ON THE AUSLANDER–REITEN CONJECTURE FOR COHEN–MACAULAY LOCAL RINGS

SHIRO GOTO AND RYO TAKAHASHI

Abstract. This paper studies vanishing of Ext modules over Cohen–Macaulay local rings. The main result of this paper implies that the Auslander–Reiten conjecture holds for maximal Cohen–Macaulay modules of rank one over Cohen–Macaulay normal local rings. It also recovers a theorem of Avramov–Buchweitz–Şega and Hanes–Huneke, which shows that the Tachikawa conjecture holds for Cohen–Macaulay generically Gorenstein local rings.

1. Introduction

Let R be a (commutative) Cohen–Macaulay local ring of (Krull) dimension d. The celebrated Auslander–Reiten conjecture states the following.

Conjecture 1.1 (Auslander–Reiten)**.** Let M be a finitely generated R-module. If $\mathrm{Ext}_R^i(M,M) = \mathrm{Ext}_R^i(M,R) = 0$ for all $i > 0$, then M is free.

The following conjecture is a special case of Conjecture 1.1.

Conjecture 1.2 (Tachikawa♣)**.** Let ω be a canonical module of R. If $\mathrm{Ext}_R^i(\omega,R) = 0$ for all $i > 0$, then R is Gorenstein.

A lot of partial results of these two conjectures are known; we refer the reader to the comprehensive list in [7, Appendix A]. In this paper, among other things, we are interested in the following two theorems [9, 3, 8], which are partial results of Conjectures 1.1 and 1.2, respectively.

Theorem 1.3 (Huneke–Leuschke)**.** *Suppose that R is Gorenstein and normal. Let M be a maximal Cohen–Macaulay R-module. If $\mathrm{Ext}_R^i(M,M) = 0$ for all $1 \le i \le d$, then M is free.*

Theorem 1.4 (Avramov–Buchweitz–Şega, Hanes–Huneke)**.** *Let ω be a canonical module of R. Suppose that R is generically Gorenstein. If $\mathrm{Ext}_R^i(\omega,R) = 0$ for all $1 \le i \le d$, then R is Gorenstein.*

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♣Precisely speaking, Conjecture 1.2 is a variant of a conjecture of Tachikawa, and this version has been presented by Avramov, Buchweitz and Şega [3].

Theorem 1.3 says that Conjecture 1.1 holds if R is Gorenstein and normal, while Theorem 1.4 says that Conjecture 1.2 holds if R is generically Gorenstein.

The main purpose of this paper is to study vanishing of Ext modules determined by a given module M and investigate when M is free. The following theorem is the main result of this paper.

Theorem 1.5. *Let R be a Cohen–Macaulay local ring of dimension $d > 0$. Let M be a maximal Cohen–Macaulay R-module of rank one.*

(1) *If $\mathrm{Ext}_R^i(M,M) = \mathrm{Ext}_R^j(M, \mathrm{Hom}_R(M,M)) = 0$ for all $1 \le i \le d-1$ and $1 \le j \le d$, then M is free.*
(2) *Suppose that R is normal. If $\mathrm{Ext}_R^i(M,M) = \mathrm{Ext}_R^j(M,R) = 0$ for all $1 \le i \le d-1$ and $1 \le j \le d$, then M is free.*

In fact, Theorem 1.5(2) is a direct consequence of Theorem 1.5(1). Notice that Theorem 1.5(1) recovers Theorem 1.4, and that Theorem 1.5(2) implies that Conjecture 1.1 holds if R is normal and M is maximal Cohen–Macaulay of rank one (compare this with Theorem 1.3).

The organization of this paper is as follows. In Section 2, we give several results on vanishing of Ext modules by using spectral sequences, and obtain a partial result of Conjecture 1.1. Theorem 1.5(1) is proved in Section 3 after establishing some preliminary lemmas. Section 4 makes several consequences of Theorem 1.5(1), including Theorem 1.5(2).

We close the section by stating our conventions.

