^{ik} - D^i{}_l n^l D^k{}_m n^m)\delta_{kj}}$$

given in (C.5). Using the relation $\sqrt{M^{il}\delta_{lk}}\delta^{kj} = \delta^{il}\sqrt{\delta_{lk}M^{kj}}$, we rewrite the right hand side of (C.5) after multiplying it with δ^{rj}

$$\begin{aligned}
\sqrt{(\gamma^{ik} - D^i{}_l n^l D^k{}_m n^m)\delta_{kr}}\delta^{rj} &= \delta^{ir}\sqrt{\delta_{rk}(\gamma^{kj} - D^k{}_l n^l D^j{}_m n^m)} \\
&= \delta^{ir}\sqrt{(\gamma^{jk} - D^j{}_m n^m D^k{}_l n^l)\delta_{kr}} \\
&= \sqrt{1 - n^s\delta_{su}n^u}\, D^j{}_r\delta^{ri}
\end{aligned} \tag{C.10}$$

Thus, we have

$$(\sqrt{1 - n^r \delta_{rs} n^s})\, D^i{}_r\delta^{rj} = \sqrt{1 - n^s\delta_{su}n^u}\, D^j{}_r\delta^{ri}, \tag{C.11}$$

which shows $D^i{}_r\delta^{rj} = D^j{}_r\delta^{ri}$. Furthermore, after some manupilations, we also find $\delta_{ik}D^k{}_j = \delta_{jk}D^k{}_i$.

C.2.3 The derivation of (2.141)

Let us derive (2.141) using the above relations. Before the calculation, let us remind that the Pisson bracket for the lapse is defined as

$$\{F,G\} = \int d^3z \left[\frac{\delta F}{\delta n_l(t,\mathbf{z})}\frac{\delta G}{\delta \Pi_{\mathbf{n}}{}^l(t,\mathbf{z})} - \frac{\delta F}{\delta \Pi_{\mathbf{n}}{}^l(t,\mathbf{z})}\frac{\delta G}{\delta n_l(t,\mathbf{z})}\right]. \tag{C.12}$$

Thus, by choosing $F = \Pi_{\mathbf{n}i}$ and $G = H$, we have

$$\begin{aligned}\{\Pi_{\mathbf{n}i}, H\} = &-\int d^3x\{(\delta^i_j + ND^i{}_j)n^j\mathcal{H}_i(\mathbf{x}), \Pi_{\mathbf{n}k}(\mathbf{y})\}\\ &-\frac{m^2}{\kappa^2}\int d^3x\sqrt{\gamma}\left\{\sqrt{1-n^r\delta_{rs}n^s}(\mathbf{x}), \Pi_{\mathbf{n}k}(\mathbf{y})\right\}\\ &-\frac{m^2}{\kappa^2}\int d^3xN(\mathbf{x})\sqrt{\gamma}\left\{\mathrm{tr}\left(\sqrt{(\gamma^{ik}\delta_{kj} - D^i{}_l n^l D^k{}_m n^m\delta_{kj})}\right)(\mathbf{x}), \Pi_{\mathbf{n}k}(\mathbf{y})\right\}\end{aligned} \tag{C.13}$$

In the third line, we should notice that a variation of a square root of an arbitrary matrix M is given by $\delta\,\mathrm{tr}\sqrt{M} = 1/2\,\mathrm{tr}(\sqrt{M}^{-1}\delta M)$. Thus,