Convention 1.6. Throughout the rest of this paper, let R be a commutative noetherian local ring of Krull dimension $d \ge 0$ with maximal ideal $\mathfrak{m}$ and residue field k. All R-modules are assumed to be finitely generated.

2. Results by spectral sequences

In this section, we explore vanishing of Ext modules by using some spectral sequences obtained by double complexes.

Let M be an R-module. For an integer i we denote by $\mu_i(M)$ the *i-th Bass number* of M, namely, $\mu_i(M) = \dim_k \mathrm{Ext}_R^i(k,M)$. The t-th Bass number of M with $t = \operatorname{depth} M$ is the *(Cohen–Macaulay) type* of M and denoted by $r(M)$. Also, by $\nu(M)$ we denote the *minimal number of generators* of M, that is, $\nu(M) = \dim_k(M \otimes_R k)$.

We begin with showing an equality of Bass numbers and minimal numbers of generators made by vanishing of Ext modules.

Lemma 2.1. *Let M, N be R-modules, and let n be a nonnegative integer. Assume that $\operatorname{depth}_R N \ge n$ and that $\mathrm{Ext}_R^i(M,N) = 0$ for all $1 \le i \le n$. Then $\mu_n(\mathrm{Hom}_R(M,N)) = \mu_n(N)\,\nu(M)$.*

Proof. The isomorphism $\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(M,N)) \cong \mathbf{R}\mathrm{Hom}_R(M \otimes_R^{\mathbf{L}} k, N)$ induces two spectral sequences converging the same cohomology:

$$E_2^{p,q} = \mathrm{Ext}_R^p(k, \mathrm{Ext}_R^q(M,N)) \Rightarrow H^{p+q}, \qquad {}'E_2^{p,q} = \mathrm{Ext}_R^p(\mathrm{Tor}_q^R(M,k), N) \Rightarrow H^{p+q}.$$

We have $H^n \cong E_2^{n,0}$ since $\operatorname{Ext}_R^i(M,N) = 0$ for $1 \le i \le n$, while $H^d \cong {}'E_2^{n,0}$ since $\operatorname{depth}_R N \ge n$. Hence we get an isomorphism $\operatorname{Ext}_R^n(k, \operatorname{Hom}_R(M,N)) \cong \operatorname{Ext}_R^n(M \otimes_R k, N)$, which shows the assertion. ■

Using Lemma 2.1, we readily obtain the following proposition, which gives a partial result of Conjecture 1.1.

Proposition 2.2. *Suppose that R is Cohen–Macaulay and admits a canonical module ω. Let M be a maximal Cohen–Macaulay R-module. Assume that $R \cong \operatorname{Hom}_R(M,M)$ and that $\operatorname{Ext}_R^i(M,M) = 0$ for all $1 \le i \le d$. Then one has*

$$r(R) = r(M)\,\nu(M),$$

and the following hold.

(1) *If $r(R)$ is a prime number, then M is isomorphic to either R or ω.*

(2) *If R is Gorenstein, then $M \cong R$.*

Proof. Lemma 2.1 implies $r(R) = r(M)\,\nu(M)$. The assertion (1) follows from this equality and [5, Propostion 3.3.11]. The assertion (2) is an immediate consequence of (1). ■

To show our next proposition, we need an isomorphism of Ext modules.

Lemma 2.3. *Suppose that R is Cohen–Macaulay and admits a canonical module ω. Set $(-)^\dagger = \operatorname{Hom}_R(-,\omega)$. Let M be an R-module and N a maximal Cohen–Macaulay R-module. Suppose that either M or $N^\dagger$ is locally free on the punctured spectrum of R. Then $\operatorname{Ext}_R^i(M,N) \cong \operatorname{Ext}_R^i(N^\dagger \otimes_R M, \omega)$ for all $0 \le i \le d$.*

Proof. There are isomorphisms

$$\mathbf{R}\operatorname{Hom}_R(M,N) \cong \mathbf{R}\operatorname{Hom}_R(M, \mathbf{R}\operatorname{Hom}_R(N^\dagger,\omega)) \cong \mathbf{R}\operatorname{Hom}_R(N^\dagger \otimes_R^{\mathbf{L}} M, \omega),$$

which induce a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_R^p(\operatorname{Tor}_q^R(N^\dagger, M), \omega) \Rightarrow H^{p+q} = \operatorname{Ext}_R^{p+q}(M,N).$$