$$\begin{aligned}&\delta_{\mathbf{n}}\,\mathrm{tr}\left(\sqrt{(\gamma^{ik}\delta_{kj} - D^i{}_l n^l D^k{}_m n^m\delta_{kj})}\right)\\ &= \frac{1}{2}\,\mathrm{tr}\left[\left(\sqrt{(\gamma^{ik}\delta_{kr} - D^i{}_l n^l D^k{}_m n^m\delta_{kr})}\right)^{-1}\delta_{\mathbf{n}}\left(\gamma^{rs}\delta_{sj} - D^r{}_s n^s D^u{}_v n^v\delta_{uj}\right)\right]\\ &= \frac{1}{2}\,\mathrm{tr}\left[\left(1-n^a\delta_{ab}n^b\right)^{-\frac{1}{2}}D^{-1\,i}{}_r\delta_{\mathbf{n}}\left(\gamma^{rs}\delta_{sj} - D^r{}_s n^s D^u{}_v n^v\delta_{uj}\right)\right]\\ &= -\frac{1}{2\sqrt{1-n^a\delta_{ab}n^b}}\,\mathrm{tr}\left[D^{-1\,i}{}_r\delta_{\mathbf{n}}\left(D^r{}_s n^s D^u{}_v n^v\delta_{uj}\right)\right]\\ &= -\frac{1}{2\sqrt{1-n^a\delta_{ab}n^b}}D^{-1\,i}{}_r\delta_{\mathbf{n}}\left(D^r{}_s n^s D^u{}_v n^v\delta_{ui}\right)\\ &= -\frac{1}{2\sqrt{1-n^a\delta_{ab}n^b}}\left\{D^{-1\,i}{}_j\frac{\partial}{\partial n^l}(D^j{}_m n^m)\delta n^l D^k{}_s n^s\delta_{ki} + D^{-1\,i}{}_j D^j{}_m n^m\frac{\partial}{\partial n^l}(D^k{}_s n^s)\delta n^l\delta_{ki}\right\}\end{aligned} \tag{C.14}$$

From the property $\delta_{ik}D^k{}_j = \delta_{jk}D^k{}_i$, the first term in $\{\cdots\}$ is expressed as

$$\begin{aligned}D^{-1\,i}{}_j\frac{\partial}{\partial n^l}(D^j{}_m n^m)\delta n^l D^k{}_i\delta_{sk}n^s &= D^k{}_i D^{-1\,i}{}_j\delta_{sk}n^s\frac{\partial}{\partial n^l}(D^j{}_m n^m)\delta n^l\delta n^l\\ &= n^s\delta_{sj}\frac{\partial}{\partial n^l}(D^j{}_m n^m)\delta n^l.\end{aligned} \tag{C.15}$$

Similarly, the second term in $\{\cdots\}$ becomes

$$n^s\delta_{sj}\frac{\partial}{\partial n^k}(D^j{}_m n^m)\delta n^k. \tag{C.16}$$

Therefore,

$$\delta_{\mathbf{n}}\,\mathrm{tr}\left(\sqrt{(\gamma^{ik}\delta_{kj} - D^i{}_l n^l D^k{}_m n^m\delta_{kj})}\right) = -\frac{1}{\sqrt{1-n^a\delta_{ab}n^b}}n^s\delta_{sj}\frac{\partial}{\partial n^l}(D^j{}_m n^m)\delta n^l \tag{C.17}$$

The calculation of the remaining terms in (C.13) is straightforward. As a result, we find

$$\left(\mathcal{H}_i - \frac{m^2}{\kappa^2}\frac{\sqrt{\gamma}\,n^l\delta_{li}}{\sqrt{1-n^r\delta_{rs}n^s}}\right)\left[\delta^i_k + N\frac{\partial}{\partial n^k}\left(D^i_j n^j\right)\right] \approx 0. \tag{C.18}$$

C.3 The solution of (2.141)

We obtain the explicit form of the shift n^i by solving

$$\mathcal{H}_i - \frac{m^2}{\kappa^2}\frac{\sqrt{\gamma}\, n^l \delta_{li}}{\sqrt{1 - n^r \delta_{rs} n^s}} \approx 0. \tag{C.19}$$

First of all, we put (C.19) into an equivalent form as follows:

$$\sqrt{1 - n^r \delta_{rs} n^s}\mathcal{H}_i \approx \frac{m^2}{\kappa^2}\sqrt{\gamma}\, n^l \delta_{li} \tag{C.20}$$

Multiplying the left hand side with $\delta^{ij}(\sqrt{1 - n^r \delta_{rs} n^s})\,\mathcal{H}_i$ from the right, we find

$$(1 - n^r \delta_{rs} n^s)\mathcal{H}_i \delta^{ij} \mathcal{H}_j \tag{C.21}$$

Similarly, we multiply the right hand side $\delta^{ij}(m^2/\kappa^2)\sqrt{\gamma}\, n^l \delta_{li} = (\delta^{ij}(\sqrt{1 - n^r \delta_{rs} n^s})\,\mathcal{C}_i)$ to obtain

$$\frac{m^4}{\kappa^4}(\det\gamma) n^l \delta_{li} n^i. \tag{C.22}$$

Equating these two expression, (C.20) is rewritten as

$$\mathcal{H}_i \delta^{ij} \mathcal{H}_j - n^r \delta_{rs} n^s \mathcal{H}_i \delta^{ij} \mathcal{H}_j = \frac{m^4}{\kappa^4}\det\gamma n^l \delta_{li} n^i. \tag{C.23}$$

Solving this equation in terms of $n^r \delta_{rs} n^s$ yields

$$n^r \delta_{rs} n^s = \frac{\mathcal{H}_i \delta^{ij} \mathcal{H}_j}{\mathcal{H}_i \delta^{ij} \mathcal{H}_j + \frac{m^4}{\kappa^4}\det\gamma}. \tag{C.24}$$

Therefore, substituting (C.24) into (C.20), we find the explicit form of the shift:

$$n^i = -\mathcal{H}_j \delta^{ji}\left[(m^4/\kappa^4)\det\gamma + \mathcal{H}_k \delta^{kl} \mathcal{H}_l\right]^{-1/2}.$$

We can also mention the positivity of $1 - n^r \delta_{rs} n^s$ thanks to the relation (C.24):

$$1 - n^r \delta_{rs} n^s = \frac{(m^4/\kappa^4)\det\gamma}{\det\gamma + \mathcal{H}_i \delta^{ij} \mathcal{H}_j} > 0, \tag{C.25}$$

which shows that $\sqrt{1 - n^r \delta_{rs} n^s}$ is real.

C.4 The property of the matrix A

Generally,

$$\begin{pmatrix} 1 & n^T\mathbf{I} \\ -n & -nn^T\mathbf{I} \end{pmatrix}^2 = (1 - n^T\mathbf{I}n)\begin{pmatrix} 1 & n^T\mathbf{I} \\ -n & -nn^T\mathbf{I} \end{pmatrix}$$

$$\begin{pmatrix} 1 & n^T\mathbf{I} \\ -n & -nn^T\mathbf{I} \end{pmatrix}^3 = (1 - n^T\mathbf{I}n)^2\begin{pmatrix} 1 & n^T\mathbf{I} \\ -n & -nn^T\mathbf{I} \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} 1 & n^T\mathbf{I} \\ -n & -nn^T\mathbf{I} \end{pmatrix}^k = (1 - n^T\mathbf{I}n)^{k-1}\begin{pmatrix} 1 & n^T\mathbf{I} \\ -n & -nn^T\mathbf{I} \end{pmatrix} \tag{C.26}$$

Thus,

$$\begin{aligned}\mathbb{A}^k &= (1-n^T\mathbf{I}n)^{-k/2}\begin{pmatrix} 1 & n^T\mathbf{I} \\ -n & -nn^T\mathbf{I}\end{pmatrix}^k \\ &= (1-n^T\mathbf{I}n)^{k/2-1}\begin{pmatrix} 1 & n^T\mathbf{I} \\ -n & -nn^T\mathbf{I}\end{pmatrix}\end{aligned} \tag{C.27}$$