By assumption, the R-module $\operatorname{Tor}_q^R(N^\dagger, M)$ has finite length for all $q > 0$. Hence $E_2^{p,q} = 0$ if $p < d$ and $q > 0$. We obtain $H^i \cong E_2^{i,0}$ for all $0 \le i \le d$. ■

The condition on Hom and Ext appearing in Proposition 2.2 is characterized as follows under some mild conditions.

Proposition 2.4. *Let R be Cohen–Macaulay and have a canonical module ω. Set $(-)^\dagger = \operatorname{Hom}_R(-,\omega)$. Let M be a maximal Cohen–Macaulay R-module such that either M or $M^\dagger$ is locally free on the punctured spectrum of R. Then the following are equivalent.*

(1) *$R \cong \operatorname{Hom}_R(M,M)$ and $\operatorname{Ext}_R^i(M,M) = 0$ for all $1 \le i \le d$.*

(2) *$M^\dagger \otimes_R M \cong \omega$.*

When one of these equivalent conditions holds, one has $r(R) = r(M)\,\nu(M)$.

Proof. Lemma 2.3 implies that $\operatorname{Ext}_R^i(M,M) = 0$ for $1 \le i \le d$ if and only if $\operatorname{Ext}_R^i(M^\dagger \otimes_R M, \omega) = 0$ for $1 \le i \le d$, which is equivalent to saying that $M^\dagger \otimes_R M$ is a maximal Cohen–Macaulay R-module by the local duality theorem. Note that $X \cong X^{\dagger\dagger}$ for every maximal Cohen–Macaulay R-module X and that there is an isomorphism $(M^\dagger \otimes_R M)^\dagger \cong$

$\mathrm{Hom}_R(M,M)$. Now it is easy to see that (1) and (2) are equivalent. The equality $r(R) = r(M)\,\nu(M)$ is obtained by comparing the minimal numbers of generators of both sides of the isomorphism $M^\dagger \otimes_R M \cong \omega$ (see [5, Propostion 3.3.11]). ∎

Remark 2.5. One may think that Proposition 2.4(1) implies either $M \cong R$ or $M \cong \omega$. This does not hold in general; a counterexample for $d = 1$ is given in [11, Example 7.3]. Let k be a field, and let $R = k[[t^9, t^{10}, t^{11}, t^{12}, t^{15}]]$, where t is an indeterminate over k. Let Q be the quotient field of R. Then $\omega = R + Rt + Rt^3 + Rt^4 \subseteq Q$ is a canonical module of R. Let $I = R + Rt$, and set $J = I^\dagger := \mathrm{Hom}_R(I, \omega)$. We then have $J \cong R + Rt^3$ and $I^\dagger \otimes_R I \cong \omega$.

3. The main result

This section is devoted to proving the main result of this paper. For this purpose, we begin with establishing a couple of lemmas.

Lemma 3.1. *Let M be an R-module. Let x be an R- and M-regular element of R. Let m, n be positive integers, and suppose that $\mathrm{Ext}^i_R(M,M) = \mathrm{Ext}^j_R(M, \mathrm{Hom}_R(M,M)) = 0$ for all $1 \le i \le m$ and $1 \le j \le n$. Then $\mathrm{Ext}^i_{\overline{R}}(\overline{M}, \overline{M}) = \mathrm{Ext}^j_{\overline{R}}(\overline{M}, \mathrm{Hom}_{\overline{R}}(\overline{M}, \overline{M})) = 0$ for all $1 \le i \le m-1$ and $1 \le j \le n-1$, where $\overline{(-)} = - \otimes_R R/xR$.*