From this property, we can calculate

$$\begin{aligned}\mathrm{tr}\mathbb{A}^k &= (1-n^T\mathbf{I}n)^{k/2-1}(1-n^T\mathbf{I}n) = (1-n^T\mathbf{I}n)^{k/2} \\ (\,\mathrm{tr}\mathbb{A})^k &= \left[(1-n^T\mathbf{I}n)^{-1/2}(1-n^T\mathbf{I}n)\right]^k = (1-n^T\mathbf{I}n)^{k/2},\end{aligned} \tag{C.28}$$

which means the following relation holds:

$$\mathrm{tr}\mathbb{A}^k = (\,\mathrm{tr}\mathbb{A})^k \tag{C.29}$$

Appendix D

Properties of $g^{\mu_1\nu_1\cdots\mu_n\nu_n}$ and useful relations

D.1 Properties of $g^{\mu_1\nu_1\cdots\mu_n\nu_n}$

In this appendix, we list the properties of $g^{\mu_1\nu_1\cdots\mu_n\nu_n}$. Note that the properties below are held on arbitrary space-time.

D.1.1 Definition

First, we define the tensor $g^{\mu_1\nu_1\cdots\mu_n\nu_n}$ as

$$\begin{aligned} g^{\mu_1\nu_1\cdots\mu_n\nu_n} &\equiv g^{\mu_1\nu_1}g^{\mu_2\nu_2}g^{\mu_3\nu_3}\cdots g^{\mu_n\nu_n} - g^{\mu_1\nu_2}g^{\mu_2\nu_1}g^{\mu_3\nu_3}\cdots g^{\mu_n\nu_n} + \cdots \\ &= \frac{-1}{(D-n)!}E^{\mu_1\mu_2\cdots\mu_n\sigma_{n+1}\cdots\sigma_D}E^{\nu_1\nu_2\cdots\nu_n}{}_{\sigma_{n+1}\cdots\sigma_D}\,. \end{aligned} \tag{D.1}$$

Here D denotes the dimension of the space-time and the totally anti-symmetric tensor $E^{\mu_1\mu_2\cdots\mu_n}$ is defined as

$$E^{\mu_1\mu_2\cdots\mu_D} \equiv \frac{1}{\sqrt{-g}}\epsilon^{\mu_1\mu_2\cdots\mu_D}\,. \tag{D.2}$$

with the totally anti-symmetric Levi-Civita tensor density

$$\epsilon^{\mu_1\mu_2\cdots\mu_D} = \begin{cases} +1 \text{ if}(\mu_1\mu_2\cdots\mu_D) \text{ is an even permutation of}(0123\cdots) \\ -1 \text{ if}(\mu_1\mu_2\cdots\mu_D) \text{ is an even permutation of}(0123\cdots) \\ 0 \text{ otherwise} \end{cases}$$

In the following, we call the tensor $g^{\mu_1\nu_1\cdots\mu_n\nu_n}$ the pseudo-linear tensor.

Finally, we summarize the symmetric property of the pseudo-linear tensor.

$\mu_i \longleftrightarrow \mu_j$:anti-symmetric $\quad \nu_i \longleftrightarrow \nu_j$:anti-symmetric$(\mu_i,\nu_i) \longleftrightarrow (\mu_j,\nu_j)$:symmetric $\quad \{\mu_i\} \longleftrightarrow \{\nu_i\}$:

D.1.2 Useful relations

The contraction of a pair of indices μ_n and ν_n leads to the following relation:

$$g^{\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}\mu_n}{}_{\mu_n} = (D-n+1)g^{\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}\,. \tag{D.3}$$

The pseudo linear tensor can be expanded in terms of the lower rank tensor:

$$\begin{aligned} g^{\mu_1\nu_1\cdots\mu_n\nu_n} &= \delta^{\nu_1\ \nu_2\cdots\nu_n}_{\lambda_1\ \lambda_2\cdots\lambda_n} g^{\mu_1\lambda_1}\cdots g^{\mu_n\lambda_n} \\ &= \delta^{\nu_1\ \nu_2\cdots\nu_n}_{\lambda_1\ \lambda_2\cdots\lambda_n} \frac{1}{m!(n-m)!} g^{\mu_1\lambda_1\cdots\mu_m\lambda_m} g^{\mu_{m+1}\nu_{m+1}\cdots\mu_n\nu_n} . \end{aligned} \tag{D.4}$$