Proof. Set $E = \mathrm{Hom}_R(M,M)$. Fix an integer $t \ge 1$ and an R-module C such that x is a nonzerodivisor on C and $\mathrm{Ext}^i_R(M,C) = 0$ for all $1 \le i \le t$. Applying the functor $\mathrm{Hom}_R(M,-)$ to the exact sequence $0 \to C \xrightarrow{x} C \to \overline{C} \to 0$, we get an exact sequence

$$0 \to \mathrm{Hom}_R(M,C) \xrightarrow{x} \mathrm{Hom}_R(M,C) \to \mathrm{Hom}_R(M,\overline{C}) \to 0,$$

and $\mathrm{Ext}^i_R(M,\overline{C}) = 0$ for $1 \le i \le t-1$. Letting $C = M$ shows that x is a nonzerodivisor on E, and there are natural isomorphisms $\overline{E} \cong \mathrm{Hom}_R(M,\overline{M}) \cong \mathrm{Hom}_{\overline{R}}(\overline{M},\overline{M})$. Note that for a free R-resolution F of M, the complex F/xF is a free $\overline{R}$-resolution of $\overline{M}$. Hence $\mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{C}) = \mathrm{Ext}^i_R(M,\overline{C}) = 0$ for $1 \le i \le t-1$. Letting $C = M$ (resp. $C = E$), we have $\mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{M}) = 0$ (resp. $\mathrm{Ext}^i_{\overline{R}}(\overline{M},\overline{E}) = 0$) for all $1 \le i \le m-1$ (resp. $1 \le j \le n-1$). Thus the assertion of the lemma follows. ∎

Lemma 3.2. *Let R be a 1-dimensional Cohen–Macaulay local ring. Let M be a maximal Cohen–Macaulay R-module such that $\mathrm{Ext}^1_R(M, \mathrm{Hom}_R(M,M)) = 0$. Suppose that there is an isomorphism $\mathrm{Hom}_R(M,M) \cong M$ of R-modules. Then M is cyclic.*

Proof. We may assume $M \ne 0$. Let x be a system of parameters of M, and put $\overline{(-)} = - \otimes_R R/xR$. As x is M-regular, there is an exact sequence $0 \to M \xrightarrow{x} M \to \overline{M} \to 0$. Note that $\mathrm{Ext}^1_R(M,M) = \mathrm{Ext}^1_R(M, \mathrm{Hom}_R(M,M)) = 0$. Applying the functor $\mathrm{Hom}_R(M,-)$ to the exact sequence, we obtain isomorphisms

$$\overline{M} \cong \overline{\mathrm{Hom}_R(M,M)} \cong \mathrm{Hom}_R(M,\overline{M}) \cong \mathrm{Hom}_{\overline{R}}(\overline{M},\overline{M}).$$

Applying the functor $\mathrm{Hom}_{\overline{R}}(k,-)$, we have

$$\mathrm{Hom}_{\overline{R}}(k,\overline{M}) \cong \mathrm{Hom}_{\overline{R}}(k, \mathrm{Hom}_{\overline{R}}(\overline{M},\overline{M})) \cong \mathrm{Hom}_{\overline{R}}(\overline{M} \otimes_{\overline{R}} k, \overline{M}) \cong \mathrm{Hom}_{\overline{R}}(k,\overline{M})^{\oplus n},$$

where $n = \nu_R(M) = \nu_{\overline{R}}(\overline{M})$. The dimension of the k-vector space $\mathrm{Hom}_{\overline{R}}(k,\overline{M})$ is equal to $r_R(M)$, which is a positive integer. Hence we must have $n = 1$, which means that M is a cyclic R-module. ∎

Now we can prove our main result, which is nothing but Theorem 1.5(1).

Theorem 3.3. *Let R be a Cohen–Macaulay local ring of dimension $d > 0$. Let M be a maximal Cohen–Macaulay R-module of rank one. If $\mathrm{Ext}^i_R(M,M) = \mathrm{Ext}^j_R(M, \mathrm{Hom}_R(M,M)) = 0$ for all $1 \le i \le d-1$ and $1 \le j \le d$, then M is free.*

Proof. Let $\rho : R \to A$ be a flat local homomorphism of local rings. It is verified that all the conditions in the theorem are preserved under ρ. Also, a finitely generated R-module X is free if (and only if) so is the A-module $X \otimes_R A$ (see [2, Theorem 8.7(6)]). So we may assume that R is complete and k is infinite.

As M is a torsionfree R-module of rank one, it is isomorphic to some proper ideal I of R. Note then that R/I is a Cohen–Macaulay local ring of dimension $d-1$. Choose a sequence $\boldsymbol{x} = x_1, \dots, x_{d-1}$ of elements of R that is regular on R and R/I. As I is a maximal Cohen–Macaulay R-module, $\boldsymbol{x}$ is regular on I. Repeated application of Lemma 3.1 yields

$$\mathrm{Ext}^1_{R/\boldsymbol{x}R}(I/\boldsymbol{x}I, \mathrm{Hom}_{R/\boldsymbol{x}R}(I/\boldsymbol{x}I, I/\boldsymbol{x}I)) = 0.$$

An induced sequence $0 \to I/\boldsymbol{x}I \to R/\boldsymbol{x}R \to R/I + \boldsymbol{x}R \to 0$ is exact, and $I/\boldsymbol{x}I$ is a maximal Cohen–Macaulay $R/\boldsymbol{x}R$-module. Since the R-module $R/I + \boldsymbol{x}R$ has finite length, $I/\boldsymbol{x}I$ has rank one as an $R/\boldsymbol{x}R$-module. If $I/\boldsymbol{x}I$ is a free $R/\boldsymbol{x}R$-module, then I is a free R-module by [5, Lemma 1.3.5]. Thus we may assume $d = 1$.

Since M has rank one and M is isomorphic to I, we observe that I is an $\mathfrak{m}$-primary ideal of R. There exists an element $r \in I$ such that rR is a reduction of I. Note that r is an R-regular element. Let Q be the total quotient ring of R, and set

$$L = \frac{I}{r} = \left\{ \frac{y}{r} \in Q \,\middle|\, y \in I \right\}.$$

We then have isomorphisms $L \cong I \cong M$ of R-modules and inclusions $R \subseteq L \subseteq \overline{R} \subseteq Q$, where $\overline{R}$ stands for the integral closure of R (in Q). Since $\mathfrak{m}\overline{R} \cap R = \mathfrak{m}$, we see that $1 \in L \setminus \mathfrak{m}L$. Putting $S = (L :_Q L)$, we have $R \subseteq S \subseteq L$. Setting $C = L/S$, we get an exact sequence

$$0 \to S \xrightarrow{\theta} L \xrightarrow{\pi} C \to 0$$

of R-modules. There is an isomorphism $L/R \cong I/rR$, which shows that L/R has finite length as an R-module. The natural surjection $L/R \to L/S = C$ implies that C also has finite length.

Suppose that C is nonzero. Then one can choose a nonzero socle element z of the R-module C. Since 1 is part of a minimal system of generators of L, there is an R-homomorphism $\phi : L \to C$ such that $\phi(1) = z$. The assignment $s \mapsto (y \mapsto sy)$ makes an isomorphism $S = (L :_Q L) \to \mathrm{Hom}_R(L, L)$. As $L \cong M$, we have $\mathrm{Ext}^1_R(L, S) = \mathrm{Ext}^1_R(M, \mathrm{Hom}_R(M,M)) = 0$. Hence the induced map

$$\mathrm{Hom}_R(L, \pi) : \mathrm{Hom}_R(L, L) \to \mathrm{Hom}_R(L, C)$$

is surjective, and we find an element $\psi \in \mathrm{Hom}_R(L, L)$ such that $\phi = \pi\psi$. Let s be an element of S such that $\psi(y) = sy$ for all $y \in L$. Then

$$z = \phi(1) = \pi\psi(1) = \pi(s) = \pi\theta(s) = 0,$$

which is a contradiction. Consequently, we obtain $C = 0$.