For example,

$$\begin{aligned} g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} &= g^{\mu_1\nu_1} g^{\mu_2\nu_2\mu_3\nu_3} + g^{\mu_1\nu_2} g^{\mu_2\nu_3\mu_3\nu_1} + g^{\mu_1\nu_3} g^{\mu_2\nu_1\mu_3\nu_2} , \\ g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} &= g^{\mu_1\nu_1} g^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} - g^{\mu_1\nu_2} g^{\mu_2\nu_1\mu_3\nu_3\mu_4\nu_4} - g^{\mu_1\nu_3} g^{\mu_2\nu_2\mu_3\nu_1\mu_4\nu_4} - g^{\mu_1\nu_4} g^{\mu_2\nu_2\mu_3\nu_3\mu_4\nu_1} , \end{aligned} \tag{D.5}$$

(D.3) and (D.4) can be easily proven from (D.1).

We obtain other useful relations using the ADM variables e_{ij}, N, and N_i:

$$g^{0ji_1j_1i_2j_2\cdots i_nj_n} = \frac{1}{n!} \delta^{j\ j_1\ j_2\cdots j_n}_{k\ k_1\ k_2\cdots k_n} \frac{N^k}{N^2} e^{i_1k_1i_2k_2\cdots i_nk_n} , \tag{D.6}$$

$$g^{0j_1i_10i_2j_2i_3j_3\cdots i_nj_n} = \frac{1}{N^2} e^{i_1j_1i_2j_2\cdots i_nj_n} . \tag{D.7}$$

Here $e^{i_1j_1i_2j_2\cdots i_nj_n}$ is anti-symmetrization of the product $e^{i_1j_1}e^{i_2j_2}\cdots e^{i_nj_n}$ with respect to j_i.

$$e^{i_1j_1i_2j_2\cdots i_nj_n} \equiv e^{i_1j_1}e^{i_2j_2}\cdots e^{i_nj_n} - e^{i_1j_2}e^{i_2j_1}\cdots e^{i_nj_n} + \cdots . \tag{D.8}$$

Let us prove the identities (D.7). Just for convenience, we define the following tensors,

$$\begin{aligned} \tilde{g}^{\mu_1\nu_1\cdots\mu_n\nu_n} &\equiv \frac{1}{n!} g^{\mu_1\nu_1\cdots\mu_n\nu_n} = \tilde{\delta}^{\nu_1\cdots\nu_n}_{\lambda_1\cdots\lambda_n} g^{\mu_1\lambda_1}\cdots g^{\mu_n\lambda_n} \\ &= \frac{1}{n!}\left(g^{\mu_1\nu_1}g^{\mu_2\nu_2}g^{\mu_3\nu_3}\cdots g^{\mu_n\nu_n} - g^{\mu_1\nu_2}g^{\mu_2\nu_1}g^{\mu_3\nu_3}\cdots g^{\mu_n\nu_n} + \cdots\right) , \\ \tilde{e}^{i_1j_1\cdots i_nj_n} &\equiv \frac{1}{n!} e^{i_1j_1\cdots i_nj_n} = \tilde{\delta}^{j_1\cdots j_n}_{k_1\cdots k_n} e^{i_1k_1}\cdots e^{i_nk_n} . \end{aligned} \tag{D.9}$$

Therefore, we can easily prove (D.6) as follows,

$$\begin{aligned} \tilde{g}^{0ji_1j_1i_2j_2\cdots i_nj_n} &= \tilde{\delta}^{j\ j_1\cdots j_n}_{\lambda\ \lambda_1\cdots\lambda_n} g^{0\lambda}g^{i_1\lambda_1}\cdots g^{i_n\lambda_n} = \tilde{\delta}^{j\ j_1\cdots j_n}_{k\ k_1\cdots k_n} g^{0k}g^{i_1k_1}\cdots g^{i_nk_n} \\ &= \tilde{\delta}^{j\ j_1\cdots j_n}_{k\ k_1\cdots k_n} \frac{N^k}{N^2}\left(e^{i_1k_1} - \frac{N^{i_1}N^{k_1}}{N^2}\right)\cdots\left(e^{i_nk_n} - \frac{N^{i_n}N^{k_n}}{N^2}\right) \\ &= \tilde{\delta}^{j\ j_1\cdots j_n}_{k\ k_1\cdots k_n} \frac{N^k}{N^2} e^{i_1k_1}\cdots e^{i_nk_n} = \tilde{\delta}^{j\ j_1\cdots j_n}_{k\ k_1\cdots k_n} \frac{N^k}{N^2} \tilde{e}^{i_1k_1\cdots i_nk_n} . \end{aligned} \tag{D.10}$$