Therefore $S = L \cong I$, and there is an R-isomorphism $I \cong \mathrm{Hom}_R(I, I)$. It follows from Lemma 3.2 that I is a cyclic R-module. Since I is an $\mathfrak{m}$-primary ideal of R, it is a parameter ideal, whence $I \cong R$. Thus M is a free R-module. ■

4. Consequences of the main theorem

In this section, we state several consequences of our main Theorem 3.3. First of all, the proof of Theorem 3.3 given in the previous section actually shows the following statement.

Corollary 4.1. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 1$. Let I be an ideal of R of positive height which is maximal Cohen–Macaulay as an R-module. Suppose that $\mathrm{Ext}_R^i(I, I) = \mathrm{Ext}_R^j(I, \mathrm{Hom}_R(I, I)) = 0$ for all $1 \leq i \leq d-1$ and $1 \leq j \leq d$. Then $I \cong R$.*

Let I be an ideal of R containing a nonzerodivisor. Then $\mathrm{Hom}_R(I, I)$ is identified with a module-finite extension of R in the total quotient ring of R, and I is called *closed* if $R = \mathrm{Hom}_R(I, I)$. There are a lot of examples of closed ideals; see [4, third paragraph of §1 and Example 4.3]. We have an immediate consequence of Corollary 4.1.

Corollary 4.2. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 1$. Let I be a closed ideal of R which is maximal Cohen–Macaulay as an R-module. Suppose that $\mathrm{Ext}_R^i(I, I) = \mathrm{Ext}_R^j(I, R) = 0$ for all $1 \leq i \leq d-1$ and $1 \leq j \leq d$. Then $I \cong R$.*

With the notation of the proof of Theorem 3.3, suppose that R is normal. Then we have $R = L = \overline{R}$ and $R \cong \mathrm{Hom}_R(M, M)$. Thus the following holds.

Corollary 4.3. *Let R be a Cohen–Macaulay normal local ring of dimension $d > 0$. Let M be a maximal Cohen–Macaulay R-module of rank one. If $\mathrm{Ext}_R^i(M, M) = \mathrm{Ext}_R^j(M, R) = 0$ for all $1 \leq i \leq d-1$ and $1 \leq j \leq d$, then M is free.*

Corollary 4.3 is nothing but the second assertion of Theorem 1.5, and implies Conjecture 1.1 of Auslander–Reiten in the case where R is normal and M is maximal Cohen–Macaulay of rank one; we should compare this fact with Theorem 1.3 due to Huneke–Leuschke [9]. Furthermore, it is worth mentioning that Corollary 4.3 also yields the following result. Here, an R-module M is called a *rigid Cohen–Macaulay module* if M is maximal Cohen–Macaulay and $\mathrm{Ext}_R^1(M, M) = 0$.

Corollary 4.4. *Let R be a Gorenstein normal local ring of dimension two. Then every rigid Cohen–Macaulay R-module of rank one is free.*

Recall that a finitely generated R-module is called *semidualizing* if the natural homomorphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^i(C, C) = 0$ for all $i > 0$. Theorem 3.3 yields the following result on semidualizing modules, which recovers Theorem 1.4 due to Avramov–Buchweitz–Şega [3] and Hanes–Huneke [8].

Corollary 4.5. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 1$. Let C be a semidualizing R-module.*

(1) *If R is generically Gorenstein, then C has rank one.*

(2) *Suppose that C has a rank. If $\operatorname{Ext}^i_R(C,R)=0$ for all $1 \le i \le d$, then $C \cong R$.*

Proof. (1) Let $\mathfrak{p}$ be an associated prime ideal of R. Then $\mathfrak{p}$ is minimal, and $C_\mathfrak{p}$ is a semidualizing module over the Gorenstein local ring $R_\mathfrak{p}$. Hence $C_\mathfrak{p} \cong R_\mathfrak{p}$ by [6, Corollary (8.6)]. Thus C has rank 1.

(2) The R-module C is maximal Cohen–Macaulay by [10, Proposition 1.1]. The isomorphism $R \cong \operatorname{Hom}_R(C,C)$ implies that C has rank one. The assertion now follows from Theorem 3.3. ■

The one-dimensional case of Theorem 3.3 is worth stating independently because it does not contain the vanishing condition on the modules $\operatorname{Ext}^i_R(M,M)$.

Corollary 4.6. *Let R be a Cohen–Macaulay local ring of dimension one. Let M be a maximal Cohen–Macaulay R-module of rank one. If $\operatorname{Ext}^1_R(M,\operatorname{Hom}_R(M,M))=0$, then $M \cong R$.*

Using this result, we have the following. Recall that an R-module M is said to satisfy *Serre's condition* (S_n) if $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \ge \min\{n, \operatorname{ht}\mathfrak{p}\}$ for all prime ideals $\mathfrak{p}$ of R.

Corollary 4.7. *Let R be a local ring of dimension $d \ge 0$, and let M be an R-module of rank one. Suppose that R, M satisfy Serre's condition (S_2). If $\operatorname{Ext}^1_R(M,\operatorname{Hom}_R(M,M))=0$, then $R \cong \operatorname{Hom}_R(M,M)$.*

Proof. Set $S=\operatorname{Hom}_R(M,M)$ and let $\phi : R \to S$ be the natural map. We see that M is faithful, whence ϕ is injective. By virtue of Corollary 4.6, ϕ is an isomorphism when $d=1$. Let $d>1$, and assume that our assertion holds true for every local ring of smaller dimension satisfying (S_2). Then for each nonmaximal prime ideal $\mathfrak{p}$ of R the induced map $\phi_\mathfrak{p} : R_\mathfrak{p} \to S_\mathfrak{p} = \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p},M_\mathfrak{p})$ is an isomorphism. Thus there is an exact sequence

$$0 \to R \xrightarrow{\phi} S \to X \to 0$$

of R-modules with X of finite length. As the R-module S has depth at least 2, we get $d = \operatorname{depth} R = 1$ by the depth lemma. This contradicts our assumption, which completes the proof. ■

Finally, recall that our Theorem 3.3 contains the assumptions below.

(1) $\operatorname{Ext}^i_R(M,M)=0$ for $1 \le i \le d-1$,
(2) $\operatorname{Ext}^j_R(M,\operatorname{Hom}_R(M,M))=0$ for $1 \le j \le d$.

One may wonder if "$d-1$" in (1) and "d" in (2) can be replaced with "$d-2$" and "$d-1$", respectively. The following example says that it is impossible even if R is such a good ring as an isolated simple hypersurface singularity.

Example 4.8. Let k be a field.

(1) Consider the ring $R=k[[x,y,z]]/(x^2-yz)$ and the ideal $I=(x,y)$ of R. Then R is a 2-dimensional hypersurface normal local domain, and I is a maximal Cohen–Macaulay R-module of rank one. (Note that R is a simple surface singularity of type (A_1) when

$k = \mathbb{C}$.) The natural homomorphism $R \to \mathrm{Hom}_R(I, I)$ is an isomorphism as R is normal, and we have $\mathrm{Ext}^1_R(I, \mathrm{Hom}_R(I, I)) = \mathrm{Ext}^1_R(I, R) = 0$ since R is Gorenstein. However, I is not a free R-module. (This example can be extended to the d-dimensional case for any $d \geq 2$ by taking the d-th Veronese subring of $k[[x_1, \dots, x_d]]$.)

(2) Let $R = k[[x, y]]/(x^2 - y^3)$ and $\mathfrak{m} = (x, y)$. Then R is a 1-dimensional hypersurface local domain, and $\mathfrak{m}$ is a maximal Cohen–Macaulay R-module of rank one. (Note that R is a simple curve singularity of type (A_2) when $k = \mathbb{C}$.) The R-module $\mathfrak{m}$ is not free.

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Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashi-mita, Tama-ku, Kawasaki 214-8571, Japan

E-mail address: goto@math.meiji.ac.jp

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Aichi 464-8602, Japan

E-mail address: takahashi@math.nagoya-u.ac.jp

URL: http://www.math.nagoya-u.ac.jp/~takahashi/