Furthermore, we can prove (D.7) by using mathematical induction.

1. $n = 1$ case

$$g^{0ji0} = \frac{N^i}{N^2}\frac{N^j}{N^2} - \frac{1}{N^2}\left(e^{ij} - \frac{N^iN^j}{N^2}\right) = \frac{e^{ij}}{N^2} . \tag{D.11}$$

2. $n = m$ case
   If we assume,

$$g^{0j_1i_10i_2j_2i_3j_3\cdots i_{m-1}j_{m-1}} = \frac{1}{N^2} e^{i_1j_1i_2j_2\cdots i_{m-1}j_{m-1}} , \tag{D.12}$$

then we find

$$
\begin{aligned}
\tilde{g}^{0j_1i_10i_2j_2\cdots i_mj_m} =& \tilde{\delta}^{j_1\ 0\ j_2\cdots j_m}_{\lambda_1\ \lambda\ \lambda_2\cdots\lambda_m} g^{i_m\lambda_m}\tilde{g}^{0\lambda_1 i_1\lambda i_2\lambda_2\cdots i_{m-1}\lambda_{m-1}} \\
=& \frac{1}{m+1}\left[g^{i_mj_m}\tilde{g}^{0j_1i_10i_2j_2\cdots i_{m-1}j_{m-1}} - g^{i_mj_1}\tilde{g}^{0j_mi_10i_2j_2\cdots i_{m-1}j_{m-1}}\right. \\
& - g^{i_m0}\tilde{g}^{0j_1i_1j_mi_2j_2\cdots i_{m-1}j_{m-1}} - g^{i_mj_2}\tilde{g}^{0j_1i_10i_2j_m\cdots i_{m-1}j_{m-1}}\cdots \\
& \left.-g^{i_mj_{m-1}}\tilde{g}^{0j_ni_10i_2j_2\cdots i_{m-1}j_m}\right] \\
=& \frac{1}{m+1}\left[m\tilde{\delta}^{j_1\ j_2\cdots j_m}_{k_1\ k_2\cdots k_m} g^{i_mk_m}\tilde{g}^{0k_1i_10i_2k_2\cdots i_{m-1}k_{m-1}} - g^{i_m0}\tilde{g}^{0j_1i_1j_mi_2j_2\cdots i_{m-1}j_{m-1}}\right] \\
=& \frac{1}{m+1}\left[\tilde{\delta}^{j_1\ j_2\cdots j_m}_{k_1\ k_2\cdots k_m} g^{i_mk_m}\frac{1}{N^2}\tilde{e}^{i_1k_1i_2k_2\cdots i_{m-1}k_{m-1}} - g^{i_m0}\tilde{g}^{0j_1i_1j_ni_2j_2\cdots i_{m-1}j_{m-1}}\right] \\
=& \frac{1}{m+1}\left[\tilde{\delta}^{j_1\ j_2\cdots j_m}_{k_1\ k_2\cdots k_m} e^{i_mk_m}\frac{1}{N^2}\tilde{e}^{i_1k_1i_2k_2\cdots i_{m-1}k_{m-1}}\right] \\
=& \frac{1}{m+1}\frac{1}{N^2}\tilde{e}^{i_1j_1i_2j_2\cdots i_mj_m}\,. \qquad \text{(D.13)}
\end{aligned}
$$

We used the assumption (D.12) in the fourth line and also used equation (D.6) in the fifth line.

So we have proved equations (D.6) and (D.7).

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